# PROBABILITY THEORY AND THE FIRST BOUNDARY VALUE PROBLEM

Dedicated to Paul Lévy
on the occasion of his seventieth birthday

BY

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In [4] a rather general approach to the first boundary value problem for a class of functions called regular functions was presented, and the application of probability theory to the solution was indicated. In the present paper, this work is carried further, in several directions.

The place of the relativized problem, introduced by Brelot [1] into the study of harmonic functions on a Green space (see also related work in a different context by Feller [5]) is discussed. Boundary limit properties of extremal and minimal regular functions are obtained. Finally, a new characterization in probability terms of upper and lower first boundary value problem solutions is obtained, which makes possible a rather elegant characterization of the resolutive functions. This characterization implies that, in a large class of applications, including for example the case when the regular functions are the solutions of the heat equation, if the domain of the functions has a compact closure in the defining space, every continuous boundary function is resolutive.

## 1. Review of [4]

The basis for the theory of regular functions in [4], comprised in hypotheses TM1-4 and RS1-4 of that reference, can be summarized as follows. A locally compact separable Hausdorff space $R$ is given, together with a specified class of open subsets of $R$, called regular sets. The regular sets have compact closures and form a basis for the topology of $R$. If $D$ is a regular set, with boundary $D'$, and if $\xi \in D$, there is a certain probability measure $\mu(\xi, D, \cdot)$, defined on the Borel subsets of $D'$. A function $u$ on $R$ is said to be regular if it is continuous and if it is equal at each point $\xi$ of each regular set $D$ to its average over $D'$ with respect to the measure $\mu(\xi, D, \cdot)$. Corresponding definitions of subregular and superregular functions are made.

An additional hypothesis will be used, but only when mentioned explicitly, in discussing the first boundary value problem for regular functions on an open subset $D$ of $R$. It is a form of the maximum principle, which we denote by $M(D, D')$, and which states that, if $u$ is subregular on $D$ and bounded from above, its supremum on $D$ is a limiting value of $u$ at some point of the boundary $D'$ of $D$.

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If $M(D, D')$ is satisfied, the standard PWB (Perron-Wiener-Brelot) method can be applied to the first boundary value problem for regular functions on $D$. If a boundary function $f$ yields a unique finite solution, $f$ is called PWB resolutive. If every bounded continuous function on $D'$ is PWB resolutive, $D$ is called strongly PWB resolutive. If $D'$ is compact and if $D$ is strongly PWB resolutive, the value of the solution at $\xi$ corresponding to a continuous boundary function $f$ is a bounded linear functional of $f$, and is therefore the integral of $f$ with respect to a measure $\mu(\xi, D, \cdot)$ of Borel subsets of $D'$. The same relation then holds between any PWB resolutive boundary function and corresponding solution.

Throughout this paper, we shall suppose, in the language of [4], that $R$ is strongly PWB resolutive from below, that is, that there is a sequence $\{R_n, n \geqq 1\}$ of open subsets of $R$ (specified and unchanged throughout the discussion) with the following properties: the closure of $R_n$ is a compact subset of $R_{n+1}$, $\bigcup_1^\infty R_n = R$; hypothesis $M(R_n, R_n')$ is satisfied, and $R_n$ is strongly PWB resolutive.

We shall, but never unless explicitly mentioned, find it convenient sometimes to introduce a boundary $R'$ of $R$. The space $R \cup R'$ will always be separable and compact. The hypothesis of compactness, together with strong PWB resolutivity from below of $R$, means that $M(R, R')$ is satisfied, so that the first boundary value problem for regular functions on $R$ can be treated by the PWB method.

In [4], corresponding to each point $\xi$ of $R$ a stochastic process $\{z_n, n \geqq 0\}$ was defined. (Here and below the point $\xi$ is omitted from the notation.) For this process, $z_0 = \cdots = z_N = \xi$, where $N + 1$ is the first value of $n$ with $\xi \in R_n$, and, for $n > N$, $z_n$ is a random variable distributed on $R_n$. The process is a Markov process, with transition probability from $R_n'$ to $R_{n+1}'$ given by the measure $\mu(\xi, R_{n+1}, \cdot)$, for $n > N$. If a boundary $R'$ is introduced, almost every $z_n$ process path has a limit point on $R'$, so that, in the language of [4], hypothesis $M'(R, R')$ is satisfied. The set of limit points (necessarily on $R'$) of such a path will be called the limit set of the path on $R'$.

## 2. $h$-regular functions

Let $u$ be a positive (by which we always mean "nonnegative" in this paper) lower semicontinuous function defined on $R$. The open set on which $u$ is strictly positive will be called the open support of $u$.

If $u$ is a positive superregular function on $R$, the zeroes of $u$ form a closed set, containing together with any point the set of all points covered by the point relative to $R$ and any neighborhood system (see [4]). If $\xi$ is a zero of $u$, and if $\xi \in D$, where $D$ is any regular set or set $R_n$, the set of zeroes of $u$ will contain a subset of $D'$ of $\mu(\xi, D, \cdot)$ measure 1.

Throughout the rest of this paper, $h$ is a positive, nonidentically vanishing function, on $R$, which is regular and has open support $H$. If $\xi \in R_n H$, then

$H$ includes a subset of $R_n'$ of positive $\mu(\xi, R_n, \cdot)$ measure. In the following, we adopt the usual convention that $0 \cdot \pm \infty = 0$.

Let $u$ be an extended real-valued function on $R$. Then we shall call $u$ an $h$-regular function if $uh$ is regular on $R$. Every function on $R$ will be said to be 0-regular. If $u$ is $h$-regular on $R$, if $D$ is an open set with compact closure, if $M(D, D')$ is satisfied, if $D$ is strongly PWB resolutive, if $\xi \in DH$, and if $\mu^h$ is defined by

$$\mu^h(\xi, D, A) = \int_A h(\eta)\mu(\xi, D, d\eta)/h(\xi), \tag{2.1}$$

then

$$u(\xi) = \int_{D'} u(\eta)\mu^h(\xi, D, d\eta). \tag{2.2}$$

Thus, if $h$ never vanishes, treating $h$-regular functions instead of regular functions amounts to replacing $\mu$ by $\mu^h$. In fact the fundamental hypotheses TM1-4 and RS1-4 imposed on regular sets and the measures $\mu$ remain valid when $\mu$ is replaced by $\mu^h$, and we have therefore obtained no increase in generality in replacing $\mu = \mu^1$ by $\mu^h$. However there is a slight increase in generality when $h$ has zeroes in $R$, even though we have done very little more in this case than to replace $R$ by $H$.

If $u$ is an extended real-valued function, such that $uh$ is superregular on $H$, that the integral in (2.2) is well-defined (absolutely convergent), and that (2.2) with "$=$" replaced by "$\geqq$" is true with the above-stated restrictions on $D$ and $\xi$, then $u$ will be said to be $h$-superregular. If $uh$ is superregular, then $u$ is $h$-superregular, and conversely if $u$ is positive on $H$. The negative of an $h$-superregular function will be called $h$-subregular. If $u$ is $h$-regular, it is both $h$-superregular and $h$-subregular, and conversely.

Only the values of $u$ on $H$ are relevant to $h$-regularity, $h$-superregularity, and $h$-subregularity of $u$.

