MARKOFF PROCESSES AND POTENTIALS II

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The first five sections of this installment treat a situation which is related to the one considered in the first installment, pages 44 to 93 of this volume, just as the potential theory of the Laplacian in a region is related to the theory of the Newtonian potential in the whole of Euclidean space. The following section shows the relative theory to be in a sense complete; and the last section sketches a slight extension—or rather another interpretation—of the main theorems.

The numbering continues that of the first installment. References such as [1] are to the list at the end of the first installment.

10. Terminal times

The simple terminal time S, which serves only to produce convergence, will now be replaced by one defined in terms of a positive function a and a set A. Loosely speaking, the new terminal time is the moment a wandering particle is destroyed, if there is probability $a(r)\,d\tau$ that the particle, having reached the point r safely, is destroyed in the subsequent time interval $d\tau$ and if in addition the particle is sure to be destroyed the instant it touches A.

To be precise, let a be a positive function measurable over the field $\mathfrak{A}$ and let A be a nearly analytic set. Given a process X and a positive random variable Z_X, independent of the process and having the density function $e^{-\sigma}$ for positive σ, define $R_X(\omega)$ to be the infimum of the strictly positive τ for which at least one of the statements

$$(10.1) \qquad X(\tau, \omega) \in A, \qquad \int_0^\tau a\big(X(\sigma, \omega)\big)\, d\sigma \geqq Z_X(\omega),$$

is true, with the understanding that $R_X(\omega)$ is infinite if there are no such τ. We shall say that R_X is the terminal time assigned to X by a and A, with Z_X as auxiliary variable.

Let T be a stopping time for X, with $\mathcal{E}$ as auxiliary field, and suppose that X, Z_X, $\mathcal{E}$ are independent and that Ω', the set where T is less than R_X, has strictly positive probability. Take Y to be the process

$$Y(\tau, \omega) = X\big(\tau + T(\omega), \omega\big), \qquad \tau \geqq 0,\ \omega \in \Omega',$$

defined over the probability field Ω', and take R_Y to be the restriction of $R_X - T$ to Ω'. Straightforward calculation shows that R_Y is in fact that terminal time assigned to Y by a and A, the auxiliary variable being the

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restriction of

$$Z_X - \int_0^T a(X(\tau))\, d\tau$$

to Ω'. It is this property that distinguishes the terminal times defined by a and A from an arbitrary family of stopping times.

A system $\mathfrak{R}$ of terminal times assigned in this way to the various processes, with one auxiliary variable for each process, is said to be determined by a and A, with the Z_X as auxiliary variables. The simple terminal time S^λ may be considered the system determined by the constant function λ and the empty set, with S^1 as auxiliary variable. A system determined by an arbitrary positive function and the empty set was studied in detail in §4.

The auxiliary variables usually will not be specified, for the choice is of consequence only when several systems are discussed at the same time. If they are held fast, each terminal time of the system decreases as a increases or as A increases. There is even a certain continuity in the variation. As the pair (a, A) increases to (a', A') through a sequence of values, a terminal time decreases to the corresponding one of the system determined by (a', A'). A similar statement for decreasing (a, A) requires some additional hypothesis; it is true, for example, if A is held fast and the functions a are bounded.

It is worth making a few remarks that will not be used in the paper. If A is empty and a bounded away from 0 and ∞, the equation

$$\int_0^{R_X} a(X(\tau))\, d\tau = Z_X$$

holds, so that the auxiliary variables can be recovered from the terminal times. Next, altering A by a negligible set, or redefining a on the union of A with an approximately null set, changes each terminal time in $\mathfrak{R}$ only on a set of probability null. Finally, the results of the following sections hold also for limits of the systems defined above. An example of such a limit is given by uniform motion on a line, a particle having an even chance of passing the origin or being destroyed there. It is likely that the broader class can be described axiomatically and that it is the proper basis for a relative theory, but we shall consider only the systems defined constructively.

Let R_r be the time assigned to a process starting at a point r. The zero-one law implies that R_r vanishes with probability either 0 or 1; if the probability is 1, then r is said to be regular for $\mathfrak{R}$.

It is sometimes convenient to replace A by A', the set formed by adjoining to A all points regular for $\mathfrak{R}$. Doing so changes each terminal time of the system only on a set of probability null, a trivial alteration for our purposes, and after the replacement one can say, for almost every ω, that either $R_X(\omega)$ is infinite or one of the two statements

$$X(R_X(\omega), \omega) \in A', \qquad \int_0^{R_X(\omega)} a(X(\tau, \omega))\, d\tau = Z_X(\omega), \tag{10.2}$$

is true.

Before deriving further properties of systems of terminal times, we shall extend one or two results of the simple theory. Suppose the auxiliary variables of $\mathfrak{R}$ to be independent of the simple terminal time S—that is to say, for each process X, the triple X, Z_X, S is independent—and define the kernel $H_{\mathfrak{R}}(r, ds)$ by the formula

$$(10.3) \qquad H_{\mathfrak{R}}(r, D) \equiv \mathcal{P}\{R_r < S, X_r(R_r) \in D\},$$

with R_r the terminal time assigned to a process X_r starting at r.

PROPOSITION 10.1. *If the function φ is excessive relative to S, then $H_{\mathfrak{R}} \varphi$ has the same property and nowhere exceeds φ. If f is a positive function vanishing outside A, then $H_{\mathfrak{R}} Uf$ coincides with Uf.*

First assume the parameter λ of the simple theory to be strictly positive. It suffices then, by Proposition 5.3, to prove the first assertion when φ is the potential of a positive function f. Given r and τ, take R' to be the infimum of the σ greater than τ for which one of the statements

$$X_r(\sigma) \in A, \qquad \int_\tau^\sigma a(X_r(\alpha))\, d\alpha \geqq Z,$$

is true, Z being the auxiliary variable used in defining R_r. Clearly R' decreases to R_r as $\tau \to 0$; also, one may consider $R' - \tau$ to be the terminal time assigned to the process $X_r(\sigma + \tau)$, where σ is the time variable. Consequently, as $\tau \to 0$,

$$H_\tau H_{\mathfrak{R}} Uf.(r) \equiv \int_{\Omega'} d\omega \int_{R'}^{S} f(X_r(\sigma))\, d\sigma,$$

where Ω' is the set on which S exceeds R', increases to

$$H_{\mathfrak{R}} Uf.(r) \equiv \int_{\Omega''} d\omega \int_{R_r}^{S} f(X(\sigma))\, d\sigma,$$

where Ω'' is the set on which S exceeds R_r. The first assertion is now proved for strictly positive parameter, and it follows for vanishing parameter by a passage to the limit.

The second assertion is proved by rewriting the integral defining $Uf.(r)$ as in the proof of Proposition 4.2. The same argument shows that $H_{\mathfrak{R}} Uf$ increases as a and A increase; the statement remains true when Uf is replaced by any function excessive relative to S.

The transformation $L_{\mathfrak{R}}$ is defined in the same manner as L_E in §8. If the parameter is strictly positive and ζ is a measure excessive relative to S, choose a sequence of measures μ_n whose potentials increase to ζ; the potentials $\mu_n H_{\mathfrak{R}} U$ increase with n and do not exceed ζ; and $L_{\mathfrak{R}} \zeta$ is taken to be their limit. The transformation $L_{\mathfrak{R}}$ is defined for vanishing parameter in the same way as L_E. The next proposition summarizes the properties of the transformation, the proofs being the same as those in §8.

PROPOSITION 10.2. *$L_{\mathfrak{R}}$ leaves invariant the class of measures excessive relative to S and preserves majorization. If ζ_n increases to ζ, then $L_{\mathfrak{R}}\zeta_n$ increases to $L_{\mathfrak{R}}\zeta$.*

We shall say that $\mathfrak{R}$ and a second system $\mathfrak{S}$, determined by b and B, are relatively independent if, for every process X, the triple (X, Z_X, Z'_X) is independent, Z_X and Z'_X being the auxiliary variables of the two systems. If $\mathfrak{R}$ and $\mathfrak{S}$ are relatively independent and if T_X is the minimum of R_X and S_X, then the family of times T_X is in fact a system $\mathfrak{T}$ determined by $a + b$ and $A \cup B$, and it is said to be the minimum of $\mathfrak{R}$ and $\mathfrak{S}$; the verification is left to the reader. The auxiliary variables of $\mathfrak{T}$ are not always uniquely defined, nor are they usually independent of the auxiliary variables of $\mathfrak{R}$ and $\mathfrak{S}$. So the direct method of forming the minimum must at times be replaced by the construction of a system defined by $a + b$, $A \cup B$, and a new set of auxiliary variables.

One device in reducing the complexity of a proof is to regard $\mathfrak{R}$ as the minimum of two systems, the one determined by a and the other by A.

There is an important relation connecting two relatively independent systems of terminal times $\mathfrak{R}$ and $\mathfrak{S}$ with their minimum $\mathfrak{T}$. The notation is chosen to agree with the most frequent specialization, but should not be confused, for the moment, with the notation of preceding sections. Let $H_\tau(r, ds)$ be the transition probabilities relative to $\mathfrak{S}$,

$$H_\tau(r, D) \equiv \mathcal{P}\{\tau < S_r, X_r(\tau) \in D\}, \tag{10.4}$$

X_r being a process starting at r and S_r the time assigned to it by $\mathfrak{S}$; let $K_\tau(r, ds)$ be the transition probabilities relative to $\mathfrak{T}$; and let $H_{\mathfrak{R}}(r, d\sigma, ds)$ be the family of measures on the product space $I \times \mathcal{K}$ defined by the formula

$$H_{\mathfrak{R}}(r, C, D) \equiv \mathcal{P}\{R_r < S_r, R_r \in C, X_r(R_r) \in D\}, \tag{10.5}$$

with I the interval $0 \leqq \tau < \infty$. On separating the sample paths into three classes according to the relative magnitudes of R_r and S_r, one obtains the equation

$$\begin{aligned} H_\tau(r, D) = K_\tau(r, D) + \int_{I_\tau \times \mathcal{K}} H_{\mathfrak{R}}(r, d\sigma, ds) H_{\tau-\sigma}(s, D) \\ + \mathcal{P}\{R_r = \tau < S_r, X_r(\tau) \in D\}, \end{aligned} \tag{10.6}$$

where I_τ is the interval $0 \leqq \sigma < \tau$. The last term on the right vanishes except for countably many values of τ, so that the equation becomes

$$U = V + H_{\mathfrak{R}} U \tag{10.7}$$

after integration on τ. Here the kernel U is

$$\begin{aligned} U(r, D) &\equiv \int_0^\infty H_\tau(r, D)\, d\tau \\ &\equiv \int_\Omega d\omega \int_0^{S_r} \chi(X_r(\tau))\, d\tau, \end{aligned} \tag{10.8}$$

with χ the characteristic function of the set D; the kernel V is defined similarly in terms of $\mathfrak{T}$; and the kernel $H_{\mathfrak{R}}$ is

$$\begin{aligned} H_{\mathfrak{R}}(r, D) &\equiv H_{\mathfrak{R}}(r, I, D) \\ &\equiv \mathcal{P}\{R_r < S_r, X_r(R_r) \in D\}. \end{aligned} \tag{10.9}$$

It is clear that each is a kernel in the sense of §3.

The notation of the relative theory will now be fixed. The positive function a and the nearly analytic set A are held fast, and $\mathfrak{R}^\lambda$ is a system of terminal times defined by $a + \lambda$ and A, with λ a positive parameter. Most of the assertions to be made are quite independent of the auxiliary variables of $\mathfrak{R}^\lambda$, and in the proofs we may choose any convenient set. In particular, $\mathfrak{R}^\lambda$ may be considered the minimum of $\mathfrak{R}^0$ and the simple terminal time S^λ, provided S^λ and $\mathfrak{R}^0$ are taken to be relatively independent.

The transition probabilities relative to $\mathfrak{R}^\lambda$ are denoted by $K_\tau^\lambda(r, ds)$,

$$K_\tau^\lambda(r, D) \equiv \mathcal{P}\{\tau < R_r^\lambda, X_r(\tau) \in D\}, \tag{10.10}$$

where R_r^λ is the time assigned to a process X_r starting at r; they obviously satisfy the equations

$$K_\sigma^\lambda K_\tau^\lambda = K_{\sigma+\tau}^\lambda, \tag{10.11}$$

$$K_\tau^\lambda = e^{-\lambda\tau} K_\tau^0. \tag{10.12}$$

The corresponding kernel for potentials is $V^\lambda(r, ds)$,

$$\begin{aligned} V^\lambda(r, D) &\equiv \int_0^\infty K_\tau^\lambda(r, D)\, d\tau \\ &\equiv \int_\Omega d\omega \int_0^{R_r^\lambda} \chi(X_r(\tau))\, d\tau, \end{aligned} \tag{10.13}$$

where χ is the characteristic function of D. An integration by parts, or the proper specialization of (10.7), gives the relation

$$V^{\lambda+\alpha}(r, D) = V^\lambda(r, D) - \alpha \int_0^\infty e^{-\alpha\tau} K_\tau^\lambda V_\cdot^\lambda(r, D)\, d\tau. \tag{10.14}$$

valid for α greater than $-\lambda$. This relation may also be written

$$V^{\lambda+\alpha} = V^\lambda - \alpha V^{\lambda+\alpha} V^\lambda. \tag{10.15}$$

If $\mathfrak{T}$ is a system relatively independent of $\mathfrak{R}^\lambda$, the kernel $K_{\mathfrak{T}}^\lambda$ is defined to be

$$K_{\mathfrak{T}}^\lambda(r, D) \equiv \mathcal{P}\{T_r < R_r^\lambda, X_r(T_r) \in D\}. \tag{10.16}$$

We write K_E^λ, however, if $\mathfrak{T}$ is determined by the set E and the null function
The kernel N^λ is defined to be

$$N^\lambda(r, D) \equiv \mathcal{P}\{R_r^0 < S^\lambda, X_r(R_r^0) \in D\}, \tag{10.17}$$

where $\mathfrak{N}^0$ and the simple terminal time S^λ are assumed relatively independent. On specializing $\mathfrak{N}$, $\mathfrak{S}$, $\mathfrak{T}$ in relation (10.7) to $\mathfrak{N}^0$, S^λ, $\mathfrak{N}^\lambda$, we obtain

$$U^\lambda = V^\lambda + N^\lambda U^\lambda,$$

where U^λ is the kernel for potentials relative to S^λ. If λ is strictly positive, the equation may be written

$$V^\lambda = U^\lambda - N^\lambda U^\lambda, \tag{10.18}$$

for then U^λ is bounded. This relation enables one to carry over many results from the simple to the relative theory.

Let $\mathcal{K}$ be the set of points in $\mathcal{H}$ which are not regular for $\mathfrak{N}$; the set is independent of the parameter, since a point is regular for $\mathfrak{N}^\lambda$ if and only if it is regular for $\mathfrak{N}^0$. The function $K_\tau^\lambda(r, D)$ vanishes identically in λ, τ, D unless r belongs to $\mathcal{K}$. Also, $V^\lambda(r, D)$ vanishes identically in λ, r if D is disjoint from $\mathcal{K}$. In verifying the last assertion we assume, as we may, that A includes the complement of $\mathcal{K}$; then, by Proposition 10.1,

$$V^\lambda(r, D) \equiv U^\lambda(r, D) - N^\lambda U^\lambda_\cdot(r, D) = 0,$$

for λ strictly positive. The assertion for vanishing parameter is a consequence, because V^λ increases to V^0 as λ decreases to 0.

It is necessary to arrange the points of $\mathcal{K}$ according to their regularity for $\mathfrak{N}^\lambda$. For this purpose, take $\Phi_\mathfrak{N}$ to be the function

$$\Phi_\mathfrak{N}(r) = \mathcal{P}\{R_r^\lambda < S^1\},$$

R_r^λ being the time assigned to a process X_r starting at r, and the simple terminal time S^1 being independent of the system $\mathfrak{N}^\lambda$. This function is excessive relative to S^1, by Proposition 10.1. We define $\mathcal{K}_\beta^\lambda$, for β less than 1, by the inequality $\Phi_\mathfrak{N} < \beta$; the set $\mathcal{K}$ itself comprises the points where $\Phi_\mathfrak{N}$ is less than 1. The $\mathcal{K}_\beta^\lambda$ increase with β, their union is $\mathcal{K}$, and each one is nearly Borel and nearly open in the sense of §7. For fixed β, the set $\mathcal{K}_\beta^\lambda$ decreases as λ increases, but the collection of sets $\mathcal{K}_\beta^\lambda$, with $0 \leqq \beta < 1$, does not vary with λ. To see this, let Z be a positive random variable which is independent of S^1 as well as of $\mathfrak{N}^0$ and which has the density function $\lambda e^{-\lambda\sigma}$ for positive σ. Then R_r^λ may be considered the minimum of R_r^0 and Z, and a simple computation shows that $\mathcal{K}_\beta^\lambda$ is precisely the set $\mathcal{K}_\gamma^0$, where γ is $\beta - \lambda + \lambda\beta$. The result shows incidentally that $\mathcal{K}_\beta^\lambda$ is the empty set unless β exceeds $\lambda/(1 + \lambda)$.

Proposition 10.3. *Given λ and β, one can find two numbers, γ less than 1 and α strictly positive, so that the probability of the joint event*

$$R_r^\lambda \geqq \alpha, \qquad X_r(\tau) \in \mathcal{K}_\gamma^\lambda \quad \textit{for } 0 \leqq \tau \leqq \alpha,$$

exceeds α for every r belonging to $\mathcal{K}_\beta^\lambda$.

Let γ and α be subject to the restrictions mentioned, and let r be a point of $\mathcal{K}_\beta^\lambda$. Take T to be the minimum of α and the time X_r hits the complement of $\mathcal{K}^\lambda_\gamma$, take Ω' to be the set where T is less than α, and let Y be the process $X_r(\tau + T)$ defined over the probability field Ω'. It is clear that $\Phi_{\mathfrak{R}}(Y(0))$ is at least γ almost everywhere on Ω' and that R_Y, the time assigned to Y by $\mathfrak{R}^\lambda$, may be taken to be the restriction of $R^\lambda_r - T$ to Ω', the auxiliary variable being chosen properly. Matters being so, the extended Markoff property, with stopping time T, implies that

$$\mathcal{P}\{R^\lambda_r \geqq 2\alpha\} \leqq \mathcal{P}\{R^\lambda_r \geqq \alpha,\ T \geqq \alpha\} + \mathcal{P}'\{R_Y > \alpha\}.$$

The proof is completed by first choosing α less than $(1 - \beta)/4$ and so small that the left member of the inequality exceeds $(1 - \beta)/2$, then choosing γ so close to 1 that the second term of the right member is less than $(1 - \beta)/4$. The choices are possible, because the inequalities

$$\mathcal{P}\{R^\lambda_r < S^1\} < \beta, \qquad \mathcal{P}\{R^\lambda_s < S^1\} \geqq \gamma,$$

respectively imply the inequalities

$$\mathcal{P}\{R^\lambda_r > 2\alpha\} \geqq 1 - \beta e^{2\alpha}, \qquad \mathcal{P}\ \{R^\lambda_s > \alpha\} \leqq \frac{1-\gamma}{1-e^{-\alpha}}.$$

The full notation of the relative theory has been explained, but in practice the symbol λ will be omitted except when the parameter is varied. The symbols H_τ and U will have the same meaning as in §§3–9, with the same value of the parameter as in the relative theory. Note that the kernel N, which relates U to the kernel V of the relative theory by equation (10.18), is defined in terms of $\mathfrak{R}^0$ and S^λ. Note also that the sets $\mathcal{K}_\beta$ are defined only for β less than 1, so that to say a set is included in some $\mathcal{K}_\beta$ is the same as to say it is included in $\mathcal{K}_\beta$ for some β less than 1; and the truth of such a statement does not depend upon the value of λ.

