MAPPINGS ON S^1 INTO ONE-DIMENSIONAL SPACES

BY M. K. FORT, JR.

Throughout this paper, P is the cartesian plane, S^1 is the unit circle in P, D is the closed disk in P whose boundary is S^1, and Y is a metric space. Two theorems concerning homotopy properties of mappings on S^1 into Y are proved in this paper.

LEMMA 1. *If $f: S^1 \to Y$ is inessential, then there exists a continuous extension $F: D \to Y$ of f such that none of the components of the inverse sets $F^{-1}(y)$, $y \in Y$, separates the plane P.*

Proof. Let $f: S^1 \to Y$ be inessential. Then, there exists a continuous extension $g: D \to Y$ of f. We define A to be the set of all components of sets $g^{-1}(y)$, $y \in Y$. It is well known that A is an upper semicontinuous decomposition of D. If a and b are members of A, we define $a < b$ to mean that a is contained in a bounded component of $P - b$.

Let a, b, and c be members of A, and suppose $a < b$ and $b < c$. Then, $a \subset \beta$ and $b \subset \gamma$, where β and γ are bounded components of $P - b$ and $P - c$ respectively. Since c is connected, γ must be simply connected. It follows that β, being a bounded component of $P - b$, is contained in γ. Thus $a \subset \gamma$ and $a < c$. This proves that $<$ partially orders A.

Next, suppose that a, b_1, and b_2 are members of A, $a < b_1$, and $a < b_2$. We will prove that either $b_1 \leqq b_2$ or $b_2 \leqq b_1$. There exist bounded components β_1 and β_2 of $P - b_1$ and $P - b_2$ respectively such that $a \subset \beta_1$ and $a \subset \beta_2$. If β_1 and β_2 have a common boundary point p, then $p \in b_1 \cap b_2$ and $b_1 = b_2$. Otherwise, either $\bar{\beta}_1 \subset \beta_2$ or $\bar{\beta}_2 \subset \beta_1$. In the first case, $b_1 \subset \beta_2$ and $b_1 < b_2$, and in the second case, $b_2 \subset \beta_1$ and $b_2 < b_1$.

Let $a \in A$. By Zorn's lemma, there exists $b \in A$ such that $a \leqq b$ and b is maximal with respect to the relation $<$. It follows from the result proved in the preceding paragraph that such a maximal b must be unique.

Now let $x \in D$. There is an element $a \in A$ such that $x \in a$, and there is a unique maximal $b \in A$ such that $a \leqq b$. We define $F(x) = g[b]$. We must show that F has the desired properties.

For $a \in A$, we define a^* to be the union of a and the bounded components of $P - a$. Each such a^* is a continuum which does not separate the plane. The set $\{F^{-1}(y) \mid y \in Y\}$ is seen to be the same as $\{a^* \mid a \text{ maximal in } A\}$. This latter set is easily proved to be upper semicontinuous, since A is, and hence F is continuous. If $x \in S^1$ and $x \in a \in A$, then a is maximal in A, and $F(x) = g[a] = g(x) = f(x)$. This proves that F is an extension of f.

A *dendrite* is a locally connected continuum which does not contain a simple closed curve. A space is *contractible* if and only if the identity mapping of

Received June 1, 1956.

the space into itself is homotopic (in the given space) to a constant mapping on the space.

LEMMA 2. *If K is a dendrite, then K is contractible.*

Proof. Let K be a dendrite. Then, there exists a sequence $A_1, A_2, \cdots$ of arcs in K and a set E of end points of K such that

$$K = E \cup \left(\bigcup_{n=1}^{\infty} A_n \right),$$

and

$$A_n \cap \left(\bigcup_{j=1}^{n-1} A_j \right) = \{p_n\},$$

where p_n is one of the end points of A_n, and (diameter A_n) $\to 0$ as $n \to \infty$. (See [4], p. 89.)

Let $T_n = \bigcup_{j=1}^{n} A_n$ for n a positive integer, and let $T_0 = \{p_1\}$. We now define a retraction r_n of K onto T_n for each nonnegative integer n. If $x \in K - T_n$, there exists a unique point x^* in K which can be joined to x by an arc, all of whose points except x^* are in $K - T_n$. We define $r_n(x) = x^*$ for $x \in K - T_n$, and $r_n(x) = x$ for $x \in T_n$.

It is easy to see that the sequence $r_0, r_1, \cdots$ of mappings converges uniformly to the identity mapping on K.

For each positive integer n, there is a homeomorphism h_n on $[0, 1]$ onto A_n such that $h_n(1) = p_n$. We define

$$R_n(t, x) = \begin{cases} x, & \text{if } x \in T_{n-1}\,; \\ x, & \text{if } x = h_n(s) \quad \text{for some } s, \qquad t \leqq s \leqq 1; \\ h_n(t), & \text{if } x = h_n(s) \quad \text{for some } s, \qquad 0 \leqq s < t. \end{cases}$$

Finally, we define

$$C(t, x) = \begin{cases} x, & \text{for } t = 0; \\ R_n(2^n t - 1, r_n(x)), & \text{for } 2^{-n} < t \leqq 2^{-n+1}. \end{cases}$$

It is easy to verify that the function C is a homotopy which contracts K onto the point p_1.

