# SMOOTHING METHODS FOR CONTOURS

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## 1. Introduction

The method of using contours in the investigations of problems involving surface area and the calculus of variations was introduced by L. Cesari [3] and has been used by both of the present authors in developing the theory of surfaces from a point of view which differs in many respects from the classical point of view in that it depends less on analytical techniques. The method of contours uses chiefly topological ideas and hence would appear to be more closely related to the true nature of a surface which is essentially topological. We shall describe here how contours are defined and give several methods of smoothing contours so that certain types of problems in surface theory can be more conveniently treated.

Let $J$ be a simply connected Jordan region in the plane, and let $T: J \to E_3$ be a continuous map. $T$ defines a surface $S$ under the standard Fréchet definition. Although a theory of contours can be developed for multiply connected regions or even 2-manifolds, we shall be concerned in this paper only with simply connected Jordan regions. If $[S]$ is the set of points in $E_3$ occupied by the surface, let $f$ be any real valued continuous function defined in $E_3$, and if $p = T(w)$, $w \in J$, $p \in E_3$, we define $F: J \to$ reals by $F(w) = f(T(w))$. For any real value of $t$, $-\infty < t < \infty$, define $C(t)$ to be the subset of $J$ for which $F(w) = t$, $D^-(t) = \{w \in J \mid F(w) < t\}$, $D^+(t) = \{w \in J \mid F(w) > t\}$. (Some of these sets may be empty.) Evidently $C(t)$ is compact, and $D^-(t)$, $D^+(t)$ are open in $J$ for all $t$ and have their boundaries contained in $C(t)$. For a fixed value of $t$ let $\{\alpha\}$ be the collection of all components of $D^-(t)$ and for each $\alpha$ let $\{\gamma\}_\alpha$ be the family of all components of the set $\bar{\alpha} - \alpha \subset C$. If we denote the union of all the $\gamma$ for all $\alpha \in \{\alpha\}$ by $\kappa(t)$, we say that $\kappa(t)$ is the *contour* associated with the value $t$ for the mapping $T$ and the function $f$. Since for the same surface $S$, different representations $T$ and different functions $f$ may be considered, it can be seen that a large variety of types of contours can exist. R. E. Fullerton [6] has shown that if $S$ is nondegenerate, a representation $T$ of $S$ can be chosen in such a way that a countable dense set of the contours are sums of arcs and simple closed curves, provided that $S$ has finite area. Other authors [1, 2, 3, 5, 8, 9] have consistently used contours in questions arising from the calculus of variations for surfaces. Cesari [3] has defined a generalized length for the image $T(\kappa)$ in the following manner. Let $\gamma \in \{\gamma\}_\alpha$ for some $\alpha \in \{\alpha\}$, and let $A(\gamma, \alpha)$ be the set of all points $w \in J$ for which either $w \in \alpha$ or $w$ is separated from $\gamma$ by other components $\gamma' \in \{\gamma\}_\alpha$. Let $\{\eta\}_{A,\gamma}$ be the set of all ends of $A(\gamma, \alpha)$ ending on $\gamma$, and let $\{\eta_i\}$, $i = 1, 2, \cdots, n$, be any finite set for which $\eta_i < \eta_j$

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if $i < j$, except possibly that $\eta_1 = \eta_n$ where the ordering on the ends is as defined in [3]. If $w_{\eta_i}$ are the points of $\gamma$ corresponding to $\eta_i$, consider the sum $\sum_{i=1}^{n} | T(w_{\eta_i}) - T(w_{\eta_{i-1}}) |$. The length $l(\gamma)$ is defined to be the upper bound of these sums for all such choices of the $\{\eta_i\}$, and the length $l(t)$ of $T(\kappa(t))$ is defined as the upper bound of all finite sums of lengths $l(\gamma)$ taken over all $\gamma$ which correspond to the sets $\{\alpha\}$. If $f$ is Lipschitzian on $E_3$ and if $S$ has a finite area, it is known [3] that $l(t)$ is finite for almost all $t$. Also it is known that the length $l(t)$ is a Fréchet invariant of $S$. If $\omega$ is a prime end of $A$ with its terminal set $E_\omega \subset \gamma$ and if $l(\gamma) < \infty$, then $E_\omega$ is a subset of a continuum of constancy of $T$ in $J$. These facts follow from the Cavalieri type inequality proved in [3].

The contours $\kappa(t)$ may be of a rather complex topological nature, and it is convenient in some cases to substitute for the contour a somewhat simpler smoothed contour whose image does not exceed the original contour in length and which has somewhat simpler properties. Several methods of smoothing will be considered here, and the equivalence of the methods will be established. We first consider the problem of smoothing a portion of a component $\gamma$ of the contour between two prime ends $\omega_1$ and $\omega_2$. The question of smoothing an entire contour is then considered, and methods applying to open and closed type components will be considered. The first smoothing method depends upon choosing an appropriate subset of the hyperspace $\tilde{\Gamma}$ of maximal continua of constancy and defining the smoothed contour as the intersection of the point set in $J$ which this defines with the original component $\gamma$. The second method involves consideration only of the original topology on $J$ and consists of eliminating certain inessential portions of the original contour. A third method can be defined by considering the complementary domains of the set $A(\alpha, \gamma)$ and gives a smoothing for an entire component $\gamma$ of the contour at one time.

L. Cesari, who has discussed in [4] the first of these smoothing methods for contours, has announced applications of it to the representation problem for surfaces.

