THE IRREDUCIBLE REPRESENTATIONS OF A SEMIGROUP RELATED TO THE SYMMETRIC GROUP

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1. Introduction

A. H. Clifford [2] has studied the representations of a class of semigroups. His results lead to a complete classification of the representations of a particular class of semigroups having considerable independent interest. These semigroups are the semigroups $\mathfrak{T}_n$ defined as follows.

Consider a finite set consisting of say n elements; for the sake of definiteness we may consider the set $\{1, 2, \cdots, n\}$. Let $\mathfrak{T}_n$ be the set of all single-valued mappings of this set onto or into itself. For f, $g \in \mathfrak{T}_n$ let fg be the element of $\mathfrak{T}_n$ such that $fg(i) = f(g(i))$ $(i = 1, \cdots, n)$. With this definition of multiplication, $\mathfrak{T}_n$ is obviously an associative system, *i.e.*, a semigroup. The order of $\mathfrak{T}_n$ is n^n; $\mathfrak{T}_n$ contains the symmetric group $\mathfrak{S}_n$, properly if $n > 1$; $\mathfrak{T}_n$ is noncommutative if $n > 1$.

By the term (α, β) matrix, we shall mean a matrix with α rows and β columns and complex entries. A representation of a semigroup G is a homomorphism M of G into the multiplicative semigroup of all (α, α) matrices (α an arbitrary positive integer) such that $M(x) \neq 0$ for some $x \in G$. If the set $\{M(x)\}_{x \in G}$ is an irreducible set of matrices (*i.e.*, if every (α, α) matrix is a linear combination of matrices $M(x)$), then M is said to be an irreducible representation of G. The identity representation is the mapping that carries every $x \in G$ into the identity matrix.

In the present paper we give an explicit determination of all irreducible representations of $\mathfrak{T}_n$. The idea of studying $\mathfrak{T}_n$ was suggested to us by D. D. Miller (oral communication). The problem of obtaining representations of semigroups as distinct from groups seems to have been first studied by Suškevič [6]. A. H. Clifford [2] has, as noted above, given a construction of all representations of a class of semigroups closely connected with $\mathfrak{T}_n$. Ponizovskiĭ: [5] has pointed out some simple properties of $\mathfrak{T}_n$. In the present paper we also relate the irreducible representations of $\mathfrak{T}_n$ to the semigroup algebra $\mathfrak{L}_1(\mathfrak{T}_n)$ (notation as in [3]).

2. Definitions

Let f be an element of $\mathfrak{T}_n$. Then f splits the set $\{1, 2, \cdots, n\}$ into a number, p, of nonvoid, disjoint subsets, each of the form $\{x : f(x) = a\}$ for some a in the range of f. Obviously f is determined by these sets and the corresponding a's. We will set down a unique notation for the elements

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of $\mathfrak{T}_n$. For a nonvoid subset s of $\{1, 2, \cdots, n\}$, let s^* be the least element of s. Now write the sets $\{x:f(x) = a\}$ in the order $s_1, s_2, \cdots, s_p$, where $s_1^* < s_2^* < \cdots < s_p^*$. We can represent f by the symbol $\begin{pmatrix} s_1 \, s_2 \cdots s_p \\ a_1 \, a_2 \cdots a_p \end{pmatrix}$, meaning by this that every element of s_i is mapped by f into a_i $(i = 1, 2, \cdots, p)$. It is easy to see that every element f of $\mathfrak{T}_n$ occurs once and only once among the $\begin{pmatrix} s_1 \, s_2 \cdots s_p \\ a_1 \, a_2 \cdots a_p \end{pmatrix}$, where $1 \leqq p \leqq n$, the sets $s_1, \cdots, s_p$ are a decomposition of $\{1, 2, \cdots, n\}$ of the kind described, and $a_1, a_2, \cdots, a_p$ are any distinct integers lying between 1 and n. From now on, the expression $s_1, s_2, \cdots, s_p$ will always mean a decomposition of $\{1, 2, \cdots, n\}$ into nonvoid, disjoint subsets with $s_1^* < s_2^* < \cdots < s_p^*$. The letters t and w will be used similarly. Also $a_1, a_2, \cdots, a_p$ will always mean any ordered sequence of distinct integers from 1 to n; the letters c and d will be used similarly

For $p = 1, 2, \cdots, n$, let $\mathfrak{B}_p$ be the set of all elements of $\mathfrak{T}_n$ whose range contains just p elements: that is, all $\begin{pmatrix} s_1 \, s_2 \cdots s_p \\ a_1 \, a_2 \cdots a_p \end{pmatrix}$ for a fixed p. Strictly speaking, $\mathfrak{B}_p$ depends upon n as well as p. However, only one value of n will be treated at any one time, unless otherwise specified. The set $\mathfrak{B}_n$ is obviously the symmetric group $\mathfrak{S}_n$. The set $\mathfrak{B}_1$ is a semigroup with the trivial multiplication $fg = f$. No other $\mathfrak{B}_p$ is a subsemigroup of $\mathfrak{T}_n$. It will be convenient to have the semigroup $\mathfrak{B}_p \cup \{z\}$, where multiplication is defined by

$$zz = fz = zf = z \quad \text{for all} \quad f \in \mathfrak{B}_p,$$

$$fg = \begin{cases} fg \text{ as in } \mathfrak{T}_n \text{ if } fg \in \mathfrak{B}_p \\ z \text{ if } fg \text{ non} \in \mathfrak{B}_p. \end{cases}$$

3. Preliminary theorems

We make a first reduction of our problem by showing that irreducible representations of $\mathfrak{T}_n$ must behave in certain special ways.

