# SOME DUALITY THEOREMS[1]

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## 1. Basic notions

Certain concepts used in the theory of group representations apply equally to matrix-valued functions defined on a set $S$. For instance, if $f: S \to M_1$ and $g: S \to M_2$ where $M_i$ is the total matrix algebra over some field ($i = 1, 2$), then the Kronecker product $f \times g$ is defined just as for representations. Similarly, the concept of irreducibility also carries over. $f$ will be called irreducible if $f$ maps $S$ onto an irreducible set of matrices.[2]

Suppose $G$ is a compact topological group, and $R_1$, $R_2$ representations of $G$. According to a basic theorem, the Kronecker product $R_1 \times R_2$ "decomposes" into irreducible components. More precisely, there exist irreducible representations $P_1, P_2, \cdots, P_k$ of $G$, positive integers $m_1, m_2, \cdots, m_k$, and a nonsingular matrix $A$, such that

$$(1) \qquad R_1 \times R_2 = A \begin{pmatrix} \left.\begin{matrix} P_1 & & & \\ & P_1 & & \\ & & \ddots & \\ & & & P_1 \end{matrix}\right\} m_1 \text{ times} & & & \\ & \left.\begin{matrix} P_2 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_2 \end{matrix}\right\} m_2 \text{ times} & & \\ & & \ddots & \\ & & & \begin{matrix} P_k & & & \\ & P_k & & \\ & & \ddots & \\ & & & P_k \end{matrix} \end{pmatrix} A^{-1},$$

where the big matrix above is to be completed with zero matrices. We shall denote this matrix by $\Delta_{i=1}^{k} m_i P_i$.

Systems of matrix-valued functions which satisfy algebraic relations of the type (1) will be of interest. For this purpose we make the following definition.

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[1] This paper is part of a doctoral dissertation written at the University of Pennsylvania under the sponsorship of Professor N. J. Fine.

[2] For a general discussion of matrix-valued functions on a set $S$, see [1], Ch. VI where they are discussed under the name of "$S$-modules".

DEFINITION 1. Let $F = \{f_1, f_2, \cdots\}$ be a countable family of functions defined on a set $S$. $F$ will be called a *Kronecker system on* $S$ if

(a) $f_i : S \to G_i$ where $G_i$ is some group of matrices over the complex field.

(b) For each pair $(i, j)$ there exists a unique sequence of nonnegative integers $\{m_k^{ij}\}$ only a finite number of which are nonzero, and a nonsingular matrix $A_{ij}$ such that $f_i \times f_j = A_{ij}\, \Delta_{k=1}^{\infty}\, (m_k^{ij} f_k) A_{ij}^{-1}$.

(c) If $f \in F$, then $\bar{f} \in F$, where $\bar{f}$ is the complex conjugate of $f$.

DEFINITION 2. A Kronecker system $F$ on a set $S$ will be called *irreducible* if each of its elements is irreducible.

We remark that a general algebraic proposition implies that if $F$ is irreducible, the decomposition assumed in (b) of Definition 1 is necessarily unique up to the choice of the constant matrix $A_{ij}$ (see [1], p. 175).

DEFINITION 3. Let $F = \{f_1, f_2, \cdots\}$ be a Kronecker system on $S$. The *dual of* $F$ will be the totality $\mathfrak{M}$ of all mappings $M$ defined on $F$ such that

(a) If $f_i : S \to G_i$, then $M(f_i) \in G_i$.

(b) If $f_i \times f_j = A_{ij}\, \Delta_{k=1}^{\infty}\, (m_k^{ij} f_k) A_{ij}^{-1}$, then
$M(f_i) \times M(f_j) = A_{ij}\, \Delta_{k=1}^{\infty} \left(m_k^{ij} M(f_k)\right) A_{ij}^{-1}$.

(c) $M(\bar{f}_i) = \overline{M(f_i)}$.

LEMMA 1. *Let $F$ be a Kronecker system on a set $S$, and $\mathfrak{M}$ its dual. If $M_1, M_2 \in \mathfrak{M}$ and $f \in F$, define $(M_1 M_2)(f) = M_1(f) M_2(f)$. Under this operation, $\mathfrak{M}$ is a group.*

The proof of this lemma is a simple exercise (see [5]).

LEMMA 2. *Let $F = \{f_1, f_2, \cdots\}$ be a Kronecker system on a set $S$ such that, for every $i$, $f_i : S \to G_i$, a compact group of matrices. Then its dual group $\mathfrak{M}$ may be topologized in a natural way. Under this topology, $\mathfrak{M}$ is a compact topological group.*

*Proof.* The Cartesian product $G = G_1 \times G_2 \times \cdots$ is compact by Tychonoff's theorem. Map $\mathfrak{M}$ into $G$ by: $M \to (M(f_1), M(f_2), \cdots)$. We shall identify $\mathfrak{M}$ with a subset of $G$ in this way, omitting the identification map. Assign to $\mathfrak{M}$ the induced topology. Then $\mathfrak{M}$ becomes a topological subgroup of $G$, for the algebraic operation in $\mathfrak{M}$ coincides with that of $G$. Finally, to establish compactness, we need only show that $\mathfrak{M}$ is closed in $G$. This follows directly from the compactness of the groups $G_i$ and from the fact that the mappings

$$M \to \bar{M}; \quad (M, N) \to M \times N; \quad (M_1, M_2, \cdots, M_n) \to A\, \Delta_{k=1}^{n}\, (m_k M_k) A^{-1}$$

are all continuous on their respective spaces.