Let $u_1$ be positive and $h$-superregular on $R$, and let $u_2$ be positive and $h$-regular on $R$. We adopt the convention throughout this paper that $u_1/u_2$ is defined arbitrarily at the zeroes of $u_2$ on $R - H$, is defined as $+\infty$ at a zero of $u_2$ on $H$ which is not a zero of $u_1$, and is defined as 0 at a common zero of $u_1$ and $u_2$ on $H$. With this convention, $u_1/u_2$ is $(hu_2)$-superregular on $R$. If $u_1$ is positive and regular, this conclusion cannot always be strengthened. However if we suppose also that $u_1$ vanishes whenever $u_2$ does on $H$, $u_1/u_2$ will then be $(u_2h)$-regular. As a particular case, we remark that $1/h$ is $h$-superregular, and is even $h$-regular if $h$ never vanishes.

It will be convenient to qualify the notation systematically by the superscript $h$ when $\mu$ is replaced by $\mu^h$, but no superscript will be used when $h = 1$. The family of measures $\{\mu^h(\xi, R_n, \cdot), n \geqq 1, \xi \in H\}$ defines a stochastic process $\{z_n^h, n \geqq 0\}$ just as in the case $h = 1$, except that the initial point must be in $H$. The sample paths will be called $h$-paths from the initial point. Almost no $h$-path from a point ever leaves $H$.

Let $D$ be an open subset of $R$, with compact closure. Suppose that $M(D, D')$ is satisfied, and that $D$ is strongly PWB resolutive. Then, if $h$ never vanishes on $D \cup D'$, $M^h(D, D')$ is satisfied, and the PWB method can be applied to the first boundary value problem for $h$-regular functions on $D$. If $u$ is the regular function on $D$ which is the PWB solution for the boundary function $fh$ on $D'$, then $u/h$ is the PWB solution corresponding to the boundary function $f$. Thus $D$ is strongly PWB$^h$ resolutive. It follows that $R$ is strongly PWB$^h$ resolutive from below, if $h$ never vanishes.

If $R$ is provided with a boundary $R'$, in such a way that $R \cup R'$ is separable and compact, then (for arbitrary $h$) $M^h(R, R')$ is satisfied, in the sense that, if $u$ is subregular and bounded from above, and if $u \leqq c$ in some neighborhood of each point of $R'$ that is a limit point of $H$, then $u \leqq c$ on $H$. To see this, we shall assume, as is no restriction, that $c = 0$. Then $uh \leqq 0$ in some neighborhood of each point of $R'$. Hence $uh \leqq 0$ on $R_n'$, if $n$ is sufficiently large, so that $u \leqq 0$ on $R_n' H$ for large $n$. Hence $u \leqq 0$ on $H$, using the definition of $h$-subregularity.

If $u$ is positive and superregular on $R$, it has a limit along almost every 1-path from each point $\xi$ of $R$, according to [4]. Moreover, if $u_1(\xi)$ is the expected value of the limit, $u_1$ is a regular function, $u_1 \leqq u$, and $u_1$ has the same limit as $u$ along almost every 1-path from each point of $R$. If $h$ is strictly positive, the above remains true if regularity is replaced by $h$-regularity and 1-paths by $h$-paths. This is not really a generalization, since 1-paths for $\mu^h$ are $h$-paths for $\mu$. If $h$ may vanish, the result remains true with the obvious restriction that only points of $H$ are used as initial points of $h$-paths. The functions involved are defined arbitrarily on $R - H$. The proof needs no change. Similarly, in the case $h = 1$, and therefore for every $h$, it is known from [4] that the following assertions are true. Let $\mathbf{D}^h$ be the class of functions $u$, $h$-regular on $R$, for which $u$, considered successively on $R_1'$, $R_2'$, $\cdots$ defines, in relation to the measures $\mu^h(\xi, R_1, \cdot)$, $\mu^h(\xi, R_2, \cdot)$, $\cdots$ a uniformly integrable sequence of functions, for each point $\xi$ of $H$. Then each member of $\mathbf{D}^h$ has a limit on almost every $h$-path from each point of $H$, and the value of $u$ at the initial point is the expected value of this limit. The function $u_1$ above is in the class $\mathbf{D}^h$. Note that only the values of $u$ on $H$ are involved in the defining condition of $\mathbf{D}^h$. Conversely, according to [4], as generalized in replacing 1 by $h$, it is possible to prescribe the limits along $h$ paths and find a unique member of $\mathbf{D}^h$ (neglecting values on $R - H$) with these limits. This is really how $u_1$ above was obtained. Prescribing the limits along $h$-paths from $\xi$ means assigning a random variable $x_\xi$ as the limit of $u$ along almost all $h$-paths from each point $\xi$ of $H$. If these random variables satisfy certain necessary consistency conditions, the class $\{x_\xi, \xi \in H\}$ is called a stochastic boundary function, and, if $\mathbf{E}\{|x_\xi|\} < \infty$ for every point $\xi$ of $H$, the function $u$ defined by $u(\xi) = \mathbf{E}\{x_\xi\}$ for $\xi \in H$, and arbitrarily on $R - H$, is in the class $\mathbf{D}^h$. This function $u$ is called the stochastically ramified Dirichlet solution determined by the stochastic boundary function. It has the limit $x_\xi$ along almost every one of the given $h$-paths from $\xi$ in $H$.

Roughly speaking, $h$-paths move in $R$ to regions where $h$ is not small. For example, since $1/h$ is a positive $h$-superregular function, it has a finite limit along almost every $h$-path from a point of $H$. This means that $h$ must have a *strictly* positive (or infinite) limit on almost every such $h$-path.

To clarify this discussion, we make the following comments, without going into detail. If $u$ is positive and $h$-superregular, the integral

$$\int_{R_n'} u(\eta)h(\eta)\mu(\xi, R_n, d\eta)$$

defines a function $v_n$ on $R_n$, which vanishes on $R_n - R_n H$, because if $\xi$ is a zero of $h$ in $R_n$, $h$ vanishes $\mu(\xi, R_n, \cdot)$ almost everywhere on $R_n'$. Moreover, on $R_n$, $v_n$ is regular, $v_n \leqq uh$, and $v_{n+1} \leqq v_n$. Let $v$ be the limit of the monotone sequence of functions obtained in this way. Then $v$ is regular on $R$, $v$ vanishes on $R - H$, and $v \leqq uh$. The function $u_1 = v/h$ is $h$-regular and vanishes on $R - H$, according to our conventions. This function is the member of the class $\mathbf{D}^h$ (stochastically ramified Dirichlet solution) with the same limit as $u$ on almost every $h$-path from each point of $H$.

The following theorem illustrates the tendency of $h$-paths to go where $h$ is not small. We shall say that $\Gamma$ is a condition on the final character of paths if $\Gamma$ is a set of points in the space of sequences $(\xi_1, \xi_2, \cdots)$, $\xi_n \in R_n'$, of the following character. For each positive integer $n$, the set $\Gamma$ is in the Borel field of sets determined by restrictions of the form $\xi_k \in A$, where $A$ is a Borel subset of $R_k'$, and $k \geqq n$. For example, if $f$ is a Baire function on $R$, the condition on paths that $f$ converge on them is such a condition. That is, in more formal language, the set $\Gamma$ defined by the condition that $\lim_{n\to\infty} f(\xi_n)$ exist is a set of the stated type. The probability that an $h$-path from $\xi$ satisfies a condition $\Gamma$, that is, that the sample path is a point of $\Gamma$, is well-defined, if $\xi \in H$, and defines a function of $\xi$ which is $h$-regular.