11. Excessive functions

A positive function φ is excessive relative to $\mathfrak{R}$ if it is measurable over the field $\mathcal{A}$ and if $K_\tau\varphi$ increases to φ as τ decreases to 0. In this section and the next two, *excessive* stands for *excessive relative to* $\mathfrak{R}$, X_r is a process starting at the point r, and R_r is the time assigned to it by $\mathfrak{R}$.

If ψ is excessive relative to the simple terminal time S, the function which coincides with ψ on $\mathcal{K}$ and vanishes elsewhere is excessive relative to $\mathfrak{R}$. A system $\mathfrak{T}$ of terminal times, relatively independent of $\mathfrak{R}$, determines two excessive functions,

$$(11.1) \qquad \Psi_{\mathfrak{T}}(r) = \mathcal{P}\{T_r < R_r\}, \qquad \Theta_{\mathfrak{T}}(r) = \mathcal{P}\{T_r \leqq R_r,\ R_r > 0\},$$

where T_r is the time assigned to X_r by $\mathfrak{T}$. We shall omit the proofs, which are quite simple, for the assertions are implied by later more general statements. The last two functions will be denoted by Ψ_E and Θ_E when $\mathfrak{T}$ is determined by the set E and the null function.

Two circumstances make the relative theory more difficult than the simple theory of §§3–9. The function which is 1 on $\mathcal{K}$ and 0 elsewhere may not be a potential, and the time $\mathfrak{R}$ assigns to a process may coincide on a set of strictly positive probability with the time the process hits a given set. There is accordingly a complication in detail—for example, sets which are included in some $\mathcal{K}_\beta$ and have compact closure replace sets with compact closure, and a statement concerning all τ usually becomes one concerning the τ preceding the time assigned to a process by $\mathfrak{R}$, or perhaps preceding the time the process leaves $\mathcal{K}$. The statement of a proposition may require a further qualification or its proof an additional argument or two, on leaving the simple theory, but these are elaborations of the old pattern. In presenting the relative theory I shall therefore state the definitions and principal theorems, giving a proof only when it differs substantially from the corresponding one of the simple theory.

The remarks at the beginning of §5 are still valid. The potential relative to $\mathfrak{R}$ of a positive function f is the function Vf,

$$(11.2)\qquad \begin{aligned} Vf.(r) &\equiv \int_0^\infty K_\tau f.(r)\, d\tau \\ &\equiv \int_\Omega d\omega \int_0^{R_r} f(X_r(\tau))\, d\tau, \end{aligned}$$

so that, if $\mathfrak{T}$ and $\mathfrak{R}$ are relatively independent,

$$(11.3)\qquad K_{\mathfrak{T}}\, Vf.(r) \equiv \int_{\Omega'} d\omega \int_{T_r}^{R_r} f(X_r(\tau))\, d\tau,$$

with Ω' the set where T_r, the time assigned to X_r by $\mathfrak{T}$, is less than R_r. The next proposition is a consequence of these equations, the behavior of a system of terminal times as the determining set and function increase, and the reasoning of the first few pages of §4 and §5.

PROPOSITION 11.1. *The potential Vf of a positive function f is excessive and depends only on the restriction of f to $\mathcal{K}$. Let $\mathfrak{T}$ be relatively independent of $\mathfrak{R}$ and determined by b and B; then $K_{\mathfrak{T}}\, Vf$ is excessive relative to $\mathfrak{R}$, nowhere exceeds Vf, coincides with Vf if f vanishes outside B, and increases to $K_{\mathfrak{T}'}\, Vf$ if b and B increase to b' and B' through sequences and if $\mathfrak{T}'$ is determined by b' and B'.*

If f is bounded and λ strictly positive, one may write

$$(11.4)\qquad Vf = Uf - NUf,$$

according to (10.18). The second term on the right is excessive relative to the simple terminal time S, by Proposition 10.1 or 11.1, so that Vf is nearly Borel measurable and—rather exceptionally—$Vf.(X(\tau))$ is continuous on the right for all τ with probability 1.

The analogues of the first two propositions of §5, which have the same proof as before, give the following approximation theorem.

THEOREM 11.2. *If λ is strictly positive and φ excessive relative to $\mathfrak{R}$, there is a sequence of positive bounded functions f_n whose potentials Vf_n increase to φ.*

It follows that an excessive function is nearly Borel measurable and that $K_{\mathfrak{T}}\, \varphi$ behaves as if φ were a potential.

It is now proved, just as in Proposition 5.5, that

$$\inf_E \varphi \leqq \varphi(r) \leqq \sup_E \varphi,$$

if φ is excessive and the point r is regular for E but not for $\mathfrak{R}$,—that is to say, if r belongs to $\mathcal{K}$ and is regular for E. This result and the beginning of the proof of Theorem 5.6 show that, given a strictly positive number α and a process X whose initial distribution is concentrated on $\mathcal{K}$, one can find a stopping time T for X which is strictly positive with probability 1 and which has the property that

$$|\, \varphi(X(\tau)) - \varphi(X(0)) \,| \leqq \alpha \textit{ for } 0 \leqq \tau < T.$$

In repeating the argument with the original process replaced by $X(\tau + T)$, one must neglect the ω for which $X(T(\omega), \omega)$ lies outside $\mathcal{K}$. The latter half of the proof of Theorem 7.2 of [7], on being modified accordingly, yields the continuity of $\varphi(X(\tau))$ on the right only up to the moment the process leaves $\mathcal{K}$. The proof of finiteness of $\varphi(X(\tau))$ carries over for τ less than the terminal time; here the terminal time may be replaced by a somewhat greater time, as we shall see in a moment, but usually it may not be replaced by the time the process first leaves $\mathcal{K}$. Consider, by way of example, uniform motion to the right on the reals, taking a null, A the origin, φ the function which vanishes to the left of the origin and is infinite elsewhere. That the time a process leaves $\mathcal{K}$ usually cannot be increased, in the statement about continuity on the right, becomes clear on considering Brownian motion on the line, taking A to be a closed interval and φ to be 0 in the interval and 1 outside. These results are summed up in the next theorem.

THEOREM 11.3. *Let φ be excessive relative to $\mathfrak{R}$, X a process, R the time assigned to it by $\mathfrak{R}$, and T the time it hits the complement of $\mathcal{K}$. Then φ is nearly Borel measurable; $\varphi(X(\tau))$ is with probability* 1 *continuous on the right in the interval* $[0, T)$; *and $\varphi(X(\tau))$ is with probability* 1 *finite in the interval* $[0, R)$, *provided the expectation of $\varphi(X(0))$ is finite.*

The minimum of two excessive functions is now easily seen to be excessive.

The discussion of the semimartingale defined by an excessive function and a process will be given in detail, for it differs a good deal from the one in §5.

Suppose that $Y(\tau, \omega)$ is a positive function on the product space $I \times \Omega$, with I the interval $0 \leqq \tau < \infty$, that it is a decreasing function of τ, and that the expectation of $Y(0, \omega)$ is finite. Let $(\mathfrak{F}_\tau)_{\tau \epsilon I}$ be any increasing family of subfields of $\mathfrak{F}$, and define $Y'(\tau)$ to be the conditional expectation

$$Y'(\tau) = \mathcal{E}\{Y(\tau) \,|\, \mathfrak{F}_\tau\}.$$

The family $(Y'(\tau), \mathfrak{F}_\tau)$ is then a lower semimartingale, for $Y'(\tau)$ is measurable over $\mathfrak{F}_\tau$, by definition, and

$$\begin{aligned}\mathcal{E}\{Y'(\tau) \mid \mathfrak{F}_\sigma\} &= \mathcal{E}\{Y(\tau) \mid \mathfrak{F}_\sigma\} \\ &\leqq \mathcal{E}\{Y(\sigma) \mid \mathfrak{F}_\sigma\} = Y'(\sigma), \qquad \sigma < \tau.\end{aligned}$$

Now let λ be strictly positive, f a positive bounded function, X a process, R the time assigned to it by $\mathfrak{R}$, and define

$$Y(\tau) = \varepsilon(\tau, R) \int_\tau^R f(X(\sigma))\, d\sigma,$$

where $\varepsilon(\tau, R)$ is 1 if $\tau < R$ and 0 otherwise. Taking $\mathfrak{F}_\tau$ to be the field generated by the random points $X(\sigma)$, with σ not exceeding τ, one has

$$Y'(\tau) = \mathcal{P}\{R > \tau \mid \mathfrak{F}_\tau\} Vf.(X(\tau)). \tag{11.5}$$

With this specialization, $Y(\tau)$ is continuous in τ, and $Y'(\tau)$ can be shown to be continuous on the right with probability 1 if the proper version of the conditional probability is chosen. The context suggests that this fact, or perhaps even Theorem 11.3, is a consequence of martingale theory and general properties of the $Y(\tau)$ and $\mathfrak{F}_\tau$.

It is necessary to analyze the conditional probability in the last equation; in the discussion we shall permit λ to have any value and write simply a instead of $a + \lambda$. Let R' be the time X hits A, and let R'' be the terminal time assigned to X by a and the empty set, with auxiliary variable Z the one used in defining R. Since R is the minimum of R' and R'',

$$\mathcal{P}\{R > \tau \mid \mathfrak{F}_\tau\} = \varepsilon(\tau, R') \exp\left\{-\int_0^\tau a(X(\sigma))\, d\sigma\right\} \tag{11.6}$$

is one version of the conditional probability. The factor $\varepsilon(\tau, R')$ is obviously continuous on the right in τ, and we shall prove that the exponential is with probability 1 continuous in the interval $0 \leqq \tau < T$, where T is again the time X hits the complement of $\mathfrak{K}$. Indeed, the exponential is discontinuous only at a number ρ with the properties

$$\int_0^\rho a(X(\sigma))\, d\sigma < \infty, \qquad \int_0^{\rho+\alpha} a(X(\sigma))\, d\sigma = \infty \quad \textit{for} \quad \alpha > 0,$$

and for each ω there is at most one such number. Let Ω', the set of ω for which such a number $\rho(\omega)$ exists, have strictly positive probability, and consider the process $X(\tau + \rho(\omega), \omega)$ defined over Ω'. It is easily seen that the terminal time assigned to this process by a and the empty set vanishes with probability 1, even identically so if the auxiliary variable is the restriction to Ω' of the random variable $Z - \int_0^\rho a(X(\sigma))\, d\sigma$. For almost all ω in Ω' the point $X(\rho(\omega), \omega)$ is therefore regular for the system defined by a, so also for $\mathfrak{R}$, and the assertion concerning the exponential is proved. In the remainder of the section we shall assume, as we may, that every point regular for $\mathfrak{R}$ belongs to A. Then

$\rho(\omega)$ is at least as great as $R'(\omega)$ for almost all ω in Ω'. Let us extend ρ to all of Ω by taking $\rho(\omega)$ to be the infimum of the τ for which the exponential vanishes; $\rho(\omega)$ may be less than $R'(\omega)$ if ω is not in Ω'. The right member of (11.6) is, with probability 1, continuous on the right for all τ, and even continuous and strictly positive for τ less than ρ and R'.

The family of random variables defined by (11.5) remains a semimartingale, even for vanishing λ, when the potential Vf is replaced by any function φ, excessive relative to $\mathfrak{R}$, for which the expectation of $\varphi(X(0))$ is finite; the semimartingale is separable, for almost all the sample functions are continuous on the right, by Theorem 11.3 and what has just been proved. As a consequence of the last paragraph and the behavior of semimartingales, the function $\varphi(X(\tau))$ almost certainly is finite and has finite limits from the left, so long as τ is less than ρ and R'. This statement is slightly stronger than the last one in Theorem 11.3. It is also true that $\varphi(X(\tau))$ has, with probability 1, a finite limit as τ increases to the minimum of ρ and R', provided the exponential factor does not decrease to 0. The last event can happen only if both $\rho(\omega)$ and $R'(\omega)$ are infinite or if $\rho(\omega)$ is finite and ω not in Ω'; for almost all such ω, the terminal time $R''(\omega)$ is strictly less than $\rho(\omega)$. It follows that, with probability 1, the function $\varphi(X(\tau))$ has a finite limit as τ increases to R, the terminal time assigned to the process by $\mathfrak{R}$.

Suppose the expectation of $\varphi(X(0))$ to be infinite, and let T be the minimum of R and the time X hits the set where φ is finite. With probability 1, the function $\varphi(X(\tau))$ is infinite for $0 \leqq \tau < T$, is finite for $T < \tau < R$, and has finite limits from the left for $T < \tau \leqq R$. The proof is like one in §5.

PROPOSITION 11.4. *Let $\mathfrak{R}$ be the minimum of the relatively independent systems $\mathfrak{S}$ and $\mathfrak{T}$, and let ψ be excessive relative to $\mathfrak{T}$. Then the function which coincides with ψ on $\mathcal{K}$ and vanishes elsewhere is excessive relative to $\mathfrak{R}$.*

In the proof we assume, as one easily sees we may, that ψ is bounded. For each r and τ the transition measure $N_\tau(r, ds)$ relative to $\mathfrak{T}$ majorizes $K_\tau(r, ds)$, and, for r in $\mathcal{K}$, the mass of the second measure increases to 1 as $\tau \to 0$. Consequently $K_\tau \psi$ nowhere exceeds ψ; thus, $K_\tau \psi$ is a decreasing function of τ, because the kernels K_τ form a semigroup. And, for r in $\mathcal{K}$,

$$\lim_{\tau \to 0} K_\tau \psi.(r) = \lim N_\tau \psi.(r) = \psi(r),$$

because ψ is bounded. For points outside $\mathcal{K}$ there is nothing to verify.

A set E is said to be approximately null relative to $\mathfrak{R}$ if $V(r, E)$ vanishes identically, negligible relative to $\mathfrak{R}$ if, for every process and with probability 1, the time the process hits E is at least as great as the terminal time assigned to the process. The second property implies the first, by the argument at the end of §5, and a set is certainly negligible relative to $\mathfrak{R}$ if it is included in A. The place of a point regular for a set E or for a system of terminal times $\mathfrak{T}$ is taken, in the relative theory, by one regular for E or $\mathfrak{T}$ but not for $\mathfrak{R}$. If E is a nearly analytic set such that every point which is regular for E is also regular for $\mathfrak{R}$, then almost all sample paths of a process meet E at most countably many times before hitting the complement of $\mathcal{K}$. If E is nearly analytic, the points belonging to E but either regular for $\mathfrak{R}$ or not regular for E form a set which is approximately null relative to $\mathfrak{R}$. An excessive function is determined by its values outside a set approximately null relative to $\mathfrak{R}$, and the set where an excessive function is infinite is negligible relative to $\mathfrak{R}$ if and only if it is approximately null relative to $\mathfrak{R}$. These statements are proved just as were the corresponding ones in the simple theory.

We shall discuss next some properties of excessive functions that were not considered in the simple theory. Let φ be excessive relative to $\mathfrak{R}$, and let $\mathfrak{T}$ be a system relatively independent of $\mathfrak{R}$ and determined by (b, B). Clearly, $K_{\mathfrak{T}}\,\varphi$ increases with b, B, and φ. It also decreases as a and A increase, since the kernel $K_{\mathfrak{T}}$ decreases. In this statement φ is taken to vanish outside the set $\mathcal{K}$ of the moment and to be excessive for the least values of a and A considered; it is then excessive for the other values of a and A, according to the preceding proposition. If the pair (b, B) increases to (b', B') through a sequence, $K_{\mathfrak{T}}\,\varphi$ increases to $K_{\mathfrak{T}'}\,\varphi$, where $\mathfrak{T}'$ is a system determined by (b', B'); one has only to use Theorem 11.3, supposing all the systems to have the same auxiliary variables. A similar statement for (b, B) decreasing, or for (a, A) varying in either sense, requires an additional hypothesis.

Let $\mathfrak{T}$ and $\mathfrak{T}'$ be systems determined by the pairs (b, B) and (b', B'). We shall say that $\mathfrak{T}$ *dominates* $\mathfrak{T}'$ if b nowhere exceeds b' and B is a subset of B'; when these conditions are satisfied and the two systems have the same auxiliary variables, every terminal time of $\mathfrak{T}$ is at least as great as the corresponding terminal time in $\mathfrak{T}'$. On the other hand, $\mathfrak{T}$ is said to *majorize* $\mathfrak{T}'$ if the two systems are relatively independent and if each terminal time of $\mathfrak{T}$ is, with probability 1, at least as great as the corresponding one in $\mathfrak{T}'$; this is evidently so when B' includes both B and the set where b is strictly positive. The terms will be used sparingly.

The treatment of what may be called combinatorial properties should be compared with the one by Choquet in II §7 of [3].

Suppose $\mathfrak{R}$, $\mathfrak{T}$, $\mathfrak{T}_1, \cdots, \mathfrak{T}_n$ to be relatively independent, that is to say, a process and the auxiliary variables corresponding to it in the several systems always form an independent family. Define kernels by the formula

$$K(i, \cdots, k) \equiv K_{\mathfrak{S}}\,; \qquad \mathfrak{S} \equiv \min\{\mathfrak{T}, \mathfrak{T}_i, \cdots, \mathfrak{T}_k\}, \qquad 1 \leqq i < \cdots < k \leqq n.$$

We shall study the quantity

$$\Delta_r(\mathfrak{T}, \mathfrak{T}_1 \cdots, \mathfrak{T}_n) \equiv -K_{\mathfrak{T}}\,\varphi.(r) - \sum_{1 \leqq m \leqq n} (-1)^m \sum K(i_1, \cdots, i_m)\varphi.(r),$$

where φ is excessive relative to $\mathfrak{R}$. The properties of Δ are summed up in the next theorem, in which r is to be restricted by the requirement that all terms in the sum are finite.

THEOREM 11.5. *The quantity* $\Delta_r(\mathfrak{T}, \mathfrak{T}_1, \cdots, \mathfrak{T}_n)$ *is symmetric in the* $\mathfrak{T}_i$. *It vanishes if r is regular for* $\mathfrak{T}$ *or if one of the* $\mathfrak{T}_i$ *majorizes* $\mathfrak{T}$. *The inequalities*

$$0 \leqq \Delta_r(\mathfrak{T}, \mathfrak{T}_1, \cdots, \mathfrak{T}_n) \leqq \varphi(r), \tag{11.7}$$

$$\Delta_r(\mathfrak{T}, \mathfrak{T}_1, \cdots, \mathfrak{T}_n) \leqq \Delta_r(\mathfrak{T}, \mathfrak{T}_1, \cdots, \mathfrak{T}_{n-1}), \qquad n > 1, \tag{11.8}$$

hold generally. The inequality

$$\Delta_r(\mathfrak{T}, \mathfrak{T}_1, \cdots, \mathfrak{T}_n) \leqq \Delta_r(\mathfrak{T}', \mathfrak{T}_1', \cdots, \mathfrak{T}_n') \tag{11.9}$$

holds if $\mathfrak{T}$ *dominates* $\mathfrak{T}'$ *and* $\mathfrak{T}_i'$ *dominates* $\mathfrak{T}_i$ *for each i.*

Suppose first that λ is strictly positive and that φ is the potential of the positive bounded function f. One can then write

$$K(i, \cdots, k)\varphi.(r) = \int_{\Omega'} d\omega \int_{T'}^{R} f(X(\tau))\, d\tau,$$

where X is a process starting at r, R is the time assigned by $\mathfrak{R}$, T' is the minimum of the times assigned by $\mathfrak{T}, \mathfrak{T}_i, \cdots, \mathfrak{T}_k$, and Ω' is the set where T' is less than R. This expression leads to the equation

$$\Delta_r(\mathfrak{T}, \mathfrak{T}_1, \cdots, \mathfrak{T}_n) = \int_{\Omega^*} d\omega \int_{T^*}^{T} f(X(\tau))\, d\tau,$$

where T^* is the maximum of the times assigned by the various systems $\mathfrak{T}_i$ ' T is the time assigned by $\mathfrak{T}$, and Ω^* is the set defined by the inequalities $T^* < T < R$; the argument is precisely the one used to derive the formulas of Poincaré for the probabilities of composite events. All the assertions of the theorem are now obvious, except perhaps the last inequality, and that becomes so when one takes the auxiliary variables of corresponding systems to be the same. It is also clear that Δ increases if f is increased or if $\mathfrak{R}$ is replaced by a system which dominates $\mathfrak{R}$.