## 2. The hyperspace $\tilde{\Gamma}(T, J)$ and the first smoothing method

A continuum of constancy for $T$ in $J$ is a continuum $g \subset J$ such that $p_1, p_2 \in g$ implies $T(p_1) = T(p_2)$. The set is a maximal continuum of constancy for $T$ in $J$ if it is maximal with respect to this property. Let $\Gamma(T, J)$ be the set of all maximal continua of constancy $g$ for $T$ in $J$. Evidently $J = \bigcup_{g \in \Gamma} g$, and any two $g$ are disjoint or coincident. Also it is known that $\Gamma$ forms an upper semicontinuous decomposition of $J$. This allows the definition of a topology on $\Gamma(T, J)$ (Whyburn[11], Rado[7]) under which it becomes a topological space which we shall denote by $\tilde{\Gamma}(T, J)$. Under this topology, a point $g \in \Gamma$ is a limit point of a set $\Delta \subset \tilde{\Gamma}$ provided that every open set $G \subset J$ for which $g \subset G$ also contains all points of some other continuum $g' \in \Delta$. Since $\Gamma(T, J)$ is an upper semicontinuous collection, this topology is well defined and has the further property that if $\Delta \subset \tilde{\Gamma}$ is a connected subset of $\tilde{\Gamma}$, then $\bigcup_{g \in \Delta} g \subset J$ is a connected subset of $J$.

Let $\omega_1$, $\omega_2$ be two prime ends of $A(\alpha, \gamma)$ with their terminal sets $E_{\omega_1}$, $E_{\omega_2}$ subsets of $\gamma$, $\omega_1 < \omega_2$. Consider the set of all prime ends $\omega$ with $\omega_1 \leqq \omega \leqq \omega_2$, and denote this set by $[\omega_1, \omega_2]$. Define $\tilde{\gamma}(\omega_1, \omega_2)$ to be the set of all $g \subset \tilde{\Gamma}$ such that for some $\omega \in [\omega_1, \omega_2]$, $g \cap E_\omega \neq 0$ in $J$. Let $K = \bigcup g$ for $g \in \tilde{\gamma}(\omega_1, \omega_2)$. Evidently $K \cap A = 0$, $K \subset C \subset J$, and $K$ is a continuum including all points of $\gamma$ which lie in terminal sets of prime ends between $\omega_1$ and $\omega_2$. Also $\tilde{\gamma}(\omega_1, \omega_2)$ is a continuum in $\tilde{\Gamma}(T, J)$ under the hyperspace topology since otherwise $K$ itself would not be a continuum. Since it has been assumed that $l(\gamma) < \infty$, each set $E_\omega$ is contained in some $g \in \Gamma(T, J)$. Hence there exist $g_1$, $g_2 \in \Gamma(T, J)$ such that $E_{\omega_1} \subset g_1$, $E_{\omega_2} \subset g_2$ ($g_1$ may equal $g_2$). By a result of Cesari [4] there exist arcs (possibly indefinite) $s_1$, $s_2$, defining $\omega_1$, $\omega_2$ respectively, which intersect in $A$ in only one point. Then $s_1 \cup s_2 \cup \gamma$ is the boundary of an open set $A' \subset A$. Let $\tilde{K}_0 \subset \tilde{\gamma}(\omega_1, \omega_2)$ be an irreducible continuum in $\tilde{\Gamma}(T, J)$ joining $g_1$ and $g_2$. If $g_1 = g_2$, then we have $\tilde{K}_0 = g_1$. If $g_1 \neq g_2$, then $\tilde{K}_0$ is a nondegenerate continuum. Let $K_0 = \bigcup g$ for $g \subset \tilde{K}_0$. Then $K_0 \subset K$, $K_0 \cap A = 0$, and $A$ is contained in a component $A_0$ of $J - (K_0 \cup K')$ where $K'$ is the union of the $g \in \Gamma$ which intersect points of $\gamma$ other than those which are terminal elements of $\omega \in [\omega_1, \omega_2]$. We define the smoothed contour $k_0(\omega_1, \omega_2)$ to be the set $B^* \cap K_0$ where $B^*$ is the boundary of the open set $B$ bounded by $s_1 \cup s_2 \cup K_0$, $B \subset A_0 \subset J$. If we denote the set of ends and prime ends of $B$ ending on $k_0$ by $\{\eta\}_0$, $\{\omega\}_0$, this set can be ordered in the usual way to make $\omega_1$ the first element and $\omega_2$ the last, since evidently by definition $\omega_1$, $\omega_2 \in \{\omega\}_0$. Various properties of the smoothed contour $k_0$ have been developed by Cesari in [4]. For example it is proved that if in the new open set $A_0$, the prime ends $\omega_0' < \omega_0''$ end on the same continuum of constancy $g \in \Gamma$, then all prime ends $\omega_0 \in \{\omega\}_0$ with $\omega_0' \leqq \omega_0 \leqq \omega_0''$ also end on $g$. Also results are obtained concerning the relationships of generalized lengths of the images of boundaries of complementary domains to $A_0$. Such results will be discussed later in the paper when other equivalent smoothing definitions are given. In particular if $\omega_1$, $\omega_2$ in the above construction are the first and last elements of the set of prime ends from $A$ ending on $\gamma$, then the smoothed contour between $\omega_1$ and $\omega_2$ can be defined to be the contour obtained by smoothing all of $\gamma$. As will be seen in section 6 this method of smoothing all of $\gamma$ does not necessarily coincide with the third method which we shall describe. Also in case $\omega_1 = \omega_2$ and the contour $\gamma$ gives rise to a cyclic ordering of $\{\omega\}$, then the definitions given above will yield $E_{\omega_1} = E_{\omega_2}$ as a smoothed contour, and hence the method of smoothing $\gamma$ will be difficult to define. Simple examples show that it is not sufficient to choose two prime ends $\omega_1$, $\omega_2$ which are different in this case and to smooth first the interval from $\omega_1$ to $\omega_2$ and then from $\omega_2$ to $\omega_1$, since different choices of $\omega_1$, $\omega_2$ may yield different smoothed contours, and also the smoothing process will not yield the desired set. Hence other definitions in the large seem necessary.