THEOREM 2.1. *If, for some point $\xi \in H$, almost no h-path satisfies a specified condition $\Gamma$ on the final character of paths, then, if $h_1$ is positive and regular on $R$, $h/h_1$ has the limit 0 on almost every $h_1$-path from $\xi$ satisfying condition $\Gamma$.*

Since the theorem is trivially true if $h_1(\xi) = 0$, we suppose in the proof that $h_1(\xi) > 0$. We shall prove this theorem as an application of a simple inequality between $h$-path measures and $h_1$-path measures. Let $\Gamma_n$ be a set in $(\xi_1, \xi_2, \cdots)$ space determined by a condition of the form $(\xi_n, \cdots, \xi_{n+m}) \in A_m$, where $A_m$ is a Borel set of the product space $R_n' \times \cdots \times R_{n+m}'$. Then, if $p^h(\xi)$ is the probability that a sample $h$-path from $\xi$ lies in $\Gamma_n$, where we suppose that $\xi \in R_n$, we see that

$$\tag{2.3} \begin{aligned} p^h(\xi) &= \int \underset{A_m}{\cdots} \int \mu^h(\xi, R_n, d\eta_0) \cdots \mu^h(\eta_{m-1}, R_{n+m}, d\eta_m) \\ &\geqq \int \underset{A_m'}{\cdots} \int \mu^{h_1}(\xi, R_n, d\eta_0) \cdots \mu^{h_1}(\eta_{m-1}, R_{n+m}, d\eta_m) \frac{h_1(\xi)h(\eta_m)}{h(\xi)h_1(\eta_m)}. \end{aligned}$$

Here $A'_m$ is the subset of $A_m$ for which $\prod_{k=0}^{m} h(\xi_{n+k})h_1(\xi_{n+k}) > 0$. In probability notation, this inequality can be written in the form

$$p^h(\xi) \geqq \mathbf{E}\left\{\left(\frac{h}{h_1}\right)(z_{n+m}^{h_1})\phi\right\}\frac{h_1(\xi)}{h(\xi)}, \tag{2.4}$$

where $\phi$ has the value 1 when a sample seqence of $z_n^h, \cdots, z_{n+m}^h$ lies in $A'_m$ and $\phi$ has the value 0 otherwise. Now $h/h_1$ is positive and $h_1$-superregular, and therefore has a limit $x$ along almost every $h_1$-path from $\xi$, so that

$$p^h(\xi) \geqq \frac{\mathbf{E}\{x\phi\}h_1(\xi)}{h(\xi)}. \tag{2.5}$$

More generally, it follows from this special inequality that the inequality is even true if $\Gamma_n$ is any set in $(\xi_1, \xi_2, \cdots)$ space, in the Borel field determined by restrictions of the form $\xi_k \in A$, where $A$ is a Borel subset of $R'_k$, and $k \geqq n$. In particular, if $p^h(\xi) = 0$, the truth of the theorem becomes obvious.

As an application, take $h = 1$, and suppose that $u$ is a regular function, $0 \leqq u \leqq 1$, with limits 0 or 1 on almost every 1-path from each point of $R$. Then, according to the theorem, $1/h_1$ has the limit 0 along almost every $h_1$-path, from a point $\xi$ of $R$, on which $u$ does not have the limit 0 or 1. In particular, if $h_1 = u$, and if $u(\xi) > 0$, this means that $u$ has limit either 0 or 1 along almost every $u$-path from $\xi$. Since, as we have already proved, the first limit is possible only with probability 0, $u$ has the limit 1 along almost every $u$-path from $\xi$. More generally, the same argument shows that, if $\xi \in R$, and if $S$ is a Borel set of numbers with the property that the bounded positive regular function $u$ has a limit in $S$ on almost all 1-paths from $\xi$, then $u$ will have a limit in $S$ (less the origin if the origin is in $S$) along almost every $u$-path from $\xi$.

## 3. $h$-minimal functions

We shall call a function $u$ on $R$ $h$-minimal if it is positive and $h$-regular, and if any other $h$-regular function $u_1$ on $R$, satisfying the inequality $0 \leqq u_1 \leqq u$ on $H$, is a constant multiple of $u$ on $H$. All functions on $R$ will be considered 0-minimal. If $u$ vanishes identically, it is trivially $h$-minimal. If $h = 1$, an $h$-minimal function will be called minimal. If $u$ is $h$-minimal, if $h_1$ is positive and regular, and if $uh$ vanishes whenever $h_1$ vanishes, then $uh/h_1$ is $h_1$-minimal.

If $u$ is $h$-minimal, and if $H$ is not connected, then $u$ either vanishes identically on $H$ or vanishes identically on all but one connected open component of $H$. For this reason $R$ has usually been supposed connected in discussing $h$-minimal functions in the classical special cases.

THEOREM 3.1. *An $h$-regular function $u$ is $h$-minimal if and only if the members of the class $\mathbf{D}^{uh}$ are all constant on the open support of $uh$.*

Let $U$ be the open support of $u$. If $0 \leqq u_1 \leqq u$ on $H$, and if $u_1$ is $h$-regular, then $u_1/u$ is $(uh)$-regular, and is bounded on $UH$. Hence $u_1/u$ is in the class

$\mathbf{D}^{uh}$. It follows that, if all members of this class are constant functions on $UH$, $u_1 =$ const. $u$ on $UH$. Since this equality is trivially true on $H - UH$, we find that $u_1 =$ const. $u$ on $H$, so that $u$ must be $h$-minimal. Conversely, if $u$ is $h$-minimal, and if $v$ is a $(uh)$-regular function, bounded on $UH$, then, if $c$ is a strictly positive bound of $|v|$ on $UH$, the function $(v + c)u/(2c)$ is $h$-regular and $\leqq u$ on $H$. It follows that $(v + c)u$ is a constant multiple of $u$ on $H$, so that $v$ is a constant function on $UH$. More generally, if $v$ is any member of $\mathbf{D}^{uh}$, it is the limit of a sequence of $(uh)$-regular functions, each of which is bounded on $UH$ (obtained by modifying $v$ on $UH$ so that $v$ becomes the member of $\mathbf{D}^{uh}$ whose limits on $(uh)$-paths are those of $v$ changed to 0 when at least $n$ in modulus). Hence $v$ is a constant function on $UH$, as was to be proved.

THEOREM 3.2. *If $u$ is $h$-minimal, one of the following two assertions is true.*

(a) *$u$ is not in the class $\mathbf{D}^h$ (a fortiori is is unbounded on $H$) and has the limit* 0 *on almost every $h$-path from each point of $H$.*

(b) *There is a strictly positive number $a$ such that $0 \leqq u \leqq a$ on $H$, that $u$ has one of the limits* 0, *$a$ on almost every $h$-path from each point $\xi$ of $H$, and that $u(\xi)/a$ is the probability that the limit is $a$.*

Since $u$ is positive and $h$-regular, it has a limit along almost every $h$-path from each point $\xi$ of $H$. Moreover, if $u_1(\xi)$ is the expected value of this limit, $u_1(\xi) \leqq u(\xi)$ and $u_1 \in \mathbf{D}^h$. Since $u$ is $h$-minimal, $u$ is a constant multiple $c$ of $u$ on $H$. If $u$ is not in the class $\mathbf{D}^h$, $c < 1$. But $u_1$ and $u$ have the same boundary limits. Hence $u_1 = 0$ on $H$, and we are in case (a). On the other hand, if $u \in \mathbf{D}^h$, and if $\delta$ is a strictly positive constant, min $[u, \delta]$ has a stochastic boundary function with stochastically ramified Dirichlet solution $u_\delta$ having this stochastic boundary function, and for which $0 \leqq u_\delta \leqq u$ on $H$. Then $u_\delta = c_\delta\, u$ on $H$, for some constant $c_\delta$. If $u = 0$, there is nothing to prove. Otherwise $c_\delta > 0$, and, if $\delta$ is sufficiently small, $c_\delta < 1$. It follows that, if $\delta$ is sufficiently small, the only limits of $u$ on almost any $h$-path from a point of $H$ are 0, $\delta/c_\delta = a$. Since $u \in \mathbf{D}^h$, $u(\xi)$ is the expected value of its limit along $h$-paths from $\xi$, and the theorem is now completely proved.