The theorem is proved for an arbitrary excessive function, but with λ still strictly positive, by a passage to the limit using Theorem 11.2. The results carry over to vanishing λ because $K_{\mathfrak{T}}^{\lambda}$, for example, increases to $K_{\mathfrak{T}}^{0}$ as $\lambda \to 0$, and because a function which is excessive for one value of λ is excessive for all greater values. The proof is now complete.

For a moment, while discussing an inequality already used in §6, we shall write $K(\mathfrak{T})$ instead of $K_{\mathfrak{T}}$. Consider a sequence of systems $\mathfrak{T}_i$ determined by the pairs (b_i, B_i), and let $\mathfrak{T}$ be a system determined by the sum of the b_i and the union of the B_i. If the $\mathfrak{T}_i$ are relatively independent, a terminal time in $\mathfrak{T}$ may be taken to be the infimum of the corresponding ones in the systems $\mathfrak{T}_i$. Let $\mathfrak{T}_i'$ and $\mathfrak{T}'$ have similar meanings, and suppose that $\mathfrak{T}_i'$ dominates $\mathfrak{T}_i$ for each i. Under these hypotheses, one has the inequality

$$K(\mathfrak{T})\varphi.(r) - K(\mathfrak{T}')\varphi.(r) \leqq \sum [K(\mathfrak{T}_i)\varphi.(r) - K(\mathfrak{T}_i')\varphi.(r)], \tag{11.10}$$

provided the second term on the left is finite, so that the expressions make sense. Assume first that λ is strictly positive and that φ is the potential of the positive bounded function f. There is no loss of generality in supposing the $\mathfrak{T}_i$ to be relatively independent and the auxiliary variables of $\mathfrak{T}$, $\mathfrak{T}_i$ to be the same as those of $\mathfrak{T}'$, $\mathfrak{T}_i'$. When this is done, the inequality can be written

$$\int_{\Omega'} d\omega \int_T^{R'} f(X(\tau))\, d\tau \leqq \sum \int_{\Omega_i'} d\omega \int_{T_i}^{R_i'} f(X(\tau))\, d\tau,$$

where X is a process starting at r, T_i is the time assigned by $\mathfrak{T}_i$, R_i' is the minimum of the times assigned by $\mathfrak{R}$ and $\mathfrak{T}_i'$, Ω_i' is the set on which T_i is less than R_i', and T, R', Ω' have similar meanings. The inequality in this form is nearly obvious since T, for example, may be taken to be the infimum of the T_i. The proof is completed in the same way as that of the theorem. Another proof, based on the positiveness of $\Delta_r(\mathfrak{T}, \mathfrak{T}_1, \mathfrak{T}_2)$, is given in Choquet's memoir.

12. Special sets

In this section we suppose the sets $\mathcal{K}_\beta$ to be defined by giving λ the value 0, although any positive value would do as well. The index β, it will be recalled, runs over the positive numbers strictly less than 1.

A set is said to be special if it is nearly open, has compact closure, and is included in some $\mathcal{K}_\beta$; the last condition is a kind of uniform separation from the complement of $\mathcal{K}$. These sets take over the role of open sets with compact closures, the intersection of one of the latter sets with some $\mathcal{K}_\beta$ being indeed the simplest example of a special set. The details of the relative theory depend upon the way special sets are defined; the definition we have given works well if the transition probabilities are sufficiently regular.

PROPOSITION 12.1. *Under hypothesis (C) of §9, for every special set D there is another special set D' such that $V(r, D')$ is bounded away from 0 on D.*

We shall first prove a preliminary assertion: For every compact set F and every number α less than 1, there exist an open set G with compact closure and a strictly positive number ρ such that the sample paths of a process starting at a point of F remain in G until time ρ, with probability at least α. The parameter λ is to have a strictly positive value during the proof, and γ is a number less than 1 that will be fixed at the end.

Consider the potential Uf as f runs through a sequence of positive functions in $\mathcal{C}(\mathcal{K})$ which increase everywhere to λ. The potential belongs to $\mathcal{C}(\mathcal{K})$, under (C), and increases everywhere to 1; the convergence is therefore uniform on F. Choose f so that Uf exceeds γ at all points of F and is bounded by 1 everywhere. The function $H_\rho\, Uf$ tends uniformly to Uf as $\rho \to 0$, for

$$Uf - H_\rho Uf = \int_0^\rho H_\tau f\, d\tau \leqq \rho \max f,$$

and we fix ρ so that $H_\rho\, Uf$ also exceeds γ on F. Take G to be the set where Uf exceeds $\gamma/2$. Given a process X starting at a point r of F, let T be the minimum of ρ and the time the process first leaves G. Then

$$\int_\Omega e^{-\lambda T}\, Uf.(X(T))\, d\omega \geqq H_\rho\, Uf.(r) > \gamma,$$

and the integrand is bounded by $\gamma/2$ on the set where T is less than ρ. Consequently,

$$\mathcal{P}\{T = \rho\} + \frac{\gamma}{2}\,\mathcal{P}\{T < \rho\} > \gamma,$$

$$\mathcal{P}\{T = \rho\} > \frac{\gamma}{2 - \gamma}.$$

The preliminary assertion is now proved by choosing γ suitably.

Let D be a special set. Proposition 10.3 and what has just been proved imply the existence of another special set D' and a strictly positive number α with the following property: If R_r is the terminal time assigned to a process X_r starting at a point r regular for D, then the probability of the joint event

(12.1) $$R_r > \alpha, \qquad X_r(\tau) \in D' \text{ for } \tau < \alpha,$$

is at least α. This result, which is stronger than the proposition, will be needed occasionally. It implies the proposition, of course, for $V(r, D')$ must be at least α^2 on D.

The parameter λ was required at times to be strictly positive in the simple theory. This condition is more than is needed, and we shall replace it in the relative theory by one or the other of the following conditions.

(*D*) *Let D be a special set, X a process, and Ω' the set where the terminal time assigned to X by $\mathfrak{R}$ is infinite. Then, for almost all ω in Ω', the point $X(\tau, \omega)$ lies outside D for all sufficiently large τ.*

(*E*) *The function $V(r, D)$ is bounded in r whenever D is a special set.*

The first asserts that almost all sample paths of a process finally leave a given special set, either because they are terminated or because they wander to infinity; the second asserts that the sample paths of a process spend, on the average, a finite time in a special set. Both statements are true if λ is strictly positive. For many transition measures they are true even when λ, a, A all vanish; instances are the transition measures of Brownian motion in three dimensions or higher, and those leading to Riesz potentials in two dimensions or higher. Later in the section we shall discuss the verification of (*D*) and (*E*) when the transition measures are ergodic.

THEOREM 12.2. *Let (D) or (E) hold, and let φ be excessive relative to $\mathfrak{R}$. Then there are positive functions f_n whose potentials Vf_n increase to φ.*

We shall prove in a moment that there is a sequence of positive functions g_n such that Vg_n is bounded for each n and increases with n to infinity at every point of $\mathcal{K}$. The finiteness of the potentials implies that $K_\tau\, Vg_n \to 0$ as $\tau \to \infty$.

The function φ_n, the minimum of φ and Vg_n, is excessive and increases to φ as n becomes large. Also, the potential of $(\varphi_n - K_\tau \varphi_n)/\tau$ increases to φ_n as $\tau \to 0$, by the proof of Proposition 5.2. The theorem follows from these two facts.

The existence of the functions g_n is clear under (E), and the following argument proves their existence under (D). Let χ be the characteristic function of a special set D, and define ψ by the formula

$$\psi(r) = 1 - \int_\Omega \exp\left\{ - \int_0^R \chi(X(\tau))\, d\tau \right\} d\omega,$$

where X is a process starting at the point r and R is the terminal time assigned to X. The integral within the curly brackets is strictly positive with probability 1, if r belongs to D, so that ψ is strictly positive in D. On the other hand, the integral is finite with probability 1 for every r, under hypothesis (D), so that ψ is less than 1 everywhere. The finiteness also justifies the following calculations, which are like those in the proof of Proposition 4.4.

$$\begin{aligned}\psi(r) &= \int_\Omega d\omega \int_0^R \chi(X(\tau)) \exp\left\{ - \int_\tau^R \chi(X(\sigma))\, d\sigma \right\} d\tau \\ &= \int_0^\infty d\tau \int_{\Omega^\tau} \chi(X(\tau)) \exp\left\{ - \int_\tau^R \chi(X(\sigma))\, d\sigma \right\} d\omega,\end{aligned}$$

where Ω^τ is the set on which R exceeds τ. By the simple Markoff property, the last expression can be written

$$\begin{aligned}\psi &= \int_0^\infty d\tau \int_{\mathfrak{X}} K_\tau(r, ds)\chi(s)[1 - \psi(s)] \\ &= V[(1 - \psi)\chi].\end{aligned}$$

Thus $(1 - \psi)\chi$ is a positive function whose potential is bounded on $\mathfrak{X}$ and strictly positive on D. The existence of the functions g_n is now clear, for $\mathfrak{K}$ is the union of countably many special sets.

Since $(1 - \psi)\chi$ is strictly positive on D, the calculation shows incidentally that (D) very nearly implies (E). If $\mathfrak{X}$ is discrete, a special set comprises only finitely many points, so that then (D) does in fact imply (E).

The next two propositions are the extensions of Propositions 4.4 and 6.6. The proofs are omitted, as they differ only trivially from the former ones.

Let $\mathfrak{T}$ be a system determined by the positive function h and relatively independent of $\mathfrak{R}$. Given a positive function φ, define ψ by the formula

$$\psi(r) \equiv \int_{\Omega^*} \varphi(X(T))\, d\omega,$$

where X is a process starting at r and Ω^* is the set on which T, the time assigned by $\mathfrak{T}$, is less than R, the time assigned by $\mathfrak{R}$.

PROPOSITION 12.3. *Suppose that φ is bounded and that, for every process X*

and with probability 1, *the integral* $\int_0^R h(X(\tau))\,d\tau$ *is finite. Then* ψ *is the potential, relative to* $\mathfrak{R}$, *of the function* $(\varphi - \psi)h$.

The hypothesis on h is satisfied, in particular, if either (D) or (E) holds and if h is bounded and vanishes outside a special set.

PROPOSITION 12.4. *Let* (D) *or* (E) *hold, let* E *be a nearly open set, and let* φ *be excessive relative to* $\mathfrak{R}$. *Then there are positive functions* f_n, *each one vanishing outside* E, *whose potentials* Vf_n *increase to* $K_E\,\varphi$.

The following proposition gives a sufficient condition for the truth of (D) in terms of the simple and relative kernels for potentials.

PROPOSITION 12.5. *Statements* (C) *and* (E) *together imply* (D).

Let D be a special set, X a process starting at the point s, and R the terminal time assigned to X. Choose the special set D' and the strictly positive number α so that the joint event (12.1) has probability at least α whenever r is regular for D. Now, define a sequence of stopping times for X by taking T_1 to be the time X hits D and T_{n+1} to be the infimum of the τ greater than $T_n + \alpha$ for which $X(\tau)$ belongs to D, with the understanding that T_{n+1} is infinite if there are no such τ. We must prove that Ω', the set where all T_n are less than R, has probability null. To do this, consider the inequality

$$
\begin{aligned}
V(s, D') &= \int_\Omega d\omega \int_0^R \chi(X(\tau))\,d\tau \\
&\geqq \sum_n \int_\Omega \varepsilon(T_n, R)\,d\omega \int_{T_n}^{T_n'} \chi(X(\tau))\,d\tau,
\end{aligned}
\tag{12.2}
$$

where χ is the characteristic function of D', the function $\varepsilon(\sigma, \tau)$ has the value 1 or 0 according as σ is less than τ or not, and T_n' is the minmum of T_{n+1} and R. The point $X(T_n)$ is almost surely regular for D if T_n is finite, the factor $\varepsilon(T_n, R)$ depends on the behavior of the process only infinitesimally past the time T_n, and T_{n+1} is at least $T_n + \alpha$ if T_n is finite. Hence each term of the sum is at least $\alpha^2\mathcal{P}\{\Omega'\}$, by the extended Markoff property and the choice of α and D'. It follows that Ω' has probability null, because the first member of (12.2) is finite under (E). Only the fact that $V(s, D')$ is everywhere finite has been used in the proof, not the full strength of (E).

The remainder of the section deals with verifying (D) and (E) for a certain class of transition measures. A more detailed treatment for Brownian motion in the plane is to be found in [10], where it is also proved that the classical Green's function of a domain coincides with the kernel for potentials relative to the system of terminal times defined by the complement of the domain.

The notation is that of the simple theory, the parameter λ appearing as a superscript.

The transition measures are said to be ergodic if Φ_B^0 is identically 1 whenever

B is a Borel set and not approximately null, that is to say, if almost all sample paths of every process meet such a set.

PROPOSITION 12.6. *The transition measures are ergodic if and only if $U^0(r, B)$ is identically infinite in r for every set B which is not approximately null. If the transition measures are ergodic and if E is nearly analytic and not negligible, then Φ_E^0 is identically* 1.

Suppose first that the transition measures are ergodic and that B is not approximately null. Take E to be the set where $U^1(s, B)$ exceeds 2α, with α strictly positive and so small that E is not empty. The set E is nearly open, therefore not approximately null, and $U^1(s, B)$ is at least 2α if s is regular for E. Take ρ so large that

$$\int_0^\rho P_\tau(s, B)\, d\tau > \alpha$$

whenever s is regular for E. Now, given a process X starting at the point r, let T_1 be the time X hits E, and let T_{n+1} be the infimum of the τ greater than $T_n + \rho$ for which $X(\tau)$ belongs to E. All the T_n are finite with probability 1, because E includes a Borel set which is not approximately null. We have, writing χ for the characteristic function of B,

$$\begin{aligned} U^0(r, B) &= \int_\Omega d\omega \int_0^\infty \chi(X(\tau))\, d\tau \\ &\geqq \sum_n \int_\Omega d\omega \int_{T_n}^{T_{n+1}} \chi(X(\tau))\, d\tau, \end{aligned}$$

and each term in the sum is at least α, as one sees by the extended Markoff property. So $U^0(r, B)$ is identically infinite.

In the rest of the proof we assume $U^0(r, B)$ to be infinite for all r whenever B is not approximately null. A preliminary result concerning a nearly open set E will be derived first. Let ψ^λ be the function

$$\psi^\lambda(r) = 1 - \int_\Omega \exp\left\{-\int_0^{S^\lambda} \chi(X(\tau))\, d\tau\right\} d\omega,$$

where χ is the characteristic function of E and X is a process starting at r. By Proposition 4.4, this function satisfies the equation

$$U^\lambda\{(1 - \psi^\lambda)\chi\} = \psi^\lambda$$

if λ is strictly positive. On letting λ decrease to 0 and noting that ψ^λ increases to ψ^0, which is bounded by 1, we obtain

$$U^0\{(1 - \psi^0)\chi\} \leqq 1.$$

This inequality and the hypothesis at the beginning of the paragraph imply that ψ is 1 at all points of E, excepting perhaps an approximately null set; the exceptional set is empty, however, since it is also nearly open. The function ψ^0 has the value 1 also at points which are regular for E. Thus, if r is regular for E, the integral $\int_0^\infty \chi(X(\tau))\,d\tau$ is infinite with probability 1, so that for almost all ω the point $X(\tau, \omega)$ belongs to E for certain arbitrarily large values of τ.

Let F be compact and not approximately null. The function $U^\lambda(r, F)$ is strictly positive for all λ and r, under the present hypotheses, and for each strictly positive λ it is bounded and has F for a determining set. The function Φ_F^1 exceeds some multiple of $U^1(r, F)$, according to the part of Theorem 6.11 proved without using (B), so that it too is strictly positive. Let E be the nearly open set where Φ_F^1 exceeds 2α, with α a given strictly positive number, and choose ρ so that a process starting at a point regular for E hits F by time ρ with probability at least α. Given such a process X, define a sequence of stopping times by taking T_0 to be null and T_{n+1} to be the infimum of the τ greater than $T_n + \rho$ for which $X(\tau)$ belongs to E. The T_n are all finite with probability 1, by the preceding paragraph, and repeated use of the extended Markoff property shows that

$$\Phi_F^0(r) \geqq \sum \alpha(1 - \alpha)^n = 1,$$

if r is the point at which X starts. Since E increases to $\mathcal{R}$ as α decreases to 0, the function Φ_F^0 must be identically 1.

Let us now assume only that F is nearly analytic and not negligible. Since Φ_F^1 does not vanish identically, there is a strictly positive α such that the set E where Φ_F^1 exceeds 2α is not empty. Clearly, E includes a set which is compact and not approximately null, so that almost all sample paths of an arbitrary process meet E. The argument of the preceding paragraph, with E fixed and r any point of $\mathcal{R}$, shows that Φ_F^0 is identically 1. The proposition is now completely proved.

It follows at once from the proposition that (D) holds if the transition measures are ergodic and the set A is not negligible, for then every terminal time of the system $\mathfrak{R}$ is finite with probability 1. The condition that A is not negligible may be replaced by the condition that a is strictly positive on a set which is not approximately null.

Proposition 12.7. *Statement (E) holds if the transition measures are ergodic, if A is not negligible, and if, for every compact set F, the intersection $F \cap \mathcal{K}_\beta$ is empty for β sufficiently small but strictly positive.*

In the proof we shall take a and λ to be null, for doing so only increases the terminal times. Let $N_\tau(r, ds)$ be the transition probabilities relative to the system of terminal times defined by $A \cup \mathcal{K}_\beta$, with β strictly positive. By the definition of $\mathcal{K}_\beta$, there are strictly positive numbers α and ρ such that the sample paths of a process starting at a point outside $\mathcal{K}_\beta$ meet A by time ρ with probability at least α; on the other hand, every point of $\mathcal{K}_\beta$ is regular for $\mathcal{K}_\beta$. Consequently, $N_\rho(r, \mathcal{H})$ is bounded by $1 - \alpha$, so that

$$N_{\tau+\rho}(r, \mathcal{H}) = \int_{\mathcal{H}} N_\rho(r, ds) N_\tau(s, \mathcal{H}) \leqq (1 - \alpha) \sup_s N_\tau(s, \mathcal{H}).$$

It follows that $N_\tau(r, \mathcal{H})$ decreases exponentially as τ increases and that the total mass of $W(r, ds)$, the kernel for potentials relative to the system determined by $A \cup \mathcal{K}_\beta$, has a bound independent of r.

Given a compact set F, choose β strictly positive so that $\mathcal{K}_{2\beta}$ and F are disjoint. We shall prove $V(r, F)$ to be bounded by majorizing it in terms of $W(r, \mathcal{H})$.

Let X be a process starting at the point r, and define a sequence of stopping times by taking T_0 to be null, T_{2n+1} to be the infimum of the τ greater than T_{2n} for which $X(\tau)$ belongs to $\mathcal{K}_\beta$, and T_{2n} to be the infimum of the τ greater than T_{2n-1} for which $X(\tau)$ belongs to F. If F is not approximately null, all these times are finite with probability 1; the reader may suppose F to be so restricted, since otherwise there is nothing to prove, but the following calculations are not disturbed by the stopping times being infinite. The T_n tend to infinity with probability 1, for else the function $\Phi_A^1(X(\tau))$ would not have limits from the left with probability 1. We shall also need T'_{2n}, the infimum of the τ greater than T_{2n} for which $X(\tau)$ belongs to $A \cup \mathcal{K}_\beta$.

Let ν_n be the measure

$$\nu_n(B) \equiv \mathcal{P}\{X(T_{2n}) \in B, T_n < R\},$$

where R is the time X hits A. A process starting at a point of F hits A at or before the time it hits $\mathcal{K}_\beta$, with probability at least $\beta/(1 - \beta)$; for, if γ is this probability, one obtains the inequality

$$\gamma + (1 - \beta)\gamma \geqq 2\beta$$

on classifying the paths that hit A before time S^1 according to whether they hit A or $\mathcal{K}_\beta - A$ first. Now, the measure ν_n is concentrated on F, for n strictly positive, so that the mass of ν_{n+1} does not exceed $\beta/(1 - \beta)$ times the mass of ν_n, by the extended Markoff property and what has just been proved. Therefore the mass of ν, the sum of all the ν_n, has a bound independent of r.

