SOLVABLE FACTORIZABLE GROUPS

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Let H and K be subgroups of a finite group G, and let $G = HK$. During recent years, a number of theorems of the following type have been proved (see [2], [3], [4], [5]): if H and K satisfy certain conditions, then G is solvable. In this paper, four additional theorems of this kind will be given. It will be shown that G is solvable under any of the following conditions: (i) H nilpotent, K Abelian or Hamiltonian; (ii) H nilpotent of odd order, K contains a subgroup L of index 2 such that all subgroups of L are normal in K; (iii) H cyclic, K contains a subgroup L of index 2 or 3 such that all subgroups of L are normal in K; (iv) H dihedral or dicyclic, K dihedral or dicyclic.

The proofs of these theorems follow the pattern of earlier proofs by Huppert and Itô [2], [3], [4], [5]. Part (i) generalizes a theorem of Itô [5] for the case where K is Abelian. Part (ii) should be compared with the theorem of Huppert and Itô [4, Satz 4] where H is only nilpotent, but L is cyclic (all subgroups of L being automatically normal in K in this case). Part (iv) is a generalization of a theorem of Huppert [2, Satz 3] that G is solvable if H and K are both dihedral. Its proof requires a theorem (Theorem 4) concerning the primitive S-rings of Schur and Wielandt (see [6], [8], [9]), to the effect that there are no primitive S-rings over a generalized dicyclic group. Finally, it should be mentioned that a generalization (Theorem 5) of (iv) is proved. The extent of this generalization depends on knowledge of permutation groups not yet available.

All groups are to be finite. The following notation will be used: $H \subset G$, $H < G$, $H \lhd G$, to denote that H is a subgroup, proper subgroup, or normal subgroup of G respectively, G_p for a Sylow p-subgroup of G, ${}_pG$ for a Sylow p-complement of a nilpotent group G, $G'(p)$ for the p-commutator subgroup, $Z(G)$ for the center of G, $N(H)$ for the normalizer of H in G, $o(G)$ for the order of G, $[G:H]$ for the index of H in G, H^g for $g^{-1}Hg$, and $\{A, \cdots\}$ for the subgroup generated by $A, \cdots$.

THEOREM 1. *If $G = HK$, where H is nilpotent and K is Abelian or Hamiltonian, then G is solvable.*

Proof. Any subgroup or factor group of an Abelian or Hamiltonian group is again Abelian or Hamiltonian. The proof in [5] now applies without change.

THEOREM 2. *If $G = HK$, where H is nilpotent of odd order, and K has a subgroup L of index 2 such that every subgroup of L is normal in K, then G is solvable.*

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Proof. (See [4, Satz 4]). It is easy to verify that any subgroup or factor group of K is either Abelian, Hamiltonian, or of the same type as K. Now induct on $o(G)$.

If there exists a normal solvable subgroup N of G, then $G/N = (HN/N)(KN/N)$, where the first factor is again nilpotent of odd order, and the second is Abelian, Hamiltonian, or of the type of K. Hence by Theorem 1 or induction, G/N is solvable, G is solvable, and the theorem is true. Therefore it suffices to prove the existence of a normal solvable subgroup.

If $D = H \cap L > 1$, then $D \lhd K$, hence the normal closure $\bar{D}$ of D is contained in H. Therefore $\bar{D}$ is a normal solvable subgroup of G. Hence $H \cap L = 1$. If $H \cap K > 1$, then $o(H \cap K) = 2$, which contradicts the fact that H is of odd order. Hence $H \cap K = 1$.

Case 1. H is maximal. There is a Sylow p-subgroup G_p such that $H_p \subset G_p$. If $H_p < G_p$, then by the nilpotence of H, the normalizer $N(H_p)$ is G. Hence H_p is a normal solvable subgroup and we are done. Therefore $H_p = G_p$ for all primes p such that $p \mid o(H)$, and $(o(H), o(K)) = 1$.