3.1. THEOREM. *The two-sided ideals of $\mathfrak{T}_n$ are exactly the sets*

$$\bigcup_{j=1}^{p} \mathfrak{B}_j \qquad (p = 1, 2, \cdots, n).$$

Proof. Let $\mathfrak{J}$ be a two-sided ideal in $\mathfrak{T}_n$, that is, $\mathfrak{J}\mathfrak{T}_n \cup \mathfrak{T}_n\mathfrak{J} \subset \mathfrak{J}$, and $0 \neq \mathfrak{J} \subset \mathfrak{T}_n$. Let p be the largest integer such that $I \cap \mathfrak{B}_p \neq 0$, and let $f = \begin{pmatrix} s_1 \, s_2 \cdots s_p \\ a_1 \, a_2 \cdots a_p \end{pmatrix}$ be in $\mathfrak{J}$. Let $\begin{pmatrix} t_1 \, t_2 \cdots t_q \\ c_1 \, c_2 \cdots c_q \end{pmatrix}$ be any element of $\mathfrak{T}_n$ with $q \leqq p$. Let $w_1, w_2, \cdots, w_q$ be the sets $\{a_1\}, \{a_2\}, \cdots, \{a_{q-1}\}$, $\{a_1, a_2, \cdots, a_{q-1}\}'$ ($'$ denotes complement in $\{1, 2, \cdots, n\}$), ordered as prescribed in §2. Finally, let d_i $(i = 1, 2, \cdots, q)$ be defined as c_j, where j is such that $a_j \in w_i$. Then we have

$$\begin{pmatrix} t_1 \, t_2 \cdots t_q \\ c_1 \, c_2 \cdots c_q \end{pmatrix} = \begin{pmatrix} w_1 \, w_2 \cdots w_q \\ d_1 \, d_2 \cdots d_q \end{pmatrix} \begin{pmatrix} s_1 \, s_2 \cdots s_p \\ a_1 \, a_2 \cdots a_p \end{pmatrix} \begin{pmatrix} t_1 \, t_2 \cdots t_q \\ s_1^* \, s_2^* \cdots s_q^* \end{pmatrix}.$$

(Recall that the product fg of transformations f and g is the transformation obtained by carrying out g and *then* f.) Conversely, it is clear that every set $\bigcup_{j=1}^{p} \mathfrak{B}_j$ is a two-sided ideal in $\mathfrak{T}_n$.

3.2 THEOREM. *Let M be an irreducible representation of $\mathfrak{T}_n$. The set $\{f : f \in \mathfrak{T}_n, M(f) = 0\}$ is either void or one of the sets*

$$\bigcup_{j=1}^{p} \mathfrak{B}_j \qquad (p = 1, \cdots, n-1).$$

Proof. If the set $\{f : f \in \mathfrak{T}_n, M(f) = 0\}$ is not void, then clearly it is a two-sided ideal in $\mathfrak{T}_n$. The result now follows from Theorem 3.1.

3.3 THEOREM. *Let M be an irreducible representation of $\mathfrak{T}_n$, and let p be the least integer such that $M(f) \neq 0$ for some $f \in \mathfrak{B}_p$. Then the set of matrices $\{M(f)\}_{f \in \mathfrak{B}_p}$ is irreducible.*

Proof. Let m be the degree of M. Since M is irreducible, the set of all matrices $\sum_{f \in \mathfrak{T}_n} \alpha_f M(f)$ (the α_f are arbitrary complex numbers) is the algebra of all (m, m) matrices. Since $\bigcup_{j=1}^{p} \mathfrak{B}_j$ is a two-sided ideal in $\mathfrak{T}_n$, the set A of all matrices $\sum \alpha_f M(f)$, summed over all f in $\bigcup_{j=1}^{p} \mathfrak{B}_j$, is a two-sided ideal in the algebra of all (m, m) matrices. Since $M(f)$ is different from 0 for some $f \in \mathfrak{B}_p$, A is not the zero ideal. Since the algebra of all (m, m) matrices is simple, A is the algebra of all (m, m) matrices, and this proves the theorem.

3.4 LEMMA. *Let q be an integer such that $2 \leqq q \leqq n - 1$, and let g be any element of $\mathfrak{B}_{q-1}$. Then there are elements f and h in $\mathfrak{B}_q$ such that $hf = g$.*

Proof. Let the range of g be $\{a_1, \cdots, a_{q-1}\}$, so written that $g^{-1}(a_{q-1})$ contains more than one element: $g^{-1}(a_{q-1}) = \{b\} \cup s$, where $s \neq 0$ and b non $\in s$. Let f be defined by

$$f(x) = \begin{cases} j & \text{if } x \in g^{-1}(a_j), \quad \leqq j \leqq q-2, \\ q-1 & \text{if } x = b, \\ q & \text{if } x \in s. \end{cases}$$

Let h be defined by

$$h(x) = \begin{cases} a_x & \text{if } 1 \leqq x \leqq q-1, \\ a_{q-1} & \text{if } x = q, \\ c & \text{if } q+1 \leqq x \leqq n, \end{cases}$$

where c is different from $a_1, \cdots, a_{q-1}$ and $1 \leqq c \leqq n$. Then $g = hf$.

3.5 THEOREM. *Let M' be an irreducible representation of the semigroup $\mathfrak{B}_p \cup \{z\}$ $(1 \leqq p < n)$ that is not the identity representation. Then there is one and only one representation M of $\mathfrak{T}_n$ such that $M(f) = M'(f)$ for $f \in \mathfrak{B}_p$. Furthermore, $M(g) = 0$ for $g \in \bigcup_{j=1}^{p-1} \mathfrak{B}_j$.*

Proof. Suppose that M is such a representation. If $g \in \mathfrak{B}_{p-1}$, then, by Lemma 3.4, $g = hf$, where $h, f \in \mathfrak{B}_p$. Hence

$$M(g) = M(h)M(f) = M'(h)M'(f) = M'(hf) = M'(z) = 0,$$

since it is clear that $M'(z)$ must be 0. Repeated applications of Lemma 3.4 show that $M(g) = 0$ for all $g \in \bigcup_{j=1}^{p-1} \mathfrak{B}_j$.

