LOCAL CONNECTEDNESS IN THE STONE-ČECH COMPACTIFICATION[1]

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Introduction

This is a study of when and where the Stone-Čech compactification of a completely regular space may be locally connected. As to when, Banaschewski [1] has given strong necessary conditions for βX to be locally connected, and Wallace [19] has given necessary and sufficient conditions in case X is normal. We show below that Banaschewski's necessary conditions are also sufficient and may be restated as follows: βX is locally connected if and only if X is locally connected and pseudo-compact (Corollary 2.5). Moreover, the requirement that βX be locally connected is so strong that it implies that every completely regular space containing X as a dense subspace is locally connected (Corollary 2.6).

As to where βX is locally connected, we note first (1.15) that the completion $\langle aX \rangle$ of X in its finest uniformity is a subspace of βX. Then βX is never locally connected at any point not in $\langle aX \rangle$ (Theorem 2.2) and is locally connected at a point of X if and only if X is locally connected there (Corollary 1.5). In the remaining case, we have only that if X is locally connected, then βX is locally connected at every point of $\langle aX \rangle$ (Theorem 2.1).

These results, together with some lemmas, are given in the first two sections. Two lemmas worthy of independent mention are Lemma 1.4: An open subset U of βX is connected if and only if $U \cap X$ is connected, and Lemma 1.14: X is locally connected if and only if every normal covering has a normal refinement consisting of connected sets. (The first of these was obtained by Wallace in [19] for normal spaces.)

In our last section we discuss Wallace's conditions which are stated in terms of Property S, a name which is given in the literature to three related but different concepts. We show that Property S in the sense of Wallace is equivalent to local connectedness and countable compactness; then our Corollary 2.5 appears as a direct generalization of Wallace's result.

1. The lemmas

In this paper, we are concerned almost exclusively with subspaces of compact Hausdorff spaces. These are the completely regular spaces, and throughout the paper *"space" will abbreviate "completely regular space" unless an exception is made explicitly.*

For any space X, let $C(X)$ denote the set of all continuous real-valued functions on X, and let $C^*(X)$ denote the set of all bounded functions in $C(X)$.

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1.1. Every completely regular space is a subspace of an essentially unique compact space βX such that every $f \in C^*(X)$ has a (unique) extension $\bar{f} \in C(\beta X) = C^*(\beta X)$. The space βX, which is usually called the Stone-Čech compactification of X, is unique in the sense that if Y is any compact space containing X as a dense subspace and such that every $f \in C^*(X)$ has a continuous extension over Y, then there is a homeomorphism of βX onto Y keeping X pointwise fixed [3, 16].

If $A \subset X$, we use A^β to denote the closure of A in βX. Two subsets A, B of X are said to be *completely separated* if there is an $f \in C^*(X)$ such that $f[A] = 0$ and $f[B] = 1$. Two subsets of X have disjoint closures in βX if and only if they are completely separated [3].

1.2. We denote by υX the subspace of βX consisting of all $p \in \beta X$ over which every $f \in C(X)$ has a continuous real-valued extension. If $X = \upsilon X$, then X is called a *Q-space*. The space υX is unique in the sense that if X is dense in a Q-space Y such that every $f \in C(X)$ has a (unique) extension $\bar{f} \in C(Y)$, then there is a homeomorphism of υX upon Y keeping X pointwise fixed [5, 8]. By a theorem of M. H. Stone [16, Theorem 88], every $f \in C(X)$ has a (unique) continuous extension $\hat{f}$ over βX into the one point compactification $R \cup \{\infty\}$ of the real line R. A point q of βX fails to be in υX if and only if there is an $f \in C(X)$ such that $\hat{f}(q) = \infty$ [5].

1.3. In [1], Banaschewski showed that if βX is locally connected (i.e., every point has a base of connected open neighborhoods), then (i) X is locally connected, and (ii) X cannot have an infinite family of open subsets whose closures are pairwise disjoint and have a closed union. In [6], it was noted that (ii) is equivalent to X being *pseudo-compact* (i.e., every $f \in C(X)$ is bounded). Equivalently, $\upsilon X = \beta X$.