By the extended Markoff property,

$$\begin{aligned} V(r, F) &= \int_\Omega d\omega \int_0^R \chi(X(\tau))\, d\tau \\ &= \sum_n \int_\Omega \varepsilon(T_{2n}, R)\, d\omega \int_{T_{2n}}^{T'_{2n}} \chi((\tau))\, d\tau \\ &\leqq \sum_n \int_{\mathcal{H}} W(s, F) \nu_n\, (ds) \end{aligned}$$

$$\leqq \nu(\mathcal{K}) \sup_s W(s, \mathcal{K}),$$

where χ is the characteristic function of F and $\varepsilon(\sigma, \tau)$ has the value 0 or 1 according as σ exceeds τ or not. The last member has a bound independent of r, by the preceding results, and the proof is complete.

We have in fact proved a little more than (E), for F is an arbitrary compact set.

The hypothesis that A is not negligible may be replaced by the hypothesis that a is strictly positive on a set that is not approximately null. Only a few details of the proof need be changed.

In many examples the function Φ_A^1 is lower semicontinuous. Then one need not state explicitly the hypothesis that F and $\mathcal{K}_\beta$ are disjoint when β is small. For Φ_A^1 is strictly positive everywhere, so that it takes on a strictly positive minimum on the compact set F if it is lower semicontinuous; and β may be taken to be any number less than the minimum. There is a similar argument if (C) holds and if A includes an open set. The function Φ_A^1 then majorizes some potential $U^1 f$, with f a positive continuous function having a compact support included in A, and the potential is strictly positive and continuous.

13. Two theorems on excessive functions

The principal results of §6 will now be extended to the relative theory. E denotes a nearly analytic set and φ a function excessive relative to $\mathfrak{R}$; only the supplementary hypotheses on E and φ will be mentioned.

PROPOSITION 13.1. *If all points of $E \cap \mathcal{K}$ are regular for E, then $K_E\, \varphi$ is the least function which majorizes φ on E and is excessive relative to $\mathfrak{R}$.*

The proof is like that of Proposition 6.1.

THEOREM 13.2. *If (D) holds, then $K_E\, \varphi$ coincides, except perhaps at the points belonging to $E \cap \mathcal{K}$ but not regular for E, with the infimum of the functions which majorize φ on E and are excessive relative to $\mathfrak{R}$.*

It suffices, according to remarks like those in §6, to prove that the infimum does not exceed $K_E\, \varphi$ at any point of $\mathcal{K}$ outside E. We shall suppose, without losing generality, that E is included in $\mathcal{K}$; for a function excessive relative to $\mathfrak{R}$ vanishes outside $\mathcal{K}$, and replacing E by its intersection with $\mathcal{K}$ has no effect on $K_E\, \varphi$.

Suppose first that E is included in some special set D and that the restriction of φ to E is bounded. The first part of the proof of Theorem 6.4 remains valid, if the sets A_n are replaced by their intersections with D, and establishes the theorem under the additional hypotheses.

The next proposition, which takes the place of Proposition 6.5, follows from the restricted form of the theorem and the fact that every subset of $\mathcal{K}$ is the union of countably many sets, each included in a special set.

PROPOSITION 13.3. *Let E be a nearly analytic subset of $\mathcal{K}$, and let Ψ_E vanish at the point r outside E. Then there is a function which is excessive relative to $\mathfrak{R}$, infinite at every point of E, and less than 1 at r.*

Suppose next that φ is finite at every point of E. Choose an increasing sequence of special sets D_k which exhaust $\mathcal{K}$, and let E_k be the part of $E \cap D_k$ where φ is less than k. The second part of the proof of Theorem 6.4 carries over without change.

Finally, an arbitrary excessive function φ is treated as in the last part of the former proof, with the help of Proposition 13.3.

We shall now prepare the way to discussing the analogue of Theorem 6.11.

PROPOSITION 13.4. *Let ψ be a positive function such that the inequality $K_\tau \psi \leqq \psi$ holds for every τ. Then $K_\tau \psi$ increases as τ decreases; and $\bar{\psi}$, the limit as $\tau \to 0$, is excessive relative to $\mathfrak{R}$, nowhere exceeds ψ, and coincides with ψ except on a set which is approximately null relative to $\mathfrak{R}$.*

The fact that the kernels K_τ form a semigroup implies at once that $K_\tau \psi$ increases as τ decreases; so $\bar{\psi}$ is excessive and nowhere exceeds ψ. In proving that $\bar{\psi}$ differs from ψ only on an approximately null set, we shall assume λ to be strictly positive; this is permissible, because $\bar{\psi}$ is unchanged and the kernels K_τ are diminished when λ is increased.

First, suppose ψ to be bounded. Then

$$V\psi = \lim_{\tau \to 0} (VK_\tau)\psi = \lim_{\tau \to 0} V(K_\tau \psi) = V\bar{\psi},$$

all terms being finite, so that the assertion is proved. Now, let ψ be unbounded, and take ψ_n to be the minimum of ψ and n. It is clear that ψ_n also satisfies the hypothesis of the proposition; so $\bar{\psi}_n$ coincides with ψ_n approximately everywhere, by what has already been proved. Consequently ψ', the limit of $\bar{\psi}_n$ as $n \to \infty$, coincides with ψ approximately everywhere. It is also excessive, being the limit of an increasing sequence of excessive functions; it is majorized by ψ; and

$$\psi \geqq \psi' \geqq K_\tau \psi \geqq K_\tau \psi', \qquad \tau > 0.$$

The proposition is proved by letting τ decrease to 0 here.

A good deal of the discussion of determining sets in §6 was irrelevant to Theorem 6.11. We shall now say that a closed set F in $\mathcal{H}$ is a determining set for a function φ, excessive relative to $\mathfrak{R}$, if $K_G \varphi$ coincides with φ whenever G is a neighborhood of F. Simple examples show that an excessive function may have a least determining set which is compact and disjoint from $\mathcal{K}$.

Hypothesis (B) implies that $K_G K_E = K_E$ whenever G is a neighborhood of the nearly analytic set E, as one can see from the discussion of the hypothesis in §6. Thus, under (B), a closed set F is always a determining set for $K_F \varphi$.

Given a function φ, excessive relative to $\mathfrak{R}$, let φ_E be the infimum of the functions which majorize φ on some variable neighborhood of E and are excessive relative to $\mathfrak{R}$. By Proposition 13.1, one can alternatively define φ_E to be the infimum of $K_G\varphi$ as G ranges over the open neighborhoods of E.

THEOREM 13.5. *Let (B) hold, let E be an analytic set, and let φ be finite approximately everywhere relative to $\mathfrak{R}$. Then φ_E coincides, except perhaps at points belonging to $\mathcal{K} \cap E$ but not regular for E and at points outside E where φ_E is infinite, with the supremum of the functions which are excessive relative to $\mathfrak{R}$, have a compact determining set included in E, and are majorized everywhere by φ. If E is open, there are no exceptional points, and φ_E is the same as $K_E\varphi$.*

Let G be an open neighborhood of E and let ψ be an excessive function which nowhere exceeds φ and has a closed subset of E for a determining set. Then

$$K_G\varphi \geqq K_G\psi = \psi,$$

so that φ_E majorizes ψ and hence the supremum mentioned in the theorem. This part of the proof makes no use of (B) or of the finiteness of φ.

If E is open, φ_E coincides with $K_E\varphi$ by the second definition of φ_E. On the other hand, $K_E\varphi$ is the limit of $K_F\varphi$ as F runs through an increasing sequence of compact sets which exhaust E, by Theorem 11.3, and the function $K_F\varphi$ has F for a determining set by (B). So the last sentence of the theorem has been proved, without using the restriction on φ.

It is also clear that φ_E majorizes $K_E\varphi$ if E is an arbitrary analytic set, so that

$$\varphi(r) \geqq \varphi_E(r) \geqq K_E\varphi.(r) = \varphi(r)$$

if r is regular for E. Since $K_E\varphi.(r)$ is the limit of $K_F\varphi.(r)$ as F runs through a certain sequence of compact subsets of E given by Proposition 2.1, the assertions of the theorem concerning points of E have been proved.

Suppose now E to be compact, and choose a decreasing sequence of open neighborhoods G_n of E whose closures are compact and shrink to E. The functions $K_{G_n}\varphi$ decrease everywhere to φ_E, which is therefore measurable over the field $\mathfrak{A}$. Since φ_E satisfies the hypotheses of Proposition 13.4, it coincides approximately everywhere with an excessive function ψ. We shall prove that ψ has E for a determining set and that ψ has the same value as φ_E at every point outside E where φ_E is finite. Thus, for compact sets, we shall prove a little more than is stated in the theorem.

The function ψ majorizes $K_E\varphi$, since the latter is an excessive function nowhere exceeding φ_E. Consequently,

$$\varphi(s) \geqq \varphi_E(s) \geqq \psi(s) \geqq K_E\varphi.(s) = \varphi(s)$$

if s is regular for E, so that $\psi(s)$ and $\varphi_E(s)$ are the same.

Let G be any open neighborhood of E. The equation

$$K_{G_n}\varphi = K_G K_{G_n}\varphi$$

holds for large n, because G finally includes G_n, and one obtains, on passing to the limit using dominated convergence,

$$\varphi_E(s) = K_G\, \varphi_E.(s) \tag{13.1}$$

at every point s where φ_E is finite. This equation will be used to prove that φ_E and ψ have the same value at a point s outside E where φ_E is finite. In the proof we assume G to be chosen so that s does not belong to the closure of G. Let X be a process starting at s, T the time X hits G, T' the infimum of the σ greater than τ for which $X(\sigma)$ belongs to E, and Ω' the set where T' is less than the terminal time assigned to X and $X(\sigma)$ lies outside G for σ less than τ. It is clear that Ω' increases to the set where T is less than the terminal time, as $\tau \to 0$, and that T' coincides with T on Ω'. Also,

$$K_\tau\, \varphi_E.(s) = K_\tau\, K_G\, \varphi_E.(s) \geqq \int_{\Omega'} \varphi_E(X(T))\, d\omega,$$

and, as $\tau \to 0$, the integral increases to $K_G\, \varphi_E.(s)$, which is the same as $\varphi_E(s)$. Consequently, φ_E agrees with ψ at s.

Let r be a point at which φ_E is finite and which either lies outside E or is regular for E. The measure $K_G(r, ds)$ attributes no mass to the set where ψ differs from φ_E, so that (13.1) implies

$$\psi(r) = K_G\, \psi.(r).$$

This equation must hold everywhere, since it holds approximately everywhere and both members are excessive. Consequently, ψ has E for a determining set. The proof for a compact set is now complete.

The proof for an analytic set makes use of Choquet's extension theorem. Let r be a point at which φ is finite, and consider $\varphi_E(r)$ as a function of the analytic set E. The function is finite, being bounded by $\varphi(r)$. It is alternating of order infinity, as one sees from Theorem 11.5 and the second definition of φ_E. It is continous on the right in E, by the second definition. If E is open, then $\varphi_E(r)$ is the supremum of $\varphi_F(r)$ as F ranges over the compact subsets of E, by what has already been proved; and the restriction that E be open may be omitted, according to the theorem of Choquet. Now fix an analytic set E and a point r outside E at which $\varphi_E(r)$ is finite. It may be assumed, without loss of generality, that $\varphi(r)$ is also finite; for $\varphi_G(r)$ is finite, G being some open neighborhood of E, and $(\varphi_G)_E$ is the same function as φ_E. This being so, choose the compact subset F of E so that $\varphi_F(r)$ is close to $\varphi_E(r)$. The number $\varphi_F(r)$ coincides, according to the preceding paragraph, with the value at r of an excessive function which nowhere exceeds φ and has F for a determining set. The proof of the theorem is now complete.

I have not been able to prove that the function φ_E is measurable over the field $\mathfrak{A}$, or that the theorem holds for nearly analytic sets, without further restrictions on the transition measures. It will be observed, too, that the theorem extends one of the versions mentioned at the end of §6, rather than Theorem 6.11 itself. We shall therefore indicate the straightforward version of Theorem 6.11. Let X be a process starting at the point r, T the time X hits E, R the terminal time assigned to X, and define

$$\tilde{\varphi}_E(r) = \int_{\Omega'} \lim_{\tau \nearrow T} \varphi(X(\tau))\, d\omega + \int_{\Omega''} \varphi(X(T))\, d\omega.$$

Here the set Ω' comprises both the ω for which

$$T(\omega) < \infty, \qquad T(\omega) < R(\omega), \qquad X(T(\omega), \omega) \in E,$$
$$X(\tau, \omega) \quad \textit{is continuous at} \quad T(\omega),$$

and the ω for which $T(\omega)$ and $R(\omega)$ are infinite and the path $X(\omega)$ meets every neighborhood of E. The set Ω'' comprises the ω for which $T(\omega)$ is strictly less than $R(\omega)$ and either $X(T(\omega), \omega)$ does not belong to E or $X(\tau, \omega)$ is not continuous at $T(\omega)$. The existence of the limit in the first integral was proved just before Proposition 11.4. The function $\tilde{\varphi}_E$ is easily proved to be excessive relative to $\mathfrak{R}$, hence measurable over $\mathfrak{A}$, even for E only nearly analytic. If φ is bounded, then φ_E and $\tilde{\varphi}_E$ agree except at the points belonging to E but not regular for E. And under (B) the function $\tilde{\varphi}_E$ coincides, except on the same set of points, with the supremum of the bounded functions which are excessive relative to $\mathfrak{R}$, nowhere exceed φ, and have a compact subset of E for a determining set. The proofs are like those in §6.

14. Excessive measures

A measure ζ, defined on the field $\mathfrak{A}$ and vanishing outside $\mathfrak{K}$, is said to be excessive relative to $\mathfrak{R}$ if $\zeta(D)$ is finite for every special set D and if ζ majorizes ζK_τ for every τ. It follows, as in §8, that ζK_τ increases to ζ as τ decreases to 0, provided ζ is excessive, and that the class of excessive measures is closed under the operations discussed at the beginning of §8.

The potential relative to $\mathfrak{R}$ of a measure μ is the measure μV, which remains the same when μ is replaced by its restriction to $\mathfrak{K}$. The potential is excessive relative to $\mathfrak{R}$ if it is finite on special sets; in particular, it is excessive if (E) holds and μ is bounded. The potential determines μ if (E) holds and μ is a bounded measure concentrated on $\mathfrak{K}$. The proof is the same as that of Proposition 7.1 for λ strictly positive. If λ vanishes, we use (10.15) to write

$$\mu V^1 = \mu V^0 - \int_0^\infty e^{-\tau} \mu V^0 K_\tau^0\, d\tau, \tag{14.1}$$

both measures on the right being finite on special sets because (E) holds; thus μV^0 determines μV^1, hence μ itself.

The projection, relative to $\mathfrak{R}$, of the measure μ on the nearly analytic set E is the measure μK_E. The class $\mathfrak{N}$ comprises those measures concentrated on $\mathfrak{K}$ whose projections on special sets are bounded. Such measures are obviously finite on special sets. Their potentials have the same property if (E) holds; for $\mu V.(D)$ and $\mu K_D\, V.(D)$ have the same value, and the second is finite, if D is a special set, because μK_D is a bounded measure and $V(r, D)$ a bounded function. When (C) holds, a measure μ belongs to $\mathfrak{N}$ if it is concentrated on $\mathcal{K}$ and if its potential is finite on special sets. To see that μK_D is bounded, D being a special set, choose a special set D' as in Proposition 12.1 so that $V(r, D')$ is bounded below by a strictly positive number α on D, hence at every point of $\mathcal{K}$ which is regular for D; then

$$\mu K_D.(\mathcal{K}) \leqq \frac{1}{\alpha} \mu K_D V.(D') \leqq \frac{1}{\alpha} \mu V.(D'),$$

the last member being finite by hypothesis. Thus, when (C) and (E) both hold, a measure belongs to $\mathfrak{N}$ if and only if its potential is finite on special sets.

Proposition 14.1. *If (E) holds, a measure μ in $\mathfrak{N}$ is determined by μV.*

The proof is the same as the proof of Proposition 7.6, except that the sequence of open sets G_n is to be replaced by an increasing sequence of special sets whose union is $\mathcal{K}$.

Suppose that ζ is excessive relative to $\mathfrak{R}$ and that $\zeta K_\tau.(D) \to 0$ as $\tau \to \infty$, whenever D is a special set. A calculation like the one proving Proposition 5.2 shows that the potential of $(\zeta - \zeta K_\tau)/\tau$ increases to ζ. This result and an argument like the one proving Proposition 8.2 give the next theorem.

Theorem 14.2. *If (E) holds, a measure excessive relative to $\mathfrak{R}$ is the limit of an increasing sequence of potentials relative to $\mathfrak{R}$.*

Let ζ be excessive relative to $\mathfrak{R}$, and let E be a nearly analytic set. The measure $M_E \zeta$ is defined in the following way, under the assumption that (E) holds. Choose a sequence of measures μ_n whose potentials increase to ζ; the measures $\mu_n K_E V$ increase with n, and $M_E \zeta$ is taken to be their limit. Just as in §8, one sees that $M_E \zeta$ can also be defined by setting

$$\int M_E \zeta.(dr)f(r) = \lim_{n\to\infty} \int \zeta(dr)f_n(r), \tag{14.2}$$

where f is an arbitrary positive function and the functions f_n are chosen so that Vf_n increases to $K_E Vf$, the choice being possible by Theorem 12.2. The operator M_E has properties similar to those of L_E in the simple theory. In particular, M_E coincides with ζ on E, and $M_E \zeta_n$ increases to $M_E \zeta$ if ζ_n in-increases to ζ. The operator will be defined generally in the proof of the next proposition.

Proposition 14.3. *If E is nearly open, then $M_E \zeta$ is the least measure which is excessive relative to $\mathfrak{R}$ and which majorizes ζ on E.*

Assume first that (E) holds. Since $M_E \zeta$ coincides with ζ on E, one has only to prove that $M_E \zeta$ is majorized by every measure ξ which is excessive relative to $\mathfrak{R}$ and which majorizes ζ on E. Now, the functions f_n appearing in (14.2) can be taken to vanish outside E, according to Proposition 12.4; with such a choice of the f_n,

$$\int M_E \zeta.(dr)f(r) = \lim \int \zeta(dr)f_n(r)$$

$$\leqq \lim \int \xi(dr)f_n(r) = \int M_E \xi.(dr)f(r),$$

and the assertion is proved.

One next proves, using this result and arguing as in §8, that $M_E \zeta$ increases as λ decreases, provided E is a nearly analytic set and λ is restricted to values for which (E) is true and ζ is excessive relative to $\mathfrak{R}$.

Suppose that ζ is excessive relative to $\mathfrak{R}^0$, that (E) holds for $\mathfrak{R}^0$, and that E is a nearly analytic set. Then $M_E^0 \zeta$ is defined, and we shall prove it to be the limit of $M_E^\lambda \zeta$ as $\lambda \to 0$. The limit measure is certainly majorized by $M_E^0 \zeta$, according to the preceding paragraph. If ζ is a potential, say μV^0, then

$$M_E^0 \zeta = \mu K_E^0 V^0 = \lim_{\lambda \to 0} \mu K_E^\lambda V^\lambda \leqq \lim_{\lambda \to 0} M_E^\lambda \zeta,$$

because $K_E^\lambda V^\lambda$ increases to $K_E^0 V^0$ and because $M_E^\lambda \zeta$ majorizes $\mu K_E^\lambda V^\lambda$, since ζ majorizes μV^λ. The assertion is therefore true if ζ is a potential. For an excessive measure ζ we have, writing ζ as the limit of an increasing sequence of potentials $\mu_n V^0$,

$$M_E^0 \zeta = \lim_n M_E^0(\mu_n V^0) = \lim_n \lim_\lambda M_E^\lambda(\mu_n V^0)$$

$$= \lim_\lambda \lim_n M_E^\lambda(\mu_n V^0) = \lim_\lambda M_E^\lambda \zeta,$$

because $M_E^\lambda(\mu_n V^0)$ increases as λ decreases and as n increases.