Since M' is irreducible and $M'(z) = 0$, there is a linear combination $\sum_{f \in \mathfrak{B}_p} \alpha_f M'(f)$ equal to the identity matrix I. Let g be any element of $\mathfrak{T}_n$. Then $fg \in \bigcup_{j=1}^{p} \mathfrak{B}_j$ if $f \in \mathfrak{B}_p$, and $M(g) = IM(g) = \sum_{f \in \mathfrak{B}_p} \alpha_f M(fg)$. Since we have just shown that M is completely determined by M' on $\bigcup_{j=1}^{p} \mathfrak{B}_j$, it follows that M is unique if it exists at all.

We now show that there is an M of the kind required. Let $M''(f) = M'(f)$ for $f \in \mathfrak{B}_p$ and $M''(f) = 0$ for $f \in \bigcup_{j=1}^{p-1} \mathfrak{B}_j$. Obviously M'' is a representation of the semigroup $\bigcup_{j=1}^{p} \mathfrak{B}_j$. Choose a fixed linear combination $\sum_{f \in \mathfrak{B}_p} \alpha_f M''(f)$ that is equal to I. Now let $M(g) = \sum_{f \in \mathfrak{B}_p} \alpha_f M''(fg)$, for all $g \in \mathfrak{T}_n$. Since $fg \in \bigcup_{j=1}^{p} \mathfrak{B}_j$ for $f \in \mathfrak{B}_p$ and $g \in \mathfrak{T}_n$, $M(g)$ is well defined. To show that M is a representation of $\mathfrak{T}_n$, we need to know that

3.5.1 $$M(g) = \sum_{e \in \mathfrak{B}_p} \alpha_e M''(ge) \qquad g \in \mathfrak{T}_n.$$

To prove this, take e in $\mathfrak{B}_p$. Then

$$\begin{aligned} M(g)M''(e) &= \sum_{f \in \mathfrak{B}_p} \alpha_f M''(fg)M''(e) = \sum_{f \in \mathfrak{B}_p} \alpha_f M''(fge) \\ &= \sum_{f \in \mathfrak{B}_p} \alpha_f M''(f)M''(ge) = IM''(ge) = M''(ge). \end{aligned}$$

From this it follows that $\sum_{e \in \mathfrak{B}_p} \alpha_e M(g)M''(e) = \sum_{e \in \mathfrak{B}_p} \alpha_e M''(ge)$. Since $\sum_{e \in \mathfrak{B}_p} \alpha_e M''(e) = I$, we have 3.5.1.

Now let g, h be any elements of $\mathfrak{T}_n$. Using 3.5.1, we have

$$\begin{aligned} M(g)M(h) &= \sum_{f \in \mathfrak{B}_p} \alpha_f M''(fg) \sum_{e \in \mathfrak{B}_p} \alpha_e M''(he) \\ &= \sum_{f \in \mathfrak{B}_p} \sum_{e \in \mathfrak{B}_p} \alpha_f \alpha_e M''(fghe) \\ &= \sum_{f \in \mathfrak{B}_p} \alpha_f M''(fgh) \sum_{e \in \mathfrak{B}_p} \alpha_e M''(e) \\ &= M(gh). \end{aligned}$$

Hence M is a representation of $\mathfrak{T}_n$. Finally, if $g \in \mathfrak{B}_p$, then

$$M(g) = \sum_{f \in \mathfrak{B}_p} \alpha_f M''(fg) = \sum_{f \in \mathfrak{B}_p} \alpha_f M''(f)M''(g) = IM''(g) = M'(g).$$

This completes the proof.

The next theorem is not strictly necessary but may be of some interest.

3.6 THEOREM. *Let M be any representation of $\mathfrak{T}_n$, and let f, g be in $\mathfrak{B}_p$, $1 \leqq p \leqq n$. Then* rank $M(f)$ = rank $M(g)$.

Proof. We may suppose without loss of generality that $M(\varphi)$ is nonsingular for $\varphi \in \mathfrak{B}_n$. Let $\{a_1, a_2, \cdots, a_p\}$ be the range of f, and let $\{u_1, u_2, \cdots, u_p\}$ be elements of $\{1, 2, \cdots, n\}$ such that $f(u_i) = a_i$ $(i = 1, 2, \cdots, p)$. Let $\{a_{p+1}, \cdots, a_n\}$ be $\{a_1, a_2, \cdots, a_p\}'$, and similarly $\{u_{p+1}, \cdots, u_n\} = \{u_1, u_2, \cdots, u_p\}'$. Let φ be the element of $\mathfrak{B}_n$ such that $\varphi(i) = u_i$ $(i = 1, 2, \cdots, n)$ and ψ the element of $\mathfrak{B}_n$ such that

$\psi(a_i) = i$ $(i = 1, 2, \cdots, n)$. Let $f' = \psi f \varphi$. Then

$$f'(i) = \begin{cases} i \text{ if } 1 \leqq i \leqq p, \\ j, \text{ for some } j(i),\ 1 \leqq j \leqq p, \text{ if } p + 1 \leqq i \leqq n. \end{cases}$$

We define a g' for the element g in the same way. The equalities $f'g' = g'$, $g'f' = f'$ are easy to verify. Hence rank $M(g') \leqq$ rank $M(f')$ and rank $M(f') \leqq$ rank $M(g')$, and so we have rank $M(f') =$ rank $M(g')$. The matrices $M(\varphi)$ and $M(\psi)$ are nonsingular, since φ and ψ are in the symmetric group $\mathfrak{B}_n$. Therefore rank $M(f) =$ rank $M(f')$ and rank $M(g) =$ rank $M(g')$. This completes the proof.