## 2. Kronecker systems and group representations.

N. J. Fine has proved recently [2] that a family of complex-valued functions defined on a measure space and satisfying certain algebraic conditions is essentially the set of characters of a compact abelian group. In this section, we extend these results to matrix-valued functions on non-abelian groups. In particular, we shall investigate conditions under which Kronecker systems of functions can be regarded as representations of their dual groups.

THEOREM 1. *Let* $(S, \Sigma, \mu)$ *be a measure space,* $\mu$ *a complete measure, and* $\mu(S) = 1$. *Suppose that* $\mathcal{R} = \{R_1, R_2, \cdots\}$ *is a countable family of functions defined on* $S$ *such that*

(a) $\mathcal{R}$ *is an irreducible Kronecker system.*

(b) $R_i : S \to U_i$, *the unitary group of degree* $n_i$, $(i = 1, 2, \cdots)$.

(c) *The set of all coefficients* $r^i_{\alpha\beta}(s)$ *of the matrices* $R_i(s)$ $(1 \leqq \alpha, \beta \leqq n_1;$ $i = 1, 2, \cdots)$ *is an orthogonal system with respect to* $\mu$.

*Then, there exists a mapping* $\varphi$ *of* $S$ *into its compact dual group* $\mathfrak{M}$ *such that*

(d) $\varphi(S)$ *is a dense subset of* $\mathfrak{M}$, *in fact, thick with respect to the normalized Haar measure* $\nu$ *on* $\mathfrak{M}$.

(e) *The functions* $\{R_i \varphi^{-1}\}$ *can be extended to form a full system of inequivalent, irreducible, unitary representations of* $\mathfrak{M}$.

(f) *For any* $\nu$*-measurable subset* $H$ *of* $\mathfrak{M}$, $\varphi^{-1}(H)$ *is* $\mu$*-measurable and* $\mu(\varphi^{-1}(H)) = \nu(H)$.

(g) $\varphi$ *is one-to-one if and only if the functions of* $\mathcal{R}$ *separate points of* $S$.

*Proof.* Without loss of generality, we may assume that the system $\mathcal{R}$ separates points of $S$. For otherwise, we could use the standard device of defining an equivalence relation among the points of $S$ by: $s_1 \sim s_2$ if and only if $R_i(s_1) = R_i(s_2)$ for $i = 1, 2, \cdots$. The given functions and measure can then be transferred to the set of equivalence classes.

Given $s \in S$, the mapping $M_s$ defined on $\mathcal{R}$ by: $M_s(R_i) = R_i(s)$ $(i = 1, 2, \cdots)$ is clearly an element of the dual group $\mathfrak{M}$. Since $\mathcal{R}$ separates points of $S$, $M_{s_1} = M_{s_2}$ if and only if $s_1 = s_2$. We may therefore identify $S$ with a subset of $\mathfrak{M}$. The composition of the two identification maps just defined is the mapping $\varphi$ in the statement of the theorem. During the course of this proof, we shall omit $\varphi$ and simply consider $S$ as a subset of $\mathfrak{M}$. In this way, if $H \subset \mathfrak{M}$, then $\varphi^{-1}(H)$ is identified with $H \cap S$.

A point $M$ of $\mathfrak{M}$ is of the form $(M(R_1), M(R_2), \cdots)$. Let $P_i$ be the projection of $M$ onto its $i^{\text{th}}$ component. $P_i : \mathfrak{M} \to U_i$ by: $P_i(M) = M(R_i)$. $P_i$ is continuous, being a projection. Furthermore, $P_i$ is a homomorphism since

$$P_i(M_1 M_2) = (M_1 M_2)(R_i) = M_1(R_i)M_2(R_i) = P_i(M_1)P_i(M_2).$$

Therefore the system $\mathcal{P} = \{P_i\}$ is a set of unitary representations of the compact group $\mathfrak{M}$. We shall show that $\mathcal{P}$ is a full system of inequivalent, irreducible representations of $\mathfrak{M}$, i.e. $\mathcal{P}$ contains exactly one member from each equivalence class of irreducible representations of $\mathfrak{M}$.

First, $P_i$ is an extension of the given function $R_i$ to all of $\mathfrak{M}$. For, given

$s \in S$, $P_i(s) = M_s(R_i) = R_i(s)$. Thus, $P_i$ agrees with $R_i$ on the subset $S$. It follows that $P_i$ is irreducible. For on the subset $S$ it coincides with $R_i$ which is assumed irreducible on $S$. The reducibility of $P_i$ would then imply the reducibility of $R_i$, contrary to assumption. Furthermore, $P_i$ and $P_j$ are inequivalent when $i \neq j$. If not, there would exist a constant matrix $A$ such that $P_j = AP_iA^{-1}$. Then the coefficients of $P_j$ would be linear combinations of those of $P_i$. But this is impossible, for on the subset $S$, the coefficients in question are assumed orthogonal. We have shown, therefore, that the set $\mathcal{P}$ is a system of inequivalent, irreducible, unitary representations of $\mathfrak{M}$. It remains to show that every irreducible representation of $\mathfrak{M}$ is equivalent to some element of $\mathcal{P}$.

Denote the coefficients of $P_i$ by $p^i_{\alpha\beta}$. Considered as functions on the compact group $\mathfrak{M}$, the set of all such coefficients ($1 \leqq \alpha, \beta \leqq n_i$; $i = 1, 2, \cdots$) is an orthogonal system in the Haar measure $\nu$ on $\mathfrak{M}$. We take $\nu$ to be normalized.

Let $\mathcal{A}$ be the set of all complex linear combinations of the functions $p^i_{\alpha\beta}$. We shall show that

(i) $\mathcal{A}$ is an algebra over the complex field.
(ii) $\mathcal{A}$ is closed under complex conjugation.
(iii) The functions of $\mathcal{A}$ separate points of $\mathfrak{M}$.
(iv) Given any point $M \in \mathfrak{M}$, not all functions of $\mathcal{A}$ vanish at $M$.

