UNDERPOLYNOMIALS AND INFRAPOLYNOMIALS[1,2]

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Introduction

If E is a point set of the z-plane containing at least n points, a polynomial $g(z) \equiv z^n + g_1 z^{n-1} + \cdots + g_n$ is called an *underpolynomial* of $f(z) \equiv z^n + f_1 z^{n-1} + \cdots + f_n$ on E provided we have $g(z) \not\equiv f(z)$ and

$$(1) \qquad |g(z)| < |f(z)| \qquad \text{on } E \text{ where } f(z) \neq 0,$$

$$(2) \qquad g(z) = f(z) \qquad \text{on } E \text{ where } f(z) = 0.$$

The polynomial $f(z)$ is called an *infrapolynomial* on E if it has no underpolynomials on E, a concept due to Fekete and von Neumann [1]. Infrapolynomials as such have been studied also by Fekete [1] and the present writers [3].

The importance of infrapolynomials lies primarily in the fact that a polynomial of the form $f(z) \equiv z^n + \cdots$ which minimizes (among all polynomials of that form) one of the classical norms ($p > 0$)

$$(3) \qquad \sup [\,|f(z)|, z \text{ on } E],$$

$$(4) \qquad \int_E |f(z)|^p \, |dz|,$$

$$(5) \qquad \iint_E |f(z)|^p \, dS,$$

must clearly be an infrapolynomial on E; of course for (4) or (5) to have a meaning, E must be rectifiable or have positive area. The extremal polynomials with norms (4) and (5), $p = 2$, are orthogonal on E, hence particularly important; they include the widely studied Legendre, Tchebycheff, and Jacobi polynomials if suitable weight functions are introduced.

If a set E consists of $n + 2$ points, an arbitrary function $F(z)$ to be approximated on E by a polynomial of degree n can be replaced on E by an equal polynomial $P(z)$ of degree $n + 1$, so the problem of best approximation to $F(z)$ on E is essentially the problem of studying the polynomial $z^{n+1} + \cdots$ of least norm on E [compare Motzkin and Walsh, 1, §8].

The object of the present paper is to investigate systematically the properties of the class of infrapolynomials of given degree on a bounded set. The strong inequality is important in (1) in its effect on the norm (3) but not

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on the norms (4) and (if E is the closure of an open set) also (5), so in §1 we study the relation of (1) to the inequality

$$(6) \qquad | g(z)| \leqq |f(z)| \text{ on } E.$$

If $f(z) \equiv z^n + \cdots$ is given, a polynomial $g(z) \equiv z^n + \cdots$ satisfying (6) is called a *weak underpolynomial* of $f(z)$ on E; a weak underpolynomial is not necessarily an underpolynomial. In §§2 and 3 we show that the class of infrapolynomials on a closed bounded set E is closed and connected. In §4 we study some special properties, and in §5 the finite generation (i.e. on finite subsets of E) of such polynomials. Convexity of the class is proved in §6 whenever it exists, and factorization is discussed in §7. Infrapolynomials on a real set have special properties of separation (§8), analogous to those previously established by the present writers [2] for a finite set, and also of finite generation (§9).

We remark that in this paper we study the class of all infrapolynomials, not excluding those which have zeros on the given point set; although the admission of the latter makes some proofs considerably more complicated, it is essential to a complete, rounded theory.

For frequent use in the sequel, we state several lemmas.

LEMMA 1. *For fixed n the set of all polynomials $f(z) \equiv z^n + \cdots$ whose zeros z_k satisfy such a relation as $| z_k | \leqq M$, is compact.*

That is to say, any infinite sequence of such polynomials admits a subsequence which converges uniformly on any bounded point set of the plane to a polynomial of the original set. If the polynomials are represented in real space of $2n$ dimensions by the real and pure imaginary parts of their zeros, they obviously form a compact set. The condition $| z_k | \leqq M$ is satisfied automatically if the $f(z)$ are infrapolynomials on a closed bounded set E containing at least n points, for then (Fejér; Fekete and von Neumann) the zeros of each $f(z)$ lie in the convex hull of E. A similar result, similarly proved but by Lagrange's interpolation formula, is

LEMMA 2. *The set of all polynomials $f(z) \equiv a_0 z^n + a_1 z^{n-1} + \cdots$ satisfying $| f(z_k) | \leqq M$ on a fixed set $\{z_k\}$ consisting of at least $n + 1$ points, is compact.*

Use of Lagrange's interpolation formula shows also that convergence on the set $\{z_k\}$ of a sequence of the polynomials of Lemma 2 implies uniform convergence on any closed bounded set of the plane.

Other lemmas refer to easily proved results on infrapolynomials themselves.

LEMMA 3. *Every factor of an infrapolynomial is also an infrapolynomial. More explicitly, if $f(z) \equiv g(z)h(z)$ is an infrapolynomial on a set E, $f(z) \equiv z^n + \cdots$, $h(z) \equiv z^\nu + \cdots$, $\nu < n$, so also is $h(z)$.*

If $h(z)$ is not an infrapolynomial on E, we have for some $h_1(z) \equiv z^\nu + \cdots$

the relations $|h_1(z)| < |h(z)|$ on E where $h(z) \neq 0$, and $h_1(z) = h(z) = 0$ on E where $h(z) = 0$. Then we also have $|g(z)h_1(z)| < |g(z)h(z)|$ on E where $f(z) \neq 0$, and $g(z)h_1(z) = g(z)h(z) = 0$ on E where $f(z) = 0$, so $f(z)$ has the underpolynomial $g(z)h_1(z)$, contrary to hypothesis.

LEMMA 4. *If $f(z) \equiv z^n + \cdots$ is an infrapolynomial on a set E, and if z_0 does not belong to E, then $f(z)(z - z_0)$ is an infrapolynomial on $E' = E + z_0$.*

If $f(z)(z - z_0)$ has an underpolynomial $F(z)$ on E', then $(z - z_0)$ is a factor of $F(z)$: $F(z) \equiv f_1(z)(z - z_0)$. We have $|f_1(z)(z - z_0)| < |f(z)(z - z_0)|$ on E' where $f(z)(z - z_0) \neq 0$, so we have $|f_1(z)| < |f(z)|$ on E where $f(z) \neq 0$. We have $f_1(z)(z - z_0) = f(z)(z - z_0) = 0$ on E' where $f(z)(z - z_0) = 0$, so we have $f_1(z) = f(z) = 0$ on E where $f(z) = 0$.

Lemma 4 is false if we omit the requirement that z_0 shall not belong to E, for if E consists of the vertices (z_0, z_1, z_2) of a triangle and we choose (Theorem 12) the infrapolynomial $f(z) \equiv z - \alpha$, where α is an interior point of the triangle, the polynomial $(z - \alpha)(z - z_0)$ is not an infrapolynomial on E; indeed $(z - \alpha')(z - z_0)$ is an underpolynomial, if α' is the projection of α on the side $z_1 z_2$.

LEMMA 5. *If $f(z)(z - z_0) \equiv z^n + \cdots$ is an infrapolynomial on a set E, and if z_0 is a point of E, then $f(z)$ is an infrapolynomial on $E' = E - z_0$.*

We omit the proof of this converse of Lemma 4. It is a consequence of Lemmas 4 and 5 that *if $g(z) \equiv z^k + \cdots$ has only simple zeros, constituting the set E_1 on E, then a necessary and sufficient condition that $f(z)g(z) \equiv z^n + \cdots$ be an infrapolynomial on E is that $f(z)$ be an infrapolynomial on $E - E_1$.*

We leave to the reader the proof of

LEMMA 6. *An infrapolynomial $f(z) \equiv z^n + \cdots$ on a set E is also an infrapolynomial on every set $E' = E + E_1$, where E_1 is arbitrary.*

If $f_1(z)$ is an infrapolynomial on a set E_1 and $f_2(z)$ an infrapolynomial on a set E_2, then $f_1(z)f_2(z)$ is not necessarily an infrapolynomial on $E_1 + E_2$. We exhibit the counterexample E_1: $(1 \pm i)$, E_2: $(-1 \pm i)$, $f_1(z) \equiv z - 1$, $f_2(z) \equiv z + 1$; then $f_1(z)$ and $f_2(z)$ are infrapolynomials on E_1 and E_2 respectively, yet z^2 is an underpolynomial of $f_1(z)f_2(z)$ on $E_1 + E_2$.