3.6.1 *Note.* Theorem 3.6, Lemma 3.4, and Theorem 3.5 show that if M is any representation of $\mathfrak{T}_n$ as in 3.5, then all matrices $M(f)$ are singular for $f \in \mathfrak{B}_p$ $(1 < p < n)$.

We now summarize the results of this section.

3.7 THEOREM. *Let M be an irreducible representation of $\mathfrak{T}_n$. Then there is a $\mathfrak{B}_p$ $(1 \leqq p \leqq n)$ such that $M(f) = 0$ for all $f \in \bigcup_{j=1}^{p-1} \mathfrak{B}_j$ ($\bigcup_{j=1}^{0} \mathfrak{B}_j$ is void) and $M(f) \neq 0$ for some $f \in \mathfrak{B}_p$. The matrices $\{M(f)\}_{f \in \mathfrak{B}_p}$ are an irreducible set, and all have the same nonzero rank. If $1 < p < n$, all $M(f)$ for $f \in \mathfrak{B}_p$ are singular. Setting $M(z) = 0$, we obtain from M an irreducible representation of $\mathfrak{B}_p \cup \{z\}$. Conversely, every irreducible representation of $\mathfrak{B}_p \cup \{z\}$ that is not the identity representation determines a unique irreducible representation of $\mathfrak{T}_n$ that is 0 on $\bigcup_{j=1}^{p-1} \mathfrak{B}_j$.*

3.8 The semigroups $\mathfrak{B}_p \cup \{z\}$ are completely simple, and Clifford [2] has given a general method for obtaining the representations of such semigroups. Since we wish to write the irreducible representations of $\mathfrak{B}_p \cup \{z\}$ as explicitly as possible, it seems advisable to write out all of the details.

4. Necessary conditions for an irreducible representation of $\mathfrak{B}_p \cup \{z\}$

Throughout this section, n and p are arbitrary but fixed. For general n and p, $\mathfrak{B}_p$ is a complicated object. To render it tractable, we consider elements of two special kinds.

4.1 DEFINTION. Let

$$u(a_1, a_2, \cdots, a_p) = \begin{pmatrix} \{1\} & \{2\} \cdots & \{p-1\} & \{p, p+1, \cdots, n\} \\ a_1 & a_2 \cdots & a_{p-1} & a_p \end{pmatrix},$$

and let

$$v(s_1, s_2, \cdots, s_p) = \begin{pmatrix} s_1 & s_2 \cdots s_p \\ 1 & 2 \cdots p \end{pmatrix}.$$

Thus $u(a_1, a_2, \cdots, a_p)$ is an element of $\mathfrak{B}_p$ that depends only on the numbers $a_1, a_2, \cdots, a_p$, and $v(s_1, s_2, \cdots, s_p)$ is an element of $\mathfrak{B}_p$ that depends only on the sets $s_1, s_2, \cdots, s_p$.

4.2 We now have:

4.2.1 $$u(a_1, a_2, \cdots, a_p)v(s_1, s_2, \cdots, s_p) = \begin{pmatrix} s_1 & s_2 \cdots & s_p \\ a_1 & a_2 \cdots & a_p \end{pmatrix};$$

4.2.2 $$u(a_1, a_2, \cdots, a_p)u(1, 2, \cdots, p) = u(a_1, a_2, \cdots, a_p);$$

4.2.3 $$u(1, 2, \cdots, p)^2 = u(1, 2, \cdots, p);$$

4.2.4 $$u(1, 2, \cdots, p)v(s_1, s_2, \cdots, s_p) = v(s_1, s_2, \cdots, s_p);$$

4.2.5 $$v(s_1, s_2, \cdots, s_p)u(1, 2, \cdots, p) = \begin{cases} u(1, 2, \cdots, p) & \text{if } s_p^* = p, \\ z & \text{if } s_p^* > p. \end{cases}$$

Equalities 4.2.1–4.2.5 can be checked directly from 4.1.

4.3 We now suppose that we are given a fixed but arbitrary representation M of $\mathfrak{B}_p \cup \{z\}$. Irreducibility will not be assumed until needed. The representation M may have many equivalent forms. Since $u(1, 2, \cdots, p)$ is idempotent (4.2.3), $M(u(1, 2, \cdots, p))$ is an idempotent matrix, and hence can be put into the form

4.3.1 $$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Without loss of generality, we may suppose that $M(u(1, 2, \cdots, p))$ has this form. Let k be the degree of the identity matrix I in 4.3.1, and let l be such that the matrix 4.3.1 has degree $k + l$. We now write

4.3.2 $$M(u(a_1, a_2, \cdots, a_p)) = \begin{pmatrix} A(a_1, a_2, \cdots, a_p) & B(a_1, a_2, \cdots, a_p) \\ C(a_1, a_2, \cdots, a_p) & D(a_1, a_2, \cdots, a_p) \end{pmatrix},$$

where A is a (k, k) matrix, B is a (k, l) matrix, C is an (l, k) matrix, and D is an (l, l) matrix. From 4.2.2, we see that $M(u(a_1, \cdots, a_p))M(u(1, 2, \cdots, p)) = M(u(a_1, \cdots, a_p))$. Since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix},$$

it follows that $B(a_1, \cdots, a_p) = 0$ and $D(a_1, \cdots, a_p) = 0$. We next write

4.3.3 $$M(v(s_1, \cdots s_p)) = \begin{pmatrix} A(s_1, \cdots s_p) & B(s_1, \cdots s_p) \\ C(s_1, \cdots s_p) & D(s_1, \cdots s_p) \end{pmatrix},$$

where the sizes of the blocks in 4.3.3 are just as in 4.3.2. From 4.2.4, we find that $C(s_1, \cdots, s_p) = 0$ and that $D(s_1, \cdots, s_p) = 0$. Equality 4.2.5 shows that