## 3. The second method

Again as before let $\gamma$ be a component of $\alpha^* - \alpha$ with $l(\gamma) < \infty$, let $A(\gamma, \alpha)$ be defined as above, and let $\omega_1$, $\omega_2$ be two prime ends of $A$, $\omega_1 < \omega_2$. If $E_{\omega_1} \cap E_{\omega_2} \neq 0$, we define the smoothed contour $k_0'$ to be the element $g \in \Gamma$ for which $E_{\omega_1} \cup E_{\omega_2} \subset g$. Thus for the nontrivial case assume $E_{\omega_1} \cap E_{\omega_2} = 0$. A second method of smoothing the contour $\gamma$ between $\omega_1$ and $\omega_2$ can be described as follows. Let $\mathcal{K}(\omega_1, \omega_2)$ be the set of all elements of $\Gamma$ which intersect any $E_\omega$ for which $\omega_1 \leqq \omega \leqq \omega_2$. Let $s_1$, $s_2$ be defined as in section 2. Let $\omega'$, $\omega'' \in \{\omega\}_A$ be such that $\omega_1 \leqq \omega' < \omega'' \leqq \omega_2$, and assume that $E_{\omega'} \cap E_{\omega''} \neq 0$. Then there exists an element $g \in \mathcal{K}(\omega_1, \omega_2)$ for which $E_{\omega'} \cup E_{\omega''} \subset g$ since $E_{\omega'}$, $E_{\omega''}$ are continua of constancy for $T$ in $J$. Delete from $\mathcal{K}(\omega_1, \omega_2)$ all elements $g' \neq g$ such that there exists $\omega \in \{\omega\}_A$ for which $E_\omega \subset g'$ and for which $\omega' < \omega < \omega''$. Let this be done for all pairs of prime ends $\omega'$, $\omega''$ of the above type. For any such pair $\omega'$, $\omega''$ let $H(\omega', \omega''; \omega_1, \omega_2)$ be the subset of $\mathcal{K}(\omega_1, \omega_2)$ described above. Define $\sigma(\omega_1, \omega_2)$ as the intersection in $\Gamma$ of all the sets $H(\omega', \omega''; \omega_1, \omega_2)$ described above. As in the first method, define $K_0' - \bigcup_{g \in \sigma} g \subset J$, and let $k_0' = B^* \cap K_0'$ where $B$ is the component of the set bounded by $S_1 \cup S_2 \cup K_0$ which intersects $A$. It can be seen that $k_0' \subset C(t) \subset J$, and if $A_0'$ is the component of $J - k_0'$ which intersects $A(\alpha, \gamma)$, then $A_0' \subset A \subset J$. We shall define $k_0'$ to be the smoothed contour corresponding to $\gamma$ by the second method. The set of prime ends of $A_0'$ between $\omega_1$ and $\omega_2$, we denote by $\Omega_0'$.

THEOREM. $k_0 \subset k_0'$.

*Proof.* It must be shown that $\sigma(\omega_1, \omega_2) \subset \gamma(\omega_1, \omega_2)$ in $\Gamma$ and hence that $K_0 \subset H(\omega', \omega''; \omega_1, \omega_2)$ for every pair $\omega'$, $\omega''$ as defined above. Assume that this is not the case. Then there exist prime ends $\omega'$, $\omega''$, $\omega_1 \leqq \omega' < \omega'' \leqq \omega_2$, such that $E_\omega \subset g \subset K_0$ for some $\omega$ between $\omega'$ and $\omega''$. Let $E_{\omega'} \cup E_{\omega''} \subset g'$ and $E_\omega \subset g$. Then either $g' \in \sigma(\omega_1, \omega_2)$, or $g' \notin \sigma$. If $g' \notin \sigma$, let $\omega'$, $\omega''$ be defined by two (possibly indefinite) arcs $b_1$, $b_2 \subset A$ such that $b_1 \cap b_2 \cap A$ is a single point $\bar{w}$ and $b_1$, $b_2$ have their end points or end sets both on $g'$. Then by definition of the order on $\{\omega\}_A$, if $b$ is an arc (possibly indefinite) defining $\omega$, $b \cap A$ is separated from $E_{\omega_1}$, $E_{\omega_2}$ by $b_1 \cup b_2 \cup g'$. Since $k_0$ contains $E_{\omega_1}$ and $E_{\omega_2}$, we have $k_0$ separated by the set $b_1 \cup b_2 \cup g'$, since $E_\omega$ is contained in one component of its complement, and either $E_{\omega_1}$ or $E_{\omega_2}$ or both are contained in another. Hence $g' \in \sigma$. However, if this is true, then if $g$ is deleted from $K_0$, evidently $K_0$ would still be a continuum since evidently it is connected between $g_1$ and $g'$ and between $g'$ and $g_2$, where $E_{\omega_1} \subset g_1$, $E_{\omega_2} \subset g_2$. Thus $\sigma - g$ is still connected in $\tilde{\Gamma}$ contrary to the fact that $\tilde{K}_0$ is irreducible. Hence, $k_0 \subset k_0'$.

## 4. The interval topology on $\sigma(\omega_1, \omega_2)$

We shall show that on the set $\sigma(\omega_1, \omega_2)$ described in section 3, it is possible to define a linear ordering and a corresponding interval topology under which

$\sigma$ becomes an arc. In the next section it will be shown that this topology is equivalent to the hyperspace topology on $\sigma$ and hence that $\sigma$ has the same topological properties under either topology. We shall prove that $\sigma$ is an arc under the order topology by a sequence of lemmas.