1. Strong and weak infrapolynomials

Let us say that $f(z) \equiv z^n + \cdots$ is a *strong infrapolynomial* on E if no $g(z) \equiv z^n + \cdots \not\equiv f(z)$ exists such that (6) holds on E; by contrast the infrapolynomial previously defined may be called a *weak infrapolynomial* on E. For closed bounded sets the distinction between strong and weak infrapolynomials disappears:

THEOREM 1. *On a closed bounded set E containing at least n points, a weak infrapolynomial of degree n is also a strong infrapolynomial.*

We suppose, then, that $f(z) \equiv z^n + \cdots$ is given, and no $g(z) \equiv z^n + \cdots$ exists satisfying (1) and (2); we prove that no $g(z)(\not\equiv f(z))$ exists satisfying (6), by assuming the contrary and reaching a contradiction.

In the case $n = 1$, we take $f(z) \equiv z - a$, $g(z) \equiv z - b$; then on E we have $| [f(z) + g(z)]/2 | < | f(z) |$ from (6) contrary to hypothesis, unless at some point z_0 of E we have $g(z_0) = f(z_0)$. In the latter case $z_0 - b = z_0 - a$, whence $g(z) \equiv f(z)$, contrary to hypothesis.

If $n > 1$ we use induction. We first assume that $f(z)$ and $g(z)$ have a common factor $\varphi(z)$. With $f(z) \equiv f_1(z)\varphi(z)$, $g(z) \equiv g_1(z)\varphi(z)$, where

$$\varphi(z) \equiv z^k + \cdots,$$

inequality (6) implies

$$| g_1(z) | \leqq | f_1(z) | \tag{7}$$

at every point of E other than the zeros of $\varphi(z)$ on E, so (7) holds, by the continuity of $f_1(z)$ and $g_1(z)$, at every point of E other than the isolated zeros of $\varphi(z)$ on E, namely on the closed bounded set E_1 containing at least $n - k$ points. The polynomials $f_1(z)$ and $g_1(z)$ are of degree $n - k$, and Theorem 1 (assumed true for polynomials of this degree) asserts that there exists some $g_2(z) \equiv z^n + \cdots$ with

$$| g_2(z) | < | f_1(z) |, \qquad z \text{ on } E_1,$$

except at the zeros of $f_1(z)$ on E_1, at which the equality sign holds. Then also

$$| g_2(z)\varphi(z) | < | f_1(z)\varphi(z) | \equiv | f(z) |$$

for z on E except in the zeros of $f(z)$, at which the equality sign holds. This contradiction completes the proof for $n > 1$ if $f(z)$ and $g(z)$ have a common factor.

With $n > 1$, if $f(z)$ and $g(z)$ have no common factor, we deduce from (6) with $g_2(z) \equiv [f(z) + g(z)]/2$

$$g_2(z) = f(z) \neq 0, \qquad z \text{ on } E_2 \text{ in } E, \tag{8}$$

$$| g_2(z) | < | f(z) |, \qquad z \text{ on } E - E_2, \tag{9}$$

where the set E_2 thus defined contains at most $n - 1$ points; if (8) holds in more than $n - 1$ points, we have $g_2(z) \equiv f(z) \equiv g(z)$.

There exists a polynomial $h(z) \equiv 0 \cdot z^n + \cdots$ for which $h(z) \equiv f(z)$ on E_2. At each point of E_2 we have $| g_2(z) - h(z) | = 0 < | f(z) |$, so in some open neighborhood E_3 of E_2 likewise

$$| g_2(z) - h(z) | < | f(z) |.$$

On $E \cdot E_3$ for $0 < \varepsilon \leqq 1$ we have

$$| \varepsilon[g_2(z) - h(z)] + (1 - \varepsilon)g_2(z) | = | g_2(z) - \varepsilon h(z) | < | f(z) |. \tag{10}$$

On the closed set $E - E_3$ inequality (9) is valid, whence for suitably chosen ε, $0 < \varepsilon < 1$,

$$| g_2(z) - \varepsilon h(z) | < | f(z) |. \tag{11}$$

Then by (10), inequality (11) holds at every point of E; this contradiction completes the proof of Theorem 1.

Although we have considered strong and weak infrapolynomials primarily on a closed bounded set E, those restrictions on E are obviously not necessary. We postpone to a later paper the consideration of unbounded sets, but now formulate the

COROLLARY. *If E is a bounded infinite set, a polynomial $f(z) \equiv z^n + \cdots$ is an infrapolynomial on E when and only when it is an infrapolynomial on the closure $\bar{E}$ of E.*

If $f(z)$ has no underpolynomial on $\bar{E}$, it has no underpolynomial on E, for an underpolynomial $g(z)$ on E would imply $| f(z) | \geqq | g(z) |$ on E and hence on $\bar{E}$, which contradicts Theorem 1. Conversely, if $f(z)$ has an underpolynomial on $\bar{E}$, it has the same underpolynomial on E.

Henceforth for a closed bounded set E we ignore the distinction between strong and weak infrapolynomials.

2. Closure of the set of infrapolynomials

By a zero of a polynomial $f(z)$ *of the third kind* on a (closed) set E, we understand a zero of $f(z)$ at a nonisolated point of E. We prove

THEOREM 2. *Let $f(z) \equiv z^n + \cdots$ be an infrapolynomial on a compact infinite set E. Set $f(z) \equiv g(z)h(z)$ where the zeros of $g(z) \equiv z^q + \cdots$ are precisely the zeros of $f(z)$ of the third kind on E, with the same multiplicities. Then $h(z)$ is an infrapolynomial on E.*

Conversely, if $g(z) \equiv z^q + \cdots$ has q zeros of the third kind on E (not necessarily distinct) and $h(z)$ is an infrapolynomial on E, then $f(z) \equiv g(z)h(z)$ is also an infrapolynomial on E.

The first part of Theorem 2 is immediate, by Lemma 3.

Conversely, suppose $f(z)$ has an underpolynomial $f_1(z)$ on E; we reach a contradiction. The polynomial $f_1(z)$ must vanish at all the zeros of $g(z)$ on E, and at each zero *to an order at least as high as that of the zero of* $g(z)$; otherwise $| f_1(z)/g(z) | \to \infty$ as z approaches such a zero of $g(z)$. Then $g(z)$ is a factor of $f_1(z)$: $f_1(z) \equiv g(z)h_1(z)$. On E we have $| f_1(z) | < | f(z) |$ except at the zeros of $f(z)$, where we have $f_1(z) = f(z)$; thus if E_1 denotes the set of zeros of $g(z)$, on $E - E_1$ we have

$$| h_1(z) | \leqq | h(z) |. \tag{12}$$

By allowing z on $E - E_1$ to approach an arbitrary point of E_1 we have (12) also on E_1, hence throughout E, so (by Theorem 1) $h(z)$ is not an infrapolynomial on E. This contradiction completes the proof.

It follows from Theorem 2 that *any polynomial all of whose zeros are zeros of the third kind on E is an infrapolynomial on E.*

We are now in a position to prove the main result of §2:

THEOREM 3. *If E is a closed bounded set containing at least n points, the set of infrapolynomials on E of degree n is closed.*

We prove a rephrasing of Theorem 3: *If the sequence of infrapolynomials $f_k(z)$ on E all of degree n converges uniformly to the polynomial $f_0(z)$ on E, then $f_0(z)$ is an infrapolynomial on E.*

Uniformity of convergence on every bounded set is a consequence of convergence on E by Lemma 2. All zeros of the $f_k(z)$ and of $f_0(z)$ lie in the convex hull of E. The zeros of $f_0(z)$ are precisely the limit points of the zeros of the $f_k(z)$ (Hurwitz). The zeros of the $f_k(z)$ can be separated into sequences approaching the respective distinct zeros of $f_0(z)$, whether the latter are simple or not. The sequences of zeros of the $f_k(z)$ approaching zeros of the third kind of $f_0(z)$ can be suppressed by dividing out the corresponding factors $g_k(z)$ (with unity the coefficient of the highest power of z) of the polynomials $f_k(z) \equiv g_k(z)h_k(z)$; the remaining factors $h_k(z)$ of the $f_k(z)$ are infrapolynomials on E (Lemma 3); to prove Theorem 3 it is, by the second part of Theorem 2, sufficient to prove that $h_0(z)$ is an infrapolynomial on E, where $f_0(z) \equiv g_0(z)h_0(z)$ and $h_0(z)$, the (necessarily uniform) limit of the $h_k(z)$ on E, has no zeros of the third kind on E. Let E_0 be the subset of E on which $h_0(z)$ vanishes, so that E_0 consists only of isolated points of E.