4.3.4 $$A(s_1, \cdots, s_p) = \delta_{s_p^*, p}\, I,$$

where $\delta_{s_p^*,p}$ is the Kronecker δ-function. From 4.3.1, we see that

4.3.5. $C(1, 2, \cdots, p) = 0, \qquad B(\{1\}, \{2\}, \cdots, \{p-1\}, \{p, \cdots, n\}) = 0.$

4.4 If φ and ψ are 1-to-1 mappings of the set $\{1, \cdots, p\}$ onto itself (*i.e.*, elements of $\mathfrak{S}_p$), then 4.1 implies that

4.4.1 $u(\varphi(1), \cdots, \varphi(p))u(\psi(1), \cdots, \psi(p)) = u(\varphi(\psi(1)), \cdots, \varphi(\psi(p))).$

Then, as in 4.3, we see that

4.4.2 $A(\varphi(1), \cdots, \varphi(p))A(\psi(1), \cdots, \psi(p)) = A(\varphi(\psi(1)), \cdots, \varphi(\psi(p))).$

Thus the matrices $A(a_1, \cdots, a_p)$ for which $\{a_1, \cdots, a_p\} = \{1, \cdots, p\}$ produce a representation of $\mathfrak{S}_p$.

For a positive integer a, let $a' = \min(a, p)$. Then, for $1 \leqq a \leqq n$, $u(1, \cdots, p)$ carries a into a'. From this it is easy to see that

4.4.3 $$u(1, \cdots, p)u(a_1, \cdots, a_p) = \begin{cases} u(a_1', \cdots, a_p') & \text{if } a_1', \cdots, a_p' \text{ are all different,} \\ z & \text{otherwise.} \end{cases}$$

In the usual way, 4.4.3 implies that

4.4.4 $$A(a_1, \cdots, a_p) = \begin{cases} A(a_1', \cdots, a_p') & \text{if } a_1', \cdots, a_p' \text{ are all different,} \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $A(e_1, \cdots, e_p)$ were defined in 4.3.2 only for sequences $e_1, \cdots, e_p$ with no repetitions. We now define $A(e_1, \cdots, e_p)$ as 0 if $e_i = e_j$ for some distinct i and j. With this convention, 4.4.4 becomes

4.4.5 $$A(a_1, \cdots, a_p) = A(a_1', \cdots, a_p'),$$

and 4.4.2 can be extended to

4.4.6 $$A(a_1, \cdots, a_p)A(c_1, \cdots, c_p) = A(a_{c_1'}', \cdots, a_{c_p'}').$$

4.5 We now discuss the matrices $C(a_1, \cdots, a_p)$ defined in 4.3.2. If φ is a 1-to-1 mapping of $\{1, \cdots, p\}$ onto itself, then

4.5.1 $$u(a_1, \cdots, a_p)u(\varphi(1), \cdots, \varphi(p)) = u(a_{\varphi(1)}, \cdots, a_{\varphi(p)}).$$

This is easy to verify. Our usual steps give us

4.5.2 $$C(a_{\varphi(1)}, \cdots, a_{\varphi(p)}) = C(a_1, \cdots, a_p)A(\varphi(1), \cdots, \varphi(p))$$

and equivalently

4.5.3 $$C(a_1, \cdots, a_p) = C(a_{\varphi(1)}, \cdots, a_{\varphi(p)})A(\varphi^{-1}(1), \cdots, \varphi^{-1}(p)).$$

For each ordered sequence $a = a_1, a_2, \cdots, a_p$, we define the function $\rho_a(i)$ $(i = 1, 2, \cdots, p)$ so that $a_{\rho_a(1)} < a_{\rho_a(2)} < \cdots < a_{\rho_a(p)}$. Since the a_j are all distinct, we can do this. Plainly ρ_a is uniquely defined. Now we have

4.5.4 $$C(a_1, \cdots, a_p) = C(a_{\rho_a(1)}, \cdots, a_{\rho_a(p)})A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p)),$$

which follows immediately from 4.5.3. Thus, if the representation $A(\varphi(1), \cdots, \varphi(p))$ of $\mathfrak{S}_p$ is known, and if $C(a_1, \cdots, a_p)$ is known for monotonically increasing $a_1, \cdots, a_p$, the matrices $C(a_1, \cdots, a_p)$ are known for all $a_1, \cdots, a_p$.

4.6 We now discuss the matrices $B(s_1, \cdots, s_p)$ defined in 4.3.3. For every family of sets $s = s_1, s_2, \cdots, s_p$, we define the function

$$\sigma_s(i) \qquad (i = 1, 2, \cdots, n)$$

so that $i \in s_{\sigma_s(i)}$. The equality

$$4.6.1 \quad v(s_1, \cdots, s_p)u(a_1, \cdots, a_p) = \begin{cases} u(\sigma_s(a_1), \cdots, \sigma_s(a_p)) & \text{if all } \sigma_s(a_i) \text{ are different,} \\ z & \text{otherwise,} \end{cases}$$

is not hard to verify. For $1 \leqq b_1 < b_2 < \cdots < b_p \leqq n$, 4.6.1 and our usual steps give us

$$4.6.2 \quad \begin{aligned} &B(s_1, \cdots, s_p)C(b_1, \cdots, b_p) \\ &\qquad = -\delta_{s_p^*,p}A(b_1', \cdots, b_p') + A(\sigma_s(b_1), \cdots, \sigma_s(b_p)). \end{aligned}$$

The condition that the sequence $b_1, \cdots, b_p$ be monotonic increasing is not required in 4.6.2. However this special case of 4.6.2 is all that will be needed. We agree that $b_1, \cdots, b_p$ will always mean a monotone strictly increasing sequence of integers lying between 1 and n.