COROLLARY. *In case* (b), *if $v$ is positive and $h$-superregular on $R$, and if $c$ is the infimum of $av/u$ on the open support of $uh$, then $v$ has the limit $c$ on almost every $h$-path from a point of $H$ on which $u$ has the limit $a$. In particular, $c = 0$ if $v$ is $h$-minimal and is not a multiple of $u$ on $H$.*

(We stress that the theorem and corollary are not essentially more general as stated than for the special case $h = 1$. They are stated this way only to facilitate reference and to avoid later misunderstanding.) To prove the corollary, let $v_\infty$ be the function in $\mathbf{D}^h$ with the same limit as $v$ on almost every $h$-path from a point along which $u$ has the limit $a$, and the limit 0 along almost every other $h$-path from the point. Let $v_\delta$ be the function in $\mathbf{D}^h$ with the same stochastic boundary function as min $[v, \delta]$. Then $v_\delta \leqq \delta u/a$ on $H$, so that, for some constant $c_1$, $v_\delta = c_1 u$ on $H$, and $v \geqq v_\infty \geqq v_\delta$ on $H$. It follows that, for large $\delta$, $c_1$ does not depend on $\delta$, and $v_\delta$ has the limit $c_1 a$ on almost every $h$-path, from a point of $H$, on which $v$ has a limit $\leqq \delta$ and $u$ has the limit $a$. Hence $v$ has the limit $c_1 a$ on almost every $h$-path from a point of $H$ on which $u$ has the limit $a$, and $v \geqq v_\infty = c_1 u$ on $H$. Then the first assertion of the corollary is true, with $c = c_1 a$. If $v$ is $h$-minimal, either $v = v_\infty = c_1 u$ on $H$ or $v_\infty = 0$, so that the second assertion is also true.

THEOREM 3.3. *If $u$ is $h$-superregular and positive, and if $h$ is minimal, then $u$ has the limit* $\inf_{\xi \in H} h(\xi)$ *on almost every $h$-path from each point of $H$. In particular, $h$ has the limit* $\sup_{\xi \in H} h(\xi)$ *on almost every $h$-path from each point of $H$.*

Let $u_1$ be the member of $\mathbf{D}^h$ with the same stochastic boundary function as $u$. Then $u_1 \leqq u$. According to Theorem 3.1, the members of the class $\mathbf{D}^h$ are constant on $H$, and the first assertion of the theorem is now obviously true. If $u = 1/h$, we obtain the second assertion of the theorem as an immediate consequence of the first.

On comparing Theorems 3.2 and 3.3 we see how differently $h$ behaves on 1-paths and on $h$-paths. We have already remarked that $h$-paths go to the parts of $R$ where $h$ is not small, and this fact is particularly clear when $h$ is minimal.

THEOREM 3.4. *Let $v_1$ and $v_2$ be $h$-minimal, and suppose the following:*

(a) *$v_2$ vanishes on $H$ whenever $v_1$ does;*

(b) *$v_2$ is not a multiple of $v_1$ on $H$.*

*Then $v_2/v_1$ is unbounded on the open support of $v_1 h$, and has the limit 0 on almost every $(v_1 h)$-path from a point of this open support.*

In fact, $v_2/v_1$ is $(v_1 h)$-minimal because $v_2$ is $h$-minimal and (a) is true. Moreover, if $v_2/v_1$ is bounded on the open support of $v_1 h$, $v_2 \leqq$ const. $v_1$ on $H$, using (a). But then $v_2$ is a multiple of $v_1$ on $H$, since $v_1$ is $h$-minimal. Since this conclusion is false, according to hypothesis (b), $v_2/v_1$ is not bounded on the open support of $v_1 h$, and the theorem now follows either from Theorem 3.2 or or Theorem 3.3.

If $h$ never vanishes, it is natural to inquire under what conditions the $h$-regular function $1/h$ is uniquely determined by its limits on $h$-paths, that is, when this function is in the class $\mathbf{D}^h$. This question is answered by the following theorem, which implies among other things that, if $h$ is minimal, never vanishes, and is not identically a constant, $1/h$ is never in the class $\mathbf{D}^h$.

THEOREM 3.5. *If $h$ never vanishes, the function $1/h$ is in the class $\mathbf{D}^h$ if and only if $h$ has a strictly positive limit on almost every 1-path from each point of $R$.*

By definition of $\mathbf{D}^h$, $1/h \in \mathbf{D}^h$ if and only if $1/h$ on $R_1'$, $R_2'$, $\cdots$ is uniformly integrable with respect to the measures $\mu^h(\xi, R_1, \cdot)$, $\mu^h(\xi, R_2, \cdot)$, $\cdots$ for each point $\xi$ of $R$. That is, (uniformly as $n$ varies with $\xi \in R_n$)

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \int_{\{h(\eta)\leqq\varepsilon\}} \frac{1}{h(\eta)}\,\mu^{h}(\xi, R_n, d\eta) = \lim_{\varepsilon\to 0}\int_{\{h(\eta)\leqq\varepsilon\}} \mu(\xi, R_n, d\eta)/h(\xi) \\
&= \lim_{\varepsilon\to 0} \mathbf{P}\{h[z_n(\omega)] \leqq \varepsilon\}/h(\xi).
\end{aligned} \tag{3.1}
$$

Now $h$ has a finite limit on almost every 1-path from $\xi$,

$$\lim_{n\to\infty} h(z_n) = x, \tag{3.2}$$

and it is clear that the condition (3.1) is equivalent to the condition that $x$ vanishes with probability 0.

In the following theorem, we suppose that $R$ is immersed in a compact separable space $R \cup R'$, in which $R$ has boundary $R'$.

THEOREM 3.6. *In case* (b) *of Theorem* 3.2, *there is a point of $R'$ with the property that the probability that an h-path from $\xi$ in $H$ has the point as a limit point is $h(\xi)/a$.*

To see this, let $A$ be the union of a sequence of closed subsets of $R'$, and let $p(\xi, A)$ be the probability that an $h$-path from $\xi$ in $H$ on which $u$ has the limit $a$ has a limit point in $A$. Then, applying the evaluation in Theorem 3.2(b), $p(\cdot, A) \leqq u/a$. The function $p(\cdot, A)$ is $h$-regular, and has the limit 1 on almost every $h$-path from a point of $H$ with the two stated properties, the limit 0 on almost every other. In fact this function is the stochastically ramified Dirichlet solution corresponding to these prescribed limits. Since $u$ is $h$-minimal, it follows that $p(\cdot, A) = cu$ on $H$, for some constant $c \leqq 1/a$. Applying the Corollary to Theorem 3.2, we find that, if $p(\xi, A) > 0$, then $p(\xi, A) \geqq u(\xi)/a$. Hence, either $p(\cdot, A) = 0$ or $p(\cdot, A) = u/a$. Since $R'$ is compact, and since

$$p(\cdot, \textstyle\bigcup_1^\infty A_j) \leqq \sum_1^\infty p(\cdot, A_j),$$

there must be a point $\xi'$ on $R'$ with the property that $p(\cdot, A) = u/a$, for every open (relative to $R'$) set containing $\xi'$. Hence $p(\cdot, \{\xi'\}) = u/a$, and this concludes the proof of the theorem.