If G is p-normal, then by the theorem of Grün [10, p. 141] $G/G'(p) \cong H/H'(p) > 1$. Hence $G'(p)$ is a proper normal subgroup of G. Since $p \nmid o(K)$, $K \subset G'(p)$. Therefore $G'(p) = H^*K$ for some subgroup H^* of H, and by induction $G'(p)$ is solvable.

If G is not p-normal, then there exists an $x \in K$ such that $Z(G_p) \subset G_p^x$, but $Z(G_p) \neq Z(G_p^x)$, where $G_p = H_p$. Since the case $Z(G_p) \lhd G$ is trivial, we must have $H = N(Z(G_p))$. This latter group contains H and ${}_pH^x$. If ${}_pH > 1$, then ${}_pH^x \subset H$, hence ${}_pH^x = {}_pH$, and $N({}_pH) \supset \{H, x\} = G$, so ${}_pH$ is a normal solvable subgroup of G. Thus it may be assumed that ${}_pH = 1$, i.e. H is a p-group. If $2 \mid o(L)$, then there exists an $x \in L$ of order 2. By the assumption on K, $x \in Z(K)$, and x has p^α conjugates for some α. Therefore [1, p. 322] there exists a proper normal subgroup N of G. If $N \subset H$, then N is solvable. If $N \not\subset H$, then $HN = G$, and $o(G/N) = p^\beta$. Since $p \nmid o(K)$, $K \subset N$, and therefore $N = H^*K$, $H^* \subset H$. By induction N is solvable. If, finally, $2 \nmid o(L)$, then $2 \mid o(G)$, $4 \nmid o(G)$. Hence there is a normal subgroup S of index 2. Since H and L are of odd order, $HL \subset S$, and since $H \cap L = 1$, $HL = S$. By Theorem 1, S is solvable, since L is Abelian or Hamiltonian.

Case 2. H is not maximal. Then there exists a maximal subgroup HR, where $R \subset K$. If $D = R \cap L > 1$, then $D \lhd K$ and the normal closure $\bar{D}$ of D is contained in HR. By induction or Theorem 1, HR is solvable, hence D is a normal solvable subgroup of G. Therefore $R \cap L = 1$, and $o(R) = 2$, $K = RL$. Since $H \cap K = 1$, also $HR \cap L = 1$.

Represent G as a permutation group on the cosets of L. If the representation is not faithful, then L contains a solvable normal subgroup of G. Since $HR \cap L = 1$, HR is regularly and faithfully represented. Since $o(HR)$ is twice an odd number, there is an odd permutation in HR. Hence there is a subgroup N such that $[G:N] = 2$. Since N contains all elements of odd order, $H \subset N$, and $N = HK^*$, $K^* \subset K$. By Theorem 1 or induction, N is solvable, and the theorem is proved.

THEOREM 3. *If $G = HK$, where H is cyclic, and K has a (normal) subgroup L of index 2 or 3 such that all subgroups of L are normal in K, then G is solvable.*

Proof. Induct on $o(G)$. Any subgroup or factor group of K is either Abelian, Hamiltonian, or of the same type as K. Hence if there exists a normal solvable subgroup N of G, we are done, just as in the proof of Theorem 2.

If $D = H \cap K > 1$, then $D \lhd H$, and the normal closure $\bar{D}$ of D is a normal solvable subgroup, being contained in K. Thus $H \cap K = 1$.

If K is not maximal, then there is a group $U = H^*K$, with $1 < H^* \lhd H$. Then U is solvable by induction, while the normal closure of H^* is contained in U and is therefore solvable. Hence K is maximal.

If H is maximal, then $N(H) = H$. If, for some $x \notin H$, $D = H \cap H^x > 1$, then $D \lhd \{H, H^x\} = G$. Hence all $H \cap H^x = 1$, and by a theorem of Frobenius [7, p. 202], $K \lhd G$. Since K is solvable, we are done. Hence H is not maximal.