LEMMA 1. *Let $g \in \sigma(\omega_1, \omega_2)$. Then in the set $\Omega_0'$ there exists a first element $\omega_g'$ and a last $\omega_g''$ ending on $g$. Also if $\omega''' \in \Omega_0'$, $\omega_g' \leqq \omega''' \leqq \omega_g''$, then $E_{\omega'''} \subset g$.*

*Proof.* Let us first consider the existence of $\omega_g'$. If $\omega_1$ is such that $E_{\omega_1} \subset g$, then $\omega_1 = \omega_g'$. If $\omega_1 \neq \omega_g'$, we can define a cut in $\Omega_0'$ as follows. In the lower class we put all the elements of $\Omega_0'$ which precede every element $\omega$ with $E_\omega \subset g$. The second class will contain all other elements of $\Omega_0'$. This defines a prime end $\omega_g' \subset \Omega_0'$ by definition of a prime end. Suppose that $E_{\omega_g'} \not\subset g$. Since $g$ and $E_{\omega_g'}$ are closed in $J$, there exists an open set $G \subset J$, $E_{\omega_g'} \subset G$, $g \cap G = 0$. It is known [4, 10] that for any $\varepsilon > 0$, there exists a cross cut $c$ in $A_0'$ ending on the boundary which is within $\varepsilon$ of $E_{\omega_g'}$ and which defines two ends $\eta'$, $\eta''$ with $\eta' < \omega_g' < \eta''$. However since $c \subset G$, $c$ does not end on $g$. Hence if $\omega \in \Omega_0'$ has $E_\omega \subset g$, $\eta''$ separates $\omega_g'$ from $\omega$. Hence $\omega_g' < \eta''$, and the cut defined above does not determine $\omega_g'$, contrary to assumptions. Hence $E_{\omega_g'} \cap g \neq 0$, and $E_{\omega_g'} \subset g$. By a similar argument, $E_{\omega_g''} \subset g$.

If $\omega''' \in \Omega_0'$ and $\omega_g' < \omega''' < \omega_g''$, then $E_{\omega'''} \subset g'$ for some $g' \in \sigma(\omega_1, \omega_2)$. If $g' \neq g$, $g' \notin H(\omega_g', \omega_g''; \omega_1, \omega_2)$, and hence $g' \notin \sigma(\omega_1, \omega_2)$. Thus $E_{\omega'''} \subset g$.

In the above lemma it may happen that $\omega_g' = \omega_g''$.

LEMMA 2. *Let $g \in \sigma(\omega_1, \omega_2)$. If $\Omega$ is the set of prime ends from $A$ ending on $\gamma$, there exist elements $\omega_g^*$, $\omega_g^{**} \in \Omega$ such that in the ordering on $\Omega$, $\omega_g^*$ is the first prime end with $E_{\omega_g^*} \subset g$ and $\omega_g^{**}$ is the last end with this property. Furthermore if $\omega_g'$, $\omega_g''$ are the prime ends defined in Lemma 1, $E_{\omega_g'} \cap E_{\omega_g^*} \neq 0$, and $E_{\omega_g''} \cap E_{\omega_g^{**}} \neq 0$.*

*Proof.* The existence of $\omega_g^*$ and $\omega_g^{**}$ is established by the same method as was used in Lemma 1 to establish the existence of $\omega_g'$ and $\omega_g''$.

To prove the second part of the lemma, note that since $g \cap k_0' \supset g \cap \gamma$, $E_{\omega_g^*} \cap \gamma \subset g \cap \gamma \subset g \cap k_0' \neq 0$ since $E_{\omega_g^*} \cap \gamma \neq 0$. Hence there must exist elements $\omega \in \Omega_0'$, $\omega_g' \leqq \omega \leqq \omega_g''$, with $E_{\omega_g^*} \cap E_\omega \neq 0$. This implies that if $E_{\omega_g'} \cap E_{\omega_g^*} = 0$, $\omega_g' < \omega$ for every $\omega \in \Omega_0'$ with $E_\omega \cap E_{\omega_g^*} \neq 0$. If $E_{\omega_g'} \cap E_{\omega_g^*} = 0$, there exists an indefinite arc $c$ in $A_0'$ defining $\omega_g'$ and an indefinite arc $c' \subset A \subset A_0$ defining $\omega_g^*$ such that $c \cap c' = 0$, since otherwise, for every two such arcs, $c \cap c'$ would include a set of points converging to a point of $E_{\omega_g^*}$ and $E_{\omega_g^*} \cap E_{\omega_g'} \neq 0$. This implies that if $c''$ is an indefinite arc defining $E_{\omega_g^{**}}$ which intersects $c'$ in a single point of $A$, then $c' \cup c'' \cup g$ bound a subset $B$ of $A_0'$ which does not include $c$. Let $B_1$ be the component of $A_0' - \overline{c' \cup c''}$ which contains $c$. By the way in which $A_0'$, $k_0'$ were constructed, $g \cap B_1^* \subset E_{\omega_g^*} \cup E_{\omega_g^{**}}$, since $\omega_g^*$ and $\omega_g^{**}$ were the first and last prime ends ending on $g$, and if there were other points of $g \cap B_1^*$, there would have to be prime ends in $\{\omega\}_A$ with $E_\omega \subset g \cap B_1^*$, contrary to the fact that $\omega_g^*$ and $\omega_g^{**}$

are the first and last. Hence $E_{\omega_g'} \cap g = 0$, contrary to hypothesis. Thus $E_{\omega_g'} \cap E_{\omega_g^*} \neq 0$. Similarly, $E_{\omega_g''} \cap E_{\omega_g^{**}} \neq 0$. Hence in the sense of the above lemma there exists a correspondence between $\omega_g^*$ and $\omega_g'$ and between $\omega_g^{**}$ and $\omega_g''$.