Essentially this theorem, although stated nonprobabilistically, and in the case when $R$ is a Green space and the regular functions are the harmonic functions, was obtained by Naïm [6].

Theorem 3.6 can be used to exclude the possibility of bounded minimal functions in many applications. For example, suppose that $h = 1$, that $R$ is an open connected set of Euclidean $N$-space, $N \geqq 2$, and that $R'$ is its relative boundary. Then, if the regular functions are the harmonic functions, the probability paths from a point of $R$ can be taken as the ordinary Brownian paths from the point, up to their first meeting with $R'$, if any. Alternatively, in keeping with the present study, the point $z_n$ on $R_n'$ is the first point in which a Brownian path meets $R_n'$, where $n$ is so large that the initial point lies in $R_n$. In this application, case (b) of Theorem 3.2 cannot arise unless the point at $\infty$ is a boundary point of positive harmonic measure, because each finite boundary point has zero harmonic measure. (In probability language, this corresponds to the fact that almost no Brownian path ever passes through a specified point of $N$-space.) This result on minimal harmonic functions is due to Naïm [6]. Actually in applications like this, what is proved is far stronger than that there are no bounded $h$-minimal functions, namely that there are no $h$-minimal functions in the ordinarily far larger class $\mathbf{D}^h$.

Before concluding our study of minimality, we make a few remarks connecting this subject with convexity. The class of positive $h$-regular functions on $R$, with value 1 at a specified point of $H$, is a convex set, and it is trivial to verify that the extreme points of this set (considering the functions only on $H$) are the $h$-minimal functions. The class of $h$-regular functions $u$ on $R$ satisfying the inequality $0 \leqq u \leqq 1$ on $H$ is also convex, and we shall now prove the following theorem.

THEOREM 3.7. *The function $u$ is an extremal of the convex set of positive $h$-regular functions $\leqq 1$ if and only if it is a function in the set whose limit on almost every $h$-path from each point of $H$ is either* 0 *or* 1.

If $u$ is an extremal function in the class, not vanishing identically, and if $\delta$ is strictly positive, define $u_1$ as the $h$-regular function in the class $\mathbf{D}^h$, defined arbitrarily on $R - H$, with the same limit as $u$ along almost every $h$-path from each point $\xi$ in $H$, if the limit is $\leqq \delta$, and the limit 1 along almost every other $h$-path from $\xi$. We write $u$ in the form

$$u = \delta u_1 + (1 - \delta) \frac{u - \delta u_1}{1 - \delta}. \tag{3.3}$$

Unless $u$ has the form described in the theorem, $\delta$ can be chosen so small that $u_1(\xi) > u(\xi)$ at some point $\xi$ of $H$, and the above expression for $u$ then contradicts the hypothesis that $u$ is extremal. Conversely, if $u$ has the property described in the theorem, and if $u = t_1 u_1 + t_2 u_2$, where $u_1$, $u_2$ are members of the convex class in question, and $0 < t_i$, $t_1 + t_2 = 1$, then $u_1$ and $u_2$ must both have limit 1 [0] on an $h$-path along which $u$ has the limit 1 [0]. Hence $u$, $u_1$, and $u_2$, as bounded functions with the same stochastic boundary function, are identical on $H$, so that $u$ is extremal.

## 4. The probability of hitting a set

In this section we shall assume that $R$ has a boundary $R'$ and that $R \cup R'$ is compact and separable. Let $A$ be a closed subset of $R'$, or a countable union of such sets. Then, if $\xi \in H$, the probability $u_A^h(\xi)$ that an $h$-path from $\xi$ has a limit point on $A$ is well-defined. The function $u_A^h$, defined arbitrarily on $R - H$, is $h$-regular, and, if $\xi \in H$, has limit 1 [0] on almost every $h$-path from $\xi$ with limit point on $A$, limit 0 on almost every other $h$-path from $\xi$. In fact $u_A^h$ is the stochastically ramified Dirichlet solution determined by this boundary behavior.

In order to be able to connect the probability and PWB analysis of the first boundary value problem even more closely than in [4], we shall sometimes impose another hypothesis on $R$ and on the regular functions we are considering. This hypothesis will enable us to calculate $u_A^h$. In the typical application, our $h$-paths from $\xi$ can be replaced by continuous, or at least right-continuous paths, and what we have called $z_n^h$ is the first point in which an $h$-path from $\xi$ meets $R_n'$. In such an application, it is useful to consider the probability $u_G^h(\xi)$ that an $h$-path from $\xi$ meets an open set $G$, after leaving $\xi$. Under the usual hypotheses, this probability defines an $h$-superregular function of $\xi$, $h$-regular on $R$ less the closure of $G$, equal to 1 on $G$. Moreover the function has the limit 0 along almost all $h$-paths, from a point of $H$, which do not meet $G$ near $R'$, the limit 1 on almost all other $h$-paths from the point. This function is the equilibrium potential of $G$ in potential theoretic studies. We need the existence of $u_G^h$ for $G$ the intersection of $R$ with a neighborhood (relative to $R \cup R'$) of any specified closed subset of $R'$, and it does not appear that our present hypotheses are sufficient to imply the existence of this function. From the point of view of the first boundary value (Dirichlet) problem, $u_G^h$ is obtained by solving the Dirichlet problem for $h$-regular functions on $R$ less the closure of $G$ relative to $R$, with boundary value 1 on the boundary points of $G$ in $R$, and 0 at other boundary points. The solution yields $u_G^h$ in $R$ less the closure of $G$ relative to $R$. The function $u_G^h$ is then defined as 1 in $G$, and is defined at the boundary points of $G$ in $R$ to be $h$-superregular in $R$. Alternatively, the function $u_G^h$ can be found, in many applications, by solving a Dirichlet problem for regular functions rather than for $h$-regular functions. In fact, let $v$ be the solution of the Dirichlet problem for regular functions in $R$ less the closure of $G$ relative to $R$, with boundary value $h$ on the boundary points of $G$ in $R$, and 0 at the other boundary points. The function $v$ is defined as $h$ in $G$, and is defined at the boundary points of $G$ in $R$ to be superregular in $R$. Then $u_G^h = v/h$. Either of these two procedures can be carried out, for example, if $R$ is a Green space and if the regular functions are the harmonic functions.

Rather than attempting to add hypotheses to insure the existence of $u_G^h$, we shall simply formulate its existence as a hypothesis, as follows. This hypothesis will never be presupposed without explicit mention.