Let $H < U < G$. Then $U = HR$, $R \subset K$. If R contains a normal subgroup of K, then as before, there exists a normal solvable subgroup. Hence R contains no normal subgroup of K, $o(R) = 2$ or 3, $K = LR$. Furthermore, $U \cap L = 1$, $G = UL$, and U is a maximal subgroup of G. If $[K:L] = 2$, then $[U:H] = 2$, and, since L is nilpotent, the theorem follows from [4, Satz 4]. Therefore assume that $[K:L] = 3$.

Represent G as a permutation group on the cosets of K. The representation is faithful, otherwise K contains a normal subgroup of G. Thus H is a regular subgroup of a primitive permutation group. If G is not solvable, then by [1, p. 341] and [6, Satz III] (or [8]) the representation is 2-fold transitive. Let V be the subgroup of K fixing two letters. If $V = 1$, then by the theorem of Frobenius, $H \lhd G$, hence let $V > 1$. Let $o(H) = h$. Then K is transitive on $h - 1$ letters, hence V contains no normal subgroup of K. Thus $o(V) = 3$, and $o(L) = [K:V] = h - 1$.

Now represent G as a permutation group on the cosets of U. The subgroup U is solvable and maximal, hence $N(U) = U$. Let N be the maximum normal subgroup of U contained in H. Then $[U:N] = 3$ or 6. If $D = N \cap N^x > 1$ for some x outside U, then $D \lhd \{U, U^x\} = G$, and we are done. Hence, for all such x, $N \cap N^x = 1$. Let $W = U \cap U^x$. Then $W/(W \cap N) \cong WN/N$, hence $[W:W \cap N] \mid 6$, and similarly $[W:W \cap N^x] \mid 6$. Therefore, since $N \cap N^x = 1$, $o(W) \mid 36$. Now, using an obvious notation, by [7, p. 63],

$$[UxU:U] = [U:U^x \cap U] = [U:W] = o(U)/o(W),$$

where $o(W) \mid 36$. Since double cosets are disjoint,

$$h - 1 = o(L) = [G:U] = 1 + \sum \lambda_i(3h/n_i),$$

where the λ_i are nonnegative integers, and the summation is over all positive divisors n_i of 36. Reducing to a common denominator and simplifying, we have

$$12 - 24/h = \lambda,$$

where λ is an integer. Since $o(G) = o(U)o(L) = 3h(h - 1)$, this yields

$$o(G) = 6, 18, 36, 90, 168, 396, \text{ or } 1656.$$

Since the list of simple groups is complete through 1656, it follows that G is the simple group of order 168. But then H is cyclic of order 8, whereas the Sylow 2-subgroup of G is not cyclic. This proves the theorem.

Before proceeding further, it is necessary to recall some of the definitions and facts concerning S-rings (see [6], [8], and [9]). Let G be a group with identity e. Let $I(G)$ be the group ring of G over the integers I. If $\sigma = \sum c_g g \in I(G)$, let $\sigma^* = \sum c_g g^{-1}$. An *S-ring* T over G is a subring of $I(G)$ with a basis $(\tau_0 = e, \tau_1, \cdots, \tau_n)$, such that $\tau_i = \sum g$, $g \in S_i$, $S_i \cap S_j$ empty for $i \neq j$, $\bigcup S_i = G$, and such that $\tau_i^* = \tau_j$ for some j. The S-ring T is *primitive* if the subgroup generated by S_i is G for $i \neq 0$. There is always at least one primitive S-ring over G, namely the *trivial* S-ring T_0 with basis $(e, G - e)$.

A group G is a *B-group* if every primitive overgroup (of degree $o(G)$) of the regular representation of G is 2-fold transitive. Schur [6, p. 604] (see also [9, pp. 384–385]) has proved the following theorem:

If the only primitive S-ring over G is the trivial one, then G is a B-group.

A dicyclic group is one with generators x, y and relations $x^{2n} = 1$, $y^2 = x^n$, $y^{-1}xy = x^{-1}$. More generally, a group G will be called *generalized dicyclic* if G is generated by an Abelian subgroup H of index 2 and an element y of order 4, such that H_2 is cyclic and $y^{-1}hy = h^{-1}$ for all $h \in H$.