If $h_0(z)$ is not an infrapolynomial on E, it has an underpolynomial $p_0(z)$:

$$| p_0(z) | < | h_0(z) | \text{ on } E - E_0, \qquad | p_0(z) | = | h_0(z) | = 0 \text{ on } E_0 .$$

The function $| h_0(z) | - | p_0(z) |$ is continuous on the closed set $E - E_0$, so for a suitably chosen $\varepsilon(>0)$ we have

$$| p_0(z) | < | h_0(z) | - \varepsilon \text{ on } E - E_0 .$$

Then for sufficiently large k we have

$$| p_0(z) | < | h_k(z) | \text{ on } E - E_0 ,$$

$$| p_0(z) | = 0 \leqq | h_k(z) | \text{ on } E_0 ,$$

so $p_0(z)$ is an underpolynomial of $h_k(z)$ on E, contrary to hypothesis.

3. The set of infrapolynomials is connected

To study connectedness we need several preliminary results. The polynomial of best approximation in the sense of Tchebycheff with weight function depends continuously on the weight function:

THEOREM 4. *Let the function $f(z)$ be continuous on the closed bounded set E,* *let* $w_j(z)$ $(j = 0, 1, 2, \cdots)$ *be a function nonnegative on* E *and positive in at least* $n + 1$ *points of* E, *with* $\lim_{k\to\infty} w_k(z) = w_0(z)$ *uniformly on* E. *Let* $t_j(z)$ *be the (necessarily unique) polynomial of degree* n *of best approximation to* $f(z)$ *on* E *in the sense of Tchebycheff with weight function* $w_j(z)$:

$$\max\,[w_j(z)|\,f(z) - t_j(z)\,|,\, z \text{ on } E] = \mu_j \tag{13}$$

is the minimum of the corresponding expression over all polynomials of degree n. *Then we have* $\lim_{k\to\infty} t_k(z) = t_0(z)$ *uniformly on every compact set.*

The polynomials $t_k(z)$ are bounded in $n + 1$ points of E, so by Lemma 2 there exists a subsequence $t_{k_j}(z)$ converging uniformly on E to some polynomial $t_0^*(z)$ also of degree n. We set $\mu_0^* = \max\,[w_0(z)|\,f(z) - t_0^*(z)\,|,\, z \text{ on } E]$, and from the uniform convergence of the $w_k(z)$ and $t_{k_j}(z)$ on E there follows

$$\lim_{k_j\to\infty} \mu_{k_j} = \mu_0^*\,;$$

by the extremal property of $t_0(z)$ we have

$$\mu_0^* \geqq \mu_0\,.$$

If ε (>0) is given, the extremal property of $t_k(z)$ implies for k sufficiently large

$$\begin{aligned}\mu_k &\leqq \max\,[w_k(z)|\,f(z) - t_0(z)\,|,\, z \text{ on } E]\\ &\leqq \max\,[w_0(z)|\,f(z) - t_0(z)\,|,\, z \text{ on } E]\\ &\quad + \max\,[|\,f(z) - t_0(z)\,|,\, z \text{ on } E]\cdot\max\,[|\,w_k(z) - w_0(z)\,|,\, z \text{ on } E] \leqq \mu_0 + M\varepsilon,\end{aligned}$$

where M (independent of ε) is suitably chosen. Then we have

$$\mu_0^* \leqq \mu_0\,,$$

whence $\mu_0^* = \mu_0$. The uniqueness of $t_0(z)$ as extremal polynomial now implies $t_0^*(z) \equiv t_0(z)$. Thus every subsequence of the $t_k(z)$ admits a new subsequence approaching $t_0(z)$ uniformly on E, so the sequence $t_k(z)$ approaches $t_0(z)$ uniformly on E, and on any compact set.

Theorem 4 applies at once to the choice $f(z) \equiv z^{n+1}$, and we conclude $\lim_{k\to\infty} [f(z) - t_k(z)] = [f(z) - t_0(z)]$ uniformly on E, namely that the extremal polynomial $z^{n+1} + \cdots$ with Tchebycheff norm and weight function $w_k(z)$ on E approaches uniformly on E the extremal polynomial $z^{n+1} + \cdots$ with Tchebycheff norm and weight function $w_0(z)$ on E.

Theorem 4 refers to uniform convergence of a continuous weight function to another continuous weight function, whereas we shall need to consider also a noncontinuous weight function as limit.

THEOREM 5. *Let* $h(z) \equiv z^n + \cdots$ *be an infrapolynomial on a closed bounded set* E *containing at least* n *points, and for* $\varepsilon > 0$ *let* $h_\varepsilon(z) \equiv z^n + \cdots$ *be the polynomial* $z^n + \cdots$ *of least Tchebycheff norm on* E *with weight function*

$w_\varepsilon(z) \equiv 1/\{\max [| h(z) |, \varepsilon]\}$. *Then we have* $\lim_{\varepsilon\to 0} h_\varepsilon(z) = h(z)$ *uniformly on* E.

Theorem 5 is of interest as complementary to a remark due to Fekete [1], namely that if the infrapolynomial $h(z)$ has no zeros on E, it is the polynomial of least Tchebycheff norm on E with weight function $1/| h(z) |$; in Theorem 5, where $h(z)$ may have zeros on E, we have replaced this weight function by a continuous truncated one, and we recover $h(z)$ by a limiting process.

From the extremal property of $h_\varepsilon(z)$ we deduce for z on E

$$\max \frac{| h_\varepsilon(z) |}{\max [| h(z) | , \varepsilon]} \leqq \max \frac{| h(z) |}{\max [| h(z) | , \varepsilon]} \leqq 1,$$

$$| h_\varepsilon(z) | \leqq \max [| h(z) |, \varepsilon].$$

If E contains but n points, we have $h_\varepsilon(z) \equiv h(z) \equiv 0$ on E; in any other case, it follows by Lemma 2 that there exists a subsequence $h_{\varepsilon_k}(z)$ of the polynomials $h_\varepsilon(z)$ with $\varepsilon_k \to 0$, converging uniformly on E to some polynomial $h_0(z) \equiv z^n + \cdots$.

Let E_ε denote the subset of E on which we have $| h(z) | < \varepsilon$; then on $E - E_\varepsilon$ we have $| h_\varepsilon(z) | \leqq | h(z) |$, $| h_0(z) | \leqq | h(z) |$. This last inequality is valid for z in $E - E_{\varepsilon_k}$, hence is valid for every fixed z in E where $h(z) \neq 0$. If z does not lie in any $E - E_{\varepsilon_k}$, we have $| h_{\varepsilon_k}(z) | < \varepsilon_k$, $h_0(z) = 0 = h(z)$. Then $h_0(z)$ is a weak underpolynomial of $h(z)$ on E and therefore identical with $h(z)$. That is to say, every subsequence of the polynomials $h_\varepsilon(z)$ admits a subsequence converging to $h(z)$ uniformly on E, so the theorem follows.

In this theorem we may replace the constant ε by a positive function $\eta_\varepsilon \equiv \eta_\varepsilon(z)$, provided $\max [| h(z) |, \eta_\varepsilon(z)]$ is continuous and $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$.

The principal result of §3 is

THEOREM 6. *If E is a closed bounded set consisting of at least $n + 1$ points, the set of infrapolynomials on E of degree n is connected.*

Any two weight functions $w_1(z)$ and $w_2(z)$, positive and continuous on E, can be connected continuously by a one-parameter family of weight functions likewise positive and continuous on E; we need merely consider

$$\mu w_1(z) + (1 - \mu)w_2(z), \qquad 0 \leqq \mu \leqq 1.$$

Hence the polynomials $z^n + \cdots$ of least Tchebycheff norm on E with respective weight functions $w_1(z)$ and $w_2(z)$ (the polynomials may be arbitrary infrapolynomials not vanishing on E) can be connected continuously by a one-parameter family of polynomials each of least Tchebycheff norm on E and hence an infrapolynomial. Since the weight function $w_\varepsilon(z)$ of Theorem 5 is positive and continuous on E, and since Theorem 5 shows that the extremal polynomial $h_\varepsilon(z)$ can be connected continuously with the given infrapolynomial $h(z)$ by a one-parameter family of infrapolynomials, Theorem 6 follows. Indeed, we have shown that two given infrapolynomials can be connected to each other by a set of polynomials which depend continuously (Theorems 4 and 5) on one parameter, of which all polynomials but the first and last are polynomials of least Tchebycheff norm with suitable positive and continuous weight function on E.