(i) It is enough to show that the product of any two functions $p^i_{\alpha\beta}$ and $p^j_{\gamma\delta}$ is an element of $\mathcal{A}$. This product occurs in the matrix $P_i \times P_j$. Now,

$$P_i(M) \times P_j(M) = M(R_i) \times M(R_j)$$
$$= A_{ij}\,\Delta_k(m^{ij}_k M(R_k))A^{-1}_{ij} = A_{ij}\,\Delta_k(m^{ij}_k P_k(M))A^{-1}_{ij}.$$

The coefficients on the left are linear combinations of the coefficients from a finite number of the $P_k$, hence elements of $\mathcal{A}$. Therefore $p^i_{\alpha\beta}\,p^j_{\gamma\delta} \in \mathcal{A}$.

(ii) Given any function $p^i_{\alpha\beta}$ it is enough to show $\overline{p^i_{\alpha\beta}} \in \mathcal{A}$. By assumption, for each $i$ there is a $j$ such that $R_j = \bar{R}_i$. Since $M(\bar{R}_i) = \overline{M(R_i)}$ for every $M \in \mathfrak{M}$, we have

$$P_j(M) = M(R_j) = M(\bar{R}_i) = \overline{M(R_i)} = \overline{P_i(M)}.$$

Comparing coefficients, $\overline{p^i_{\alpha\beta}} = p^j_{\alpha\beta} \in \mathcal{A}$.

(iii) $P_i$ is the projection of $\mathfrak{M}$ onto its $i^{\text{th}}$ coordinate. Two distinct points $M_1$ and $M_2$ must differ in some coordinate, say the $k^{\text{th}}$. This means $p^k_{\alpha\beta}(M_1) \neq p^k_{\alpha\beta}(M_2)$ for some pair $(\alpha, \beta)$, $1 \leqq \alpha, \beta \leqq n_k$. Therefore $\mathcal{A}$ separates points of $\mathfrak{M}$.

(iv) For every $M \in \mathfrak{M}$ and any $i$, $P_i(M)$ is a nonsingular matrix. Not all of its coefficients $p^i_{\alpha\beta}(M)$ can vanish.

The four properties of $\mathcal{A}$ just established are precisely the conditions of the Stone-Weierstrass Theorem. We may conclude that the set $\mathcal{A}$ is uniformly dense in the space of all continuous, complex-valued functions defined on $\mathfrak{M}$.

Now suppose there were an irreducible representation $Q$ of $\mathfrak{M}$ not equivalent to any element of $\mathcal{P}$. Its coefficients must be orthogonal (with respect to the Haar measure $\nu$) to the functions in $\mathcal{A}$. However, this is impossible since these coefficients, being continuous, can be uniformly approximated by functions of $\mathcal{A}$. Therefore no such $Q$ can exist, and assertion (e) of the theorem is proved.

Since $\mathcal{P}$ is a full set of irreducible representations, $\mathcal{P}$ contains a representation, say $P_1$, such that $P_1(M) = 1$ for every $M \in \mathfrak{M}$. It follows from assumption (c) that

$$\int_S r^i_{\alpha\beta}\, d\mu = \int_S p^i_{\alpha\beta}\, d\mu = \begin{cases} 1, & (i = 1), \\ 0, & (i > 1). \end{cases}$$

By the orthogonality relations for compact groups, it is also true that

$$\int_{\mathfrak{M}} p^i_{\alpha\beta}\, d\nu = \begin{cases} 1, & (i = 1), \\ 0, & (i > 1), \end{cases}$$

where $\nu$ is the normalized Haar measure on $\mathfrak{M}$. Therefore in all cases

$$\int_S p^i_{\alpha\beta}\, d\mu = \int_{\mathfrak{M}} p^i_{\alpha\beta}\, d\nu.$$

Since any continuous function $f$ on $\mathfrak{M}$ can be uniformly approximated by linear combinations of the $p^i_{\alpha\beta}$, a standard argument shows that

$$\int_S f\, d\mu = \int_{\mathfrak{M}} f\, d\nu \tag{2}$$

Now let $\bar{S}$ denote the closure of $S$. If $\bar{S}$ were a proper subset of $\mathfrak{M}$, there would exist by Urysohn's Lemma a continuous nonnegative function $f$ vanishing on $S$ but not vanishing identically on $\mathfrak{M}$. Then, from (2)

$$0 = \int_S f\, d\mu = \int_{\mathfrak{M}} f\, d\nu > 0,$$

a contradiction. Therefore $\bar{S} = \mathfrak{M}$, so that $S$ is dense in $\mathfrak{M}$.

Assertions (d) and (f) are proved by the technique employed in [2]. Since the argument of [2] carries over nearly word for word, we shall only outline it here. If $H$ is a closed subset of $\mathfrak{M}$, one can approximate the characteristic function of $H$ by a sequence of continuous functions. It then follows easily from equation (2) above that $\nu(H) = \mu(H \cap S)$. This relation is then extended to all Borel sets. Finally it is extended to all measurable subsets $H$ of $\mathfrak{M}$ using the regularity of Haar measure and the completeness of $\mu$. This finishes the proof of the theorem.

It has been brought to the attention of the author that there is a paper of Kreĭn [4] containing a result similar to Theorem 1. Kreĭn's point of departure is somewhat different from ours, however; it is what he calls a "block algebra." This is a commutative algebra over the complex field with a unit element and an involution, and which can be grouped into square matrices satisfying conditions similar to ours. He proves that such an algebra is the set of representative functions of a compact group. He does not deal with questions of measure.