$J^h$. *If $G$ is an open subset of $R$, which is the intersection with $R$ of a neighborhood of a closed subset of $R'$, there is a function $u_G^h$ on $R$ with the following properties:*

(a) *$u_G^h$ is $h$-superregular on $R$, $h$-regular on $H$ less the common part of $H$ and the closure of $G$.*

(b) *$0 \leqq u_G^h \leqq 1$ on $H$, and $u_G^h$ has the value 1 at every point of $GH$.*

(c) *If $\xi \in H$, $u_G^h$ has the limit 1 on almost every $h$-path from $\xi$ which meets $G$ infinitely often, the limit 0 on almost every other $h$-path from $\xi$.*

(d) *If $u$ is positive and $h$-superregular on $R$, $\geqq 1$ on $G$, then $u \geqq u_G^h$ on $H$.*

We shall not need this hypothesis for every $G$, but only for a sequence of such sets, corresponding to any prescribed closed boundary set $A$, whose closed covers shrink to $A$. Let $\hat{G}_n$ be a neighborhood (relative to $R \cup R'$) of the closed subset $A$ of $R'$, and suppose that $\hat{G}_1 \supset \hat{G}_2 \supset \cdots$, $\bigcap \hat{G}_n = A$. Let $G_n = \hat{G}_n R$. Under hypothesis $J^h$, (d) implies that $u^h_{G_1} \geqq u^h_{G_2} \geqq \cdots$. Let $u$ be the limit of the sequence. Then $u$ is $h$-regular on $R$, and $0 \leqq u \leqq 1$. Hence $u$ has a limit on almost every $h$-path from a point of $H$, and this limit must be 0 for almost every such path with no limit point in $A$. Let $u^h_A$ be the function defined at the beginning of this section. Since $u^h_{G_n}(\xi)$ is at least equal to the expected value of the limit of $u^h_{G_n}$ on $h$-paths from $\xi$,

$$u^h_{G_n}(\xi) \geqq u^h_A(\xi), \qquad \xi \in H. \tag{4.1}$$

Hence

$$u(\xi) \geqq u^h_A(\xi), \qquad \xi \in H. \tag{4.2}$$

Since $u(\xi)$ is equal to the expected value of the limit of $u$ on $h$-paths from $\xi$, and, as we have seen, this limit is 0 on almost every $h$-path which has no limit point in $A$, whereas the limit is at most 1 on other $h$-paths,

$$u(\xi) \leqq u^h_A(\xi), \qquad \xi \in H. \tag{4.3}$$

Hence

$$u = \lim_{n\to\infty} u^h_{G_n} = u^h_A \tag{4.4}$$

on $H$. It is this result which plays an essential role below.

In the following, we write $u_A$ instead of $u^1_A$. Let $H_A$ be the open support of $u_A$. According to an application of Theorem 2.1 made in Section 2, $u_A$ has the limit 1 on almost every $u_A$-path from a point of $H_A$. Another application of Theorem 2.1 (with $\Gamma$ the condition that a path have no limit point in $A$ and that $u_A$ have limit 1 on the path) shows that almost every $u_A$-path from a point of $H_A$ has a limit point in $A$. Now let $B$ be a union of a sequence of closed subsets of $R'$, and define $u_{A,B}(\xi)$ as the probability that a 1-path from $\xi$ has a limit point in $A$ and also one in $B$. Then $u_{A,B}$ is regular, and has limit 1 on almost every 1-path, from each point $\xi$ on $R$, with a limit point in each set, and has limit 0 on almost every other 1-path from $\xi$. From now on we assume that $H_A$ is not the null set. Then $v = u_{A,B}/u_A$ is $u_A$-regular, and $v(\xi)$, for $\xi \in H_A$, is the probability that a 1-path from $\xi$ has a limit point in $B$, if it is known to have one in $A$. We now prove that this probability is the probability that a $u_A$-path has a limit point in $B$. To prove this, we need only prove that $v$ has the limit 1 on almost every $u_A$ path from $\xi$, with a limit point in $B$, and that $v$ has limit 0 on almost every other $u_A$ path from $\xi$. Now the condition $\Gamma$ that $u_{A,B}$ does not have the limit 1 on a path, and that the path has a limit point in $A$ and one in $B$, has probability 0 if the paths are 1-paths from a point of $R$. Hence, applying Theorem 2.1, $1/u_A$ has the limit 0 on almost every $u_A$-path, from a point of $H_A$, satisfying $\Gamma$. That is, almost no $u_A$-path satisfies $\Gamma$, and, in view of the properties of these paths, this means that $v$ has the limit 1 on almost every $u_A$-path, from a point of $H_A$, with a limit point in $B$. The other half of the assertion on the limits of $v$ along $u_A$-paths is proved by another application of Theorem 2.1, in which $\Gamma$ is the conditions that $u_{A,B}$ does not have limit 0 on a path, and that the path has a limit point in $A$ but not in $B$.

## 5. The PWB method

In this section we shall again assume that $R$ has a boundary $R'$, and that $R \cup R'$ is compact and separable. We can then apply the PWB method to the first boundary value problem for regular functions on $R$. This was done in [4], and various interrelations between the PWB results and probability boundary value limit theorems were obtained.

In exactly the same way, since, as we have seen in Section 2, the maximum principle $M^h(R, R')$ is satisfied, the PWB method can be applied to the first boundary value problem for $h$-regular functions on $R$. The situation is no different from that when $h = 1$, if $h$ never vanishes. If $h$ may vanish, the only difference is that we consider functions on $H$ instead of on $R$, and $R'$ is accordingly replaced by the intersection of $R'$ and the boundary of $H$ relative to $R \cup R'$. The rest of the boundary becomes irrelevant. Just as in the case $h = 1$ treated in [4], if a boundary function is PWB$^h$ resolutive, the corresponding PWB$^h$ solution $u$ is a member of the class $\mathbf{D}^h$, $f$ is constant on the limit set of almost every $h$-path from a point of $H$, and this constant is the limit of $u$ on the $h$-path.

We now go into this analysis in more detail. Let $f$ be a function defined on $R$. Then $f$ determines upper and lower PWB$^h$ solutions, $h$-regular if they are finite on a dense subset of $H$. (The assertions in [4] on p. 56 in this connection are overoptimistic.) If these two solutions are finite and equal, $f$ is PWB$^h$ resolutive. In particular, if $f$ is the characteristic function of a set $A$, we denote by $v_A^h$ the upper PWB$^h$ solution for $f$. If $A$ is closed, it is clear that $v_A^h \geqq u_A^h$ on $H$, where $u_A^h$ is the function defined in Section 4, and that there is equality under hypothesis $\mathrm{J}^h$.

We shall need the following properties of $v_A^h$.

(a) $v_A^h \leqq v_B^h$ *if* $A \subset B$.

(b) $v_A^h$ *is the infimum of* $v_B^h$, *for* $B$ *open relative to* $R'$ *and* $B \supset A$.

(c) *If* $A = \bigcup A_n$, *where* $A_1 \subset A_2 \subset \cdots$, *then* $v_A^h = \lim_{n\to\infty} v_{A_n}^h$.

The proofs of the first two properties are immediate. The proof of the third only slightly less so (see [1]). In fact, if $\xi \in H$, if $v_n^h$ is in the upper class for the characteristic function of $A_n$, and if we take into account the elementary properties of $h$-superregular functions, we find that the function

$$\lim_{n\to\infty} v_{A_n}^h + \min\,[\textstyle\sum[v_n^h - v_{A_n}^h], 1]$$

is an $h$-superregular function in the upper class for the characteristic function of $A$. Since the sum can be made arbitrarily small at $\xi$, it follows that $\lim_{n\to\infty} v^h_{A_n} \geqq v^h_A$ on $H$, and the reverse inequality is trivial.

Note that $u^h_A$ has properties (a) and (c), but not (b).