Wielandt [9, Satz 2] has shown that the only primitive S-ring over a dihedral group is the trivial one. The same is true for generalized dicyclic groups by the following theorem:

THEOREM 4. *If G is a generalized dicyclic group, then there does not exist a nontrivial primitive S-ring over G, and G is a B-group.*

Proof. By Schur's theorem, the second conclusion follows from the first. Let H be the Abelian subgroup of index 2 in G, and x the unique element of order 2 in H. If g is *any* element of G outside H, then $g^2 = x$.

Let T be a primitive S-ring over G. Let τ be the basis element containing x. Then $\tau = \tau^*$. Let y be a fixed element outside H. Since $(yh)^{-1} = yxh$, τ is a sum of x, pairs of elements (yh_i, yxh_i), and pairs of elements (k_j, k_j^{-1}), with $h_i, k_j \in H$. If an element of G outside H occurs in τ^2, then it arises as

$k_j(yh) = (yh)k_j^{-1}$; hence such elements have an even coefficient in τ^2. Therefore

$$\tau^2 = \lambda e + 2\alpha + \beta, \qquad \alpha, \beta \in T,$$

where all the elements of G in β belong to H. If $\beta \neq 0$, then, since β is a sum of basis quantities τ_i, primitivity of T is violated. Hence $\tau^2 = \lambda e + 2\alpha$, where α does not involve e. Now consider the elements of H occurring in τ^2. The equations $(yh_i)(yxh_j) = (yxh_i)(yh_j)$ for $i \neq j$, $k_i k_j = k_j k_i$ for $i \neq j$, $(yh_i)^2 = (yxh_i)^2$, $xk_i = k_i x$, $xk_i^{-1} = k_i^{-1}x$, show that all the nonidentity terms in the expansion of τ^2 occur in equal pairs except those of the form k^2 for $k \in H$. Therefore these must occur in equal pairs also. But this means that for each $k \in H$, $k \neq x$, occurring in τ, kx also occurs in τ, hence $(kx)^{-1} = k^{-1}x$ is in τ.

Now let r be the number of elements of G occurring in τ. By the preceding argument, x occurs exactly $r - 1$ times in τ^2, hence the basis element τ containing x has coefficient $r - 1$ in the expansion of τ^2. Since e occurs r times in τ^2, all $r^2 = r + (r - 1)r$ elements in the expansion of τ^2 have been accounted for. Thus

$$\tau^2 = re + (r - 1)\tau.$$

Hence the subgroup generated by the elements of τ consists of the elements occurring in τ together with e. Since T is primitive, $\tau = G - e$, and T is the trivial S-ring.

THEOREM 5. *If $G = HK$, and H and K are each B-groups containing a cyclic subgroup of index 2 such that all noncyclic factor groups of subgroups of H and K are B-groups, then G is solvable.*

Proof. Let H^* be the cyclic subgroup of index 2 in H. If $H_2 \lhd H$, then H is nilpotent, and the theorem follows from [4, Satz 4]. If $H_2 \ntriangleleft H$, then since $H_2^* \lhd H$, H_2^* is the intersection of all Sylow 2-subgroups of H. Hence H_2^* is characteristic in H. Since all of the other H_p^* are characteristic in H, H^* is characteristic in H.

After these remarks, the proof of [2, Satz 3], with only minor changes, will serve as a proof of Theorem 5.

THEOREM 6. *If $G = HK$, where H is dicyclic or dihedral and K is dicyclic or dihedral, then G is solvable.*

Proof. Any factor group of a subgroup of a dicyclic or dihedral group is cyclic, dihedral, or dicyclic. By [9, Satz 2] and Theorem 4, the hypotheses of Theorem 5 are satisfied, and the conclusion follows.

Remark. Theorem 5 is applicable to a wider class of groups than merely the dihedral and dicyclic ones. For example, if H is the direct product of the symmetric group of degree 3 and a cyclic group of order 4, then it can be shown that H and all of its noncyclic factor groups of subgroups are B-groups.

H contains a cyclic subgroup of order 12 and index 2, but is not nilpotent, dihedral, or dicyclic.

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