Matters being so, we define $M_E^0 \zeta$ to be the limit of $M_E^\lambda \zeta$ as $\lambda \to 0$, assuming only that ζ is excessive relative to $\mathfrak{R}^0$ and that E is nearly analytic. This definition agrees with the original one if (E) holds for $\mathfrak{R}^0$, and the assertion of the proposition for vanishing λ follows at once from the assertion for λ strictly positive.

The next proposition is the critical one in studying the representation of excessive measures as potentials.

Proposition 14.4. *Let (C) hold, let λ be strictly positive, and let (μ_n) be a sequence of measures on $\mathfrak{K}$ whose masses are bounded uniformly in n and whose potentials $\mu_n V$ increase with n. Then μ_n converges weakly to a measure μ concentrated on $\mathfrak{K}$, and $\mu_n V$ increases to μV.*

The hypothesis on λ cannot be replaced by (E) alone, as one sees by considering uniform motion on a line. It will be shown in Proposition 14.8, however, that a strengthened form of (E) is sufficient.

The integral $\int \mu_n(dr)\varphi(r)$ increases with n whenever φ is excessive relative to $\mathfrak{R}$, because the measures $\mu_n V$ do so and because φ is the limit of an increasing sequence of potentials Vf, with f positive. According to Proposition 11.4, one may take for φ the restriction to $\mathcal{K}$ of a simple potential Uf, with f positive. It follows that the measures $\mu_n U$ increase with n, so that the hypotheses of Proposition 9.1 are satisfied. Thus, μ_n converges weakly to a certain measure μ, and $\mu_n U$ increases to μU. By Proposition 10.2, moreover, $\mu_n NU$ increases to μNU. These facts, and the representation (10.18) of V as $U - NU$, imply that $\mu_n V$ increases to μV. It remains to prove that μ has no mass outside $\mathcal{K}$. The inequality

$$\int \mu(dr)Vf.(r) \geqq \int \mu_n(dr)Vf.(r)$$

holds for every positive function f; and we obtain the inequality $\mu(\mathcal{K}) \geqq \mu_n(\mathcal{K})$ on letting f vary through such a sequence that Vf increases to 1 at every point of $\mathcal{K}$. On the other hand, the total mass of μ does not exceed the supremum of $\mu_n(\mathcal{K})$. The proof is now complete.

There is a similar proposition for a decreasing sequence of potentials.

PROPOSITION 14.5. *Let λ be strictly positive, let (C) hold, and let ζ be excessive relative to $\mathfrak{R}$. If the integrals $\int \zeta(dr)f(r)$ are bounded by some number α, as f ranges over the positive functions whose potentials Vf are bounded by* 1, *then ζ is the potential of a bounded measure.*

By Proposition 14.2, there are measures μ_n on $\mathcal{K}$ such that $\mu_n V$ increases to ζ. Now,

$$\int \mu_n(dr)Vf.(r) \leqq \int \zeta(dr)f(r) \leqq \alpha,$$

for every positive function f whose potential is bounded by 1. On letting f run through a sequence so that Vf increases to 1 at every point of $\mathcal{K}$, we see that the mass of μ_n does not exceed α. The proposition is therefore implied by the preceding one.

Let (E) hold, let ζ be excessive relative to $\mathfrak{R}$, and let f and g be positive functions. The inequality $Vf \geqq Vg$ then implies the inequality $\int \zeta(dr)f \geqq \int \zeta(dr)g$. The assertion is trivial if ζ is a potential, and it follows for all excessive measures by Proposition 14.2.

Let (C) hold, let λ be strictly positive, and let D be a special set. We shall prove that $M_D \zeta$ is the potential of a bounded measure, assuming of course that ζ is excessive relative to $\mathfrak{R}$. By Proposition 12.1, there is a special set D' such that the function $V(r, D')$ is bounded away from 0 on D; therefore Vg, with g some multiple of the characteristic function of D', majorizes the function Ψ_D everywhere. If f is any positive function whose potential is bounded by 1, then $K_D Vf$ is bounded by Ψ_D, so that

$$\int M_D \zeta.(dr)f(r) \leqq \int \zeta(dr)g(r).$$

The preceding proposition now implies that $M_D \zeta$ is the potential of a bounded measure.

In the remainder of the section we fix an increasing sequence of special sets D_n with the properties that the complement of each D_n is nearly analytic and that every special set is included in some D_n. The abbreviations

$$\tilde{M}_n = M_{F_n}, \qquad \tilde{K}_n = K_{F_n}, \qquad F_n = \mathcal{K} - D_n$$

will be used. One may take D_n to be the intersection of $\mathcal{K}_{1-1/n}$ with G_n, the G_n being the open sets described just before Theorem 9.4. With this choice, all points of $\mathcal{K}$ not regular for F_n are included in D_{n+1}; so, with a different choice of the sequence, all points of $\mathcal{K}$ not regular for F_n are included in some D_m.

The time a process X first leaves G_n increases to infinity with n, by the first part of hypothesis (A). The time X first leaves $\mathcal{K}_\beta$ increases with β, and the limit as $\beta \to 1$ is with probability 1 at least as great as the terminal time R assigned to the process, by the definition of the sets $\mathcal{K}_\beta$ and the extended Markoff property. Thus, for one choice of the D_n, and hence for all choices, the time X hits F_n increases with n to a limit which is with probability 1 at least as great as R. Denote by Ω_n the set where T_n, the time X hits F_n, is less than R, by μ the initial distribution of X, and by χ the characteristic function of a special set D. If (E) holds, the integral

$$\int_\Omega d\omega \int_0^R \chi(X(\tau))\, d\tau \equiv \mu V.(D)$$

is finite; and the finiteness of this integral, together with the behavior of the T_n, implies that the integrals

$$\int_{\Omega_n} d\omega \int_{T_n}^R \chi(X(\tau))\, d\tau \equiv \mu \tilde{K}_n V.(D)$$

approach 0 as n becomes large.

If (E) holds and μ is a measure in $\mathfrak{N}$, then $\mu\tilde{K}_n\, V.(D) \to 0$ as $n \to \infty$, whenever D is a special set. The assertion has just been proved for a bounded measure, and it is proved generally by writing μ as the sum of μ_1 and μ_2, with μ_1 bounded and $\mu_2\, V.(D)$ small.

THEOREM 14.6. *Let (C) and (E) hold. Then every measure ζ excessive relative to $\mathfrak{R}$ can be written $\mu V + \xi$, where μ belongs to $\mathfrak{N}$ and ξ is a measure, excessive relative to $\mathfrak{R}$, which coincides with $M_E\, \xi$ whenever E is a nearly analytic set whose complement is included in a special set.*

The assertions of the theorem and the uniqueness of the representation are proved nearly word for word as in §9, if λ is strictly positive. We shall take this result for granted, writing $\mu^\lambda V^\lambda + \xi^\lambda$ to make the parameter explicit, and deduce the theorem for vanishing parameter under the assumption that (E) holds for $\mathfrak{R}^0$.

We shall encounter several times in the proof a family of measures, say ν^λ, which are defined for λ strictly positive, decrease with λ, and are finite on special sets. Under these conditions there is a measure ν with the property that $\nu^\lambda(B) \to \nu(B)$ as $\lambda \to 0$, for every set B included in a special set. We shall say that ν is the limit of the measures ν^λ.

The symbols λ and ρ will denote strictly positive values of the parameter. If ρ exceeds λ, then

$$\mu^\lambda V^\lambda = [\mu^\lambda + (\rho - \lambda)\mu^\lambda V^\lambda]V^\rho$$

by (10.15); so ξ^λ, which is excessive relative to $\mathfrak{N}^\rho$, can be written $\nu V^\rho + \xi^\rho$ according to the theorem. Thus

$$\mu^\rho = \mu^\lambda + (\rho - \lambda)\mu^\lambda V^\lambda + \nu, \tag{14.3}$$
$$\mu^\rho V^\rho - \mu^\lambda V^\lambda = \nu V^\rho.$$

Consequently, the measures μ^λ and $\mu^\lambda V^\lambda$ decrease, and the measure ξ^λ increases, as λ decreases to 0. Let μ^0, α, β be the limit measures, and take γ to be $\alpha - \mu^0 V^0$. We shall prove that $\mu^0 V^0 + (\gamma + \beta)$ is the required representation of ζ.

The relations

$$\mu^0 V^0 = \lim \mu^0 V^\lambda \leqq \lim \mu^\lambda V^\lambda = \alpha \leqq \zeta$$

show that μ^0 belongs to $\mathfrak{M}^0$, because its potential is finite on special sets, and that γ is a positive measure, finite on compact sets.

The measure β is finite on special sets, being majorized by ζ. One verifies that it is excessive relative to $\mathfrak{N}^0$ by passing to the limit in the inequality $\xi^\lambda K_\tau^\lambda \leqq \xi^\lambda$. Consequently, β majorizes $M_E^0 \beta$ for every nearly analytic set E. On the other hand, $M_E^\lambda \xi^\lambda$ coincides with ξ^λ if the complement of E is included in a special set; the inequality $M_E^\lambda \beta \geqq \xi^\lambda$ follows; on passing to the limit here, one finds that $M_E^\lambda \beta$ majorizes β. So the measure β behaves properly.

It is clear that $(\mu^\lambda - \mu^0)V^\lambda$ decreases to γ. If ρ exceeds λ, then

$$(\mu^\lambda - \mu^0)V \geqq (\mu^\lambda - \mu^0)V^\lambda K_\tau^\rho \geqq \gamma K_\tau^\rho .$$

The inequality $\gamma \geqq \gamma K_\tau^0$ is obtained by letting first λ, then ρ, tend to 0. So γ is excessive relative to $\mathfrak{N}^0$.

The measure $\mu^\rho V^\rho - \alpha$ is excessive relative to $\mathfrak{N}^\rho$, since it is the limit of an increasing sequence of such measures according to (14.3). Hence, for ρ greater than λ and for every nearly analytic set E,

$$0 \leqq M_E^\rho(\mu^\lambda V^\lambda - \alpha) \leqq \mu^\lambda V^\lambda - \alpha,$$

the last member decreasing to the null measure as $\lambda \to 0$. It follows that $M_E^\rho(\mu^\lambda V^\lambda - \mu^0 V^0)$ decreases to $M_E^\rho \gamma$ as $\lambda \to 0$.

Suppose now that the complement of E is included in a special set D. We shall assume, without losing generality, that D includes all points not regular for E; it is then clear that

$$\nu V^\rho - M_E^\rho(\nu V^\rho) \leqq \nu_D V^\rho \leqq \nu_D V^0 \tag{14.4}$$

for every measure ν, with ν_D the restriction of ν to D. This observation will be applied to the potential

$$(14.5)\quad \mu^\lambda V^\lambda - \mu^0 V^0 \equiv [\mu^\lambda + (\rho - \lambda)\mu^\lambda V^\lambda - \mu^0 - \rho\mu^0 V^0]V^\rho,$$

taking λ less than ρ. Let B be included in a special set. It follows from the last two relations and the behavior of the various measures as λ varies, that the mass assigned to B by the measure

$$\mu^\lambda V^\lambda - \mu^0 V^0 - M_E^\rho(\mu^\lambda V^\lambda - \mu^0 V^0)$$

is bounded by

$$(14.6)\qquad [\mu^\rho(D) - \mu^0(D) + \rho\mu^\rho V^\rho.(D)] \sup_r V^0(r, B)$$

for all λ less than ρ. The same expression therefore bounds the mass assigned to B by $\gamma - M_E^\rho \gamma$, according to the preceding paragraph. On observing that the expression (14.6) tends to 0 with ρ, we find that $\gamma - M_E^0 \gamma$ assigns no mass to B. So the two measures γ and $M_E^0 \gamma$ coincide.

It is now clear that $\mu^0 V^0 + (\gamma + \beta)$ is the required representation of ζ. The measure $(\gamma + \beta)$ is the limit of $\tilde{M}_n^0 \zeta$, by the remark preceding the theorem, and this limit does not depend upon the particular choice of the sets D_n. So the representation is unique. A similar argument establishes the next theorem.

THEOREM 14.7. *Let (C) and (E) hold. A measure ζ excessive relative to $\mathfrak{N}$ is the potential of a measure in $\mathfrak{N}$ if and only if $\tilde{M}_n \zeta$ decreases to the null measure as $n \to \infty$.*

In the next proposition, which extends Proposition 14.4, we denote by (E^*) the statement that, for every special set D, the function $V(r, D)$ is bounded on $\mathfrak{K}$ and vanishes at infinity—that is to say, $V(r, D)$ can be made arbitrarily small by requiring r to be outside a certain compact set. Clearly, (E^*) holds if λ is strictly positive and (C) holds.

PROPOSITION 14.8. *Let (C) and (E^*) hold, and let (μ_n) be a sequence of measures on $\mathfrak{K}$ whose masses are bounded uniformly in n and whose potentials $\mu_n V$ increase with n. Then $\mu_n V$ increases to μV, with μ a bounded measure on $\mathfrak{K}$.*

Let ζ be the limit of the measures $\mu_n V$. If D is a special set, then $\zeta(D)$ is finite; for $V(r, D)$ is bounded, under (E^*), and therefore $\mu_n V.(D)$ is bounded in n. So ζ is excessive relative to $\mathfrak{N}$.

We shall prove next that $V(r, D)$ is small on the complement of some D_n, if D is a given special set. Let D be included in $\mathfrak{K}_\gamma$, let r be outside $\mathfrak{K}_\delta$, let T be the time a process starting at r hits D, let R be the terminal time assigned to the same process, and let S be a positive random variable with the density function $e^{-\sigma}$ for positive σ and independent of R. Then

$$V(r, D) \leqq \mathcal{P}\{T < R\} \sup_s V(s, D)$$

by the extended Markoff property, and

$$1 - \delta \geqq \mathcal{P}\{R > S\} \geqq (1 - \gamma)\mathcal{P}\{R > T\}$$

by the definition of the sets $\mathcal{K}_\beta$ and the extended Markoff property. Therefore

$$V(r, D) \leqq \frac{1 - \delta}{1 - \gamma} \sup_s V(s, D),$$

so that $V(r, D)$ is small if δ is near 1 and r lies outside $\mathcal{K}_\delta$. On the other hand, $V(r, D)$ is small for r outside a certain open set G with compact closure. So one has only to choose D_n to include the intersection of G and $\mathcal{K}_\delta$.

Matters being so, let α be a bound for the masses $\mu_k(\mathcal{K})$ and ε_n a bound for $V(r, D)$ on the complement of D_n. The relation

$$\tilde{M}_n \zeta.(D) = \lim_k \mu_k \tilde{K}_n V.(D) \leqq \alpha\varepsilon_n$$

shows that $\tilde{M}_n \zeta$ decreases to the null measure as $n \to \infty$. It follows that ζ is the potential of some measure μ on $\mathcal{K}$. Finally, μ is bounded because its mass is

$$\mu(\mathcal{K}) = \sup_f \int \zeta(dr) Vf.(r) \leqq \alpha,$$

where f ranges over the functions whose potentials Vf are bounded by 1.

In Proposition 14.5 one may similarly use (E^*) instead of the hypothesis that λ be strictly positive. The next theorem follows immediately from Theorems 14.6 and 14.7.

THEOREM 14.9. *If* (C) *and* (E) *hold, a measure excessive relative to* $\mathfrak{N}$ *is the potential of a measure in* $\mathfrak{N}$ *if it is majorized by such a potential.*

The only assertion of importance in the last two propositions of §9 is the weak convergence of the measures. Such convergence, at least on $\mathcal{K}$, usually does not hold in the relative theory. In many examples, of course, $\mathcal{K}$ is locally compact in the ordinary relative topology, and weak convergence on $\mathcal{K}$ can be asserted in the extended theorems.

Versions of the results of this section can also be proved assuming (D) instead of (E), the finiteness restriction on an excessive measure being changed to require that the functions $(1 - \psi)\chi$ appearing in the proof of Theorem 12.2 all be integrable. This amounts to another definition of special sets.

The analogue of Theorem 6.11 for excessive measures was not discussed in the simple theory. We shall treat it now in the relative theory.

Given an excessive measure ζ and a nearly analytic set E, define ζ_E to be the infimum of $M_B \zeta$ as B ranges over the nearly open sets that include E. It is understood that ζ_E is first defined for subsets of special sets and then extended to be a measure; an alternative definition will make matters clear. Choose a positive function f, bounded away from 0 on every special set, so that the integral $\int \zeta(ds)f(s)$ is finite; the existence of f follows from the finiteness of ζ on special sets and the representation of $\mathcal{K}$ as the union of the sets D_n. Let α be the infimum of $\int M_B\,\zeta.(ds)f(s)$, with B restricted as before, and fix a decreasing sequence of sets B_n so that the integral decreases to α as B runs through the sequence. Then ζ', the limit of $M_B\,\zeta$ as B runs through the B_n, is an excessive measure that majorizes the set function ζ_E, defined at the moment on subsets of special sets. On the other hand, if $\zeta'(C)$ were greater than $\zeta_E(C)$ for some subset C of a special set, then $\zeta'(C)$ would be greater than $M_D\,\zeta.(C)$ for some nearly open set D including E, and hence the integral $\int M_B\,\zeta.(ds)f(s)$ could be made less than α by taking B to be $D \cap B_n$ with n sufficiently large. It is now clear that ζ_E, when extended to a measure, is precisely the excessive measure ζ'. By Proposition 14.3, the measure ζ_E is also the infimum of the excessive measures that majorize ζ on some nearly open set including E.

Let ν be a measure on E whose potential is majorized by ζ, and let B be a nearly open set including E. Then

$$\nu V = M_B(\nu V) \leqq M_B\,\zeta,$$

so that νV is majorized by ζ_E. It will be proved, under further hypotheses, that ζ_E is the supremum of such potentials νV; the statements are less general, but more concrete, than the corresponding ones employing the notion of determining set.

PROPOSITION 14.10. *Let μ be a measure whose potential is excessive. Then $(\mu V)_E$ is the infimum of $M_G(\mu V)$ as G ranges over the open neighborhoods of E; is is also the supremum of νV as ν ranges over the measures on E whose potentials are majorized by μV.*

First suppose μ to be concentrated on the complement of E. If μ has total mass 1, let X be a process with μ for initial distribution, and choose a decreasing sequence of open neighborhoods G_n of E according to Proposition 2.2 so that T_n, the time X hits G_n, decreases with probability 1 to T, the time X hits E. Then

$$\int \mu K_{G_n} V.(ds)g(s) \equiv \int_{\Omega_n} d\omega \int_{T_n}^{R} g(X(\tau))\,d\tau,$$

with Ω_n the set where T_n is less than the terminal time R, decreases to

$$\int \mu K_E\, V.(ds)g(s) \equiv \int_{\Omega'} d\omega \int_{T}^{R} g(X(\tau))\,d\tau,$$

with Ω' the set where T is less than R, provided g is positive and integrable relative to the measure μV. This result implies that $\mu K_G\, V$ decreases to $\mu K_E\, V$ as G runs through some decreasing sequence of open neighborhoods of E, even when the mass of μ is infinite. Thus the infimum of $M_G(\mu V)$ is obviously $\mu K_E\, V$, and this measure majorizes $(\mu V)_E$. By Proposition 2.1, on the other hand, $\mu K_F\, V$ increases to $\mu K_E\, V$ as F runs through a certain sequence of compact subsets of E, which may even be taken as subsets of special sets, and μK_F is a measure concentrated on F. The remark preceding the proposition now implies that $(\mu V)_E$ reduces to $\mu K_E V$, so that the proposition is true in this instance.

If μ is concentrated on E, then $(\mu V)_E$ and the infimum of $M_G(\mu V)$ both coincide with μV, and there is nothing to prove.