Theorem 6 follows much more simply if $n = 1$, and also if the set E contains precisely $n + 1$ points, for in each of these two cases the totality of infrapolynomials is convex; see Theorems 12 and 13. However, in other cases this totality is not convex (compare Theorem 14), and thus some supplementary proof, such as the one given, is necessary.

Of course not every infrapolynomial $f(z)$ on a set E is a polynomial of least norm in the sense of Tchebycheff there; for instance $f(z) \equiv z$ is not such a polynomial on the set $\{z = 0, z = 1\}$, although obviously an infrapolynomial there; see [Motzkin and Walsh, 1, §1].

In the discussion of connectedness we may admit weight functions which are positive but which become infinite in an appropriate manner on E. For instance if $f(z)$ is an infrapolynomial of degree n on E, we may consider the Tchebycheff problem of minimizing

$$\max\,[|\,h(z)/f(z)\,|,\ z \text{ on } E] \tag{14}$$

over all $h(z) \equiv z^n + \cdots$; here if $f(z)$ has precisely the zeros $\alpha_1, \alpha_2, \cdots, \alpha_k$ on E, this condition is interpreted as requiring $h(z)$ to vanish in the α_j ; thus, if we set $h(z) \equiv h_1(z)(z - \alpha_1) \cdots (z - \alpha_k), f(z) \equiv f_1(z)(z - \alpha_1) \cdots (z - \alpha_k)$, this problem is that of minimizing

$$\max\,[|\,h_1(z)/f_1(z)\,|,\ z \text{ on } E],$$

and the new weight function $1/|\,f_1(z)\,|$ is continuous and different from zero on E. Since $f_1(z)$ as a factor of an infrapolynomial is also an infrapolynomial (Lemma 3), the unique solution is $h_1(z) \equiv f_1(z)$, and the unique solution of the former problem is $h(z) \equiv f(z)$; this remark is an extension of Fekete's remark to infrapolynomials which may have zeros on E. This extension is entirely natural if E is dense in itself, but for instance if E is finite, the interpretation of (14) is somewhat artificial; compare the counterexample given after Theorem 7.

4. Miscellaneous properties

In this same circle of ideas belongs also the following

THEOREM 7. *Let the closed bounded point set E be dense in itself, let*

$$h(z) \equiv z^n + \cdots$$

be an arbitrary polynomial, and let $h^*(z) \equiv z^n + \cdots$ and $h_\varepsilon(z) \equiv z^n + \cdots$ be respectively the polynomials of least Tchebycheff norm on E (cf. §3) with weight functions $1/h(z)$ and $w_\varepsilon(z) \equiv 1/\{\max\,[h(z),\ \varepsilon]\}$. Then we have

$$\lim_{\varepsilon \to 0} h_\varepsilon(z) = h^*(z)$$

uniformly on E.

By Lemma 1 every sequence ($\varepsilon \to 0$) of polynomials $h_\varepsilon(z)$ admits a subsequence converging uniformly on E to some polynomial $h_0(z) \equiv z^n + \cdots$; it is sufficient to prove, as we shall do, that every such $h_0(z)$ is identical with $h^*(z)$.

We remark for z on E (all maxima are taken over E)

$$\max \frac{|h_\varepsilon(z)|}{\max[|h(z)|, \varepsilon]} \leqq \max \frac{|h(z)|}{\max[|h(z)|, \varepsilon]} \leqq 1.$$

We proceed to establish for z on E

$$\left|\frac{h_0(z)}{h(z)}\right| = \lim_{\varepsilon\to 0} [w_\varepsilon(z) \, |h_\varepsilon(z)|] \leqq \limsup_{\varepsilon\to 0} [\max w_\varepsilon(z) \, |h_\varepsilon(z)|]. \tag{15}$$

If E_0 denotes the subset of E on which $h(z)$ vanishes, (15) is immediate on $E - E_0$ and consequently follows also at zeros of $h(z)$ of the third kind on E; there are no other zeros of $h(z)$ on E. If $g(z) \equiv z^n + \cdots$ is arbitrary, we have by (15)

$$\begin{aligned}\max \left|\frac{h_0(z)}{h(z)}\right| &\leqq \limsup_{\varepsilon\to 0} \{\max [w_\varepsilon(z) \, |h_\varepsilon(z)|]\} \\ &\leqq \limsup_{\varepsilon\to 0} \{\max [w_\varepsilon(z) \, |g(z)|]\} \\ &\leqq \max |g(z)/h(z)|,\end{aligned}$$

the last inequality by virtue of $w_\varepsilon(z) \leqq 1/|h(z)|$. Consequently $h_0(z) \equiv h^*(z)$.

Theorem 7 is false if we omit the restriction that E be dense in itself, as is shown by the following counterexample. Let E consist of the three points $(0, 1, 2)$, and choose $h(z) \equiv z^2$. We must have

$$\max |h^*(z)/h(z)| \leqq \max |h(z)/h(z)| = 1,$$

whence $h^*(z) \equiv z^2$. The polynomial $h_\varepsilon(z) \equiv z^2 + \cdots$ is an infrapolynomial, whose zeros then lie in the convex hull of E and separate the points of E (Theorem 21); the zeros of $h_\varepsilon(z)$ lie one in each of the intervals $0 \leqq x \leqq 1$, $1 \leqq x \leqq 2$, so $h_\varepsilon(z) \to h^*(z)$ is impossible.

Still another result of this nature is of interest:

THEOREM 8. *Let E be a bounded point set containing at least $n + 1$ points, and let $f(z) \equiv z^n + \cdots$ not be an infrapolynomial on E; then $f(z)$ has an underpolynomial which is an infrapolynomial.*

Let $g(z)$ be an underpolynomial of $f(z)$ on E, and let $\{z_1, z_2, \cdots, z_{n+1}\}$ be distinct points in E. The set $\{\varphi(z) \equiv z^n + \cdots\}$ of all weak underpolynomials of $g(z)$ on E is compact. Let $g_1(z) \equiv z^n + \cdots$ be a weak underpolynomial of $g(z)$ on E such that $|g_1(z_1)| \leqq$ all $|\varphi(z_1)|$. Generally, let $g_k(z)$ be a weak underpolynomial of $g(z)$ on E such that $|g_k(z_j)| = |g_j(z_j)|$

for $j = 1, 2, \cdots, k-1$, and $|g_k(z_k)| \leqq |\varphi(z_k)|$ for all $\varphi(z)$ which are weak underpolynomials of $g(z)$ on E satisfying

$$|\varphi(z_j)| = |g_j(z_j)| \qquad \text{for } j = 1, 2, \cdots, k-1.$$

We prove that $g_{n+1}(z)$ *is an infrapolynomial* on E. Otherwise $g_{n+1}(z)$ has an underpolynomial $h(z) \equiv z^n + \cdots$ on E, also an underpolynomial of $g(z)$ on E. The inequality $|h(z_1)| < |g_{n+1}(z_1)|$ is impossible, by the definition of $g_{n+1}(z)$, so $h(z_1) = g_{n+1}(z_1) = 0$. Similarly we have $h(z_2) = \cdots = h(z_{n+1}) = 0$, whence $h(z) \equiv 0$, a contradiction. This contradiction shows that $g_{n+1}(z)$ is an infrapolynomial on E; $g_{n+1}(z)$ is a weak underpolynomial of $g(z)$, hence an underpolynomial of $f(z)$ on E, as we were to prove.

This proof of Theorem 8 applies when suitably modified in more general situations, for instance in the study of *nearest polynomials* approximating to a given continuous function on E, and to the analogue of Theorem 8 in any finite number of dimensions, replacing moduli of polynomials by products of distances. For the sake of possible other generalizations we give a second proof of Theorem 8.

Set $f(z) \equiv g(z)t(z)$, where $t(z) \neq 0$ on E and all zeros of $g(z) \equiv z^m + \cdots$ belong to E, forming a set E_g.