4.7 Combining formulas 4.2.1, 4.3.2, 4.3.3, 4.3.4, 4.4.5, and 4.5.4, one can obtain the equality

$$4.7.1 \quad M\begin{pmatrix} s_1 & s_2 & \cdots & s_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}$$

$$= \begin{pmatrix} \delta_{s_p^*,p}A(a_1', \cdots, a_p') & A(a_1', \cdots, a_p')B(s_1, \cdots, s_p) \\ \delta_{s_p^*,p}C(a_{\rho_a(1)}, \cdots, a_{\rho_a(p)})A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p)) & \\ & C(a_{\rho_a(1)}, \cdots, a_{\rho_a(p)})A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p))B(s_1, \cdots, s_p) \end{pmatrix}$$

4.8 Suppose now that M is an irreducible representation of $\mathfrak{B}_p \cup \{z\}$. Since every $(k + l, k + l)$ matrix is in this case a linear combination of matrices 4.7.1, the form of 4.7.1 and 4.4.2 show that matrices $A(a_1', \cdots, a_p')$, where $\{a_1', \cdots, a_p'\} = \{1, 2, \cdots, p\}$, produce an irreducible representation of $\mathfrak{S}_p$.

5. Sufficient conditions for a representation

In this section, we will show that conditions 4.4.6, 4.6.2, and 4.3.5 are sufficient for the mapping defined by 4.7.1, along with $M(z) = 0$, to be a representation of $\mathfrak{B}_p \cup \{z\}$.

5.1 We suppose that we have (k, k) matrices $A(c_1, \cdots, c_p)$ defined for all integers $c_1, \cdots, c_p$ between 1 and p. We suppose that we have (k, l) matrices

$B(s_1, \cdots, s_p)$ defined for all $s_1, \cdots, s_p$. We suppose that we have (l, k) matrices $C(b_1, \cdots, b_p)$ defined for all monotone strictly increasing sequences $b_1, \cdots, b_p$ of integers between 1 and n. We will show that the mapping M of $\mathfrak{B}_p \cup \{z\}$ defined by

5.1.1
$$M\begin{pmatrix} s_1 & s_2 & \cdots & s_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}$$
$$= \begin{pmatrix} \delta_{s_p^*,p} A(a_1', \cdots, a_p') & A(a_1', \cdots, a_p')B(s_1, \cdots, s_p) \\ \delta_{s_p^*,p} C(a_{\rho_a(1)}, \cdots, a_{\rho_a(p)})A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p)) & C(a_{\rho_a(1)}, \cdots, a_{\rho_a(p)})A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p))B(s_1, \cdots, s_p) \end{pmatrix}$$
$$M(z) = 0,$$

is a representation of $\mathfrak{B}_p \cup \{z\}$ provided that the following conditions are satisfied. If φ and ψ are 1-to-1 mappings of $\{1, 2, \cdots, p\}$ onto itself, then

5.1.2 $$A(\varphi(1), \cdots, \varphi(p))A(\psi(1), \cdots, \psi(p)) = A(\varphi(\psi(1)), \cdots, \varphi(\psi(p)));$$

A is not identically zero; and if there are any repetitions among the numbers $c_1, \cdots, c_p$, then

5.1.3 $$A(c_1, \cdots, c_p) = 0.$$

From 5.1.2 and 5.1.3, one can easily infer the equality

5.1.4 $$A(e_1, \cdots, e_p)A(f_1, \cdots, f_p) = A(e_{f_1}, e_{f_2}, \cdots, e_{f_p}),$$

which is valid for all allowable values of $e_1, \cdots, e_p$ and $f_1, \cdots, f_p$.

For $s_1, \cdots, s_p$ and $b_1, \cdots, b_p$, let the matrix function $\gamma_{b_1, \cdots, b_p}^{s_1, \cdots, s_p}$ be defined by

5.1.5 $$\gamma_{b_1, \cdots, b_p}^{s_1, \cdots, s_p} = -\delta_{s_p^*,p} A(b_1', \cdots, b_p') + A(\sigma_s(b_1), \cdots, \sigma_s(b_p)).$$

Then the matrices B and C are to satisfy the condition

5.1.6 $$B(s_1, \cdots, s_p)C(b_1, \cdots, b_1) = \gamma_{b_1, \cdots, b_p}^{s_1, \cdots, s_p}$$

for all $s_1, \cdots, s_p$ and $b_1, \cdots, b_p$, as well as

5.1.7 $$C(1, 2, \cdots, p) = 0$$

and

5.1.8 $$B(\{1\}, \{2\}, \cdots, \{p-1\}, \{p, \cdots, n\}) = 0.$$

The sufficiency proof that we wish to give will be simplified by being broken up into a series of steps.

5.2 LEMMA. *Let $a_1, \cdots, a_p$ and $s_1, \cdots, s_p$ be given. Let $b_i = a_{\rho_a(i)}$, where ρ_a is defined as in* 4.5 $(i = 1, 2, \cdots, p)$. *Let σ_s be as in* 4.6. *Then*

5.2.1 $$\delta_{s_p^*,p} A(a_1', \cdots, a_p') + \gamma_b^s A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p)) = A(\sigma_s(a_1), \cdots, \sigma_s(a_p)).$$

Proof. Multiply both sides of 5.1.5 on the right by $A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p))$ and apply 5.1.4.

5.3 LEMMA. *Let* $a_1, \cdots, a_p, c_1, \cdots, c_p$, *and* $s_1, \cdots, s_p$ *be given. Suppose that the numbers* $\sigma_s(a_i)$ *are all distinct. Write* $d_i = c_{\sigma_s(a_i)}$ $(i = 1, 2, \cdots, p)$. *Then we have*

5.3.1 $$d_{\rho_d(i)} = c_{\rho_c(i)} \qquad (i = 1, \cdots, p)$$

and

5.3.2 $$\rho_c^{-1}(\sigma_s(a_i)) = \rho_d^{-1}(i) \qquad (i = 1, \cdots, p).$$

Proof. Equality 5.3.1 follows from the definitions of d_i and ρ. The equality

5.3.3 $$\rho_c(i) = \sigma_s(a_{\rho_d(i)})$$

follows at once from 5.3.1 and the definition of d_i. Equality 5.3.2 becomes obvious upon replacing i by $\rho_d^{-1}(i)$ in 5.3.3.