Below (Corollary 1.5 and Lemma 1.6), we improve Banaschewski's result by making it local in character. In particular, we show that βX cannot be locally connected at any point x of X unless X is locally connected there, and that βX fails to be locally connected at any point not in υX. Moreover, the converse of Banaschewski's theorem is true. Indeed if X is locally connected and pseudo-compact, and X is dense in a completely regular space Y, then Y is locally connected (Theorem 2.4).

The following lemma was obtained by Wallace for normal spaces [19].

1.4. LEMMA. *An open subset U of βX is connected if and only if $U \cap X$ is connected.*

Proof. If $U = U_1 \cup U_2$, where U_1 and U_2 are disjoint open subsets of βX, then $U \cap X = (U_1 \cap X) \cup (U_2 \cap X)$, so $U \cap X$ is disconnected if U is disconnected.

If $U \cap X$ is disconnected, then there exist nonempty disjoint open subsets V_1, V_2 of X such that $U \cap X = V_1 \cup V_2$. Since X is dense in βX, $(U \cap X)^\beta = V_1^\beta \cup V_2^\beta$ contains U. If $V_1^\beta \cap V_2^\beta$ is empty, then

$$U = (V_1^\beta \cap U) \cup (V_2^\beta \cap U)$$

is disconnected, and we are done. On the other hand, if there is a $p \in V_1^\beta \cap V_2^\beta$, construct a $\varphi \in C(\beta X)$ such that $\varphi(p) = 0$, and $\varphi[\beta X - U] = 1$. Define a function f on X by letting $f(x) = \varphi(x)$ except where $x \in V_2$ and $\varphi(x) < \frac{1}{2}$, and by letting $f(x) = \frac{1}{2}$ otherwise. It is easily verified that $f \in C^*(X)$. The continuous extension (1.1) $\bar{f}$ of f over βX coincides with φ on V_1^β, so $\bar{f}(p) = 0$. But $\bar{f} \geqq \frac{1}{2}$ on V_2^β. This contradiction shows that U is disconnected, and completes the proof of the lemma.

Clearly $\{\Omega_\alpha\}$ is a base of open neighborhoods of $x \in X$ if and only if $\{\Omega_\alpha \cap X\}$ is a base of open neighborhoods of x in X. We may conclude:

1.5. COROLLARY. *For each point x of X, βX is locally connected at x if and only if X is locally connected at x.*

1.6. LEMMA. *βX is never locally connected at any point not in vX.*

Proof. If $p \in \beta X - vX$, there is an $f \in C(X)$ such that $\hat{f}(p) = \infty$ (1.2). For $i = 0, 1, 2, 3$, let Z_i be the set of all x in X such that $n \leqq f(x) \leqq n + 1$ for some integer $n \equiv i \pmod 4$. The four sets Z_i cover X, so p is in one of their closures in βX, say $p \in Z_1^\beta$. Then p is not in Z_3^β, since Z_1 and Z_3 are obviously completely separated (1.1). Hence there is a neighborhood U of p disjoint from Z_3^β. If βX is locally connected at p, then U contains a connected open neighborhood U' of p. By Lemma 1.4, $U' \cap X$ is connected. So, by the construction above, there is an integer n such that $4n \leqq f(x) \leqq 4n + 3$ for all $x \in U' \cap X$, contrary to the fact that $\hat{f}(p) = \infty$.

1.7. COROLLARY (Banaschewski). *βX is not locally connected unless X is locally connected and pseudo-compact.*

For the remainder of the paper, we shall need some elementary facts about normal coverings and uniformities in the sense of Tukey [18].