As a corollary of Theorem 1, one can obtain the result of Fine [2] mentioned earlier. He postulates an orthonormal semigroup of functions defined on a decent measure space and closed under complex conjugation. Such a system is proved to be essentially the set of characters of a compact group and the given measure essentially the Haar measure on the group. This result follows from Theorem 1 in the following way.

It is easy to show that each function must assume values on the unit circle. But then the conditions of our theorem are satisfied. The $R_i$'s are unitary, Kronecker multiplication reduces to ordinary multiplication, and a Kronecker system is a multiplicative semigroup of functions with the properties assumed. The dual group in this case is clearly abelian.

A second corollary of Theorem 1 is the duality theorem of Tannaka. In this case the given measure space $(S, \Sigma, \mu)$ is a compact, second countable, topological group with normalized Haar measure $\mu$. $\mathcal{R}$ is a full system of inequivalent, irreducible, unitary representations of $S$. All conditions of Theorem 1 are clearly satisfied. The mapping $\varphi$ of $S$ into its compact dual group $\mathfrak{M}$ is easily seen to be a continuous homomorphism. Since $\mathcal{R}$ separates points, $\varphi$ is one-to-one. Since $S$ is compact and $\varphi(S)$ is dense in $\mathfrak{M}$, $\varphi$ is a homeomorphism onto. Therefore, $S$ and $\mathfrak{M}$ are isomorphic topological groups, which is equivalent to the statement of Tannaka's Theorem.

In the above argument, there is a slight technical difficulty which was pointed out by the referee. If $R \in \mathcal{R}$ and $\bar{R}$ is equivalent to $R$, then $\mathcal{R}$ does not satisfy condition (c) of Definition 1. Nevertheless, the proof of Theorem 1 still goes through. (c) is needed only to insure that the algebra $\mathcal{A}$ is closed under complex conjugation. But if $\bar{R} = ARA^{-1}$, the coefficients of $\bar{R}$ are linear combinations of those of $R$, hence elements of $\mathcal{A}$.

Let us observe that it is the algebraic structure of a Kronecker system which allows us to define the dual group and its representations. It is therefore of interest to divest the theorem of all considerations of measure and state it purely in algebraic terms.

THEOREM 2. *Let $\mathcal{R} = \{R_1, R_2, \cdots\}$ be an irreducible Kronecker system defined on a set $S$ such that*

(a) *$R_i: S \to G_i$, a compact matrix group (not necessarily unitary).*

(b) *If $i \neq j$, then $R_i$ is not equivalent to $R_j$, i.e. there exists no constant matrix $A$ such that $R_i = AR_jA^{-1}$.*

*Then, there is a mapping $\varphi$ of $S$ into its compact dual group $\mathfrak{M}$ such that*

(c) *The functions $\{R_i\varphi^{-1}\}$ can be extended to form a full system of inequivalent, irreducible representations of $\mathfrak{M}$.*

(d) *The subgroup generated by $\varphi(S)$ is dense in $\mathfrak{M}$.*

*Proof.* With one minor change the proof of assertion (c) is the same as in Theorem 1. The dual group $\mathfrak{M}$ is now a compact subgroup of the compact group $G_1 \times G_2 \times \cdots$. The Stone-Weierstrass argument goes through exactly as before. The only modification occurs in proving $P_i$ and $P_j$ are inequivalent when $i \neq j$. In Theorem 1 this is done by the assumed orthogonality of the coefficients $r^i_{\alpha\beta}$. Actually, the strength of orthogonality is not needed for this purpose; linear independence would suffice. Here we need only the even weaker assumption (b). If $P_i$ were equivalent to $P_j$ on $\mathfrak{M}$, they would be equivalent on the subset $S$, contradicting (b).

Now let $H$ be the closed subgroup of $\mathfrak{M}$ generated by $S$, and consider the functions $P_i$ restricted to $H$. Since $H$ is compact and these functions are extensions of the given functions $R_i$, the same reasoning as used above applies to $H$ as well as $\mathfrak{M}$. The conclusion is that we also have a full system of irreducible representations of $H$.

It now follows easily that $H = \mathfrak{M}$. For, every representation of a compact group is equivalent to a unitary representation. Therefore, there exist constant matrices $A_1, A_2, \cdots$ such that $\{Q_i = A_i P_i A_i^{-1}\}$ is a full system of irreducible unitary representations of $\mathfrak{M}$ and likewise for their restrictions to $H$. Let $\mu$ and $\nu$ denote the normalized Haar measures on $H$ and $\mathfrak{M}$ respectively. Then, by the orthogonality relations,

$$\int_H q^i_{\alpha\beta}\, d\mu = \int_{\mathfrak{M}} q^i_{\alpha\beta}\, d\nu; \qquad (Q_i = \| q^i_{\alpha\beta} \|).$$

The argument of Theorem 1 then yields $H = \mathfrak{M}$, proving (d).

It is of interest to try to drop the assumption of irreducibility in Theorem 2. We shall show that if this is done, we still obtain a Kronecker system $\mathcal{P}$ of representations of $\mathfrak{M}$. These representations are not necessarily irreducible. However, $\mathcal{P}$ is complete in the weaker sense that every irreducible representation of $\mathfrak{M}$ is contained in at least one element of $\mathcal{P}$.