Let us define the restricted projection on E of a measure μ in $\mathfrak{N}$ to be the measure

$$\mu K_E' \equiv (\mu - \mu') + \mu' K_E,$$

where μ' is the part of μ concentrated on the complement of E. It has just been proved that $(\mu V)_E$ is the potential of $\mu K_E'$, provided μV is excessive; one should note that $\mu K_E'$ is concentrated on E if that set is closed, or more generally if it includes all points regular for E but not regular for $\mathfrak{R}$.

THEOREM 14.11. *Let (C) and (E*) hold, let E be a nearly analytic subset of $\mathfrak{K}$, and let ζ be a measure excessive relative to $\mathfrak{R}$. Then ζ_E is the supremum of νV as ν ranges over the measures on E whose potentials are majorized by ζ.*

Let us write ζ as $\mu V + \xi$, according to Theorem 14.6. It suffices to discuss ξ, since potentials have already been dealt with.

If D is a special set, then $M_D\, \xi$ is the potential of a bounded measure ν. To see this, consider a sequence of measures ν_n whose potentials increase to $M_D\, \xi$. One may take the measures to be concentrated on the union of D with the set of points regular for D but not regular for $\mathfrak{R}$, replacing ν_n by $\nu_n K_D$ if necessary; the argument proving Theorem 6.6 shows that the measures may even be taken to have supports included in D. Proposition 12.1 and the finiteness of ξ on special sets together ensure that the mass of ν_n has a bound independent of n, and the existence of ν follows from Proposition 14.8.

If E is included in the special set D, then clearly

$$\xi_E = (M_D\, \xi)_E = \nu K_E' V,$$

with $\nu K_E'$ the restricted projection of ν on E. The theorem is therefore true in this instance.

It is worth going a step further, still assuming E to be included in a special set, to prove that ξ_E reduces to $M_E\, \xi$. Consider again the sequence of special sets D_n introduced after Proposition 14.5, taking n so great that D_n includes E, and write

$$M_{D_n}\, \xi = \nu_n V.$$

Then ν_n is the projection of ν_m on D_n if m exceeds n, so that ν_n', the part of ν_n on E, decreases as n increases. Thus $\nu_n' V$ decreases to an excessive measure which must be a potential, say αV, according to Proposition 14.9. On the other hand, the potential of $\nu_n - \nu_n'$ increases to some excessive measure, say β, so that ξ can be written $\alpha V + \beta$. It now follows from Theorem 14.6 and the properties of ξ that α must be the null measure. On writing

$$\xi_E = \nu_n' V + (\nu_n - \nu_n') K_E V \leqq \nu_n' V + M_E\, \xi,$$

one sees that $M_E\,\xi$ majorizes ξ_E. Since the converse relation holds trivially, the two measures are the same.

In dealing with an arbitrary nearly analytic set E, we shall use the dual of (11.10), which we proceed to discuss. Let ζ be an excessive measure, (B_n) and (E_n) two sequences of nearly analytic sets such that B_n includes E_n for every n, and take B or E to be the union of the B_n or the E_n. Then

$$M_B\,\zeta - M_E\,\zeta \leqq \sum \{M_{B_n}\,\zeta - M_{E_n}\,\zeta\}, \tag{14.7}$$

the terms making sense and the inequality being true when the measures are restricted to any special set. Assume first that (E) holds and that the sequences are finite. Relation (11.10) implies that

$$\nu K_B\,V.(C) - \nu K_E\,V.(C) \leqq \sum \{\nu K_{B_n} V.(C) - \nu K_{E_n}\,V.(C)\}$$

holds whenever C is included in a special set and ν is a bounded measure. One obtains (14.7) on letting ν run through a sequence of measures whose potentials increase to ζ. Another passage to the limit, letting the parameter λ decrease to 0, shows that hypothesis (E) is unnecessary. Now, $M_B\,\zeta$ increases to $M_D\,\zeta$ as B runs through an increasing sequence of sets whose union is D; so another passage to the limit, letting the number of sets increase, completes the proof. One can also transfer Theorem 11.5 to excessive measures by the same reasoning.

Let us return to the proof of the theorem. We represent E as the union of an increasing sequence of sets E_n, each included in a special set. We shall also assume Vf to be bounded, f being the function appearing in the second definition of ζ_E; this is permissible because (E) holds. Given a strictly positive number ε, choose a nearly open set B_n so that

$$\int M_{B_n}\,\xi.(ds)f(s) \leqq \int M_{E_n}\,\xi.(ds)f(s) + 2^{-n}\varepsilon,$$

and let B be the union of the B_n. By (14.7),

$$\int M_B\,\xi.(ds)f(s) \leqq \int M_E\,\xi.(ds)f(s) + \varepsilon.$$

Since ε is arbitrary and $M_E\,\xi$ is majorized by ξ_E, these measures must coincide. Finally, $M_F\,\xi$ increases to $M_E\,\xi$ as F runs through some increasing sequence of compact subsets of E, each included in a special set, and $M_F\,\xi$ is the potential of a measure on F. The proof is now complete.

It has been proved incidentally that ζ_E is the potential of a measure on E whenever E is included in a special set and every point regular for E but not regular for $\mathfrak{R}$ belongs to E.

The theorem implies that the transformation $\zeta \to \zeta_E$ is idempotent and that ζ_E increases to $\zeta_{E'}$ as E increases to E' through a sequence. It is not generally true that ζ_E increases to ζ'_E as ζ increases to ζ'.

Suppose $\mathcal{K}$ to be the union of a sequence of special sets each of which is open, not only nearly open. Then ζ_E is the infimum of $M_G \zeta$ as G ranges over the open sets including E. Indeed, nearly open sets, in the form of special sets, were used only to reduce the discussion of ζ_E to that of a potential, assuming E to be included in a special set, and one can now use open sets for this purpose. The simple theory of §§4–9 is one instance of this situation; another instance is treated in the next section. Possibly the nearly open sets may always be replaced by open sets in the definition of ζ_E .

15. A principle of the maximum

A part of what is commonly taken to be potential theory can be described in the following manner.

$\mathcal{K}$ is a separable locally compact Hausdorff space, $\mathcal{C}$ the Banach space of functions continuous on $\mathcal{K}$ and vanishing at infinity, $\mathcal{B}$ the set of continuous functions with compact supports, $\mathcal{C}^+$ and $\mathcal{B}^+$ the corresponding sets of positive functions. There is given a linear transformation V from $\mathcal{B}$ to $\mathcal{C}$ which satisfies:

(α) *An inequality $Vf \geqq Vg$ holds everywhere if it holds on the support of g and both f and g belong to $\mathcal{B}^+$.*

(β) *There is a sequence of functions h_n in $\mathcal{B}^+$ such that Vf_n increases everywhere to* 1.

(γ) *The range of V is dense in $\mathcal{C}$.*

We shall prove that under these conditions Vf can be expressed in terms of Markoff processes in the way suggested by the notation. The description itself will be discussed after the proof of this assertion.

That V sends positive functions into positive functions is proved by taking g to be null in (α) and noting that then the support of g is empty. The other preliminary remarks depend less dangerously on the wording.

The first two conditions imply the following statement, sometimes called the principle of the maximum.

(δ) *Let a be a positive constant, f and g functions in $\mathcal{B}^+$. Then $a + Vf$ majorizes Vg everywhere if it does so on the support of g.*

In proof, consider the sequence of functions $V(f + bh_n)$, where b is any number greater than a and the functions h_n are the ones mentioned in (β). For some value of n, according to Dini's theorem, $V(f + bh_n)$ exceeds Vg on the support of g; consequently $b + Vf$ majorizes Vg everywhere.

In the rest of the proof (δ) will be used in place of (α) and (β). We shall see, after the proof is completed, that (δ) is indeed equivalent to the conjunction of (α) and (β) whenever (γ) holds.

Let f^+ and f^- be the positive and negative parts of a function f in $\mathcal{B}$, and let a be either 0 or the maximum of Vf on the support of f^+, whichever is the greater. By (δ), the inequality $a + Vf^- \geqq Vf^+$ holds everywhere, so that a

majorizes Vf. It follows that the supremum of Vf, provided it is strictly positive, must be achieved on the support of the positive part of f.

The last statement will be strengthened to the following principle of a positive maximum: *The function f is positive at every point where Vf achieves its maximum, provided the maximum is positive.* Let Vf achieve its positive maximum at the point r, and let A be a compact neighborhood of r. According to (γ), there is a function g in $\mathcal{B}$ such that $Vg.(r)$ is strictly positive and greater than the supremum of Vg on the complement of A. For every strictly positive ε, the maximum of $V(f + \varepsilon g)$ is strictly positive and is achieved on A but not on the complement of A; so $f + \varepsilon g$ is positive at some point of A, according to the preceding paragraph. It follows that $f(r)$ must be positive, because ε and A are arbitrary and g depends only on A. I am indebted to Carl Herz for this proof, which takes the place of one requiring stronger hypotheses.

The principle of a positive maximum implies that f vanishes identically if Vf does so.

We shall discuss bounded transformations first. Suppose, then, V to be bounded when $\mathcal{B}$ is normed as a subset of $\mathcal{C}$, and extend V to all of $\mathcal{C}$ by continuity. The extended transformation satisfies (δ) even for f and g in $\mathcal{C}^+$, as one verifies by approximating f and g from below by functions in $\mathcal{B}^+$ and using Dini's theorem. It follows as before that Vf cannot take on a positive maximum at a point where f is strictly negative and that V establishes a one-to-one correspondence between $\mathcal{C}$ and its range $\mathfrak{D}$, which is dense in $\mathcal{C}$.

Define I on $\mathfrak{D}$ by the formula $IVf = -f$, and define V_λ, for λ small and positive, by the formula

$$V_\lambda \equiv \sum_{k \geqq 0} (-\lambda)^k V^{k+1}, \qquad 0 \leqq \lambda < 1/\| V \|.$$

Straightforward calculation shows that, for every f in $\mathcal{C}$,

$$V_\lambda (\lambda - I)Vf = Vf$$

on the one hand, and

$$V_\lambda^{[n]} f \equiv \sum_{0 \leqq k < n} (-\lambda)^k V^{k+1} f \rightarrow V_\lambda f,$$

$$(\lambda - I) V_\lambda^{[n]} f = f - (-\lambda V)^n f \rightarrow f,$$

on the other. Accordingly, $\lambda - I$ and V_λ are inverses, for $\lambda - I$ is a closed transformation. By the principle of a positive maximum,

$$(\lambda - I)g.(r) \geqq \lambda g(r)$$

if g is a function in $\mathfrak{D}$ taking on a positive maximum at the point r; so V_λ sends positive functions into positive functions, and its bound does not exceed $1/\lambda$.

The definition of V_λ can be extended to the intervals $0 \leqq \lambda < 2^n/\| V \|$ recursively by means of the formula

$$V_\lambda \equiv \sum (\alpha - \lambda)^k V_\alpha^{k+1}, \qquad 0 \leqq \lambda < 2\alpha, \tag{15.1}$$

and the assertions already proved for small positive λ remain valid. These assertions, together with (γ), form the hypotheses of the theorem of Hille and Yosida. According to that theorem, there is a semigroup of linear transformations K_τ on $\mathcal{C}$ which has I for infinitesimal generator and which satisfies the following relations:

$$K_\tau f \to f, \qquad \tau \to 0, \tag{15.2}$$

$$\| K_\tau \| \leqq 1, \qquad \tau > 0, \tag{15.3}$$

$$K_\tau f \geqq 0 \quad \text{if} \quad f \geqq 0, \qquad \tau > 0, \tag{15.4}$$

$$V_\lambda f = \int_0^\infty e^{-\lambda\tau} K_\tau f \, d\tau, \qquad \lambda \geqq 0. \tag{15.5}$$

In the first relation the convergence is in the form of $\mathcal{C}$.

Using the Riesz representation theorem, one can write, for every τ and r,

$$K_\tau f.(r) = \int_{\mathcal{K}} K_\tau(r, ds) f(s),$$

with $K_\tau(r, ds)$ a positive measure on $\mathcal{K}$ of mass not greater than 1. These measures behave like stationary Markoff transition measures, and $K_\tau(r, A)$ is Borel measurable in the pair (τ, r) whenever A is a Borel set. A standard theorem on semigroups ensures that the measures are determined by V, in the sense that the associated semigroup of transformations K_τ of $\mathcal{C}$ is the only one satisfying the four relations above. We shall interrupt the main argument to prove a somewhat stronger statement of uniqueness needed later on.

For the moment let $\mathcal{A}(\mathcal{K})$ be the field defined as $\mathcal{A}$ in §1, with $\mathcal{K}$ replaced by $\mathcal{K}$, and let A be a variable set in $\mathcal{A}(\mathcal{K})$. We shall prove that a family of measures $K'_\tau(r, ds)$ defined on $\mathcal{A}(\mathcal{K})$ must be the family $K_\tau(r, ds)$ if it has the following properties: $K'_\tau(r, A)$ is measurable over $\mathcal{A}(\mathcal{K})$ as a function of r; it is measurable, as a function of the pair (τ, r), over the completed field of definition of every measure which is the product of Lebesgue measure and some bounded measure on $\mathcal{A}(\mathcal{K})$; the equation

$$K'_{\sigma+\tau}(r, A) = \int_{\mathcal{K}} K'_\sigma(r, ds) K'_\tau(s, A)$$

holds identically; and the equation

$$Vf.(r) = \int_0^\infty d\tau \int_{\mathcal{K}} K'_\tau(r, ds) f(s)$$

holds identically in r for every function f in $\mathcal{B}$.

Consider the Banach space of bounded functions measurable over $\mathcal{A}(\mathcal{K})$, normed by the supremum of the absolute value. Define V_λ as a transformation of this space by formula (15.5), taking the transformations K_τ to be defined by the measures $K_\tau(r, ds)$, and define V'_λ similarly in terms of the measures $K'_\tau(r, ds)$. Both transformations have bounds not exceeding the bound of V, and V_0' coincides with V_0 because it does so on $\mathcal{B}$. It is also easy to see that both V_λ and V_λ' satisfy (15.1), at least for $| \alpha - \lambda |$ less than the number $1/\| V \|$, so that V_λ' coincides with V_λ for all values of λ. Thus, for given f and r, the two functions $K_\tau' f.(r)$ and $K_\tau f.(r)$ agree for almost all τ according to the uniqueness theorem for Laplace transforms. Let us take f to be Vg, with g in $\mathcal{B}^+$. Then $K_\tau' f.(r)$ is decreasing in τ, and $K_\tau f.(r)$ is continuous, so that there are no exceptional values of τ. It follows that $K_\tau'(r, ds)$ is the same measure as $K_\tau(r, ds)$, for every τ and r, because $V(\mathcal{B})$ is dense in $\mathcal{C}$.

The uniqueness being settled, we proceed to define true stationary Markoff transition measures. Take $\mathcal{H}$ to be $\mathcal{K}$ augmented by a single point w, and set

$$
\begin{aligned}
P_\tau(r, A) &= K_\tau(r, A), && r \in \mathcal{K}, \quad A \subset \mathcal{K}, \\
P_\tau(r, w) &= 1 - K_\tau(r, \mathcal{K}), && r \in \mathcal{K}, \\
P_\tau(w, w) &= 1.
\end{aligned}
\tag{15.6}
$$

Give $\mathcal{H}$ such a topology that it is compact and includes $\mathcal{K}$ as a subspace: if $\mathcal{K}$ is not compact, $\mathcal{H}$ is the space obtained by adjoining w as the point at infinity; but if $\mathcal{K}$ is compact, an open set in $\mathcal{H}$ is one whose trace on $\mathcal{K}$ is open. We shall suppose the functions in $\mathcal{C}$ to be extended to continuous functions on $\mathcal{H}$ by taking them to vanish at w; the Banach space $\mathcal{C}(\mathcal{H})$ is then the direct sum of the subspace $\mathcal{C}$ and the one-dimensional subspace of constants. The kernels $P_\tau(r, ds)$ clearly induce a semigroup of linear transformations P_τ on $\mathcal{C}(\mathcal{H})$ having the property that $P_\tau f$ converges uniformly to f as $\tau \to 0$.

Transition measures behaving so well always satisfy hypothesis (A). The proof is to be found in [1], [5], and [12], as will be shown in the following outline. Let $\mathcal{H}$ be a separable locally compact space, perhaps not compact, and let $P_\tau(r, ds)$ be stationary Markoff transition measures on $\mathcal{H}$. The associated transformations P_τ are assumed to leave invariant the Banach space $\mathcal{C}(\mathcal{H})$ of continuous functions vanishing at infinity and to converge strongly there to the identity transformation as $\tau \to 0$.

It follows quickly from these hypotheses that $P_\tau(r, A)$ is Borel measurable in the pair (τ, r) if A is a Borel set and that the integral $\int P_\tau(r, ds)f(s)$ is continuous in r if f is bounded and continuous; the second assertion is condition (D) of [1].

The fact that $P_\tau f$ approaches f uniformly as $\tau \to 0$, provided f belongs to $\mathcal{C}(\mathcal{H})$, implies the following statement: *Let A be compact, B a closed subset of the interior of A, and C a closed subset of the exterior of A; then*

$$
\lim_{\tau \to 0} P_\tau(r, A) = \begin{cases} 1 & \textit{for} \quad r \quad \textit{in} \quad B, \\ 0 & \textit{for} \quad r \quad \textit{in} \quad C, \end{cases}
$$

and the convergence is uniform on $B \cup C$. This statement is to be used in place of Kinney's condition $D(u, u, \cdot, b)$, and it implies Blumenthal's condition (E').

A condition is needed limiting return from a neighborhood of infinity. The following one is Blumenthal's condition (F); it is to be used in place of Kinney's $\dot{R}(\infty)$ and $R(-\infty)$, which are phrased ambiguously. *If A is a compact set, then $P_\tau(r, A) \to 0$ as r tends to infinity, and the convergence is uniform for τ restricted to a compact set.* In proof, let f be a positive function in $\mathcal{C}(\mathcal{H})$ which exceeds 1 at all points of A. The function

$$g \equiv \lambda \int_0^\infty e^{-\lambda\tau} P_\tau f \, d\tau$$

belongs to $\mathcal{C}(\mathcal{H})$ and tends uniformly to f as $\lambda \to \infty$; fix λ so that g is at least 1 at all points of A; then

$$P_\tau(r, A) \leqq P_\tau \, g.(r) \leqq e^{\lambda\tau} g(r),$$

and the assertion is proved.

Let μ be a probability measure on the Borel sets of $\mathcal{H}$. The existence of a stationary Markoff process Z, having μ for initial distribution and the $P_\tau(r, ds)$ for transition measures, is proved in [5], pages 613 through 616. Separability of a process relative to the class of closed sets is defined just as in II §2 of [5]; the proof of Theorem 2.4 of that book enables one to replace Z by an equivalent separable process Y, provided the space $\mathcal{H}$ is enlarged to its one-point compactification.

One now proves Theorem 2.1 of [1] or Theorem IV (ii) of [12]. The assertion is that almost every path of Y is included in a compact subset of the original space $\mathcal{H}$, provided the time parameter is restricted to a compact set. In particular, it was not really necessary to compactify $\mathcal{H}$ in passing from Z to Y, and left and right limits $Y(\tau+, \omega)$ and $Y(\tau-, \omega)$ are almost certainly points of $\mathcal{H}$ if they exist.

Either Theorem 2.2 of [1] or Theorem VI (i) of [12] establishes the existence of right and left limits. The proof by martingales given by Kinney is very short; the denominator $1 + x_s^2(\omega)$ appearing in the theorem and the proof should be omitted, since sample paths do not approach the point at infinity in finite time. Blumethal's proof makes no use of martingales.

That Y has no fixed discontinuities is Theorem 2.3 of [1]. The matter is also treated in Theorem I of [12], but the proof of the second part of the theorem is not sufficient.