1.8. An open covering v of a space is said to be a *refinement* of a covering u if every member of v is a subset of some member of u. The open covering $v = \{V_\beta\}$ is said to be a *star-refinement* of the open covering u if every V_β is contained in some member U of u in such a way that U contains every member of v that meets V_β. An open covering u is *normal* if there is an infinite sequence $\{u^n\}$ of open coverings beginning with $u^1 = u$, such that u^{n+1} is a star-refinement of u^n. A binary open covering $\{U, V\}$ is normal if and only if $X - U$ and $X - V$ are completely separated [18, V. 9.3].

Some insight into this concept may be gained by the following remark which is given in [9, Corollary 2.2].

An open covering u of a space X is normal if and only if there exists a metrizable space Y, an open covering v of Y, and a continuous function f on X onto Y such that $f^{-1}(v)$ is a refinement of u.

1.9. We presuppose a familiarity with Tukey's development of uniform spaces, but we will repeat some known facts about uniformities, primarily those described with the aid of nonstandard terminology. The open coverings of a uniform space that are members of its uniformity are called *large coverings*. Every large covering is normal. A filter $\mathfrak{F}$ on a uniform space μX is called a *Cauchy filter* if every large covering contains a member of $\mathfrak{F}$. A uniform space μX is *complete* if every Cauchy filter on μX converges. Every uniform space μX is a dense subspace of a unique complete uniform space $\langle \mu X \rangle$, called the completion of μX, such that every Cauchy filter on X converges to a point in $\langle \mu X \rangle$. A uniform space is called *precompact* if its completion is compact. There is a finest uniformity on a space X compatible with its topology. It consists of all normal (open) coverings of X. The associated uniform space is denoted by aX.

The next two lemmas are due essentially to Tukey and Doss.

1.10. LEMMA. *For every point x of a space X and every open neighborhood U of x, there is a closed neighborhood V of x such that $\{U, X - V\}$ is a normal covering.*

Proof. There is an $f \in C(X)$ such that $f(x) = 0$ and $f[X - U] = 1$. Let $V = \{x \in X : f(x) \leqq \frac{1}{2}\}$. Then it is easily seen that $X - U$ and V are completely separated. So, as noted in 1.8, $\{U, X - V\}$ is normal.

1.11. LEMMA. *The space X is pseudo-compact if and only if every normal (open) covering of X has a finite normal subcovering.*

Proof. In [4], Doss has shown that X is precompact in all its uniformities if and only if X is pseudo-compact. Tukey [10, p. 60] has shown that a uniform space is precompact if and only if every large covering has a finite large subcovering. (Tukey uses "largely compact" for our "precompact".) But then X is precompact in all its uniformities if and only if aX is precompact, so we have the lemma.

The next lemma is due to A. H. Stone. Although a weaker statement is made in [15, p. 979], the following is actually proved therein.

1.12. LEMMA (A. H. Stone). *Every normal covering has a normal refinement that can be written as the union of countably many collections $\{V_{n\gamma}\}$, $n = 1, 2, \cdots$, such that for each fixed n, the $V_{n\gamma}$'s have pairwise disjoint closures.*

1.13. Recall that a space X is locally connected (connected *im kleinen*) at a point x if every neighborhood of x contains a connected open neighborhood (connected neighborhood). Locally, im kleinen connectedness is a weaker property; but in the large the two are equivalent [10, p. 94]. Therefore, to show that a space is locally connected, it suffices to show that it is connected im kleinen at each of its points.

A space is locally connected if and only if components of open sets are open. The union of a family of connected sets that meet a given connected set is connected [21, p. 10, p. 45].

1.14. LEMMA. *A space X is locally connected if and only if every normal covering has a normal refinement consisting of connected sets.*

Proof. Let U be any open neighborhood of a point x of X. By Lemma 1.10, there is a closed neighborhood V of x such that $\{U, X - V\}$ is a normal covering of X. Hence the sufficiency follows.

To prove the necessity, we will show that if $\{U_\alpha\} = u$ is a normal covering of a locally connected space X, then the covering v consisting of all of the components of the elements of u is normal.

