ON INGHAM'S TRIGONOMETRIC INEQUALITY

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Ingham has recently[1] proved the following

THEOREM. *Let*

$$f(t) = \sum_{n=N}^{N'} a_n e^{-\lambda_n t i},$$

where the λ*'s are real and* $\lambda_n - \lambda_{n-1} \geqq \gamma > 0$ $(N < n \leqq N')$, *and let* $\gamma T = \pi$. *Then*

$$(1) \qquad | a_n | \leqq \frac{1}{T} \int_{-T}^{T} | f(t) | \, dt \qquad (N \leqq n \leqq N').$$

He notes that we may take $\gamma = 1$, $T = \pi$ by the substitution $\gamma t = t'$. We may then rewrite the result as the

THEOREM. *Let*

$$(2) \qquad f(t) = \sum_{r=0}^{n} a_r e^{-\lambda_r t i},$$

where the λ*'s are real and* $\lambda_r - \lambda_{r-1} \geqq 1$ $(1 \leqq r \leqq n)$. Then

$$(3) \qquad | a_r | \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} | f(t) | \, dt \qquad (0 \leqq r \leqq n).$$

His proof, to which he was led by considerations of Fourier transforms, is quite short. Its essential idea, however, as I see it, can be presented in a rather simpler way, which also leads to a more precise result. He has shown that the factor $1/T$ in (1) cannot be replaced by a factor c/T where c is an absolute constant <1, but my proof shows that the factor $1/\pi$ in (3) can be replaced by a factor $K_r < 1/\pi$ depending upon the λ's.

On multiplying (2) throughout by $e^{\lambda_r t i}$, it suffices to take $f(t)$ in the form

$$(4) \quad f(t) = \sum_{r=-m}^{n} a_r e^{-\lambda_r i t}, \quad \lambda_0 = 0, \qquad \lambda_r - \lambda_{r-1} \geqq 1 \quad (-(m-1) \leqq r \leqq n),$$

and to estimate $| a_0 |$. I prove that

$$(5) \qquad | a_0 | \leqq \frac{K_0}{\pi} \int_{-\pi}^{\pi} | f(t) | \, dt,$$

with

$$(6) \qquad K_0 = 1 - \frac{1}{2} \prod_{r=-m}^{r=n}{}' (\mu_r/\lambda_r),$$

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[1] *A further note on trigonometric inequalities*, Proceedings of the Cambridge Philosophical Society, vol. 46 (1950), pp. 535–537.

where the dash denotes the omission of $r = 0$, and μ_r is defined to be the integer such that μ_r has the same sign as λ_r, and $|\mu_r|$ is the greatest integer $\leqq |\lambda_r|$. Clearly $K_0 = \frac{1}{2}$ when all the λ's are integers.

Write

$$g(t) = \sum_{s=-m}^{n} A_s e^{\mu_s it}, \qquad \mu_0 = 0, \tag{7}$$

where we shall presently define the A's as real constants and the μ's as a steadily increasing set of real numbers. We have

$$\int_{-\pi}^{\pi} f(t)g(t)\,dt = \sum_{r,s=-m}^{n} a_r A_s \int_{-\pi}^{\pi} e^{-\lambda_r it+\mu_s it}\,dt. \tag{8}$$

We now impose the condition on the A's that the coefficients of all the a's except a_0 are zero. We have a simple expression for a_0 if we assume now that the μ's are integers. Then

$$2\pi a_0 A_0 = \int_{-\pi}^{\pi} f(t)g(t)\,dt,$$

and so

$$2\pi\,|a_0|\,|A_0| \leqq \int_{-\pi}^{\pi} |f(t)| \sum_{s=-m}^{n} |A_s|\,dt. \tag{9}$$

We assume for the time being that none of the differences $\lambda_r - \mu_s$ are zero except $\lambda_0 - \mu_0$. Then