THEOREM 3. *Let $\mathcal{R}$ be as in Theorem 2, but not necessarily irreducible. Then there is a mapping $\varphi$ of $S$ into the dual group $\mathfrak{M}$ such that*

(a) *The functions $\{R_i \varphi^{-1}\}$ can be extended to form a Kronecker system $\mathcal{P}$ of representations of $\mathfrak{M}$.*

(b) *Every irreducible representation of $\mathfrak{M}$ occurs as an irreducible component of at least one element of $\mathcal{P}$.*

*Proof.* The arguments of the preceding theorems show that $S$ may be imbedded in its compact dual group $\mathfrak{M}$ so that the set of projections $\mathcal{P} = \{P_i\}$ is a Kronecker system of representations of $\mathfrak{M}$, $P_i$ extending $R_i$ for each $i$. These representations are not necessarily irreducible however. Still, we may apply the Stone-Weierstrass Theorem as before and conclude that the ring $\mathfrak{A}$ is uniformly dense in the space of continuous functions on $\mathfrak{M}$. $\mathfrak{A}$ is again the set of all linear combinations of the coefficients $p^i_{\alpha\beta}$.

Let $\{K_1, K_2, \cdots\}$ be a full set of irreducible representations of $\mathfrak{M}$. Then

$$(3) \qquad P_i = A_i\, \Delta_j(m^j_i K_j) A_i^{-1}; \qquad (i = 1, 2, \cdots).$$

We must prove that for each $r = 1, 2, \cdots$, $K_r$ occurs in at least one of the expressions (3). Suppose this were not the case. By virtue of (3), each function $p^i_{\alpha\beta}$ is a linear combination of functions $k^i_{\alpha\beta}$. It follows that each element of $\mathfrak{A}$ is a linear combination of the $k^i_{\alpha\beta}$ where $i$ runs over some index set not containing $r$. Now the system $\{k^i_{\alpha\beta}\,;\, i = 1, 2, \cdots\}$ is orthogonal with respect to the Haar measure on $\mathfrak{M}$. Therefore, $k^r_{\alpha\beta}$ is orthogonal to every element of $\mathfrak{A}$. But this is a contradiction, for $k^r_{\alpha\beta}$ being continuous and not identically zero can be uniformly approximated by elements of $\mathfrak{A}$. Consequently for every $r$, $K_r$ must occur in one of the expressions (3). This establishes the theorem.

It is a basic result that the characters of irreducible representations of a compact group form an orthogonal system. With this in mind, it is natural to seek an analogue of Theorem 1 in which we assume orthogonality only for the traces of the given functions.

THEOREM 4. *Let* $(S, \Sigma, \mu)$ *be a measure space as in Theorem* 1, *and* $\mathfrak{R} = \{R_1, R_2, \cdots\}$ *a Kronecker system on* $S$ *such that*

(a) $\mathfrak{R}$ *is irreducible.*

(b) $R_i : S \to G_i$, *a compact group of matrices.*

(c) *If* $\psi_i$ *is the trace of* $R_i$, *the system* $\{\psi_i\}$ *is an orthogonal set of functions with respect to* $\mu$.

*Then there is a mapping* $\varphi$ *of* $S$ *into its compact dual group* $\mathfrak{M}$ *such that*

(d) *The functions* $\{R_i \varphi^{-1}\}$ *can be extended to form a full system of inequivalent irreducible representations of* $\mathfrak{M}$.

(e) *Let* $K_s$ *be the conjugate class of* $\mathfrak{M}$ *containing* $\varphi(s)$. *Then* $\bigcup_{s \in S} K_s$ *is dense in* $\mathfrak{M}$ *under the weak topology induced in* $\mathfrak{M}$ *by its characters.*

(f) *If* $H$ *is any* $\nu$*-measurable union of conjugate classes of* $\mathfrak{M}$, *then* $\varphi^{-1}(H)$ *is* $\mu$*-measurable and* $\mu(\varphi^{-1}(H)) = \nu(H)$.

*Proof.* The proof of (d) is as in Theorem 1, except for one minor change. This time, $P_i$ and $P_j$ are inequivalent when $i \neq j$ because of assumption (c). Their traces $\chi_i$ and $\chi_j$ must be extensions of $\psi_i$ and $\psi_j$ respectively. (c) implies $\chi_i \not\equiv \chi_j$ when $i \neq j$. But the traces of equivalent representations are identical.

To prove (e), we shall modify an argument used in Theorem 1. Let $Z$ denote the complex plane and $Z_0$ the countable Cartesian product $Z \times Z \times \cdots$. Define a mapping $\rho : \mathfrak{M} \to Z_0$ by:
$\rho(M) = (\chi_1(M), \chi_2(M), \cdots)$. $\rho$ is clearly continuous, and since $\mathfrak{M}$ is compact, $\mathfrak{N} = \rho(\mathfrak{M})$ is also compact.

$\mathfrak{N}$ may be thought of as the set $\mathfrak{M}$ under the weak topology induced by its characters. A subset $U$ of $\mathfrak{M}$ is open if and only if it is the complete inverse image under $\rho$ of an open subset of $\mathfrak{N}$. This is the weakest topology on $\mathfrak{M}$ under which its characters are continuous.

It is easily seen that there is a one-to-one correspondence between the continuous functions on $\mathfrak{N}$, and the continuous class functions on $\mathfrak{M}$, i.e. functions constant on each conjugate class. In fact, the correspondence is given by

$$f(M) = g(\rho(M)); \qquad (M \in \mathfrak{M}).$$

From assumption (c), and an argument already used several times, we may assert

$$\int_S \chi_i \, d\mu = \int_{\mathfrak{M}} \chi_i \, d\nu.$$

But since linear combinations of characters are uniformly dense among the continuous class functions, it follows that

$$\int_S f \, d\mu = \int_{\mathfrak{M}} f \, d\nu \tag{4}$$

for any continuous class function $f$ defined on $\mathfrak{M}$.