Define the polynomial $t^*(z) \equiv z^{n-m} + \cdots$ as minimizing

$$\max\,[|t^*(z)/t(z)|, z \text{ in } E - E_g];$$

then $t^*(z)$ is an infrapolynomial on $E - E_g$, and either $t^*(z) \equiv t(z)$, or $t^*(z)$ is an underpolynomial of $t(z)$ on $E - E_g$. Correspondingly, for

$$f^*(z) \equiv g(z)t^*(z),$$

either $f^*(z) \equiv f(z)$, or $f^*(z)$ is an underpolynomial of $f(z)$ on E. No underpolynomial of $f^*(z)$ on E can be divisible by $g(z)$. For if $f_1(z) \equiv g(z)t_1(z)$ were an underpolynomial of $f^*(z)$ on E, then $t_1(z)$ would be an underpolynomial of $t^*(z)$ on $E - E_g$. If $f^*(z)$ is not an infrapolynomial on E, choose an underpolynomial $f_1(z)$ of $f^*(z)$, and similarly define $g_1(z)$, $f_1^*(z)$, $f_2(z)$, $g_2(z)$, $f_2^*(z)$, $\cdots$, where $f_{k+1}(z)$ is an underpolynomial of $f_k^*(z)$, and $f_k^*(z)$ an underpolynomial of, or identical with, $f_k(z)$. The $g_k(z)$ are all different, for

$$g_\mu(z) \equiv g_\nu(z), \qquad \nu > \mu,$$

would imply that $f_\mu^*(z)$ has an underpolynomial $f_\nu^*(z)$ on E divisible by $g_\mu(z)$. Obviously $E_g \subset E_{g_1} \subset \cdots$. However only finitely many polynomials $g(z)$ of degree less than or equal to n have the same set of zeros (multiplicities ignored) E_g; since E_{g_k} has at most n points, the procedure must break off, that is, an infrapolynomial of $f^*(z)$ and of $f(z)$ must be reached.

Even though (Theorem 1) the existence of a weak underpolynomial of a given polynomial $f(z)$ implies the existence of an underpolynomial in the sense of (1) and (2), the totality of weak underpolynomials deserves some attention.

THEOREM 9. *If E is a closed bounded set consisting of at least $n + 1$ points, and $f(z) \equiv z^n + \cdots$ is a given polynomial, the weak underpolynomials $\varphi(z)$ of $f(z)$ on E form, in the ($2n$-dimensional) vector space of all polynomials $z^n + \cdots$, determined by their coefficients, a closed bounded convex set. If $f(z)$ is not an infrapolynomial on E, then the underpolynomials $\psi(z)$ of $f(z)$ on E form a bounded convex set, and every weak underpolynomial of $f(z)$ is a limit of underpolynomials $\psi_k(z)$. In particular $f(z)$ itself is a limit of such underpolynomials.*

The boundedness and closure of the set $\varphi(z)$ follow by Lemma 2. To establish convexity, we notice that $|\varphi_0(z)| \leqq |f(z)|$ and $|\varphi_1(z)| \leqq |f(z)|$ on E imply for $\varphi_\theta(z) \equiv \theta\varphi_1(z) + (1 - \theta)\varphi_0(z)$ the inequality $|\varphi_\theta(z)| \leqq |f(z)|$. We remark too that in this last inequality, the equality sign can hold in n points of E only if it holds in both preceding inequalities at those same n points; whence we have $\varphi_\theta(z) \not\equiv f(z)$ unless both $\varphi_0(z) \equiv f(z)$ and $\varphi_1(z) \equiv f(z)$. The remainder of the theorem follows by the same method.[3]

5. Finite generation

An infrapolynomial on a set E is said to be *finitely generated* on E if it is an infrapolynomial on some finite subset of E. We prove

THEOREM 10. *An infrapolynomial $z^n + \cdots$ on a closed bounded set E, with no zeros of the third kind there, is finitely generated on E, namely, it is also an infrapolynomial on some subset of E containing no more than $2n + 1$ points.*

We use induction on the degree n ($\geqq 1$) of the infrapolynomial $f(z)$. If $f(z)$ has no zero on E, the conclusion is known [Fekete, 1]. If $f(z)$ has at least one zero ζ on E, then for $n = 1$ the polynomial $f(z)$ is obviously an infrapolynomial on $\{\zeta\}$; for $n > 1$ the polynomial $f(z)/(z - \zeta)$ is by Lemma 5 an infrapolynomial on the closed set $E - \zeta$, and by the induction hypothesis is also an infrapolynomial on a finite subset E_1 of $E - \zeta$ of at most $2n - 1$ points. But then by Lemma 4, $f(z)$ is an infrapolynomial on the finite subset $E_1 + \zeta$ of E.

We prove also the

COROLLARY. *An infrapolynomial $f(z)$ on an infinite closed bounded set E is also an infrapolynomial on a closed bounded proper subset of E.*

By Theorem 2 we write $f(z) \equiv g(z)h(z)$, where the zeros of $g(z)$ are the zeros of $f(z)$ of the third kind on E and $h(z)$ is an infrapolynomial on E. By Theorem 10, $h(z)$ is an infrapolynomial on a finite subset E_1 of E. Given ε (>0), let E'_ε consist of E_1 plus the points of E in the closed ε-neighborhoods of the zeros of the third kind of $f(z)$ on E; we choose ε so small that $E' = E'_\varepsilon \neq E$. Then $h(z)$ is an infrapolynomial also on E', by Lemma 6, so the conclusion follows by the second part of Theorem 2.

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[3] One proves also that the set of underpolynomials of $f(z)$ on E is *whole-faced* [Motzkin, 1, p. 16; Fenchel, 1].

Theorem 10 is false if we omit the restriction that $f(z)$ shall have no zeros of the third kind on E, as is shown by the counterexample

$$f(z) \equiv z^2, \qquad E: 0 \leqq z \leqq 1.$$

No infrapolynomial on a finite subset of E can have a multiple zero at $z = 0$ (compare Theorem 21 below), so $f(z) \equiv z^2$ is not an infrapolynomial on such a subset.

Even if $f(z)$ has zeros of the third kind on E, there is a result related to Theorem 10:

THEOREM 11. *An infrapolynomial on a closed bounded set E is the limit of finitely generated infrapolynomials on E.*

We give the proof by induction on the degree n ($\geqq 1$) of the given polynomial $f(z)$. If $f(z)$ has no zero of the third kind, Theorem 10 yields our conclusion. If $f(z)$ has at least one zero ζ which is a nonisolated point of E, then for $n = 1$ the polynomial $f(z)$ is an infrapolynomial on the set $z = \zeta$; for $n > 1$ the polynomial $f(z)/(z - \zeta)$ is an infrapolynomial on E (Lemma 3), and by the induction hypothesis is the limit on E of a sequence $f_j(z)$ of infrapolynomials on the finite subsets E_j of E. If the sequence of points z_j in E but not in E_j approaches ζ, the sequence of (Lemma 4) infrapolynomials $(z - z_j)f_j(z)$ on the finite set $E_j + z_j$ approaches $f(z)$ on E.

A consequence of Theorem 11 is that *the given infrapolynomial $f(z)$ on E is the limit of a sequence of infrapolynomials $f_k(z)$ on suitably chosen finite subsets E_k of E which approach E.* For to an arbitrary finite subset of E on which $f_k(z)$ is an infrapolynomial, we may (Lemma 6) adjoin at pleasure other points of E without altering the fact that $f_k(z)$ is an infrapolynomial; the enlarged subsets E_k may be chosen to approach E.

However, *if a sequence of point sets E_k approaches a set E, it is not true that the limit of a sequence $f_k(z)$ of infrapolynomials on E_k is necessarily an infrapolynomial on E.* As a counterexample we exhibit $f(z) \equiv f_k(z) \equiv z^2$, $E_k: (|\, x \,| \leqq \varepsilon < 1, 1, 2)$ with $\varepsilon \to 0$ as $k \to \infty$, $E: (0, 1, 2)$; compare Theorem 21 below.

6. Cases of convexity of the class of infrapolynomials

Given a set E, a polynomial all of whose zeros are simple and lie on E is called *fundamental*; a polynomial none of whose zeros lies on E is called *proper*. The relation of fundamental polynomials to infrapolynomials is close.