THEOREM 5.1. *Under* $J^h$, *if* $A$ *is a subset of* $R'$, *the upper* [*lower*] *$PWB^h$ solution at* $\xi$ *in* $H$ *for the characteristic function of* $A$ *is the infimum* [*supremum*] *as* $B$ *varies of the probability that an h-path from* $\xi$ *has a limit point* [*its limit set*] *in the open relative to* $R'$ [*closed*] *subset* $B$ *of* $R'$ *containing* [*contained in*] $A$.

Since $u^h_A = v^h_A$ for $A$ a closed subset of $R'$, this theorem is true for $A$ closed, as far as the upper $\mathrm{PWB}^h$ solutions are concerned. Moreover $u^h_A = v^h_A$ if $A$ is the union of a sequence of closed subsets of $R'$, because the property (b) of $v^h_A$ is also enjoyed by $u^h_A$. In particular, $u^h_A = v^h_A$ if $A$ is open relative to $R'$. In view of property (c), the upper $\mathrm{PWB}^h$ solution $v^h_A$ is as described in the theorem, and the description of the lower $\mathrm{PWB}^h$ solution is obtained from the fact that the lower $\mathrm{PWB}^h$ solution for the characteristic function of $A$ is 1 less the upper $\mathrm{PWB}^h$ solution for the characteristic function of $R' - A$. Note that, without the use of $J^h$, all that can be proved is that $u^h_A \leqq v^h_A$, for $A$ the union of a sequence of closed subsets of $R$.

The condition of this theorem becomes very simple when $A$ is closed. In fact in this case the conclusion of the theorem implies that the upper [lower] $\mathrm{PWB}^h$ solution at $\xi$ is the probability that an $h$-path from $\xi$ has a limit point [its limit set] in $A$. This characterization thus gives the following result, to be strengthened in Theorem 5.4.

THEOREM 5.2. *Under* $J^h$, *the characteristic function of a closed subset of* $R'$ *is $PWB^h$ resolutive if and only if, for each point* $\xi$ *of* $H$, *almost every h-path from* $\xi$ *that has a limit point in* $A$ *has all its limit points in* $A$.

This theorem has an important consequence. The class of $\mathrm{PWB}^h$ resolutive boundary functions is linear, and is closed under uniform convergence and monotone bounded convergence (see [4]). Hence $R'$ is strongly $\mathrm{PWB}^h$ resolutive if and only if the characteristic function of each closed subset of $R'$ is $\mathrm{PWB}^h$ resolutive. Hence $R'$ is strongly $\mathrm{PWB}^h$ resolutive if and only if almost every $h$-path from a point of $H$ that has a limit point on any specified closed subset of $R'$ has all its limit points in the set. If we apply this result to a sequence of finer and finer partitions of $R'$, we obtain the converse half of the following theorem. The direct half was proved in [4].

THEOREM 5.3. *If* $R'$ *is strongly $PWB^h$ resolutive, almost every h-path from a point of* $H$ *is convergent. Conversely, under hypothesis* $J^h$, *if almost every h-path from a point of* $H$ *is convergent,* $R'$ *is strongly $PWB^h$ resolutive.*

We shall now apply the general notion of capacity, due to Choquet [2]. If $\xi$ is a point of $H$, $u^h_A(\xi) = v^h_A(\xi)$ defines a function of the compact subset $A$ of $R'$. In Choquet's terminology, this function is a capacity of order $\mathfrak{A}_\infty$. In fact $u_A(\xi)$ defines what Choquet calls [2, p. 209] a "fundamental scheme of the capacities of order $\mathfrak{C}_\infty$". [His $E$ is our $R'$, his $F$ is our measure space on which the $h$-paths from $\xi$ are defined, his $A$ is the set of points $(\omega, \xi)$ of $E \times F$ for which the $h$-path generated by $\omega$ has the point $\xi$ of $R'$ as limit point.] According to Choquet's definitions, the capacity of an open (relative to $R'$) set $A$ is then also our $u_A^h(\xi) = v_A^h(\xi)$. The exterior capacity of any subset $A$ of $R'$ is the infimum of the capacity of open relative to $R'$ sets including $A$, and this is $v_A^h(\xi)$, according to property (b) of upper PWB$^h$ solutions, as stated above. The interior capacity of $A$ is the supremum of the capacities of closed subsets of $A$. The fact that $h(\xi)v_A^h(\xi)$ defines a capacity has already been noted, in the case of harmonic functions on a Green space, by Naïm (see [6]). The factor $h(\xi)$ is of course immaterial.

The interior and exterior capacities of $A$ have simple probability interpretations. The interpretation of the exterior capacity $v_A^h(\xi)$ has already been given in Theorem 5.1. The interior capacity of $A$ is simply the supremum as $B$ varies of the probability that an $h$-path from $\xi$ has a limit point in the closed subset $B$ of $A$. There is then a countable sum $F$ of such sets such that the interior capacity of $A$ is the probability that an $h$-path from $\xi$ has a limit point in $F$. It is easily seen that the interior capacity of the set $A$ (as the supremum of a lattice of $h$-regular functions) is $h$-regular, and that the set $F$ can be chosen to be the same for all $\xi$ in $H$.

The lower PWB$^h$ solution at $\xi$ for the characteristic function of the set $A$ is then, in view of Theorem 5.1, at most equal to the interior capacity of $A$. In fact, in the above notation, the two are equal if and only if there is a union $F = \bigcup F_n$ of a sequence of closed subsets of $A$, such that the interior capacity of $A$ is the probability that an $h$-path from $\xi$ has a limit point in $F$, and that almost every $h$-path from $\xi$ with a limit point in $F$ has its limit set in some $F_n$.

Now Choquet proved that the class of capacitable subsets of $R'$, that is, the class of those sets for which the interior and exterior capacities are equal, is a class which includes the Borel, and even the analytic, subsets of $R'$, at least in our case, in which the initial domain of definition is the class of closed sets of a compact separable space. Hence, if $A$ is a Borel subset of $R'$, we have new descriptions of the upper and lower PWB$^h$ solutions of its characteristic function. The description above of the interior capacity of $A$ is now a description of the value of the upper PWB$^h$ solution at $\xi$. Since the lower PWB$^h$ solution is 1 less the upper PWB$^h$ solution of the characteristic function of its complement, we find that the lower solution at $\xi$ is the infimum as $B$ varies of the probability that an $h$-path from $\xi$ has its limit set in the open relative to $R'$ set $B \supset A$. In fact there is a set $G$ which is the intersection of a sequence of such relatively open sets, such that the lower PWB$^h$ solution at $\xi$ is the probability that an $h$-path from $\xi$ has its limit set in $G$.

We have restricted $A$ to be a Borel set above, because we used the capacitability of both $A$ and $R' - A$. If both are analytic, they are both necessarily Borel sets.

Using these results, we now prove the following extension of Theorem 5.2.