We assert that $\overline{\rho(S)} = \mathfrak{N}$, where the bar denotes closure. Suppose this were not so. Then, by Urysohn's Lemma, there would exist a continuous function $g$ on $\mathfrak{N}$ vanishing on $\rho(S)$ but not vanishing identically. The corresponding class function $f = g(\rho)$ would vanish on $S$ but not everywhere on $\mathfrak{M}$. As in the proof of Theorem 1, the existence of such a function contradicts relation (4). Therefore $\overline{\rho(S)} = \mathfrak{N}$, or equivalently, $S$ is dense in the weak topology described above. This establishes assertion (e).

Since $\rho$ is a continuous mapping of $\mathfrak{M}$ onto $\mathfrak{N}$, both Hausdorff spaces, it follows that $H_1$ is a compact subset of $\mathfrak{N}$ if and only if $H = \rho^{-1}(H_1)$ is a compact subset of $\mathfrak{M}$. An immediate consequence is that $H_1$ is a Borel set if and only if $H$ is a Borel set.

Now let $H_1$ be a compact subset of $\mathfrak{N}$ and $H = \rho^{-1}(H_1)$. If $\tau_1$ is the characteristic function of $H_1$, then $\tau = \tau_1(\rho)$ is the characteristic function of the (compact) set $H$. Again we use a technique of Fine [2]. By repeated use of Urysohn's Lemma, one can construct a bounded sequence of nonnegative continuous functions $\{g_n\}$ converging pointwise on $\mathfrak{N}$ to $\tau_1$. There is a corresponding sequence $\{g_n(\rho)\}$ of continuous class functions converging pointwise on $\mathfrak{M}$ to $\tau$. From (4),

$$\int_S g_n(\rho) \, d\mu = \int_{\mathfrak{M}} g_n(\rho) \, d\nu.$$

Letting $n \to \infty$,

$$\int_S \tau \, d\mu = \int_{\mathfrak{M}} \tau \, d\nu. \tag{5}$$

Since $\tau$ is the characteristic function of the set $H$, (5) is equivalent to the statement: $\mu(H \cap S) = \nu(H)$. This proves assertion (f) in the case when $H$ is a compact subset of $\mathfrak{M}$. By the usual arguments, this result can be extended to all Borel sets $H$ which are unions of conjugate classes of $\mathfrak{M}$ (complete inverse images under $\rho$ of Borel sets in $\mathfrak{N}$).

For brevity, let us call any union of conjugate classes of $\mathfrak{M}$ an invariant set. In order to complete the proof of the theorem, it will suffice to show that, given any $\nu$-measurable invariant set $H \subset \mathfrak{M}$, there exist invariant Borel sets $A$ and $B$ such that $A \subset H \subset B$ and $\nu(A) = \nu(B)$. For then, $A \cap S \subset H \cap S \subset B \cap S$, and as we have already shown, $\mu(A \cap S) = \nu(A) = \nu(B) = \mu(B \cap S)$. Hence, by the completeness of $\mu$, $H \cap S$ is $\mu$-measurable and $\mu(H \cap S) = \nu(H)$.

Let $H$, therefore, be a $\nu$-measurable invariant subset of $\mathfrak{M}$. By the regularity of Haar measure, $\nu(H) = \sup \nu(C)$ where $C$ runs over all compact subsets of $H$.[3] Hence, there exists a compact subset $C_n$ of $H$ such that $\nu(C_n) > \nu(H) - 1/n$, $(n = 1, 2, \cdots)$. Define $C_n^* = \bigcup mC_n m^{-1}$ where the union is over all $m \in \mathfrak{M}$. Clearly $C_n^*$ is invariant, and $C_n \subset C_n^* \subset H$. We assert that $C_n^*$ is closed. For let $\xi$ be a limit point of $C_n$ and $\{m_i c_n^i m_i^{-1}\}$ a sequence of points of $C_n$ converging to $\xi$. Now $\{m_i\}$ and $\{c_n^i\}$ are sequences in the compact sets $\mathfrak{M}$ and $C_n$ respectively. Without loss of generality, we may assume that they converge to points $m$ and $c_n$ of $\mathfrak{M}$ and $C_n$. By the continuity of multiplication, $\xi = \lim m_i c_n^i m_i^{-1} = mc_n m^{-1}$. But $mc_n m^{-1} \in C_n^*$ since $c_n \in C_n$. Therefore $C_n^*$ is closed (compact). Since $C_n \subset C_n^*$, $\nu(C_n^*) > \nu(H) - 1/n$. Define $A = \bigcup_{n=1}^{\infty} C_n^*$. Then $A$ is a Borel set (in fact, an $F_\sigma$) such that $A \subset H$ and $\nu(A) = \nu(H)$. By the "complementary" argument, there exists a $G_\delta$, $B$ such that $H \subset B$ and $\nu(H) = \nu(B)$. This establishes assertion (f), completing the proof of the theorem.

The following seems a natural question to ask. Given a Kronecker system, what can one say about the nature of its dual group? The next theorem gives some information in that direction. We shall need a lemma of Chevalley ([1], p. 196), which we paraphrase in terms of our definitions.