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-\lambda_r it+\mu_s it}\,dt &= \frac{2i \sin(\mu_s - \lambda_r)\pi}{i(\mu_s - \lambda_r)} \\ &= \frac{2(-1)^{\mu_s} \sin \lambda_r \pi}{\lambda_r - \mu_s} \qquad (\lambda_r \neq \mu_s) \\ &= 2\pi \qquad (\lambda_0 = \mu_0 = 0).\end{aligned}$$

Hence

$$\sum_{s=-m}^{n} \frac{A_s(-1)^{\mu_s}}{\lambda_r - \mu_s} = 0 \qquad (-m \leqq r \leqq n, r \neq 0). \tag{10}$$

This is a system of $m + n$ homogeneous linear equations in the $m + n + 1$ unknown A's, and so there is a solution in which all the A's are not zero. Such systems are well known, and a solution is given when the A's are such that

$$\sum_{s=-m}^{n} \frac{A_s(-1)^{\mu_s}}{x - \mu_s} = \frac{\prod'_{r=-m}^{r=n} (x - \lambda_r)}{\prod_{r=-m}^{n} (x - \mu_r)} \tag{11}$$

is the identity given by splitting the right-hand side into partial fractions. This is obvious on putting $x = \lambda_r$. Multiply (11) by $x - \mu_s$ and then put $x = \mu_s$. Hence

$$A_s(-1)^{\mu_s} = \prod'_{r=-m}^{n} (\mu_s - \lambda_r) \Big/ \prod''_{r=-m}^{n} (\mu_s - \mu_r), \tag{12}$$

where the double dash denotes the omission of the term with $r = s$. In particular,

$$A_0 = \prod_{r=-m}^{n}{}' (\lambda_r/\mu_r). \tag{13}$$

We wish to estimate (9). Since we see, on multiplying (11) by x and making $x \to \infty$, that

$$\sum_{s=-m}^{n} A_s(-1)^{\mu_s} = 1, \tag{14}$$

we write (9) as

$$2\pi \mid a_0 \mid \mid A_0 \mid \leqq \int_{-\pi}^{\pi} \mid f(t) \mid \sum_{s=-m}^{n} \mid A_s(-1)^{\mu_s} \mid dt. \tag{15}$$

We show now that we can choose the integers μ_s as a steadily increasing sequence and so that $A_0 > 1$ and $A_s(-1)^{\mu_s} < 0$ when $s \neq 0$. We have already taken $\mu_0 = 0$, and we now take the other μ_r such that μ_r has the same sign as λ_r and $\mid \mu_r \mid$ is the greatest integer not exceeding $\mid \lambda_r \mid$, and so here less than $\mid \lambda_r \mid$. The μ_r are all different since the successive λ's define intervals of length at least one. Then (13) shows that $A_0 > 1$. Suppose first that $s > 0$. There are $n - s$ negative factors in the denominator of (12) arising from $r = s + 1, \cdots, n$, and $n - s + 1$ negative factors in the numerator arising from $r = s, \cdots, n$. Hence $A_s(-1)^{\mu_s} < 0$. Suppose next $s = -\sigma < 0$. Then there are $n + \sigma$ negative factors in the denominator arising from $r = -\sigma + 1, \cdots, n$, and $n + \sigma - 1$ negative factors in the numerator arising from $r = -\sigma + 1, -\sigma + 2, \cdots, -1, 1, \cdots, n$. Hence again $A_s(-1)^{\mu_s} < 0$.

We now rewrite (15) as

$$2\pi \mid a_0 \mid A_0 \leqq \int_{-\pi}^{\pi} \mid f(t) \mid \left(A_0 - \sum_{s=-m}^{n}{}' A_s(-1)^{\mu_s} \right) dt.$$

Then from (14)

$$\mid a_0 \mid \leqq \left(1 - \frac{1}{2A_0} \right) \frac{1}{\pi} \int_{-\pi}^{\pi} \mid f(t) \mid dt, \tag{16}$$

and this is (5).

We have temporarily supposed in (4) that none of the λ's except $\lambda_0 = 0$ are integers. The simplest limiting process in (16) shows that (5) still holds when any of the λ's are integers, $\lambda_0 = 0$, and the μ's are as defined there.

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