Since u is normal, there is a sequence of (normal) coverings $\{u^n\}$ with $u^1 = u$ and such that u^{n+1} is a star-refinement of u^n. If v^n denotes the set of all components of elements of u^n, then since X is locally connected, v^n is an open covering (1.13). Moreover, if $V \in v^{n+1}$, then V is a component of some $U \in u^{n+1}$, and therefore V is a subset of some $U' \in u^n$ which contains every member of u^{n+1} meeting U. A fortiori, U' contains all the elements of v^{n+1} that meet V. But, as noted in 1.13, this latter is a connected set, and thus is a subset of a component of U'. Therefore v^{n+1} is a star-refinement of v^n, and hence v' is normal.

Next, we will make some remarks comparing υX with $\langle aX\rangle$.

1.15. *The completion* $\langle aX\rangle$ *of* X *in its finest uniform structure is a subspace of* υX.

Proof. As was shown by Tukey [18, VI. 5.5], every $f \in C(X)$ is uniformly continuous on aX and hence has a continuous extension over $\langle aX\rangle$, which in turn has an extension over $\upsilon\langle aX\rangle$. Thus X is dense in the Q-space $\upsilon\langle aX\rangle$, and every $f \in C(X)$ is extensible over it. From (1.2), there is a homeomorphism of $\upsilon\langle aX\rangle$ upon υX keeping X pointwise fixed, which serves to embed $\langle aX\rangle$ in υX.

1.16. Actually, under very *weak* hypotheses on X, we may identify υX with $\langle aX\rangle$. More precisely, Shirota showed in [13] that if X has a base of open sets whose cardinal number is not strongly inaccessible from $\aleph_0$ in the sense of Tarski and Ulam, then X is a Q-space if (and only if) it admits a uniformity in which it is complete. Actually this hypothesis may be weakened a bit further, but we shall not dwell on the matter since we do not use Shirota's theorem explicitly in the sequel. Moreover, Tarski [17] has shown that it is consistent with the axioms of set theory to reject the existence of strongly inaccessible cardinals.

To see that υX and $\langle aX\rangle$ coincide under the hypothesis stated above, it suffices to note that Shirota's theorem yields that $\langle aX\rangle$ is a Q-space, and that $\langle aX\rangle$ contains X as a dense subspace so that every $f \in C(X)$ has a continuous extension over $\langle aX\rangle$ (1.2).

The next lemma, which we will need explicitly below, is also due to Shirota [13].

1.17. LEMMA (Shirota). υX *is the completion of* X *relative to the uniformity defined by all countable normal coverings.*

Our last lemma, which is due to Morita, gives us a way of passing from normal coverings of X to normal coverings of $\langle aX \rangle$. For any subset A of X, we let $\bar{A}$ denote the closure of X in $\langle aX \rangle$, and we let A^* denote the interior (in $\langle aX \rangle$) of $\bar{A}$. The proof of this lemma may be obtained by reading in the order given [11, Theorem 3], [12, Lemma 1], [11, Lemma 9] and by recalling that every large covering is normal (1.9).

1.18. LEMMA (Morita). *For any normal covering* $\{U_\alpha\}$ *of* X, $\{U_\alpha^*\}$ *is a normal covering of* $\langle aX \rangle$.

2. The theorems

2.1. THEOREM. *If* X *is locally connected, then* $\langle aX \rangle$ *is locally connected, and* βX *is locally connected at each point of* $\langle aX \rangle$.

Proof. Let $u^1 = \{U_\alpha\}$ be a normal covering of $\langle aX \rangle$, and let u^n be a descending sequence of star-refinements. Let v^n denote the restriction of u^n to X. Obviously v^{n+1} is a star-refinement of u^n, so v^1 is normal—as is v^2. By Lemma 1.14, v^2 has a normal refinement $w = \{W_\beta\}$ consisting of connected open sets. By Lemma 1.18, $\{W_\beta^*\}$ is a normal covering of $\langle aX \rangle$. Each W_β is contained in a member U_α of u^1 which contains every member of u^2 meeting W_β; a fortiori U_α contains $\overline{W}_\beta$ and hence W_β^*. Finally, each $W_\beta \in w$ is connected and dense in W_β^*, so W_β^* is connected. By Lemma 1.14 again, $\langle aX \rangle$ is locally connected. The second part of the theorem follows from the above, Corollary 1.5, and the fact that $\beta\langle aX \rangle = \beta X$ (1.15).