Lemma 1 allows us to define a linear order on $\sigma(\omega_1, \omega_2)$ as follows: If $g, g' \in \sigma(\omega_1, \omega_2)$, $g' \neq g$, we say that $g < g'$ if there exist prime ends $\omega, \omega' \in \Omega_0'$, with $\omega < \omega'$ and $E_\omega \subset g$, $E_{\omega'} \subset g'$. This gives a determinate ordering since if $\omega''$, $\omega'''$ are any two elements of $\Omega_0'$ with $E_{\omega''} \subset g$, $E_{\omega'''} \subset g'$, then $\omega'' < \omega'''$, since if $\omega''' < \omega''$, then either $\omega''' < \omega$ or $\omega < \omega'''$, and in the first case $E_\omega \subset g'$, and in the second $E_{\omega'''} \subset g$. Thus any two elements $\omega, \omega' \subset \Omega_0'$ with $E_\omega \subset g$, $E_{\omega'} \subset g'$, will determine the same order on $g, g'$. The order is evidently linear since the order on $\Omega_0'$ is, and for the same reason it is transitive.

LEMMA 3. *In the order on $\sigma(\omega_1, \omega_2)$, if $g_1 < g_2$, there exists $g_3 \in \sigma(\omega_1, \omega_2)$ such that $g_1 < g_3 < g_2$.*

*Proof.* Let $g_1 < g_2$, and let $\omega_{g_1}^{**}$ be the last element $\{\omega\}_A$ ending on $g_1$ and $\omega_{g_2}^*$ the first element on $g_2$. Then since the set $\{\omega\}_A$ is such that there exists an element $\omega$ between any two distinct elements in the ordering, there exists an element $\omega$, $\omega_{g_1}^{**} < \omega < \omega_{g_2}^*$. Let $E_\omega \subset g$. Then if $\omega_{g_1}''$, $\omega_{g_2}'$ are the elements of $\Omega_0'$ corresponding to the elements $\omega_{g_1}^{**}$, $\omega_{g_2}^*$, let $\omega_g^*$ be the first element of $\{\omega\}_A$ with $E_{\omega_g^*} \subset g$, and let $\omega_g'$ be the corresponding element of $\Omega_0'$. Then evidently $\omega_{g_1}'' < \omega_g' < \omega_{g_2}'$ by the proof of Lemma 2. Thus $g_1 < g < g_2$ by definition.

LEMMA 4. *The ordered set $\sigma(\omega_1, \omega_2)$ satisfies the Dedekind cut property.*

*Proof.* Let $L$ and $R$ be a partition of the elements of $\sigma(\omega_1, \omega_2)$ with $\omega < \omega'$ if $E_\omega \subset g \in L$, $E_{\omega'} \subset g' \in R$ nonvoid and $L \cup R = \sigma(\omega_1, \omega_2)$. It must be shown that $L$ contains a greatest element or $R$ a least. There are two possible cases.

*Case* 1. There exist elements $g_1 \in L$, $g_2 \in R$ such that in the smoothing process no elements of $\{\omega\}_A$ between $\omega_{g_1}^{**}$ and $\omega_{g_2}^*$, where $\omega_{g_1}^{**}$, $\omega_{g_2}^*$ are as defined in Lemma 2, were deleted. Thus the partition $(L, R)$ defined between $\omega_{g_1}^{**}$ and $\omega_{g_2}^*$ defines a cut in the portion of $\{\omega\}_A$ and hence defines a prime end of the original set which also corresponds to an element of $\omega_0$ of $\Omega_0'$. Also $E_{\omega_0} \subset g$ where $g_1 < g < g_2$. By definition of the order, $g$ is either the first element of $R$ or the last element of $L$.

*Case* 2. The situation in Case 1 does not hold. Then for any two elements $g_1 \in L$, $g_2 \in R$ and the corresponding $\omega_{g_1}^{**}$, $\omega_{g_2}^*$, there exist elements $\omega \in \{\omega\}_A$, $\omega_{g_1}^{**} < \omega < \omega_{g_2}^*$ for which $g \supset E_\omega$ is deleted in forming $\sigma(\omega_1, \omega_2)$. Choose any such $g_1$, $g_2$ with the corresponding $\omega_{g_1}^{**}$, $\omega_{g_2}^*$, and consider all elements of the interval $[\omega_1, \omega_2] \subset \{\omega\}_A$ which lie between $\omega_1$ and $\omega_{g_1}^{**}$ and between $\omega_{g_2}^*$ and $\omega_2$. Take the union of all such sets for $g_1 \in L$, $g_2 \in R$. The union consists of the entire interval $[\omega_1, \omega_2]$ since if $\omega \in [\omega_1, \omega_2]$, there exists a $g \in \sigma(\omega_1, \omega_2)$

such that $\omega_g^* \leqq \omega \leqq \omega_g^{**}$. Also evidently if $g \in L$ and $\omega_g^* \leqq \omega \leqq \omega_g^{**}$, and $g' \in R$ and $\omega_{g'}^* \leqq \omega' \leqq \omega_{g'}^{**}$, $\omega < \omega'$. Hence a corresponding cut $(L', R')$ in $\{\omega\}_A$ is also defined by $(L, R)$, and a prime end $\omega_0$ corresponds to the cut $(L', R')$. However $E_{\omega_0} \subset g$ for some $g \in \sigma(\omega_1, \omega_2)$, since if it did not, there would exist a $g \in \sigma(\omega_1, \omega_2)$ and elements $\omega_g^*$, $\omega_g^{**}$ with $E_{\omega_0} \subset g' \in \Gamma$ and $\omega_g^* < \omega_0 < \omega_g^{**}$. Since $g$ is in either $L$ or $R$, $\omega_0$ can be neither the largest element in $L'$ nor the smallest in $R'$, and hence cannot be the prime end defined by $(L', R')$. Thus there can exist no $g$ with $\omega_g^* < \omega < \omega_g^{**}$ in $\{\omega\}_A$, and hence $\omega_0$ is either $\omega_{g_0}^*$ or $\omega_{g_0}^{**}$ for some $g_0 \in \sigma(\omega_1, \omega_2)$, and $g_0$ is the element corresponding to the cut $(L, R)$.