Lemma 3. *Let $\mathfrak{R}$ be a Kronecker system defined on a set $S$, and $\sigma(\mathfrak{R})$ the ring generated by the coefficients $r_{\alpha\beta}^i$. Denote by $\Omega$ the set of all homomorphisms of $\sigma(\mathfrak{R})$ into the complex field. If the functions $r_{\alpha\beta}^i$ are linearly independent, there is a one-to-one mapping of $\Omega$ onto the dual group $\mathfrak{M}$ as follows: To each $\omega \in \Omega$ assign the element $M \in \mathfrak{M}$ defined by: $M(R_i) = \| \omega(r_{\alpha\beta}^i) \|$.*

Theorem 5. *Let $\mathfrak{R} = \{R_1, R_2, \cdots\}$ be a Kronecker system on a set $S$ with the properties*

(a) *The coefficients $r_{\alpha\beta}^i$ are linearly independent.*

(b) *$\sigma(\mathfrak{R})$ is finitely generated.*

*Then the dual group $\mathfrak{M}$ is finite dimensional.*

*Proof.* Without loss of generality, we may assume that the coefficients of $R_1, R_2, \cdots, R_n$ generate $\sigma(\mathfrak{R})$. According to Lemma 3, each $M \in \mathfrak{M}$ corresponds to a homomorphism $\omega$ of $\sigma(\mathfrak{R})$. $\omega$ is determined by its value on the generators of $\sigma(\mathfrak{R})$. Therefore, $M$ is determined by its values on $R_1, R_2, \cdots, R_n$.

From the definition of a Kronecker system, $R_i: S \to G_i$ where $G_i$ is some group of matrices over the complex field. Previously, we associated with

[3] For the measure-theoretic concepts considered in this paper, see [3].

each element $M$ of $\mathfrak{M}$ the point $(M(R_1), M(R_2), \cdots)$ in $G_1 \times G_2 \times \cdots$. Now define a mapping

$$\lambda:(M(R_1), M(R_2), \cdots) \to (M(R_1), M(R_2), \cdots, M(R_n)).$$

$\lambda$ is clearly one-to-one and continuous both ways. Therefore $\mathfrak{M}$ is homeomorphic to a subgroup of $G_1 \times G_2 \times \cdots \times G_n$.

## 3. Finite Kronecker systems

Because certain simplifications occur when we consider finite Kronecker systems, it is possible to establish analogues of some of the theorems of the preceding section under weaker assumptions. The following theorem, for example, is an analogue of Theorem 1 which weakens the orthogonality to linear independence and does away with the assumption of irreducibility altogether.

THEOREM 6. *Let* $\mathfrak{R} = \{R_1, R_2, \cdots, R_m\}$ *be a system of functions defined on a set* $S$ *such that*

(a) $\mathfrak{R}$ *is a Kronecker system (not necessarily irreducible).*

(b) $R_i: S \to G_i$, *a compact group of matrices of degree* $n_i$.

*and either*

(c) *The coefficients* $r^i_{\alpha\beta}$ *are linearly independent functions on* $S$.

*or*

(d) $\mathfrak{R}$ *induces a finite number* $n$ *of equivalence classes in* $S$, *and* $\sum_{i=1}^m n_i^2 = n$.

*Then, the dual group* $\mathfrak{M}$ *is finite, and there exists a mapping* $\varphi$ *of* $S$ *onto* $\mathfrak{M}$ *such that* $\{R_i \varphi^{-1}\}$ *is a full system of irreducible representations of* $\mathfrak{M}$.

*Proof.* First perform the usual collapsing of each equivalence class in $S$ to a single point. We may therefore assume that $\mathfrak{R}$ separates points of $S$. Under assumption (d) this means that $S$ is identified with a finite set having $n$ elements.

We shall prove first that, under the assumptions of the theorem, the dual group $\mathfrak{M}$ is finite. As in the preceding theorems, $\mathfrak{M}$ is identified with a compact subgroup of $G_1 \times G_2 \times \cdots \times G_m$ and the projections $P_i$, $(i = 1, 2, \cdots, m)$ are representations of $\mathfrak{M}$, $P_i$ extending $R_i$. These representations may not be irreducible. However, the same argument used in the proof of Theorem 3 shows that every irreducible representation of $\mathfrak{M}$ occurs as an irreducible component of one of the $P_i$. Consequently, $\mathfrak{M}$ has only a finite number of irreducible representations.

We shall show that a compact second countable topological group $G$ having only a finite number of irreducible representations is finite. The coefficients of these representations form a complete orthogonal set in $L^2(G, \nu)$ where $\nu$ is the normalized Haar measure on $G$. Therefore, $L^2(G, \nu)$ is finite dimensional.

However, if $G$ is infinite, then $L^2(G, \nu)$ must be infinite dimensional. It suffices to prove the existence of a sequence $\{H_1, H_2, \cdots\}$ of disjoint subsets of $G$ all of positive $\nu$-measure. For then, the corresponding characteristic functions form an infinite orthogonal set in $L^2(G, \nu)$.

The Haar measure on $G$ is non-atomic. Therefore, since $\nu(G) = 1$, there exists a subset $H_1$ of $G$ such that $0 < \nu(H_1) < 1$. Its complement, $H_1^*$, has positive $\nu$-measure. By the same argument, there is a set $H_2 \subset H_1^*$ such that $0 < \nu(H_2) < \nu(H_1^*)$. Then there is a set $H_3 \subset (H_1 \cup H_2)^*$ such that $0 < \nu(H_3) < \nu((H_1 \cup H_2)^*)$. If $G$ is infinite, this process may be repeated indefinitely yielding a sequence $\{H_1, H_2, \cdots\}$ of disjoint subsets all having positive $\nu$-measure. Hence, $G$ must be finite.

In our case, the dual group $\mathfrak{M}$ is a compact Lie group with only a finite number of irreducible representations. Therefore $\mathfrak{M}$ is finite, hence compact in the discrete topology. By the usual Stone-Weierstrass argument, the coefficients $p_{\alpha\beta}^i$ span all complex-valued functions on $\mathfrak{M}$ (since all functions on $\mathfrak{M}$ are continuous, and uniform approximation is replaced by equality). For brevity, we shall denote the vector space of all complex-valued functions on a finite set $T$ by $V(T)$.