5.4 *First step.* From 5.1.1, and using 5.1.7, 5.1.8, and 5.1.2, we find that

5.4.1 $$M\begin{pmatrix} s_1 & \cdots & s_p \\ 1 & \cdots & p \end{pmatrix} M\begin{pmatrix} \{1\} & \cdots & \{p-1\} & \{p, \cdots, n\} \\ a_1 & \cdots & a_{p-1} & a_p \end{pmatrix} = \begin{pmatrix} \delta_{s_p^*, p} I & B(s_1, \cdots, s_p) \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} A(a_1', \cdots, a_p') & 0 \\ C(a_{\rho_a(1)}, \cdots, a_{\rho_a(p)}) A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p)) & 0 \end{pmatrix}.$$

Multiply the two matrices on the right side of 5.4.1; apply 5.1.6; then apply 5.2.1. This gives

5.4.2 $$M\begin{pmatrix} s_1 & \cdots & s_p \\ 1 & \cdots & p \end{pmatrix} M\begin{pmatrix} \{1\} & \cdots & \{p-1\} & \{p, \cdots, n\} \\ a_1 & \cdots & a_{p-1} & a_p \end{pmatrix} = \begin{pmatrix} A(\sigma_s(a_1), \cdots, \sigma_s(a_p)) & 0 \\ 0 & 0 \end{pmatrix}.$$

5.5 *Second step.* As in 5.4, it follows from 5.1.1 that

5.5.1 $$M\begin{pmatrix} s_1 & \cdots & s_p \\ c_1 & \cdots & c_p \end{pmatrix} = \begin{pmatrix} A(c_1', \cdots c_p') & 0 \\ C(c_{\rho_c(1)}, \cdots, c_{\rho_c(p)}) A(\rho_c^{-1}(1), \cdots, \rho_c^{-1}(p)) & 0 \end{pmatrix} M\begin{pmatrix} s_1 & \cdots & s_p \\ 1 & \cdots & p \end{pmatrix}.$$

Multiply both sides of 5.5.1 on the right by

$$M\begin{pmatrix}\{1\} & \cdots & \{p-1\} & \{p, \cdots, n\}\\ a_1 & \cdots & a_{p-1} & a_p\end{pmatrix},$$

use 5.4.2, multiply the resulting matrices, and use 5.1.4. This yields

5.5.2
$$M\begin{pmatrix}s_1 & \cdots & s_p\\ c_1 & \cdots & c_p\end{pmatrix} M\begin{pmatrix}\{1\} & \cdots & \{p-1\} & \{p, \cdots, n\}\\ a_1 & \cdots & a_{p-1} & a_p\end{pmatrix} = \begin{pmatrix} A(c'_{\sigma_s(a_1)}, \cdots, c'_{\sigma_s(a_p)}) & 0\\ C(c_{\rho_c(1)}, \cdots, c_{\rho_c(p)})A(\rho_c^{-1}(\sigma_s(a_1)), \cdots, \rho_c^{-1}(\sigma_s(a_p))) & 0\end{pmatrix}.$$

If there are any repetitions among the $\sigma_s(a_i)$, then the right side of 5.5.2 is zero. If not, we can apply Lemma 5.3 and find, in the notation of Lemma 5.3, that

5.5.3
$$M\begin{pmatrix}s_1 & \cdots & s_p\\ c_1 & \cdots & c_p\end{pmatrix} M\begin{pmatrix}\{1\} & \cdots & \{p-1\}\{p, \cdots, n\}\\ a_1 & \cdots & a_{p-1} & a_p\end{pmatrix} = \begin{cases}\begin{pmatrix} A(d'_1, \cdots, d'_p) & 0\\ C(d_{\rho_d(1)}, \cdots, d_{\rho_d(p)})A(\rho_d^{-1}(1), \cdots, \rho_d^{-1}(p)) & 0\end{pmatrix} & \text{if the } \sigma_s(a_i) \text{ are all distinct,}\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that

5.5.4
$$\begin{pmatrix}s_1 & \cdots & s_p\\ c_1 & \cdots & c_p\end{pmatrix}\begin{pmatrix}\{1\} & \cdots & \{p-1\}\{p, \cdots, n\}\\ a_1 & \cdots & a_{p-1} & a_p\end{pmatrix} = \begin{cases}\begin{pmatrix}\{1\} & \cdots & \{p-1\}\{p, \cdots, n\}\\ d_1 & \cdots & d_{p-1} & d_p\end{pmatrix} & \text{if the } \sigma_s(a_i) \text{ are all distinct,}\\ z & \text{otherwise.}\end{cases}$$

Equalities 5.5.3 and 5.5.4, together with 5.1.1 and 5.1.8, show that

5.5.5
$$M\begin{pmatrix}s_1 & \cdots & s_p\\ c_1 & \cdots & c_p\end{pmatrix} M\begin{pmatrix}\{1\} & \cdots & \{p-1\}\{p, \cdots, n\}\\ a_1 & \cdots & a_{p-1} & a_p\end{pmatrix} = M\left(\begin{pmatrix}s_1 & \cdots & s_p\\ c_1 & \cdots & c_p\end{pmatrix}\begin{pmatrix}\{1\} & \cdots & \{p-1\}\{p, \cdots, n\}\\ a_1 & \cdots & a_{p-1} & a_p\end{pmatrix}\right).$$

5.6 *Third step.* From 5.1.1 and direct multiplication of matrices, we find that

5.6.1
$$M\begin{pmatrix} t_1 & \cdots & t_p \\ a_1 & \cdots & a_p \end{pmatrix} = \begin{pmatrix} A(a_1', \cdots, a_p') & 0 \\ C(a_{\rho_a(1)}, \cdots, a_{\rho_a(p)})A(\rho_a^{-1}(1), \cdots, \rho_a^{-1}(p) & 0 \end{pmatrix} \cdot \begin{pmatrix} \delta_{t_p^*,p} I & B(t_1, \cdots, t_p) \\ 0 & 0 \end{pmatrix}.$$