2.2. THEOREM. βX *is not locally connected at any point not in* $\langle aX \rangle$.

Proof. By Lemma 1.6, we need only consider points of vX. Suppose that p is in vX, but not in $\langle aX \rangle$. Let $\mathfrak{F}$ denote the filter of all $U \cap X$, where U is a neighborhood in βX of p. Since p is not in $\langle aX \rangle$, $\mathfrak{F}$ is not a Cauchy filter on $\langle aX \rangle$, so there is a normal covering $\{U_\alpha\}$ of $X (=$ large covering of $aX)$ no element of which is in $\mathfrak{F}$. By Lemma 1.12 (and the definition of normal covering) we may replace $\{U_\alpha\}$ by a normal star-refinement $\{V_{n\gamma}\}$, $n = 1, 2, \cdots$, where for each fixed n, the $V_{n\gamma}$'s have pairwise disjoint closures. Since $\mathfrak{F}$ is a filter containing no U_α, it contains no $\bar{V}_{n\gamma}$. However, if for $n = 1, 2, \cdots$, we put $V_n = \bigcup_\beta V_{n\beta}$, then $\{V_n\}$ is a countable normal covering. But by Lemma 1.17, vX is the completion in the uniformity on X defined by all countable normal coverings, so $\mathfrak{F}$ is a Cauchy filter relative to this uniformity, whence $\mathfrak{F}$ must contain some V_n. This means that for this n, $V_n = U \cap X$ for some neighborhood U of p. Since X is dense in βX, V_n^β contains U, and hence is a neighborhood of p. If βX is locally connected at p, then there is a connected open neighborhood U' contained in V_n^β. By Lemma 1.4, $U' \cap X$ is connected and hence is contained in one of the sets $\bar{V}_{n\gamma}$. But $U' \cap X$ is in $\mathfrak{F}$, and by the above no $\bar{V}_{n\gamma}$ can be in $\mathfrak{F}$. Hence βX cannot be locally connected at p.

From Theorem 2.1, Theorem 2.2, Corollary 1.5, and the fact that $\beta\langle aX \rangle = \beta X$, we obtain the following.

2.3. COROLLARY. *X is locally connected if and only if $\langle aX\rangle$ is locally connected.*

Note that in the corollary above, we cannot replace $\langle aX\rangle$ by υX without some cardinality restriction on X as in 1.16. For if there exists a discrete space X that is not a Q-space, then X is locally connected, but υX is not locally connected.

2.4. THEOREM. *If X is locally connected and pseudo-compact, then any (completely regular) space Y containing X as a dense subspace is locally connected.*

Proof. Let X be dense in Y. For any $y \in Y$, and any open neighborhood U of y, by Lemma 1.10, there is a closed neighborhood V of y such that $\{U, Y - V\}$ is a normal covering of Y. Then $\{U \cap X, X - V\}$ forms an open covering of X which is clearly normal. Since X is locally connected, by Lemma 1.14, this open covering has a normal refinement consisting of connected (open) subsets of X. Since X is pseudo-compact, the latter has a finite subfamily $\{F_i\}$ that covers X. Let G denote the closed subset of Y which consists of the union of the closures in Y of all those F_i such that y is a limit point of F_i. None of these F_i can be contained in $X - V$; hence they are all in $U \cap X$, so G is a subset of the closure in Y of U. Now, since Y is regular, the closed neighborhoods of y form a basis at y. Moreover, $Y - G$ is a subset of the union of the closures in Y of all those F_i of which y is not a limit point, so G is a neighborhood of y. Finally, G is a union of connected sets having a point in common, and hence is connected (1.13). Hence, Y is connected im kleinen at each of its points, so Y is locally connected (1.13).

The next two corollaries follow from Theorem 2.4 and Corollary 1.5.