$S$ is also compact in the discrete topology. The same reasoning shows that the functions $r_{\alpha\beta}^i$ span $V(S)$. If we use assumption (c), they actually form a basis. Therefore, card $\{r_{\alpha\beta}^i\} = \dim V(S)$. Now the functions $p_{\alpha\beta}^i$ are also linearly independent, being extensions of the linearly independent functions $r_{\alpha\beta}^i$. Then, from what we have already shown, $\{p_{\alpha\beta}^i\}$ is a basis for $V(\mathfrak{M})$. Hence,

$$\text{card } S = \dim V(S) = \text{card } \{r_{\alpha\beta}^i\} = \text{card } \{p_{\alpha\beta}^i\} = \dim V(\mathfrak{M}) = \text{card } \mathfrak{M}.$$

Since $S \subset \mathfrak{M}$ and card $S =$ card $\mathfrak{M}$, we have $S = \mathfrak{M}$.

Using assumption (d) instead of (c),

$$\sum_{i=1}^m n_i^2 = n \ (= \text{card } S).$$

In other words, card $\{p_{\alpha\beta}^i\} = n$. Since $\{p_{\alpha\beta}^i\}$ spans $V(\mathfrak{M})$, $\dim V(\mathfrak{M}) \leqq n$. Therefore,

$$n \geqq \dim V(\mathfrak{M}) = \text{card } \mathfrak{M} \geqq \text{card } S = n.$$

Thus, card $S =$ card $\mathfrak{M}$, so that again $S = \mathfrak{M}$.

It remains to show that the representations $P_i$ are irreducible. Let $\{Q_1, Q_2, \cdots, Q_r\}$ be a full system of irreducible representations of $\mathfrak{M}$. Then,

$$P_i = A_i \, \Delta_{k=1}^r (m_k^i \, Q_k) A_i^{-1}, \qquad (i = 1, 2, \cdots, m). \tag{6}$$

The sets $\{p_{\alpha\beta}^i\}$ and $\{q_{\alpha\beta}^i\}$ are both bases for $V(\mathfrak{M})$. Therefore each $Q_j$ must appear in at least one of the expressions (6). Otherwise, the basis $\{p_{\alpha\beta}^i\}$ would be independent of some of the elements of the basis $\{q_{\alpha\beta}^i\}$. Let $c_i$ be the degree of $P_i$ and $d_i$ the degree of $Q_i$. Then

$$\sum_{i=1}^m c_i^2 = \sum_{i=1}^r d_i^2 = n.$$

Since the degree of $A_i \Delta(m_k^i Q_k)A_i^{-1}$ is $\sum m_k^i d_k$, we obtain by summing the squares of the degrees in (6),

$$n = \sum_i c_i^2 = \sum_i (\sum_k m_k^i d_k)^2. \tag{7}$$

Since each $Q_k$ occurs at least once in (6), the corresponding $d_k$ occurs at least once in (7). The multiplicities $m_k^i$ are nonnegative integers. Therefore,

$$n = \sum_i (\sum_k m_k^i d_k)^2 \geqq \sum_k d_k^2 = n. \tag{8}$$

Equality holds in (8) if and only if each of the inner summands consists of exactly one term, and each $d_k$ occurs exactly once with coefficient unity. In other words, given $i$, all multiplicities $m_k^i$ vanish except one, $m_{k_i}$; furthermore, $m_{k_i} = 1$ and $k_1, k_2, \cdots, k_m$ is a permutation of the numbers $1, 2, \cdots, m$. It follows that the expressions (6) must reduce to

$$P_i = A_i Q_{k_i} A_i^{-1}, \qquad (i = 1, 2, \cdots, m)\cdot$$

Thus, each $P_i$ is equivalent to an irreducible representation. This completes the proof.

We remark that there exist finite analogues to the other theorems of the preceding section. These can be established by the means used in the proof of Theorem 6. For instance, the following is the analogue of Theorem 3.

THEOREM 7. *Let $\mathcal{R} = \{R_1, R_2, \cdots, R_m\}$ be a Kronecker system on a set $S$ such that*

(a) *$R_i: S \to G_i$, a compact group of matrices.*

(b) *$R_i$ is not equivalent to $R_j$ when $i \neq j$.*

*Then, there exists a mapping $\varphi$ of $S$ onto a set of generators of the finite dual group $\mathfrak{M}$ such that*

(c) *The functions $\{R_i \varphi^{-1}\}$ can be extended to form a Kronecker system $\mathcal{P}$ of representations of $\mathfrak{M}$.*

(d) *Every irreducible representation of $\mathfrak{M}$ occurs as an irreducible component of at least one element of $\mathcal{P}$.*

## REFERENCES

1. C. CHEVALLEY, *Theory of Lie groups*, Princeton, 1946.
2. N. J. FINE, *On groups of orthonormal functions* (I), Pacific J. Math., vol. 5 (1955), pp. 51–59.
3. P. R. HALMOS, *Measure theory*, New York, 1950.
4. M. KREĬN, *A principle of duality for bicompact groups and quadratic block algebras*, Doklady Akad. Nauk SSSR (N.S.), vol. 69 (1949), pp. 725–728; Math. Reviews, vol. 11 (1950), p. 491.
5. T. TANNAKA, *Über den Dualitätssatz der nichtkommutativen topologischen Gruppen*, Tôhoku Math. J., vol. 45 (1939), pp. 1–12.
6. A. WEIL, *L'intégration dans les groupes topologiques et ses applications*, Paris, 1938.

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