THEOREM 5.4. *Let $f$ be a function defined on $R'$. Then, if $f$ is $PWB^h$ resolutive, $f$ is constant on the limit set of almost every $h$-path from each point of $H$. Conversely, under hypothesis $J^h$, if $f$ is a bounded Borel measurable function on $R'$, and if $f$ is constant on the limit set of almost every $h$-path from each point of $H$, $f$ is $PWB^h$ resolutive.*

The direct half of this theorem was proved in [4]. In the converse statement, the boundedness condition can be weakened in various ways. For example, without it the function obtained by cutting off $f$ when $|f| \geqq n$, say by redefining it as 0 at such points, satisfies the stated conditions and so is $\mathrm{PWB}^h$ resolutive. To prove the converse, we remark that it is sufficient to prove it for $f$ the characteristic function of a Borel set, because $f$ is the uniform limit of a sequence of linear combinations of such characteristic functions, each satisfying the conditions of the theorem. Finally, if $f$ is the characteristic function of the Borel set $A$, our hypothesis means that almost every $h$-path from each initial point $\xi$ of $H$ determines a limit set either contained in $A$ or in $R' - A$, and we see at once from the analysis of lower and upper solutions for $f$, as made above, that $f$ is then $\mathrm{PWB}^h$ resolutive.

As an application of the preceding theorems, consider the case in which the regular functions are the solutions of the heat equation (see [3]). In this case and in similar cases, the probability paths are given in the first place, and the regular functions must be defined in terms of them, not the other way around. The probability paths are usually given, as in this case, as continuous paths (although actually, under appropriate hypotheses, right continuity is enough for the analysis to be made) which finally leave every compact subset of the defining space $R$. Let $D$ be an open subset of $R$, with compact closure, and let $\xi$ be a point of $D$. Then there is a first point $z$ in which a path from $\xi$ meets the boundary $D'$ of $D$. The point $z$, which we denote by $z(\xi, D)$, depends on the path. It is a random variable whose distribution on $D'$ is the measure $\mu(\xi, D, \cdot)$. In the application we are discussing, if $D$ has certain simple geometrical properties described (incorrectly) in [3] it can be taken as one of the regular sets of our basic definition of regularity, and all our hypotheses are satisfied. In other applications, no sets need be distinguished as regular sets, and we can simply define a function as regular if it is equal at each point of each open set $D$ with compact closure to the corresponding $\mu$ average over $D'$. This gives a slightly different basic structure to the development of the theory of regular functions, but the difference does not affect the study of the first boundary value problem on $R$ itself, as long as $R$ is still by hypothesis the union of a monotone sequence $\{R_n, n \geqq 1\}$ of open sets, with $R_n$ having a compact closure contained in $R_{n+1}$. Moreover $J^1$ is trivial to verify, the function $u_G^1$ being defined at $\xi$ as the probability that a 1-path from $\xi$ meets $G$ at some time after leaving $\xi$. The boundary limit theorems and those on resolutivity are valid, and in fact the paths can either be taken as the contінous given ones, or, in the notation of this paper, if $\xi \in R_n$, $z_n$ can be taken as $z(\xi, R_n)$. In particular, this means that, in our heat equation and more general similar cases, if $D$ is any open subset of $R$, with compact closure, and if $D$, like $R$ above, is the union of a nested sequence of its open subsets, the boundary $D'$ is strongly PWB resolutive. In fact, if the original continuous paths for $R$ from a point $\xi$ of $D$ are shortened by dropping $z(\xi, D)$ and all later points, these paths become the appropriate paths for $D$. Moreover such a path converges to a point of $D'$, and in fact to $z(\xi, D)$. Thus Theorem 5.3 is applicable to show that $D'$ is strongly PWB resolutive.

## 6. The "réduite"

There has been little justification in Sections 4 and 5 for the use of a general $h$, since, as we have seen, there is little decrease in generality if $h$ is taken as the function 1, none whatever if $h$ never vanishes. In fact what increase in generality there is simply suggests that the basic hypotheses of [4] were unnecessarily restrictive. The use of a general $h$ is advantageous when several such functions appear simultaneously, as in some of the preceding sections and in the following.

If $\phi$ is positive and superregular on $R$, we define $\phi_A(\xi)$ as the infimum at $\xi$ of the values at $\xi$ of positive superregular functions which are $\geqq \phi$ in the intersection of $R$ with some neighborhood of the set $A$. Here $A$ is to be taken as a subset of the boundary $R'$, and $R \cup R'$ is supposed compact and separable. Then $\phi_A$ is regular, and $\phi_A \leqq \phi$ on $R$. This functional of $\phi$, introduced in a more special context by Martin, is called, in that context, the "réduite" of $\phi$ relative to $A$ by Brelot [1]. In particular, if $\phi = h$ is regular, $h_A = v_A^h h \geqq u_A^h h$ on $H$, and there is equality, under hypothesis $\mathrm{J}^h$, if $A$ is the union of a sequence of closed sets.

In the rest of this section, we shall suppose that $\mathrm{J}^h$ is satisfied. If there is a point $\xi$ of $H$ with $h_A(\xi) = 0$, it follows from our preceding work that almost no $h$-path from $\xi$ has a limit point in $A$ (or even in the intersection of a suitably chosen sequence of open relative to $R'$ sets containing $A$). Conversely, if $A$ is analytic, and if almost no $h$-path from some point $\xi$ of $H$ has a limit point in $A$, then $h_A(\xi) = 0$. At the other extreme, if $A$ is analytic, and if, for some $\xi$ in $H$, $h_A(\xi) = h(\xi)$, then almost every $h$-path has a limit point in $A$ (and even in the union of a certain sequence of closed subsets of $A$). Conversely, if $A$ is an arbitrary subset of $R'$ such that almost every $h$-path from some point $\xi$ of $H$ has a limit point in $A$, then $h_A(\xi) = h(\xi)$.

If $\xi \in H$, there is a closed subset $A_1$ $[A_2]$ of $R$ such that almost every $h$-path from $\xi$ has a limit point in $A_1$ [such that $h_{A_2}(\xi) = h(\xi)$] and such that no closed proper subset of $A_1$ $[A_2]$ has this property. In fact the existence of $A_i$ follows at once from the fact that the class of sets in which $A_i$ is minimal is closed under monotone limits. Under $\mathrm{J}^h$, a set $A_i$ satisfying either of the above conditions satisfies the other, and such a set will be called a determining set of $h$ relative to $\xi$. In particular, if the set contains only one point, the point is called a pole of $h$ relative to $\xi$. In the simplest applications, in which a regular function which assumes it maximum value is a constant function,

$H = R$, and a determining set relative to $\xi$ is one relative to every point of $R$.

Suppose now that $h$ is minimal, that $J^h$ is satisfied, and that $A$ is a closed subset of $R'$. Then $h_A$ is a multiple of $h$, $h_A = ch$, on $H$. To see that $c = 0$ or $c = 1$ are the only possibilities, we can either remark that $(h_A)_A = h_A$ on $H$, or that $v_A^h = h_A/h$ is $h$-regular, with limit 1 on almost every $h$-path, from a point of $H$, with limit point in $A$, limit 0 on almost every other $h$-path from the point. We deduce at once that, if $h_A = h$, there must be a point $\eta$ in $A$ such that almost every $h$-path from each point of $H$ has a limit point in every neighborhood of $\eta$, and so almost every such path has $\eta$ itself as limit point. Thus $\eta$ is a pole of $h$ relative to every point of $H$. Taking $A = R'$, we see that every minimal function has a pole. The characteristic function of a Borel subset of $R'$ is PWB$^h$ resolutive if and only if it contains either no pole or all poles of $h$, and it follows that $R'$ is strongly PWB$^h$ resolutive if and only if $h$ has only a single pole.

If $h$ may not be minimal, $R'$ cannot be strongly PWB$^h$ resolutive unless $h$ has only one determining set, relative to any specified point of $H$, but the study of determining sets has not been carried beyond this point.

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