2.5. COROLLARY. *βX is locally connected if and only if X is locally connected and pseudo-compact.*

2.6. COROLLARY. *βX is locally connected if and only if every (completely regular) space Y containing X as a dense subspace is locally connected.*

We conclude this section by remarking that under the added assumption that Y is compact, Corollary 2.6 can be obtained more simply. For, Whyburn [20] has shown that every closed continuous image of a locally connected space is locally connected, and by a theorem of Čech [3], every compact space containing X as a dense subspace is a continuous (closed) image of βX. (We are indebted to E. Michael for the reference to Whyburn's paper.)

3. Property S

The term *Property* S has two definitions in the literature which are not seriously liable to be confused. In each case, the idea is that a set having Property S should be locally connected and "smooth". The original formulation of Sierpiński [14] is metric: for every real $\varepsilon > 0$, the space is a union of finitely many connected sets of diameter less than ε. This definition is used e.g. in Bing's solution of the convex metric problem [2], and in a textbook [7]. However, in the theory of generalized manifolds [21], it seems to be convenient to use a related property that is *topological* and *relative*; a subspace Y of a regular space X has Property S if every open covering of X can be refined on Y by a finite family of connected sets.

Wallace has introduced a third property of the same name, and has given some applications of it in the theory of extension spaces [19]. He says that a topological space X has Property S provided every finite open covering of X has a finite refinement consisting of connected sets. We shall show below that this use of the terminology is unnecessary, at least for regular spaces.

3.1. THEOREM. *The following properties of a regular space X are equivalent:*

(a) *X has Property S in the sense of Wallace.*

(b) *Every finite open covering has a finite refinement consisting of connected open sets.*

(c) *X is locally connected and countably compact.*

Proof. (a) implies (c). Suppose that X has Property S. For any point x of X, let U denote an arbitrary open neighborhood of x and let V be any closed neighborhood of x contained in U. Then $\{U, X - V\}$ has a finite refinement $\{F_i\}$ consisting of connected sets. By the argument given in the proof of Theorem 2.4, the union of the closures of those F_i of which x is a limit point is a connected neighborhood of x contained in U. Thus X is connected im kleinen at each of its points, and hence is locally connected (1.13).

Suppose next that X is not countably compact, and let $D = \{d_i\}$ denote a countably infinite closed discrete subset of X. Since X is regular, a simple induction yields a sequence $\{U_i\}$ of pairwise disjoint open sets such that each U_i is a neighborhood of d_i. Clearly, $\{X - D, \bigcup_i U_i\}$ has no finite refinement consisting of connected sets.

(c) implies (b). Suppose that X is locally connected, so that components of open sets are open (1.13). We shall assume that (b) does not hold and construct an infinite closed discrete subset of X. Let $\{V_j\}$ denote a finite open covering of X that has no finite connected open refinement.

Consider the open components $\{C_{j\alpha}\}$ of the sets $\{V_j\}$. Successively for each j, delete those $C_{j\alpha}$ which are contained in the union of (1) all V_k for $k > j$, and (2) all $C_{k\alpha}$, $k < j$, such that $C_{k\alpha}$ was not deleted at the k^{th} step. The remaining $C_{j\alpha}$ still form an open covering refining $\{V_j\}$, so there are infinitely many of them. Hence there are infinitely many of them in some one set V_j. For this j, each $C_{j\alpha}$ contains a point p_α not in any other undeleted component $C_{j\beta}$. But, the infinite set $\{p_\alpha\}$ has no limit point in any of the open sets $C_{k\beta}$ which form a covering of X. Hence $\{p_\alpha\}$ is closed and discrete, so X is not countably compact.

Clearly (b) implies (a).

In [19], Wallace showed that if X is a normal space, then X has Property S if and only if βX has Property S, and noted that for compact spaces Property S is equivalent to local connectedness. Since countably compact (completely regular) spaces are pseudo-compact, Wallace's characterization follows from our Corollary 2.5 and Theorem 3.1.

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