THEOREM 12. *If E is an arbitrary set, the totality of infrapolynomials on E of degree one is the set $z - \zeta$, where ζ lies in the convex hull H of E; this totality is precisely the totality $\sum \mu_k(z - z_k)$, $\mu_k \geqq 0$, $\sum \mu_k = 1$, where the finitely many points z_k belong to E and if E is finite may be taken in the summation as all the points of E.*

We consider only the nontrivial case, that E contains more than one point.

If $z - \zeta$ is an infrapolynomial on E, then ζ lies in H. Conversely, if ζ lies in H and if $z - \zeta$ has the underpolynomial $z - \zeta'$, then $\zeta' \neq \zeta$, and by (1) all points of E lie nearer to ζ' than to ζ, so E lies in the open half plane containing ζ' bounded by the perpendicular bisector of the segment $\zeta\zeta'$, contrary to the assumption that ζ lies in H.

The set H consists precisely of those points ζ such that $\zeta = \sum \mu_k \geqq 0$, $\sum \mu_k = 1$, z_k in E, so the set of polynomials $z - \zeta$ is the corresponding set $z - \zeta = \sum \mu_k(z - z_k)$.

THEOREM 13. *A polynomial $f(z) \equiv z^n + \cdots$ of degree n is an infrapolynomial on a given set E consisting of $n + 1$ distinct points $z_1, z_2, \cdots, z_{n+1}$ if and only if it can be expressed*

$$\begin{aligned} f(z) &\equiv \sum \lambda_i \,\omega_i(z), \qquad \lambda_i \geqq 0, \qquad \sum \lambda_i = 1, \\ \omega_i(z) &\equiv \omega(z)/(z - z_i), \qquad \omega(z) \equiv \prod (z - z_i). \end{aligned} \tag{16}$$

Thus the class of infrapolynomials of degree n is identical, not only with the class (16), *but also with the extremal polynomials on E of degree n with unprescribed weights and norm*

$$\sum_1^{n+1} \mu_i |\, f(z_i) \,|, \qquad \mu_i > 0.$$

The identity of the class (16) with these extremal polynomials is known [Motzkin and Walsh, 2, Corollary to Theorem 8], so every polynomial represented by (16) is an infrapolynomial on E. Conversely, every infrapolynomial $f(z) \equiv z^n + \cdots$, if proper, minimizes the norm [max $|\, h(z)/f(z) \,|$, z on E] over the class $h(z) \equiv z^n + \cdots$ (Fekete), hence [Motzkin and Walsh, 1, §1] is of form (16). However, if $f(z)$ is improper, say $f(z_1) = 0$, then by Lemma 5 it follows that $f(z)/(z - z_1)$ is an infrapolynomial on the set $(z_2, z_3, \cdots, z_{n+1})$; Theorem 13 then follows by induction, since it is known to be true (Theorem 12) for $n = 1$.

COROLLARY 1. *A polynomial $f(z) \equiv z^n + \cdots$ is a proper infrapolynomial on the set E of Theorem* 13 *if and only if* (16) *holds with all $\lambda_i > 0$.*

The polynomial $\sum \lambda_i \,\omega_i(z)$ vanishes for $z = z_j$ if and only if $\lambda_j = 0$.

COROLLARY 2. *The locus of all zeros of all proper infrapolynomials of degree n on E: $(z_1, z_2, \cdots, z_{n+1})$ is the relative interior of the convex hull H of E less the set E.*

We use the term *relative interior* to distinguish the two cases, that H is a line segment, or not. The fact that all zeros lie interior to $H - E$ follows by (16) from [Walsh, 1, §1.3.2, Corollary]. The fact that any point ζ interior to $H - E$ is such a zero follows by remarking that there exist numbers μ_k (>0) such that $\sum \mu_k(\zeta - z_k) = 0$, where the summation extends over all points of E; whence $\sum \mu_k |\, \zeta - z_k \,|^2/(\zeta - z_k) = 0$, which is $f(\zeta) = 0$, where $f(z)$ is given by (16) with $\lambda_k = \mu_k |\, \zeta - z_k \,|^2 > 0$.

It follows from Corollary 2 that *if E is an arbitrary closed bounded set containing at least $n + 1$ points, and if ζ lies in the convex hull of E, then ζ is a zero of a suitably chosen infrapolynomial on E of degree n.* If ζ lies on E, it is the zero of the infrapolynomial $z - \zeta$ of degree one on the set $\{\zeta\}$; if ζ does not lie on E, it is either the zero of the proper infrapolynomial $z - \zeta$ of degree one on a subset $\{z_1, z_2\}$ of E, or a zero of a proper infrapolynomial of degree two on a subset $\{z_1, z_2, z_3\}$ of E. The conclusion now follows by adjoining, as necessary, to this subset of E suitable new points of E, and simultaneously adjoining the corresponding linear factors to the infrapolynomial (Lemma 4), until we have an infrapolynomial of degree n on some subset of E; this polynomial is also an infrapolynomial on E (Lemma 6). We cannot assume here that E has merely n points ($n > 1$); a point not in E is not a zero of the only infrapolynomial of degree n which must vanish in all points of E.

If E is finite and real, the class of infrapolynomials on E is the class of polynomials of least p^{th} power norm on E with $p = 1$ [Motzkin and Walsh, 2, Theorem 5.3]. *This conclusion is false for arbitrary finite nonreal E,* as is shown by the example E: $\{z_1, z_2, z_3\}$, where these three points are not collinear, and the infrapolynomial $z - \zeta$ is considered with ζ on the segment $z_1 z_2$. Infinitesimal motion of ζ toward the interior of the triangle perpendicular to $z_1 z_2$ increases $|z_1 - \zeta|$ and $|z_2 - \zeta|$ by infinitesimals of the second order, yet decreases $|z_3 - \zeta|$ by an infinitesimal of the first order, hence decreases $\sum \mu_k |z_k - \zeta|$, $\mu_k > 0$, so $z - \zeta$ (although an infrapolynomial on E) is not extremal.

We turn now to the consideration of the convexity of the class of infrapolynomials of given degree n on a given set E. In the cases $n = 0$ and E consisting of precisely n points, the class contains but a single element and is trivially convex.

Theorems 12 and 13 express the fact, in the respective cases $n = 1$, and n arbitrary with E consisting of $n + 1$ points, that *the class of infrapolynomials of degree n is the convex family having as basis the fundamental polynomials of degree n.* There exists no other case whatever where the class of infrapolynomials of degree n is convex (without regard to the fundamental polynomials as basis), no matter what the set E may be:

THEOREM 14. *Let E be a bounded set containing at least $n + 2$ distinct points. Then the class of infrapolynomials of degree n (>1) on E is not convex.*

We shall exhibit two fundamental polynomials $f_1(z)$ and $f_2(z)$ of degree n (hence infrapolynomials) and a positively weighted mean

$$\lambda f_1(z) + (1 - \lambda) f_2(z), \qquad 0 < \lambda < 1,$$

which has a zero outside the convex hull H of E and therefore is not an infrapolynomial on E. It is sufficient to find such polynomials

$$f_1(z) \equiv (z - z_1)(z - z_2), \qquad f_2(z) \equiv (z - z_3)(z - z_4)$$

of degree 2, and then to multiply them both by $(z - z_5) \cdots (z - z_{n+2})$, where the z_k are distinct points of E, to obtain the desired fundamental polynomials of degree n.

We distinguish two cases, according to whether the boundary B of H contains a line segment belonging to H, or not.

In case 1, choose z_1 and z_3 in E so that their segment s_{13} is part of B, and choose z_2 and z_4 in E not on that segment; this is always possible. Let L be a line separating z_1 and z_3, and z_2 and z_4, but not z_1 and z_2; this too is always possible but may require a change of notation. Let L_1 be half of L exterior to H bounded by the intersection ζ_0 of L with s_{13}. We assume, as we may do, that L_1 is the negative real axis with z_1 and z_2 in the upper half plane. We consider the function $\alpha(\zeta) = \arg [f_1(\zeta)/f_2(\zeta)]$ as ζ traces L_1. As $\zeta \to -\infty$, $\alpha(\zeta) \to 0$; as ζ increases monotonically from $-\infty$, each of the expressions $\arg (z_1 - \zeta)$, $\arg (z_2 - \zeta)$, $-\arg (z_3 - \zeta)$, $-\arg (z_4 - \zeta)$ increases continuously and monotonically, so their sum $\alpha(\zeta)$ does likewise, until ζ reaches ζ_0; we have $\alpha(\zeta_0) > \arg [(z_1 - \zeta_0)/(z_3 - \zeta_0)] = \pi$. Hence for some (unique) ζ_1, $-\infty < \zeta_1 < \zeta_0$, we have $\alpha(\zeta_1) = \pi$. As the positively weighted mean of degree 2 we choose $\lambda f_1(z) + (1 - \lambda)f_2(z)$, $\lambda = |f_2(\zeta_1)|/[|f_1(\zeta_1)| + |f_2(\zeta_1)|]$, which vanishes at the point ζ_1 exterior to H.