THEOREM 1. *Let f be a mapping on S^1 into a one-dimensional space Y. Then, f is inessential if and only if there exists a dendrite K and mappings f_1 and f_2 such that:*

(1) $f = f_2 f_1$,

(2) *f_1 maps S^1 onto K, and*

(3) *f_2 is a light mapping on K into Y.*

Proof. First, let us assume that K, f_1, and f_2 exist having the above properties (1), (2) and (3). By Lemma 2, there exists a contracting homotopy C

which contracts K onto a point p. We can define a homotopy H on $[0, 1] \times S^1$ by letting

$$H(t, x) = f_2(C(t, f_1(x))).$$

It follows that f is homotopic to a constant mapping, and hence is inessential.

Now, let us assume that f is inessential. By Lemma 1, there exists an extension $F: D \to Y$ of f such that none of the components of the inverse sets $F^{-1}(y)$, $y \in Y$, separates the plane P. We define P^* to be the set whose members are the components of the sets $F^{-1}(y)$, $y \in Y$, and the points of $P - D$. Then, P^* is an upper semicontinuous decomposition of P into continua, no one of which separates P. It follows from a theorem of R. L. Moore (see [3]) that the decomposition space P^* is homeomorphic to P. We let D^* be the subspace of P^* which consists of all members of P^* that are subsets of D.

It follows from the monotone-light factorization theorem (see [4], p. 141) that there exist mappings F_1 and F_2 such that:

(1) $F = F_2 F_1$,

(2) F_1 is a mapping on D onto D^*,

(3) F_2 is a light mapping on D^* into Y.

Since F_2 is light and Y is one-dimensional, it follows from a well known theorem from dimension theory (see [2], p. 91) that D^* is one-dimensional. D^* is a subset of the topological plane P^*, and it is easy to see that if D^* contained a simple closed curve, then the interior of the simple closed curve would also be a subset of D^*. Since D^* does not contain a two-dimensional set, it follows that D^* does not contain a simple closed curve. D^* is a locally connected continuum, since it is the continuous image of D, and hence D^* is a dendrite.

Now, we define $K = F_1[S^1]$. Since K is a subcontinuum of a dendrite, it is also a dendrite. (See [4], p. 89.)

We conclude the proof of our theorem by defining $f_1 = F_1 \mid S^1$ and $f_2 = F_2 \mid K$.

An alternate proof (of a quite different nature) of Theorem 1 can be obtained using results which appear in [1].

We define a mapping f on a topological space X to be *locally one-to-one* if and only if corresponding to each point p of X, there is a neighborhood N of p such that $f \mid N$ is one-to-one. The notion of "locally one-to-one mapping" is much more general than that of "local homeomorphism". For example, it is possible to map a simple closed curve onto a "figure eight curve" by a locally one-to-one mapping, but it is not possible to do this by a local homeomorphism.

THEOREM 2. *If Y is one-dimensional and $f: S^1 \to Y$ is locally one-to-one, then f is essential.*

Proof. Let us assume that f is not essential. Then, by Lemma 1, f has a continuous extension $F: D \to Y$ such that none of the components of the inverse sets $F^{-1}(y)$, $y \in Y$, separate P. We define P^*, D^*, F_1, and F_2 as in the proof of Theorem 1.

We now prove that D^* must contain a simple closed curve c. There are two cases to consider:

Case 1. Each member of D^* has at most one point in common with S^1. In this case, the set c of all members of D^* which have one point in common with S^1 is a simple closed curve in the space P^*.

Case 2. Some member of D^* has more than one point in common with S^1. In this case, there exists an arc u which is contained in S^1 and which has end points which belong to the same member of D^*. Since f is locally one-to-one, it is easy to see that there is a nondegenerate arc u_0 which is minimal with respect to these properties. The set c of all members of D^* which intersect u_0 nonvacuously is a simple closed curve in the space P^*.

It is easy to prove that the topological disk in P^* whose boundary is c must be contained in D^*, and hence D^* is two-dimensional. This is impossible, since F_2 is a light mapping of D^* into the one-dimensional space Y.

Since we have obtained a contradiction, the supposition that f is not essential must be false.

The author believes that Theorem 2 is of interest, since it gives a fairly weak local property which is sufficient for the essentiality of a mapping on S^1. E. E. Moise has constructed an example of a locally one-to-one mapping on S^2 which is not essential.

COROLLARY. *If K is a one-dimensional, locally connected continuum, then the fundamental group of K vanishes if and only if K is a dendrite.*

Proof. First, let us assume that K is a dendrite. It then follows from Lemma 2 that the fundamental group of K vanishes.

Next, let us assume that K is not a dendrite. Then K contains a simple closed curve, and there exists a homeomorphism f on S^1 into K. It follows from Theorem 2 that f is essential. Hence, the fundamental group of K does not vanish.

BIBLIOGRAPHY

1. M. L. CURTIS AND M. K. FORT, JR., *Homotopy groups of one-dimensional spaces*, Proc. Amer. Math. Soc., vol. 8 (1957), pp. 577–579.
2. W. HUREWICZ AND H. WALLMAN, *Dimension theory*, Princeton, 1941.
3. R. L. MOORE, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc., vol. 27 (1925), pp. 416–428.
4. G. T. WHYBURN, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942.

THE UNIVERSITY OF GEORGIA
ATHENS, GEORGIA