Definition. The *interval topology* on $\sigma(\omega_1, \omega_2)$ is the topology which has all sets of the form $\{g \in \sigma \mid g_1 < g < g_2\}$ for all $g_1 < g_2$ in $\sigma$ as a basis for the open subsets of $\sigma$.

Theorem. *$\sigma(\omega_1, \omega_2)$ is an arc in the interval topology.*

*Proof.* It is known that any ordered set satisfying Lemmas 2 and 4 is an arc in the interval topology.

## 5. The equivalence of the smoothing methods

We show that on the set $\sigma(\omega_1, \omega_2)$ the hyperspace topology and the interval topology are equivalent. Hence the two methods give the same smoothed contour.

Lemma. *Let $g \in \sigma(\omega_1, \omega_2)$, and let $G \subset J$ be an open subset with $g \subset G$. Let $\Omega_g \subset \Omega_0'$ be the set of elements of $\Omega_0'$ with $E_\omega \subset g$. Then there exist ends $\eta', \eta'' \in \Omega_0'$, $\eta' < \omega' < \eta''$, for all $\omega' \in \Omega_g$ and such that $E_\omega \subset G$ for all $\omega$ with $\eta' < \omega < \eta''$.*

*Proof.* Consider first the existence of $\eta''$. Assume that there exists no end $\eta''$ for which $E_\omega \subset G$ if $\omega' < \omega < \eta''$ for $\omega' \subset \Omega_g$. If $\omega' \notin \Omega_g$, $E_{\omega'} \cap g = 0$, and there exists an open set $G \supset g$ with $E_{\omega'} \cap G = 0$. In $G \cap A_0'$, there exists a cross cut defining two ends, one on each side of $g$. Now let $N_i$ be a sequence of open sets containing $g$ such that $d(p, g) < 1/i$ for every $p \in N_i$; and for each $N_i \cap A_0'$, choose an end $\eta_i$ with $\eta_i > \omega$ for all $\omega$ with $E_\omega \subset g$ and such that $w_{\eta_i} \in N_i$. Let $\eta_1 > \eta_2 > \eta_3 > \cdots > \omega$ for every prime end in $\Omega_g$, and let $w_{\eta_i} \subset G$ for each $i$. Thus if $\omega' > \omega$ for all $\omega \in \Omega_g$ ($\omega > \Omega_g$), there exists an $\eta_i$ with $\omega' > \eta_i > \Omega_g$. Since it is assumed that there exists no $\eta > \Omega_g$ with $E_\omega \subset G$ for all $\omega$, $\eta > \omega > \Omega_g$, and since the terminal points of ends are dense on the boundary, by taking a properly chosen subsequence of $\{\eta_i\}$ if necessary, there will exist a second sequence of ends $\{\eta_i'\}$, $\eta_i > \eta_i' > \eta_{i+1}$ such that $w_{\eta_i'} \notin G$ for each $i$. Since $J$ is compact, there exists a subsequence $\{\eta_{ij}'\} \subset \{\eta_i'\}$ with $\lim_{j\to\infty} w_{\eta_{ij}'} = w \notin G$, since $J - G$ is closed. Now we define a prime end in $\Omega_0'$ by constructing a cut in the following manner. For the class $L$ consider all ends $\eta$ with $\omega_1 < \eta < \omega$ for some $\omega \in \Omega_g$.

For the class $R$ take all ends $\bar{\eta}$ with $\eta_i < \bar{\eta} < \omega_2$ for some $i$. This defines a Dedekind cut in the set of ends in $A_0'$ and hence defines a prime end $\omega'$. However $E_{\omega'} \cap g \neq 0$ since $\lim_{i\to\infty} w_{\eta_i} \subset g$. Also $E_{\omega'} \cap (J - G) \neq 0$ since $\lim_{j\to\infty} w_{\eta'_{i_j}} \in J - G$. Since $E_{\omega'}$ is a continuum of constancy for the mapping $T$, $E_{\omega'} \subset g$. This implies that $g$ is not contained in $G$, contrary to hypothesis, and hence that there must exist some end in the set $\{\eta_i\}$ for which $E_\omega \subset G$ for all $\omega$ with $\eta_i > \omega > \Omega_g$. A similar proof establishes the existence of the $\eta' < \Omega_g$.

THEOREM. *Let $M \subset \sigma(\omega_1, \omega_2)$. Then $g_0 \subset \sigma(\omega_1, \omega_2)$ is a limit point of $M$ in the interval topology if and only if it is a limit point of $M$ in the topology on $\sigma(\omega_1, \omega_2)$ determined by the hyperspace topology on* $\Gamma$.

*Proof.* Assume that $g_0$ is a limit point of $M$ in the order topology. Let $N \subset \Gamma$ be an open subset of $\tilde{\Gamma}$ in the hyperspace topology with $g_0 \in N$. We must prove that $N \cap M \neq 0$. Let $G = \bigcup_{g' \in N} g' \subset J$. $G$ is an open subset of $J$ which includes $g_0$. By the lemma there exists an open interval $(\eta', \eta'') \subset \Omega_0'$ such that $E_\omega \subset G$ if $\eta' < \omega < \eta''$ and $\eta' < \Omega_g < \eta''$. By the definition of $G$ for $\eta' < \omega < \eta''$, if $E_\omega \subset g'$, then $g' \subset G$. Thus if $w_{\eta'} \in g_1$, $w_{\eta''} \subset g_2$ and if $I = \{g' \in \sigma(\omega_1, \omega_2) \mid g_1 < g' < g_2\}$, $g_0 \in I \cap N$. Since $g_0$ is a limit of $M$ in the interval topology, $I \cap M \neq 0$, and hence $N \cap M \neq 0$. This is true for any open set $N \subset \tilde{\Gamma}$ with $g_0 \in N$, and hence $g_0$ is a limit point of $M$ in the hyperspace topology.