Since limits from the right and left exist and since there are no fixed discontinuities, one can define a process X equivalent to Y by setting

$$X(\tau, \omega) = \lim_{\sigma \searrow \tau} Y(\sigma, \omega),$$

neglecting a certain set of ω of probability null. Clearly X is a Markoff process having μ for initial distribution, the $P_\tau(r, ds)$ for stationary transition measures, and sample paths continuous on the right and with limits from the left. Theorems 1.1 and 4.2 of [1] show that the $P_\tau(r, ds)$ satisfy the rest of hypothesis (A).

It is now clear that the transition measures (15.6) satisfy (A) and that one can write

$$Vf.(r) = \int_{\Omega} d\omega \int_0^R f(X(\tau))\, d\tau, \tag{15.7}$$

where X is a process having the $P_\tau(r, ds)$ for transition measures and starting at the point r, R is the time X hits w, and f is a function in $\mathcal{C}$. Thus we have arrived at the representation by processes, assuming V to be bounded, and the results of preceding sections may be used freely.

Before going on to unbounded transformations we shall discuss the relation of a bounded transformation V to the transformation W, defined on $\mathcal{C}$ by the formula $Wf = V(bf)$ with b a strictly positive continuous function on $\mathcal{K}$. The transformation W shares all the properties of V; so it determines families of measures $L_\tau(r, ds)$ and $Q_\tau(r, ds)$ similar to $K_\tau(r, ds)$ and $P_\tau(r, ds)$. We are going to show that processes having the $Q_\tau(r, ds)$ for transition measures can be derived from those having the $P_\tau(r, ds)$ for transition measures by changing the time parameter.

Given a process X with the $P_\tau(r, ds)$ for transition measures, define a new stochastic process Y by taking $Y(\tau, \omega)$ to be $X(\sigma, \omega)$, with σ and τ related by the equation

$$\tau = \int_0^\sigma b(X(\rho))\, d\rho.$$

The initial distribution of Y is the same as that of X, and the sample paths of Y have the same properties of continuity as those of X. Next, define probability measures on $\mathcal{K}$ by setting

$$Q'_\tau(r, A) = \mathcal{P}\{Y(\tau) \in A\},$$

where Y is obtained from a process X starting at r. One proves easily that each process Y is a Markoff process having the $Q'_\tau(r, ds)$ for stationary transition measures, by translating the statements to be verified into ones concerning the processes X and then using the extended Markoff property. It can be proved in this way that the $Q'_\tau(r, ds)$ satisfy hypothesis (A), but we shall obtain a little more by another argument. By the reasoning at the end of §1, the quantity $Q'_\tau(r, A)$ is defined for all sets A in the field $\mathcal{A}$; as a function of r, it is measurable over $\mathcal{A}$; and, as a function of the pair (τ, r), it is measurable over the completed field of definition of every measure which is the product of Lebesgue measure and some bounded measure on $\mathcal{A}$. In addition, for a function belonging to $\mathcal{C}$ and hence vanishing at w,

$$\begin{aligned} Wf.(r) &= \int_{\Omega} d\omega \int_0^R f(X(\tau))b(X(\tau))\, d\tau \\ &= \int_{\Omega} d\omega \int_0^S f(Y(\tau))\, d\tau \\ &= \int_0^\infty d\tau \int_{\mathcal{K}} Q'_\tau(r, ds) f(s), \end{aligned}$$

where f, X, R have the same meaning as in (15.7), Y is the process obtained from X, and S is the time Y hits w. Thus, by the uniqueness proved before, the kernel $L_\tau(r, ds)$ is obtained by restricting both arguments of the kernel $Q'_\tau(r, ds)$ to $\mathcal{K}$. It follows that $Q'_\tau(r, ds)$ coincides with $Q_\tau(r, ds)$, so that we have indeed shown how to obtain processes corresponding to W from those corresponding to V.

The probability of a process Y hitting a certain set is obviously the same as that of the corresponding process X, although the times of hitting may not be the same. If b is identically 1 on the open set G and if X starts at a point of G, then both X and Y hit the complement of G at the same time T, and $X(\tau, \omega)$ coincides with $Y(\tau, \omega)$ for all τ less than $T(\omega)$. These are the facts we shall use in a moment to avoid a tedious analytical proof.

Let us consider now an unbounded transformation satisfying conditions (γ) and (δ); we shall denote it by $\tilde{V}$ in order to preserve the notation already introduced. With any function a in $\mathfrak{B}^+$, there is associated the transformation $f \to \tilde{V}(af)$, which has for bound the maximum of $\tilde{V}a$ and which we suppose extended to all of $\mathcal{C}$ by continuity. Since $\mathcal{K}$ is the countable union of open sets with compact closures, there is an increasing sequence of functions a_n in $\mathfrak{B}^+$ such that

$$\sum \max \tilde{V}(a_{n+1} - a_n) \equiv \alpha < \infty$$

and such that a, the limit of the a_n, belongs to $\mathcal{C}$ and is strictly positive. The transformations associated with the a_n converge strongly on $\mathcal{C}$ to a transformation V whose bound does not exceed α. The restriction of V to $\mathfrak{B}$ satisfies conditions (γ) and (δ), because Vf is just $\tilde{V}(af)$ if f has compact support. All the results we have obtained for bounded transformations may therefore be applied to V and to the transformations derived from it by changing the time parameter of processes.

The function a is to some extent arbitrary; we take it to be bounded by 1 and extend it to $\mathcal{H}$ by giving it the value 1 at w, so that the extended function is discontinuous at that point. Fix a sequence of open subsets G_n of $\mathcal{K}$ so that G_0 is empty, the closure of G_n is a compact subset of G_{n+1}, the union of the G_n is $\mathcal{K}$, and construct an increasing sequence of continuous functions b_n on $\mathcal{H}$ so that b_0 is the constant 1, b_n coincides with $1/a$ on G_n, and b_n is identically 1 outside G_{n+1}. These functions satisfy the conditions imposed on the function b above, and we denote by W^n the transformation $f \to V(b_n f)$. Clearly, W^0 is the same as V; and $W^n f$, for every f in $\mathfrak{B}^+$, increases with n and coincides with $\tilde{V}f$ as soon as G_n includes the support of f. We shall investigate the behavior, as n grows large, of the transformations L_τ^n determined by the W^n.

Let f be a function of the form $\tilde{V}g$, with g in $\mathfrak{B}^+$, and let ε be a strictly positive number. Take A to be the compact subset of $\mathcal{K}$ on which f is at least ε. If n is so large that G_n includes the support of g, then

$$L_\tau^n f = L_\tau^n W^n g \leqq f,$$

so that $L_\tau^n f$ is less than ε outside A. The discussion of the behavior on A is a little more complicated. Choose h in $\mathcal{B}^+$ so that Vh exceeds 1 at every point of A; then $\tilde{V}h$ and all the $W^n h$ also exceed 1 on A. Take B to be the compact subset of $\mathcal{K}$ on which $\tilde{V}h$ is at least ε. If X is a process starting at a point outside B and having transition measures determined by W^n, its sample paths meet A with probability at most ε, according to Theorem 13.2 and the fact that $\tilde{V}h$ majorizes $W^n h$; the probability vanishes, of course, if the starting point is w. Now choose n so large that G_n includes B, let r be a point of A, let X^n be a process starting at r with transition measures determined by W^n, and let T be the time X^n hits the complement of G_n. Given an integer m greater than n, we take X^m to be the process X^n with the time parameter changed by means of the function b_m/b_n; the process X^m starts at r and has for transition measures the ones determined by W^m. Since b_m/b_n is identically 1 on G_n, the time at which X^m hits the complement of G_n is also T, and $X^m(\tau, \omega)$ coincides with $X^n(\tau, \omega)$ for all τ not exceeding $T(\omega)$. For any given τ, partition Ω into Ω', the set where T exceeds τ, and its complement Ω''. Then $X^m(\tau, \omega)$ is the same as $X^n(\tau, \omega)$ for ω in Ω'; and the set of ω in Ω'' for which either $X^n(\tau, \omega)$ or $X^m(\tau, \omega)$ belongs to A has probability less than 2ε, as one verifies by using the extended Markoff property with stopping time T. Matters being so, we have

$$\begin{aligned} | L_\tau^n f.(r) - L_\tau^m f.(r) | &\leqq \int_{\Omega''} | f(X^n(\tau)) - f(X^m(\tau)) | \, d\omega \\ &\leqq 2\varepsilon + 2\varepsilon \max f. \end{aligned}$$

These estimates prove that $L_\tau^n f.(r)$ converges uniformly in the pair (τ, r) as $n \to \infty$. The convergence holds in fact for every f in $\mathcal{C}$, because $\tilde{V}(\mathcal{B})$ is dense in $\mathcal{C}$ and each transfromation L_τ^n has a bound not exceeding 1.

Consequently, L_τ^n converges strongly to a transformation $\tilde{K}_\tau$ of $\mathcal{C}$, and the limit transformations form a semigroup satisfying (15.2), (15.3), and (15.4). In addition, one has the representation

$$\tilde{V}f.(r) = \int_0^\infty \tilde{K}_\tau f.(r) \, d\tau$$

for every f in $\mathcal{B}$; for a function in $\mathcal{B}^+$ the representation follows at once from the uniform convergence of $L_\tau^n f.(r)$, Fatou's lemma, and the equality of $\tilde{V}f$ and $W^n f$ for large n.

Stationary Markoff transition measures and processes are introduced now just as for a bounded transformation. It must be remarked that the processes corresponding to $\tilde{V}$ can be obtained from those corresponding to V by a change of the time parameter using the function $1/a$, and that hypothesis (A) carries over immediately. We have chosen the more cumbersome method of approximation in order to prove simply that the transformations induced by the transition measures send continuous functions into continuous functions.

The transformation $\tilde{V}$ has been assumed to satisfy (γ) and (δ). In order to show that $\tilde{V}$ must also satisfy (β), it suffices to prove that the bounded transformation V satisfies (β). The factorization

$$1 + \lambda V = (\lambda - I)V$$

implies that $1 + \lambda V$ is a one-to-one transformation of $\mathcal{C}$ onto itself for all positive λ; the same assertion is therefore true of the transformation $f \to f + \lambda V(hf)$, with h a fixed strictly positive function in $\mathcal{C}$. Let ψ_λ be the one function in $\mathcal{C}$ which satisfies the equation

$$\psi_\lambda + \lambda V(h\psi_\lambda) = \lambda Vh.$$

This equation is equivalent to

$$I\psi_\lambda - \lambda h\psi_\lambda = -\lambda h,$$

so that ψ_λ lies between 0 and 1 according to the principle of a positive maximum. It is also easily verified that ψ_λ increases to 1 as $\lambda \to \infty$. On the other hand, writing

$$\psi_\lambda = \lambda V[(1 - \psi_\lambda)h] \tag{15.8}$$

shows that ψ_λ is the potential of a function in $\mathcal{C}^+$. The verification of (β) is now trivial. There is an alternative argument using equation (15.8) and the latter part of §4, but then the proof of the continuity of ψ_λ is more complicated.

We have been discussing what may be called a regular theory of potentials of functions in which a principle of the maximum holds. The dual notion, a regular theory of potentials of measures in which a principle of projection holds, can be described in this manner: There is given an additive transformation $\mu \to \mu V$ of bounded measures on $\mathcal{K}$ into measures finite on compact sets; μV varies continuously in the weak topology of measures as μ does so, provided the mass of μ remains uniformly bounded, and the following conditions are satisfied:

(α') *With each bounded measure μ and each compact set F is associated at least one bounded measure μ_F, concentrated on F, such that $\mu_F V$ is majorized by μV and coincides with μV on F.*

(β') *If μ and μ_F are related as in (α'), then the mass of μ_F does not exceed that of μ.*

(γ') *A bounded measure μ is determined by μV.*

Given such a transformation, define a transformation $f \to Vf$ of $\mathcal{B}$ into $\mathcal{C}$ by taking $Vf.(r)$ to be the integral of f with respect to the measure $\varepsilon_r V$, where ε_r is the unit mass at the point r; the transformation indeed sends $\mathcal{B}$ into $\mathcal{C}$, because $\varepsilon_r V$ varies continuously with r. One verifies the relation

$$\int \mu(dr) Vf.(r) = \int \mu V.(dr) f(r), \qquad f \in \mathcal{B}, \tag{15.9}$$

by approximating the bounded measure μ by a linear combination of point masses. This relation and condition (γ') imply that $V(\mathcal{B})$ is dense in $\mathcal{C}$. That the transformation $f \to Vf$ satisfies condition (δ) follows at once from (α'), (β'), and (15.9). It is now clear that μV has a representation in terms of Markoff processes.

We have proved that a theory of potentials satisfying the description at the beginning of the section, or the one just given, is an instance of the relative theory developed in §§10–14. On the other hand, every instance of the relative theory very nearly satisfies (α), (β), (γ) if it satisfies (D) or (E). Indeed, (α) holds for all positive functions, (β) holds with the h_n positive but perhaps not continuous, and (γ) holds in the sense that a measure μ must vanish if $\int \mu(dr) Vf.(r)$ vanishes for every positive function f.

The reduction to the relative theory by introducing processes on the space $\mathcal{K}$ augmented by a single point is particularly simple, but it is not the only possible one. Suppose, for example, that the total mass of the measure $K_\tau(r, ds)$ depends perhaps on τ but not on the point r. It must then be of the form $e^{-\lambda\tau}$, with λ positive, so that the measures $e^{\lambda\tau}K_\tau(r, ds)$ are stationary Markoff transition measures on $\mathcal{K}$ satisfying hypothesis (A). The potential theory is then an instance of the simple theory of §§4–9, based on these transition measures and having λ as parameter. Matters must be so if V is bounded and sends the constant function 1 into $1/\lambda$ when extended in the obvious manner to the class of bounded continuous functions on $\mathcal{K}$. It is desirable to have such an immediate criterion for an unbounded transformation, that is to say, for vanishing λ. If V is invariant under a transitive group of homeomorphisms of $\mathcal{K}$, in the sense that $V(f \circ \sigma)$ coincides with $(Vf) \circ \sigma$ for every f in $\mathcal{B}$ and every element σ of the group, then the transformations $\mathcal{K}$ send constants into constants, since they are also invariant, and the situation is that of the simple theory, with vanishing parameter if V is unbounded. Examples are the theories of the Newtonian, Riesz, and heat potentials.

If $\mathcal{K}$ is discrete, one can avoid introducing an additional point by constructing processes on $\mathcal{K}$ and then introducing a terminal time defined by a positive function and the empty set. The new processes can be derived from the ones already defined by suppressing the jumps from $\mathcal{K}$ to w. The construction is also possible when $\mathcal{K}$ is not discrete, provided the function $1 - \varphi$ can be written Va, with a some positive function; here φ is the limit of $K_\tau 1$ as $\tau \to \infty$, while $K_\tau 1$ and Va are to be defined as integrals, so that a need not be continuous. The system of terminal times to be used with these processes is the one determined by a and the empty set.

The results of this section can be used to present hypothesis (C) in another form. Suppose the transition measures of §1 to satisfy (A) and (C). Then the transformation U^λ, for λ strictly positive, satisfies the conditions at the beginning of this section, with $\mathcal{K}$ taken as $\mathcal{H}$, and sends constants into constants; condition (β) holds trivially, and (γ) follows from the fact that a bounded measure is determined by its potential. The transformations $e^{-\lambda\tau}P_\tau$ consequently leave $\mathcal{C}(\mathcal{H})$ invariant and converge strongly on that space to the identity transformation as $\tau \to 0$. The transformations P_τ obviously share these properties. On the other hand, (C) holds whenever the P_τ leave $\mathcal{C}(\mathcal{H})$ invariant. Under (A), accordingly, (C) is equivalent to the statement that the P_τ leave $\mathcal{C}(\mathcal{H})$ invariant, and it implies that the P_τ converge strongly on $\mathcal{C}(\mathcal{H})$ to the identity transformation.

Some simplifications can be made in phrasing the arguments and results of preceding sections for the situation described at the beginning of this section. In particular, special sets may be replaced everywhere by open sets whose closures are compact subsets of $\mathcal{K}$. It suffices to prove that every compact subset of $\mathcal{K}$ is included in some $\mathcal{K}_\beta$. Were this not so, there would be a point s of $\mathcal{K}$ such that R_r, the terminal time assigned to a process X_r starting at r, tends in probability to 0 as r tends to s through a certain sequence of points in $\mathcal{K}$; we shall prove that then $Vf.(s)$ vanishes for all f in $\mathcal{B}^+$. Let α and λ be strictly positive. The kernel V_λ is defined by taking the terminal time R_r^λ to be the minimum of R_r and S^λ, so that

$$\begin{aligned} V_\lambda f.(r) &= \int_\Omega d\omega \int_0^{R_r^\lambda} f(X_r(\tau))\, d\tau \\ &\leqq \alpha \max f + \int_{\Omega'} d\omega \int_0^{S^\lambda} f(X_r(\tau))\, d\tau \\ &\leqq \alpha \max f + \lambda^{-1}\mathcal{P}\{R_r > \alpha\} \max f, \end{aligned}$$

where Ω' is the set on which R_r exceeds α. On letting r approach s, we obtain

$$V_\lambda f.(s) \leqq \alpha \max f,$$

since $\mathcal{P}\{R_r > \alpha\}$ vanishes in the limit; on letting α, then λ, approach 0 here, we find that $Vf.(s)$ must vanish. This result contradicts both (β) and (γ). So every compact subset of $\mathcal{K}$ is included in some $\mathcal{K}_\beta$.

A slight extension of the results of this section will be treated briefly, as a preparation for the next section. Let ρ be a strictly positive continuous function on $\mathcal{K}$, and take $\mathcal{C}_\rho$ to be the Banach space comprising the continuous functions f on $\mathcal{K}$ for which f/ρ vanishes at infinity, with the maximum of $|f/\rho|$ as norm. Let V' be a linear transformation from $\mathcal{B}$ into $\mathcal{C}_\rho$. We suppose that (α) holds for V', that $V'h$ increases to ρ as h runs through some sequence of functions in $\mathcal{B}^+$, and that the range of V' is dense in $\mathcal{C}_\rho$. If ρ is bounded away from 0, the introduction of the space $\mathcal{C}_\rho$ may be avoided by requiring V' to be a transformation from $\mathcal{B}$ into $\mathcal{C}$ with range dense in $\mathcal{C}$. The transformation V,

$$Vf \equiv \frac{1}{\rho} V'f$$

clearly sends $\mathcal{B}$ into $\mathcal{C}$ and satisfies (α, β, γ). Let $K_\tau(r, ds)$ be the measures associated with V, and define new measures by the formula

$$K_\tau'(r, ds) \equiv \rho(r) K_\tau(r, ds) \frac{1}{\rho(s)}.$$

These measures determine a semigroup of transformations K'_τ of $\mathcal{C}_\rho$ that satisfy relations (15.2) through (15.4) and also the relation

$$V'f = \int_0^\infty K'_\tau f \, d\tau.$$

The mass of $K'_\tau(r, ds)$ may exceed 1, however; it is only certain that the integrals $\int K'_\tau(r, ds)\rho(s)$ are bounded by ρ.

After defining in an obvious manner the notions of potential, excessive function, excessive measure, relative to the family $K'_\tau(r, ds)$, one can transfer all the results of §§10–14 to the present situation. For example, a function φ is excessive relative to the $K'_\tau(r, ds)$ if and only of φ/ρ is excessive relative to the $K_\tau(r, ds)$; thus, if φ is such a function and if X is one of the processes defined above, with R for terminal time, the composition $\varphi(X(\tau))$ almost certainly is continuous on the right in τ and has limits from the left, for τ less than R, because the statement is true of $\varphi(X(\tau))/\rho(X(\tau))$ and $\rho(X(\tau))$. In the translation, the measure $K_E(r, ds)$ corresponds to the measure

$$K'_E(r, ds) \equiv \rho(r) K_E(r, ds) \frac{1}{\rho(s)},$$

and the transformation M_E to the transformation

$$M'_E \, \zeta = \frac{1}{\rho} M_E(\rho\zeta).$$

The extension shows, for example, that condition (β) may be omitted if $\mathcal{K}$ is compact; all the results of §§10–15 still hold with slight changes in wording. The probabilistic interpretations unfortunately become somewhat forced in this reduction to the situation we have studied. A more appropriate model, which involves creation of mass as well as destruction, will be discussed in the next section.