In case 2, B contains no line segment belonging to H, and E is infinite although not necessarily closed. Let ζ_0 be a point of B at which B has but a single line of support L_0, and choose L any other line through ζ_0; on each side of L lie an infinity of points of E, and in particular there lie on opposite sides of L near ζ_0 two points, z_1' and z_3', of E such that $\arg [(z_1' - \zeta_0)/(z_3' - \zeta_0)]$ is as near π as we please. Choose now z_1 and z_2 in E on the same side of L, z_3 and z_4 in E on the opposite side, in such a way that $\alpha(\zeta_0)$ defined as before exceeds π. The proof can then be completed as in case 1.

7. Infrapolynomials factored and as factors

We study in §7 primarily infrapolynomials as factors of other infrapolynomials of various types.

THEOREM 15. *A polynomial $z^n + \cdots$ is a proper infrapolynomial on a given closed bounded set E of $n + 1$ or more points if and only if it is proper and a factor of some (proper) infrapolynomial of degree r on some subset of $r + 1$ points of E ($n \leqq r \leqq 2n$).*

Theorem 15 is valid if these parentheses are or are not incorporated, wholly or in part; but "(proper)" refers to the set of $r + 1$ points, not to E; see below, the discussion of Theorem 18.

The first part is a consequence of Lemmas 3 and 6. Conversely, Fekete [1] has shown that the given polynomial is a factor of a polynomial of the form

$$(17) \qquad \sum_{k=1}^{r+1} \lambda_k g(z)/(z - z_k), \qquad \lambda_k > 0, \quad \sum \lambda_k = 1, \quad g(z) \equiv \prod (z - z_k),$$

where the z_k lie in E; such a polynomial is known (Corollary 1 to Theorem 13) to be a proper infrapolynomial on the set $\{z_k\}$.

COROLLARY. *The zeros of a proper infrapolynomial $f(z) \equiv z^n + \cdots$ on a closed bounded set E containing at least $n + 1$ points either all lie in the two-dimensional interior of the convex hull H of E, or they are all simple and lie on a line segment of the boundary of H, separating $n + 1$ points of E.*

The set $E' : \{z_k\}$ of the proof of Theorem 15 either forms a noncollinear set, in which case the zeros of $f(z)$ lie interior to H, by Corollary 2 to Theorem 13, or E' forms a collinear set, in which case it is clear by (17) that the r zeros of (17) separate the $r + 1$ points of E', whence the zeros of $f(z)$ separate $n + 1$ points of E'.

THEOREM 16. *A polynomial $f(z) \equiv z^n + \cdots$ is an infrapolynomial on a closed bounded set E of at least $n + 1$ points if and only if $f(z) \equiv f_1(z)f_2(z)f_3(z)$, where $f_3(z)$ is the product of all linear factors of $f(z)$ vanishing at nonisolated points of E, $f_1(z)$ is the product of different linear factors vanishing at isolated points of E, and (if E' denotes the set of zeros of $f_1(z)$), $f_2(z)$ has no zeros in $E - E'$ and is a factor of some proper infrapolynomial of degree r on some set of $r + 1$ points of $E - E'$.*

By Theorem 2 and Lemma 3, $f(z)$ is an infrapolynomial on E if and only if $f(z)/f_3(z)$ is an infrapolynomial on E. By Lemmas 4 and 5, this is true if and only if $f_2(z)$ is an infrapolynomial on $E - E'$. The conclusion follows by Theorem 15.

THEOREM 17. *A polynomial $f(z) \equiv z^n + \cdots$ is an infrapolynomial on a finite set E of m ($>n$) points if and only if it is a factor of an infrapolynomial of degree $m - 1$ on E.*

The first part of Theorem 17 follows from Lemma 3. Conversely, in the notation of Theorem 16 we write $f(z) \equiv f_1(z)f_2(z)$, where $f_1(z)$ is the product of the different linear factors of $f(z)$ vanishing in points of E. Let E' denote the set of zeros of $f_1(z)$; then by Lemma 5, $f_2(z)$ is a proper infrapolynomial on $E - E'$, hence by Theorem 15 a factor of an infrapolynomial $\varphi_2(z)$ of degree r on a subset E_1 of $E - E'$ containing $r + 1$ points. If $\varphi_1(z)$ denotes the product of all different linear polynomials $z - z_j$, z_j in $E - E_1$, then $\varphi(z) \equiv \varphi_1(z)\varphi_2(z)$ is a polynomial of degree $m - 1$ which is divisible by $f(z)$ and by Lemma 4 is also an infrapolynomial on E.

Under the conditions of Theorem 17 the infrapolynomial of degree $m - 1$ on E of which $f(z)$ is a factor is also (Theorem 13) an extremal polynomial of least p^{th} powers, $p = 1$.

THEOREM 18. *A polynomial $z^n + \cdots$ is an infrapolynomial on a closed bounded set E of $n + 2$ or more points if and only if it is a factor of an infrapolynomial of degree $n + 1$ on E.*

The first part follows from Lemma 3. The second part follows from Theorem 17 and Lemma 3 if E is finite, and if E is infinite from Theorem 2 by adjoining to $f(z)$ a factor $z - z_0$, where z_0 is a limit point of E. The second part follows alternately from the Corollary to Theorem 10 by similar adjunction, if E is infinite.

In connection with Theorem 18, we remark that if $f(z)$ is an infrapolynomial on a finite set E, it may be impossible to adjoin to $f(z)$ a linear factor vanishing on E such that the product is also an infrapolynomial on E; a necessary and sufficient condition for possibility is that $f(z)$ be an infrapolynomial on a proper subset of E; compare Lemmas 4 and 5. For instance, if E consists of the noncollinear points z_1, z_2, z_3, and if $f(z) \equiv z - \zeta$, where ζ is interior to the convex hull of E, the polynomial $f_1(z) \equiv (z - \zeta)(z - z_1)$ is not an infrapolynomial on E; but if ζ lies on the segment $z_2 z_3$, then $f_1(z)$ is an infrapolynomial on E.

On the other hand, we now discuss whether a factor *not* vanishing on E can be adjoined to an infrapolynomial on E. We show that in the statement of Theorem 15 the term "(proper)", and in the statement of Theorem 16 the last "proper", cannot be altered to mean proper on E. In other words, *we exhibit a proper infrapolynomial $f(z)$ on a set E which is not a factor of any infrapolynomial $f_1(z)$ of degree r on a subset of $r + 1$ points of E which is proper for E.* Choose E: $(z_1, z_2, z_3, z_4, z_5)$, $z_1 = x + iy, x > y > 0, z_2 = \bar{z}_1$, $z_3 = -z_1$, $z_4 = -z_2$, $z_5 = 0$, $f(z) \equiv z^2 - x^2 + y^2$. Then $f(z)$ is an infrapolynomial by Lemma 3, since

$$z^2 f(z) \equiv [\omega_1(z) + \omega_2(z) + \omega_3(z) + \omega_4(z)]/4,$$

$$\omega_j(z) \equiv \omega(z)/(z - z_j), \quad \omega(z) \equiv \prod (z - z_j).$$

The choice $r = 2$ is impossible, for the zeros of $f(z)$ do not lie in the convex hull of any three points of E. With $r > 2$, $f_1(z)$ must be, by Theorem 18, identical with, or a factor of, an infrapolynomial $f_2(z)$ on E of degree 4; the choice $f_2(z) \equiv z^2 f(z)$ is not possible, as not yielding a suitable $f_1(z)$. If $f_2(z)$ exists, say $f_2(z) \equiv \sum \lambda_j \omega_j(z)$, $\lambda_j \geqq 0$, $\sum \lambda_j = 1$, there exists a (not necessarily nonnegatively) weighted sum $(\mu_1, \cdots, \mu_5)$ of $(\lambda_1, \cdots, \lambda_5)$ and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0)$ with $\mu_j \geqq 0$, $\sum \mu_j = 1$, and at least one of the numbers μ_j $(j < 5)$ vanishing, say $\mu_1 = 0$. Then $\sum \mu_j \omega_j(z)/(z - z_1)$ is (Lemma 5) an infrapolynomial on z_2, z_3, z_4, z_5, yet has a zero exterior to the convex hull of those four points. This contradiction establishes the assertion.