For the sufficiency of the condition, assume that $g_0$ is a limit point of $M$ in the hyperspace topology. Let $I = (g_1, g_2)$ be any interval of $\sigma(\omega_1, \omega_2)$ which includes $g_0$, and construct an open set in $J$ as follows. Let $g'$, $g''$, $g' < g_0 < g''$, be chosen in $I$ so that there exist ends $\eta'$, $\eta''$ with $w_{\eta'} \in g'$, $w_{\eta''} \in g''$. From a point $p \in A_0'$ construct two arcs $c'$, $c''$ defining $\eta'$, $\eta''$ respectively and with $c' \cap c'' = p$ in $A_0'$. Let $d \subset k_0'$ be the set consisting of all points of $A_0'^*$ except the points of $E_\omega$ for $\eta' < \omega < \eta''$. $c' \cup c'' \cup d$ is a closed subset of $J$ and its complement contains the set $g_0$ in one of its components $G_0$. Let $N \subset \tilde{\Gamma}$ be the subset of $\tilde{\Gamma}$ consisting of all elements $g \in \tilde{\Gamma}$ for which $g \subset G_0$. $N$ is open in $\tilde{\Gamma}$, since if it were not, there would exist a $g \in N$ which is a limit point of the complement of $N$, and hence there would exist an open subset $G' \subset G_0$ with $g \in G'$ and such that $G'$ contains an element of the complement of $N$, contrary to the definition of $N$. By the way in which $N$ and $G_0$ were constructed, however, $N \cap \sigma(\omega_1, \omega_2) \subset I$. Since $M \cap N \cap \sigma(\omega_1, \omega_2) \neq 0$ by hypothesis, we have $M \cap I \neq 0$. Hence if $g_0$ is a limit point of $M$ in the hyperspace topology, $g_0$ is a limit point of $M$ in the interval topology. Hence the two topologies on $\sigma(\omega_1, \omega_2)$ are equivalent.

COROLLARY 1. *The two methods of smoothing contours are equivalent, i.e.,* $k_0 = k_0'$.

*Proof.* $\sigma(\omega_1, \omega_2) \supset K_0(\omega_1, \omega_2)$ by a previous theorem. However, $\sigma(\omega_1, \omega_2)$ is irreducible, and since $E_{\omega_1} \subset g_1$ implies $E_{\omega_2} \subset g_2$, $g_1$, $g_2$ are in both sets.

Thus $K_0 = \sigma$ since the inclusion cannot be proper by the irreducibility of $\sigma$. Hence $k_0 = k_0'$.

COROLLARY 2. *There is only one smoothed contour between* $\omega_1$ *and* $\omega_2$.

*Proof.* The second method gives a uniquely defined contour and hence so also does the first method.

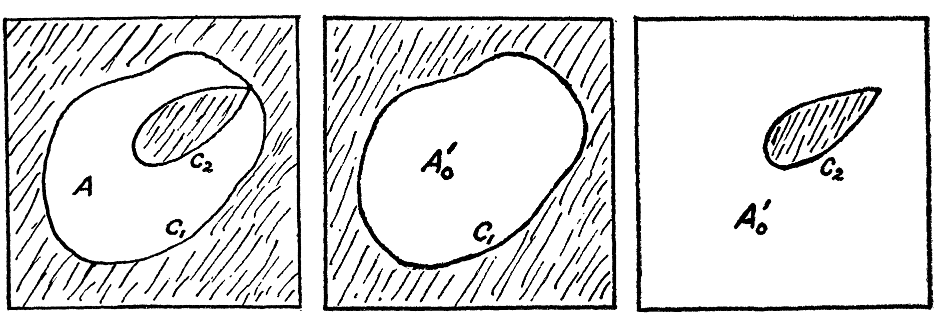
COROLLARY 3. *If* $S$ *is a nondegenerate surface, there exists a representation* $T$ *of* $S$ *such that* $k_0$ *is an arc in* $J$.

*Proof.* Let $T$ be a light representation of $S$. Then all maximal continua of constancy are points, and hence $\sigma(\omega_1, \omega_2)$ is an arc in $J$ as well as in $\Gamma$. In this case $k_0 = \sigma$, and $k_0$ is a simple arc.

## 6. The smoothing of an entire contour; the third method

The methods described in the preceding sections were applied only to cases in which a portion of a contour between two prime ends $\omega_1$ and $\omega_2$ was considered. If $\omega_1$, $\omega_2$ are the first and last prime ends for a given component $\gamma$ of a contour, then the above method gives a reasonable definition of a smoothed contour. However, if a contour is smoothed between $\omega_1$ and $\omega$ and between $\omega$ and $\omega_2$, the union of the two smoothed contours does not necessarily give the smoothed contour between $\omega_1$ are $\omega_2$. This can be verified by easily constructed examples.

Also difficulties arise in the case of a contour of "closed" type, i.e., the ordering on $\{\omega\}_A$ is cyclic. In this case even if one is careful in choosing two intervals of prime ends in such a way that the above difficulties do not arise, other difficulties may be present. Thus in the figure, (a) is the unsmoothed contour, and (b) and (c) are both reasonable smoothed contours.




(a) (b) (c)

Situation (b) would arise if two appropriately chosen ends $\eta_1$, $\eta_2$ on the portion $c_1$ of the contour were chosen and the contour smoothed from $\eta_1$ to $\eta_2$ and from $\eta_2$ to $\eta_1$. Situation (c) would arise if the same procedure were followed on $c_2$. Thus it appears that a reasonable definition of a smoothed contour in this case would depend upon which complementary domain of $\gamma$

other than $A$ were chosen. This leads to the following method of smoothing contours which we shall call the third method.