The matters dealt with in this section are also treated in [3], in quite another spirit.

16. Creation of mass

In this section the space $\mathcal{K}$ and the transition measures P_τ are assumed to satisfy the conditions set forth in §1, while A is a given nearly analytic set and c a given function, measurable oves $\mathfrak{a}$ and satisfying the following condition: *There is a positive Borel measurable function b on $\mathcal{K}$ such that $-b$ bounds c from below and such that, for every process X and with probability* 1, *the integral*

$$\int_0^\tau b(X(\sigma, \omega)) \, d\sigma$$

is finite for all τ. It will be shown that a theory very like the one of §§10–14 can be based on the pair (c, A).

The freedom in choosing b is useful in verifying some conditions imposed later. The function b will be held fast, except in a few incidental arguments, and the positive function $b + c$ will be denoted by a. Quantities of the relative theory determined by the pair (a, A) will be denoted by $\mathfrak{R}$, K_τ, V as before; the parameter λ is taken to vanish.

Given a process X, set

$$Z_X(\tau, \omega) \equiv \exp\left\{\int_0^\tau b(X(\sigma, \omega))\, d\sigma\right\}. \tag{16.1}$$

This function increases with τ and is continuous with probability 1. It is to be thought of as the mass of the particle ω at time τ, while $X(\tau, \omega)$ is the position of ω.

Define a family of measures $K_\tau^c(r, ds)$ by the formula

$$K_\tau^c(r, B) \equiv \int_{\Omega'} Z_X(\tau, \omega)\chi_B(X(\tau, \omega))\, d\omega,$$

taking X to be a process starting at r, χ_B the characteristic function of B, and Ω' the set where τ is less than the terminal time R assigned to X. These measures are by definition the transition measures relative to (c, A); one easily verifies the relation

$$K_\tau^c(r, B) = \int_{\Omega''} \chi_B(X(\tau)) \exp\left\{-\int_0^\tau c(X(\sigma))\, d\sigma\right\} d\omega,$$

with Ω'' the set where τ is less than the time X hits A, so that the dependence on the choice of b is only apparent. The measures K_τ^c may have mass greater than 1, but otherwise they behave like the transition measures relative to $\mathfrak{R}$. The kernel for potentials is taken to be

$$V^c(r, B) \equiv \int_0^\infty K_\tau^c(r, B)\, d\tau.$$

For a system $\mathfrak{T}$ of terminal times, relatively independent of $\mathfrak{R}$, the kernel $K_{\mathfrak{T}}^c(r, ds)$ is defined as

$$K_{\mathfrak{T}}^c(r, B) = \int_{\Omega^*} Z_X(T)\chi_B(X(T))\, d\omega,$$

with Ω^* the set where R exceeds the time T assigned to X by $\mathfrak{T}$.

Excessive functions or measures and potentials of functions or measures are defined relative to (c, A) by using K_τ^c and V^c in place of K_τ and V; the special sets are taken to be the special sets relative to $\mathfrak{R}$. A function or measure excessive relative to (c, A) vanishes outside the set $\mathcal{K}$ of points not regular for $\mathfrak{R}$. Some of the results of preceding sections, such as Proposition 12.1, carry over to the situation considered now, because K_τ^c majorizes K_τ; for the same reason, a function or measure excessive relative to (c, A) is also excessive relative to $\mathfrak{R}$. Instead of listing the facts that can be derived in this way, we shall show how the theory based on (c, A) can be reduced to the one treated in §§10–14 by adjoining the mass as a new variable.

We shall denote by $\mathcal{H}^\times$ the product space $\mathcal{H} \times J$, with J the interval $1 \leqq z < \infty$, and sometimes by $r^\times$ a generic point (r, z) of the product space. For given τ and $r^\times$, define a measure on $\mathcal{H}$ by setting

$$P_\tau^\times(r^\times, B) \equiv \int_\Omega \chi_B(X(\tau), zZ_X(\tau))\, d\omega, \qquad B \subset \mathcal{H}^\times,$$

taking X to be a process starting at r and χ_B to be the characteristic function of a set B in $\mathcal{H}^\times$. These measures form a family of stationary Markoff transition measures on $\mathcal{H}^\times$, and we shall prove that $P_\tau^\times(r^\times, B)$ is Borel measurable in $(\tau, r^\times)$ whenever B is a Borel set. It is enough to show that $P_\tau^\times g.(r^\times)$ is Borel measurable in (τ, r) for every function g of the form $z^{-k}f(r)$, with k a positive integer and f a function in $\mathcal{C}(\mathcal{H})$, because linear combinations of such functions are dense in $\mathcal{C}(\mathcal{H}^\times)$. First suppose b to be bounded and set

$$\psi_n(r, \tau) \equiv \frac{1}{n!} \int_\Omega f(X(\tau)) \left\{ \int_0^\tau b(X(\sigma))\, d\sigma \right\}^n d\omega, \tag{16.2}$$

with X a process starting at r and having the P_τ as transition measures. The function ψ_n is bounded by $\alpha^n \tau^n / n!$, with α a suitable constant; it is also Borel measurable, as one sees by writing the second member of (16.2) as

$$\int d\sigma_1 \cdots d\sigma_n \int P_{\sigma_1}(r, ds_1) P_{\sigma_2 - \sigma_1}(s_1, ds_2) \cdots P_{\tau - \sigma_n}(s_n, ds) f(s) \prod b(s_i),$$

where $s, s_1, \cdots, s_n$ all range over $\mathcal{H}$ and the σ_i are restricted by the inequalities $0 < \sigma_1 < \cdots < \sigma_n < \tau$. Now,

$$\begin{aligned} P_\tau^\times g.(r^\times) &\equiv \int_\Omega f(X(\tau)) z^{-k} \exp\left\{ -k \int_0^\tau b(X(\sigma))\, d\sigma \right\} d\omega \\ &= z^{-k} \sum_n (-k)^n \psi_n(r, \tau), \end{aligned}$$

the series converging uniformly in r and τ, provided τ is restricted to a compact set. Borel measurability follows at once, for a bounded b; to extend the measurability to an unbounded b, one has only to approximate b by an increasing sequence of bounded functions.

The transition measures have been shown to satisfy the conditions of the first paragraph of §1. They also satisfy hypothesis (A). To obtain a process having the $P_\tau^\times$ as transition measures and starting at the point (r, z), for example, one has only to set

$$X^\times(\tau) \equiv (X(\tau), zZ_X(\tau)),$$

taking for X a process having the P_τ as transition measures and starting at the point r of $\mathcal{H}$; the sample paths of $X^\times$ have the same properties of continuity as those of X, since the second coordinate $zZ_X(\tau)$ is almost certainly continuous. The parts of (A) concerning behavior under stopping times are also trivially reduced to the corresponding statements concerning the original transition measures P_τ.

Let us take $\mathfrak{R}^\times$ to be the system of terminal times determined by a and $A \times J$ for processes having the $P_\tau^\times$ as transition measures; here a is considered a function on $\mathcal{K}^\times$ which depends only on the first coordinate. The system $\mathfrak{R}^\times$ may obviously be identified with $\mathfrak{R}$ if processes corresponding to the $P_\tau^\times$ are constructed from those corresponding to the P_τ. Quantities relative to $\mathfrak{R}^\times$ will be denoted by $K_\tau^\times$, $V^\times$, and so on. Note that $\mathcal{K}^\times$ is precisely $\mathcal{K} \times J$ and that every $\mathfrak{R}^\times$-special set is included in an $\mathfrak{R}^\times$-special set of the form $D \times \mathrm{I}$, with I a bounded interval in J and D an $\mathfrak{R}$-special set.

We shall consider only functions on $\mathcal{K}$ or $\mathcal{K}^\times$ which are positive and vanish outside $\mathcal{K}$ or $\mathcal{K}^\times$. Given a function f on $\mathcal{K}$, denote by f^* the function on $\mathcal{K}^\times$ defined as

$$f^*(r, z) = zf(r),$$

so that $f \leftrightarrow f^*$ is a one-to-one correspondence between functions on $\mathcal{K}$ and the functions on $\mathcal{K}^\times$ that satisfy the relations

$$g(r, uz) = ug(r, z), \qquad u \geqq 1.$$

Clearly, f is excessive relative to (c, A) if and only if f^* is excessive relative to $\mathfrak{R}^\times$, and the two relations

$$(V^c f)^* = V^\times f^*,$$
$$(K_E^c f)^* = K_{E^*}^\times f^*, \qquad E^* = E \times J,$$

hold identically, f being assumed excessive relative to (c, A) in the second.

Let φ be a function excessive relative to (c, A). By the preceding correspondence, the assertions of Theorems 11.3 and 11.5 and the inequality (11.10) hold for φ, the kernels being defined relative to (c, A). The treatment of the semimartingale following Theorem 11.3 is also valid, with $\varphi(X(\tau))$ replaced by $\varphi(X(\tau))Z_X(\tau)$.

The other theorems concerning excessive functions and measures require additional hypotheses, which will be discussed now. The hypotheses on P_τ and $\mathfrak{R}$ will be denoted again by $(B), \cdots, (E)$, and those on $P_\tau^\times$ and $\mathfrak{R}^\times$ by $(B^\times), \cdots, (E^\times)$. It is to be noted that (D) and (E) imply respectively $(D^\times)$ and $(E^\times)$. Also, the argument establishing Borel measurablity of $P_\tau^\times(r, B)$ shows that $(C^\times)$ holds if (C) holds and b is a bounded continuous function; for particular transition measures, such as those of Brownian motion, much less is required of b. The hypotheses concerning the measures K_τ^c are denoted by (D^c), (E^c), and $(\dagger)$.

(D^c) *Let D be an $\mathfrak{R}$-special set, X a process with the P_τ as transition measures, and (T_n) an increasing sequence of stopping times for X, each T_n being relatively independent of the terminal time R assigned to X. If the limit of the T_n is at least R, with probability* 1, *then*

$$\int_{\Omega_n} Z_X(T_n)\, \chi_D(X(T_n))\, d\omega \to 0, \qquad n \to \infty,$$

where χ_D *is the characteristic function of* D *and* Ω_n *is the set on which* T_n *is less than* R.

(E^c) *If* D *is an* $\mathfrak{R}$*-special set, then* $V^c(r, D)$ *is bounded in* r.

(†) *There is a sequence of positive functions* h_n *whose potentials* $V^c h_n$ *are finite everywhere and increase to infinity with* n *at every point of* $\mathcal{K}$.

Obviously, (E^c) implies (E). To see that (D^c) implies (D), take T_n to be the infimum of the τ greater than n for which $X(\tau)$ belongs to D, and choose the $\mathfrak{R}$-special set D' to include every point regular for D; the probability of the joint event, that R is infinite and $X(\tau)$ belongs to D for arbitrarily great values of τ, does not exceed

$$\int_{\Omega_n} \chi_{D'}(X(T_n))\, d\omega,$$

which tends to 0 under (D^c) because Z_X is bounded below by 1.

If (C) holds, then for every special set D there is another special set D' such that $V^c(r, D')$ is bounded away from 0; this assertion follows from Proposition 12.1, because V^c majorizes V.

If (C) and (E^c) both hold, then so does (D^c). Indeed, given a special set D, consider a potential $V^c f$ which is bounded above and which exceeds 1 on D; the existence of such a potential follows from (E^c) and the preceding paragraph. One has, using the notation of (D^c),

$$\begin{aligned}\int_{\Omega_n} Z_X(T_n)\chi_D(X(T_n))\, d\omega &\leqq \int_{\Omega_n} Z_X(T_n)V^c f.(X(T_n))\, d\omega \\ &= \int_{\Omega_n} d\omega \int_{T_n}^{R} Z_X(\tau) f(X(\tau))\, d\tau,\end{aligned}$$

and the last expression vanishes in the limit, as $n \to \infty$, because the integral

$$\int_{\Omega} V^c f.(X(0))\, d\omega \equiv \int_{\Omega} d\omega \int_0^R Z_X(\tau) f(X(\tau))\, d\tau$$

is finite. Proposition 12.5 could have been proved in the same manner.

Clearly, (†) follows from (E^c), and an argument like the last one in the proof of Theorem 12.2 shows that it also follows from (D^c).

Under (†) a function excessive relative to (c, A) is the limit of an increasing sequence of potentials, while under (E^c) a measure excessive relative to (c, A) is the limit of an increasing sequence of potentials. The proofs are like those of Theorems 12.2 and 14.2.

We shall now derive the analogue of Theorem 14.6, assuming ($C^\times$) and (E^c) to hold. A measure on $\mathcal{H}$ or $\mathcal{H}^\times$ will be tacitly understood to vanish outside $\mathcal{K}$ or $\mathcal{K}^\times$ and to be finite on sets which are special relative to $\mathfrak{R}$ or $\mathfrak{R}^\times$. Consider a measuer ν on $\mathcal{H}^\times$ that satisfies the relations

$$(16.3) \qquad \nu(B,\ uC) = u^{-1}\nu(B,\ C), \qquad u \geqq 1,$$

where uC stands for the set comprising the numbers uz with z in C. Such a measure must have the form

$$\nu(dr, dz) = z^{-2}\mu(dr)dz, \tag{16.4}$$

with μ a measure on $\mathfrak{K}$ and dz Lebesgue measure on J; it must also be finite on every set of the form $D \times J$, with D an $\mathfrak{N}$-special set. One easily verifies that ν satisfies (16.3) if and only if its potential $\nu V^{\times}$ does so; here the potential is assumed finite on $\mathfrak{N}^{\times}$-special sets, and the finiteness of ν on such sets follows from $(C^{\times})$.

Given a measure μ on $\mathfrak{K}$, let us take μ^* to be the measure on $\mathfrak{K}^{\times}$ defined by the right member of (16.4), so that $\mu \leftrightarrow \mu^*$ is a one-to-one correspondence between measures on $\mathfrak{K}$ and the measures on $\mathfrak{K}^{\times}$ that satisfy (16.3). Clearly, μ is excessive relative to (c, A) if and only if μ^* is excessive relative to $\mathfrak{N}^{\times}$, and the relations

$$(\mu V^c)^* = \mu^* V^{\times}, \qquad (\mu K_E^c V^c)^* = \mu^* K_{E^*}^{\times} V^{\times}, \qquad E^* = E \times J$$

hold. The second of these relations implies that

$$(M_E^c \zeta)^* = M_{E^*}^{\times} \zeta^*, \qquad E^* = E \times J,$$

ζ being excessive relative to (c, A) and M_E^c being defined as in §14.

Let ζ be excessive relative to (c, A). Then ζ^*, which is excessive relative to $\mathfrak{N}^{\times}$, can be represented according to Theorem 14.6 as $\xi + \nu V^{\times}$, with ξ a measure having special properties. To prove that ξ and ν both satisfy (16.3), consider the representation of the measure $\zeta^* \circ h_u$, where h_u is the homeomorphism $(r, z) \to (r, uz)$ of $\mathfrak{K}^{\times}$ into itself, defined for u not less than 1. This measure is also excessive relative to $\mathfrak{N}^{\times}$, and

$$\xi \circ h_u + (\nu V^{\times}) \circ h_u = \zeta^* \circ h_u = u^{-1} \zeta^* = u^{-1} \xi + u^{-1} \nu V^{\times}.$$

Now, the first and the last members of this chain both satisfy the conditions imposed in Theorem 14.6 on the representation of an excessive measure. Since that representation is unique, ξ and $\nu V^{\times}$ must satisfy (16.3); by a remark above, the measure ν satisfies the same relations. What has been said so far enables one to write ζ as $\alpha + \mu V^c$, with α excessive relative to (c, A) and

$$\alpha^* = \xi, \qquad \mu^* = \nu.$$

An additional argument is needed to establish the characteristic property of the measure α. Let D be an $\mathfrak{N}$-special set, F the complement of $D \times J$ in $\mathfrak{K}^{\times}$, and F_n the union of F with $D \times J_n$, where J_n is the interval $n \leqq z < \infty$. By the characteristic property of ξ,

$$M_{F_n}^{\times} \xi.(B) = \xi(B)$$

for every n and every set B in $\mathfrak{K}^{\times}$. Let us take for B a set comprising only points whose second coordinates are bounded above, say by m. Since a process with the $P_\tau^{\times}$ as transition measures has a second coordinate that increases with time, the equation

$$\beta K^{\times}_{F_n} V^{\times}_{\cdot}(B) \quad = \quad \beta K^{\times}_{F} V^{\times}_{\cdot}(B)$$

holds for all n greater than m and for all measures β; hence, by the definition of the transformation $M^{\times}_{E}$,

$$M^{\times}_{F_n} \xi.(B) = M^{\times}_{F} \xi.(B).$$

Consequently $M^{\times}_{F} \xi$ coincides with ξ or, expressed in other terms, $M^{c}_{E} \alpha$ coincides with α whenever the complement of E is included in some $\mathfrak{N}$-special set. Thus $\alpha + \mu V^{c}$ is the desired representation of ζ; the uniqueness of the representation of course follows from the uniqueness of the representation of ζ^{*}.

It is perhaps easier to repeat former proofs, under new hypotheses, than to translate results already established for $P^{\times}_{\tau}$ and $\mathfrak{N}^{\times}$ into the new setting. For example, the proof of Theorem 13.2 under (D^{c}), the proof of Theorem 13.5 under (B) and the hypothesis that (D^{c}) holds whenever a strictly positive constant is added to c, and the proof of Theorem 14.11 under (C) and a strengthened form of (E^{c}) proceed almost word for word as before. On the other hand, it is desirable to discuss systematically the situations considered in §§1–14, supposing a group or semigroup to act on $\mathcal{K}$ and limiting attention to functions and measures transforming in a given way under the action of the group; the theory based on the pair (c, A) need not then be treated separately.

Our model for creation of mass is not general enough to serve as a basis for results like those of the preceding section. The defect can be remedied by introducing the mass axiomatically rather than by the explicit formula (16.1). Of course, one would still seek an explicit construction of the mass in particular situations; such a construction exists for Brownian motion on the line, as I have learned from K. Itô and H. P. McKean.

A similar generalization of the relative theory, by introducing systems of terminal times axiomatically, has already been mentioned. It is desirable, in fact, to go a step further. The statements that have been proved in this paper involve the behavior of sample paths only before the terminal time; our hypotheses require the paths to behave well for all time. Creation of mass has been treated quite differently from destruction of mass; it could have been treated similarly by speaking of the creation of particles at a system of initial times. So one might properly take as foundation the following situation.

There is given a space $\mathcal{K}$ and a family of kernels $K_{\tau}(r, ds)$. The equation $K_{\sigma} K_{\tau} = K_{\sigma+\tau}$ is to hold identically, but the mass of $K_{\tau}(r, ds)$ is unrestricted. A realization of the family K_{τ} and an initial measure μ is a triple (R, S, X), with R and S positive functions on a measure space Ω and $X(\tau, \omega)$ defined as a point of $\mathcal{K}$ for ω in Ω and τ in $[R(\omega), S(\omega))$. The function S is to be not less than R, and the measure of the set of ω satisfying the conditions

$$R(\omega) \leqq \tau_i < S(\omega), \qquad X(\tau_i, \omega) \in A_i, \qquad\qquad i = 0, 1, \cdots, n,$$

for all given sequences

$$\tau_0 = 0 < \tau_1 < \cdots < \tau_n, \qquad A_0 \subset \mathcal{K}, \cdots, A_n \subset \mathcal{K},$$

is to be

$$\int_{A_0} \mu(dr_0) \int_{A_1} K_{\Delta_1}(r_0, dr_1) \cdots \int_{A_n} K_{\Delta_n}(r_{n-1}, dr_n),$$

where Δ_i stands for $\tau_i - \tau_{i-1}$. Unfortunately, there is no discussion of such realizations similar to the discussions by Blumenthal and Kinney of realizations of Markoff transition measures.

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