As an application of our results on factorization we prove

THEOREM 19. *An improper infrapolynomial $f(z) \equiv z^n + \cdots$ on a finite set E of $n + 1$ or more points is the limit of proper infrapolynomials on E.*

Let E contain precisely m points. If $m = n + 1$, the conclusion is obvious by Theorem 13 and its Corollary 1. If $m > n + 1$, $f(z)$ is by Theorem 17 a factor of an infrapolynomial $f_1(z)$ of degree $m - 1$, and the terms of a sequence of proper infrapolynomials of degree $m - 1$ converging to $f_1(z)$ admit factors (necessarily proper infrapolynomials) converging to $f(z)$.

We note some other properties of the class of infrapolynomials of degree n on a set E of m ($>n$) points, which can be similarly deduced more or less directly, via Theorem 17, from the same properties for $m = n + 1$ (the latter class is explicitly given in Theorem 13): closure and connectedness (independently of §§2 and 3), and the fact that every infrapolynomial is the limit of other infrapolynomials having no zeros in common with it. For $m = n + 1$, this last property can be established by considering the polynomials in the space of $2n$ real dimensions.

Theorem 19 is false if the restriction that E be finite is omitted. Indeed, a compact infinite set E may even have no proper infrapolynomials; this latter situation occurs if and only if E is convex.

8. Real sets; separation properties

A number of our previous results can be made more precise for real sets E, a topic to which we now turn. Methods previously developed by the present writers [2] for the study of infrapolynomials on a finite real set apply with suitable modifications to an infinite real set, as we proceed to show.

THEOREM 20. *Let E be a closed bounded real point set containing at least $n + 1$ points, let E^* denote the set of limit points of E, and let $f(z) \equiv z^n + \cdots$ be an infrapolynomial on E. If ξ not in E^* is a zero of $f(z)$, then at least one point of E at which $f(z)/(z - \xi)$ does not vanish lies in each of the intervals $-\infty < z \leqq \xi$ and $\xi \leqq z < +\infty$. If ξ and η not in E^* (with $\eta \geqq \xi$) are two zeros of $f(z)$, then at least one point z_0 of E at which $f(z)/(z - \xi)(z - \eta)$ does not vanish satisfies $\xi \leqq z_0 \leqq \eta$.*

Of course all zeros of $f(z)$ are real, since they lie in the convex hull

$$\xi_1 \leqq z \leqq \xi_2$$

of E. The first part of Theorem 20 is proved by considering the auxiliary polynomial $f(z)(z - \xi + \varepsilon)/(z - \xi)$, and the second part by considering $f(z)[(z - \xi)(z - \eta) - \varepsilon]/(z - \xi)(z - \eta)$, $\varepsilon > 0$. Details are so similar to a proof previously given [loc. cit., Theorem 2] that they are omitted.

Theorem 20 shows the impossibility of the following orderings of points of E and the zeros of $f(z)$, as well as of the reversed orderings. That Cases I and II are impossible follows from the first part of Theorem 20, and the remainder from the second part.

Case I. $f(\xi) = 0$, ξ not in E, all z (if any) in E with $\xi < z$ are zeros of $f(z)$.

Case II. $f(z)$ has a multiple zero ξ in $E - E^*$, all z (if any) with $\xi < z$ are zeros of $f(z)$.

Case III. $f(z)$ has a zero of multiplicity greater than one at ξ not in E.

Case IV. $f(z)$ has a zero of multiplicity greater than two at ξ in $E - E^*$.

Case V. $f(z)$ has zeros at ξ and η ($>\xi$) not in E; all points of E (if any) between ξ and η are zeros of $f(z)$.

Case VII. $f(z)$ has a multiple zero at ξ in $E - E^*$ and a zero at η not in E; all points of E (if any) between ξ and η are zeros of $f(z)$.

Case VII. $f(z)$ has multiple zeros at ξ and η ($\neq \xi$) in $E - E^*$; all points of E (if any) between ξ and η are zeros of $f(z)$.

We add two remarks. Every multiple zero of $f(z)$ lies on E and if of order greater than two on E^*. It is a consequence of Case II that ξ_1 if not on E^* cannot be a double zero of $f(z)$, and a consequence of Case I that $f(z)$ can have no zero in an interval $\xi_1 < z < \xi$ which contains no point of E if $f(\xi_1) = 0$.

With the aid of Theorem 20 we have

THEOREM 21. *With the hypothesis of Theorem* 20 *on* E *and* E^*, *a necessary and sufficient condition that* $f(z) \equiv z^n + \cdots$ *be an infrapolynomial on* E *is that the ordered zeros* $y_1, y_2, \cdots y_\nu$ *of* $f(z)$ *not on* E^* *separate a subset* E' *of distinct points* $x_1, x_2, \cdots, x_{\nu+1}$ *of* E *in the sense*

$$x_1 \leqq y_1 \leqq x_2 \leqq \cdots \leqq x_\nu \leqq y_\nu \leqq x_{\nu+1}. \tag{18}$$

The first part of Theorem 21 is established by use of the cases of impossibility already enumerated, by examining the number $N_f(x)$ of zeros of $f(x)$ not in E^* and not greater than x, and the number $N_{E'}(x)$ of points of E' not greater than x, as x increases monotonically. For every x we have

$$N_f(x) \leqq N_{E'}(x) \leqq N_f(x) + 1. \tag{19}$$

As x increases monotonically from $-\infty$, we always adjoin each new x in E to E' if and only if after adjunction the relation (19) holds. Thanks to the closure of E, a first x in E succeeds each zero of $f(z)$ in $E - E^*$. A simple zero of $f(z)$ at an isolated point of E is called *a zero of the first kind*; all other zeros of $f(z)$ not on E^* are *zeros of the second kind*.

The proof of the first part of Theorem 21 is now practically identical with a proof previously given [loc. cit., §7] if the phrase "zero of $T_{n+1}(x)$" is replaced by "zero of $f(z)$ not in E^*." Details are left to the reader.

The second part of Theorem 21 is readily proved. From (18) it follows [3, Theorem 5.3] that $f(z)$ when deprived of its factors corresponding to zeros of the third kind is an infrapolynomial on the set $\{x_k\}$, and hence (Lemma 6) also on the set E, so (Theorem 2) $f(z)$ is an infrapolynomial on E.

It is a consequence of Theorem 21 that *any line segment of the convex hull of* E *containing no point of* E *can contain at most one* (*necessarily simple*) *zero of* $f(z)$. This is an important fact for polynomials minimizing norms (3) and (4), proved by Achyeser [1] for norm (4) on a set E consisting of two disjoint closed segments, $p = 1$.

9. Real sets; finite generation

The results of §5 can be somewhat sharpened for real sets.

THEOREM 22. *If* E *is a closed bounded real set consisting of at least* $n + 1$ *points, a polynomial* $f(z) \equiv z^n + \cdots$ *is a finitely generated infrapolynomial on* E *if and only if its zeros separate a subset of* E.

If the zeros y_j of $f(z)$ satisfy (18), where $\nu = n$ and the x_j lie on E, it follows from Theorem 21 that $f(z)$ is an infrapolynomial on the set $\{x_j\}$, so $f(z)$ is finitely generated on E. On the other hand, if $f(z)$ is an infrapolynomial on some finite subset E_1 of E, then also by Theorem 21 we have (18), where the y_j are the zeros of $f(z)$, $\nu = n$, and the x_j lie on E_1 and on E.

It is of interest to remark that for real E, Theorem 11 can be readily given a constructive proof, by use of Theorem 21. If $f(z)$ is an infrapolynomial on E, we consider an auxiliary infrapolynomial $f_k(z)$, whose zeros are the zeros of the first and second kinds of $f(z)$, together with ν distinct zeros in ν suitably chosen points of E approaching each ν-fold zero of the third kind of $f(z)$ on E; then $f_k(z)$ is an infrapolynomial on a finite subset of E and approaches $f(z)$.

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