DEFINITION. Let $\gamma$ be a component of a contour and let $\mathcal{K}$ be the set of all elements $g \in \Gamma$ for which $g \cap \gamma \neq 0$. Let $A_0, D_1, D_2, \cdots$ be the complementary domains of $K = \bigcup g$ for $g \in \mathcal{K}$. Let $\sigma_i$ be the set of all elements of $\mathcal{K}$ which intersect both $A^*$ and $D_i^*$. Let $A_i \supset A$ be the component of $J - \bigcup_{g \in \sigma_i} g$ which includes $A$. Then the smoothed contour $k_i$ corresponding to $D_i$ is defined as $A_i^* \cap \bigcup_{g \in \sigma_i} g$. The smoothed contour corresponding to all of $\gamma$ will be the set $\bigcup_i k_i$. If there are no complementary domains of $K$ except $A$, we define the smoothed contour to be $J^*$. The entire smoothed contour corresponding to the value $t$ will be the union over $\gamma \in \{\gamma\}_\alpha$ and $\alpha \in \{\alpha\}$ of all the smoothed contours corresponding to each such $\gamma$ and $\alpha$.

It will be noted that in case there exist first and last prime ends of $A$ ending on $\gamma$, the above definition does not necessarily give the smoothed contour $k_0(\omega_1, \omega_2)$ but a subset of this smoothed contour which essentially is obtained by leaving out of $\sigma(\omega_1, \omega_2)$, the elements $g \in \Gamma$ for which $g \cap D_i = 0$ for all $i$. For such $g$, $g \cap J^* \neq 0$. However, the following theorem shows how the above smoothed contour is related to the first and second types.

THEOREM. *Let $A_0$, $D_i$, $\sigma_i$ be defined as above. Let $i$ be fixed, and let there exist two prime ends $\omega_1^i \neq \omega_2^i$ such that $E_{\omega_1^i} \subset g_1^i$, $E_{\omega_2^i} \subset g_2^i$, $g_2^i \neq g_1^i$, and such that $\omega_1^i$, $\omega_2^i$ are respectively the first and last prime ends of $A$ for which $E_\omega \cap g \neq 0$, $g \in \sigma_i$. Then $\sigma_i = \sigma(\omega_1^i, \omega_2^i)$, and $k_0(\omega_1^i, \omega_2^i) = k_i$.*

*Proof.* We shall prove that $\sigma_i = \sigma(\omega_1^i, \omega_2^i)$, and from this it follows that $k_0(\omega_1^i, \omega_2^i) = k_i$. If $g \in \mathcal{K}$ and if there exists $g' \in \mathcal{K}$ and two elements $\omega'$, $\omega'' \in \{\omega\}_A$, $\omega_1^i \leqq \omega' < \omega'' \leqq \omega_2^i$, $E_{\omega'} \subset g'$, $E_{\omega''} \subset g''$, such that $E_\omega \subset g$ implies $\omega' < \omega < \omega''$, then evidently $g \cap D_i^* \neq 0$, since indefinite arcs $c'$, $c''$ can be drawn from $A$, $c' \cap c'' = p \in A$, where $c'$, $c''$ define $\omega'$, $\omega''$ and such that $c' \cup c'' \cup g'$ bound an open set in $J$ including $g$ and not intersecting $D_i^*$. Thus $g \notin \sigma_i$. This proves that $\sigma_i \subset \sigma(\omega_1^i, \omega_2^i)$.

To prove the inclusion $\sigma(\omega_1^i, \omega_2^i) \subset \sigma_i$, it must be shown that if $g \in \sigma(\omega_1^i, \omega_2^i)$, then $g \in \sigma_i$. The contrary assumption would imply that there exists a $g_0 \in \sigma(\omega_1^i, \omega_2^i)$ such that $g_0 \cap D_i^* = 0$. By the construction of $\sigma(\omega_1^i, \omega_2^i)$, we have $D_i^* \cap K_0(\omega_1^i, \omega_2^i)$ is a connected set, where $K_0(\omega_1^i, \omega_2^i) = \bigcup_{g \in \sigma(\omega_1^i, \omega_2^i)} g$. However, if $g \cap D_i^* = 0$, $D_i^* \cap [K_0(\omega_1^i, \omega_2^i) - g_0] = D_i^* \cap K_0(\omega_1^i, \omega_2^i)$. Thus in $\tilde{\Gamma}$ the set of $\tilde{K}_0 \subset \tilde{\Gamma}$ consisting of all elements $g \in \sigma(\omega_1^i, \omega_2^i) - g_0$ is a continuum containing $g_1^i$ and $g_2^i$. However, it has already been proved that $\sigma(\omega_1^i, \omega_2^i)$ is an irreducible continuum. This proves then that $g_0 \cap D_i^* \neq 0$, $g \in \sigma(\omega_1^i, \omega_2^i)$, and hence that $g_0 \in \sigma_i$. Thus $\sigma(\omega_1^i, \omega_2^i) = \sigma_i$, and $k_0(\omega_1^i, \omega_2^i) = k_i$. Hence in this case all three of the smoothing methods are equivalent.

COROLLARY. *If $g_1^i$, $g_2^i \in \sigma_i$, $g_1^i \neq g_2^i$, and if $\omega_1^i$, $\omega_2^i$ are such that $E_{\omega_1^i} \subset g_1^i$,*

$E_{\omega_2^i} \subset g_2^i$, $\omega_1^i < \omega_2^i$, *then the portions of* $\sigma_i$ *consisting of all* $g \in \sigma_i$ *with* $E_\omega \subset g$, $\omega_1^i \leqq \omega \leqq \omega_2^i$ *is equal to* $\sigma(\omega_1^i, \omega_2^i)$.

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