The first matrix on the right side of 5.6.1 is clearly equal to

$$M\begin{pmatrix} \{1\} & \cdots & \{p-1\} & \{p, \cdots, n\} \\ a_1 & \cdots & a_{p-1} & a_p \end{pmatrix}.$$

We therefore have

$$M\begin{pmatrix} s_1 & \cdots & s_p \\ c_1 & \cdots & c_p \end{pmatrix} M\begin{pmatrix} t_1 & \cdots & t_p \\ a_1 & \cdots & a_p \end{pmatrix}$$

5.6.2
$$= M\begin{pmatrix} s_1 & \cdots & s_p \\ c_1 & \cdots & c_p \end{pmatrix} M\begin{pmatrix} \{1\} & \cdots & \{p-1\} & \{p, \cdots, n\} \\ a_1 & \cdots & a_{p-1} & a_p \end{pmatrix} \cdot \begin{pmatrix} \delta_{t_p^*,p} I & B(t_1, \cdots, t_p) \\ 0 & 0 \end{pmatrix}.$$

Formula 5.5.3 now shows that the right side of 5.6.2 is equal to

5.6.3
$$\begin{pmatrix} \delta_{t_p^*,p} A(d_1', \cdots, d_p') & A(d_1', \cdots, d_p')B(t_1, \cdots, t_p) \\ \delta_{t_p^*,p} C(d_{\rho_d(1)}, \cdots, d_{\rho_d(p)})A(\rho_d^{-1}(1), \cdots, \rho_d^{-1}(p)) & C(d_{\rho_d(1)}, \cdots, d_{\rho_d(p)})A(\rho_d^{-1}(1), \cdots, \rho_d^{-1}(p))B(t_1, \cdots, t_p) \end{pmatrix}$$

if all the $\sigma_s(a_i)$ are distinct and is zero otherwise. We also have

5.6.4
$$\begin{pmatrix} s_1 & \cdots & s_p \\ c_1 & \cdots & c_p \end{pmatrix} \begin{pmatrix} t_1 & \cdots & t_p \\ a_1 & \cdots & a_p \end{pmatrix} = \begin{cases} \begin{pmatrix} t_1 & \cdots & t_p \\ d_1 & \cdots & d_p \end{pmatrix} & \text{if all } \sigma_s(a_i) \text{ are distinct,} \\ z & \text{otherwise.} \end{cases}$$

Formula 5.1.1 shows that 5.6.3 is equal to

$$M\begin{pmatrix} t_1 & \cdots & t_p \\ d_1 & \cdots & d_p \end{pmatrix}.$$

Therefore 5.6.4 and 5.6.2 imply that M is a representation of $\mathfrak{B}_p \cup \{z\}$.

We now summarize the results of this section.

5.7 THEOREM. *Let* A, B, C *be matrix functions as described in* 5.1 *that satisfy conditions* 5.1.2, 5.1.3, 5.1.6, 5.1.7, *and* 5.1.8. *Then the mapping* M *defined in* 5.1.1 *is a representation of* $\mathfrak{B}_p \cup \{z\}$.

6. Construction of certain representations

In this section, we will exhibit a class of representations of $\mathfrak{B}_p \cup \{z\}$. These representations are in general reducible. They will be used in §7 to find *all* of the irreducible representations of $\mathfrak{B}_p \cup \{z\}$. Throughout this section, we suppose that we have a matrix function A satisfying the conditions of Theorem 5.7. We will obtain matrix functions B and C satisfying the conditions of Theorem 5.7.

6.1 In order to write condition 5.1.6 in compact form, it is convenient to order all of the sequences $b_1, \cdots, b_p$ and all of the families of sets $s_1, \cdots, s_p$. Let there be $u + 1$ families of sets $s_1, \cdots, s_p$ and $v + 1$ sequences $b_1, \cdots, b_p$. Let $\{1\}, \{2\}, \cdots, \{p-1\}, \{p, \cdots, n\}$ correspond to the index 0, and order all remaining $s_1, \cdots, s_p$ in any way at all in a sequence with indices from 1 to u. Write $B_j = B(s_1, \cdots, s_p)$ if $s_1, \cdots, s_p$ has index j $(0 \leqq j \leqq u)$. Similarly, let the sequence $1, \cdots, p$ correspond to the index 0, and order all remaining sequences $b_1, \cdots, b_p$ in any way at all in a sequence with indices from 1 to v. Write $C_i = C(b_1, \cdots, b_p)$ if $b_1, \cdots, b_p$ has index i $(0 \leqq i \leqq v)$. Finally, write γ_i^j for $\gamma_{b_1, \cdots, b_p}^{s_1, \cdots, s_p}$ if $s_1, \cdots, s_p$ has index j and $b_1, \cdots, b_p$ has index i $(0 \leqq j \leqq u, 0 \leqq i \leqq v)$. Condition 5.1.6 in this notation is

6.1.1 $$B_j C_i = \gamma_i^j \qquad (0 \leqq j \leqq u, 0 \leqq i \leqq v).$$

6.2 We first prove

6.2.1 $$\gamma_0^j = 0, \qquad \gamma_i^0 = 0 \qquad (0 \leqq j \leqq u, 0 \leqq i \leqq v).$$

If $j = 0$, then clearly $\sigma_s(b_h) = b_h'$ $(1 \leqq h \leqq p)$. Formula 5.1.5 shows at once that $\gamma_i^0 = 0$. If $i = 0$, then $b_h = b_h' = h' = h$ $(1 \leqq h \leqq p)$. Then if $s_p^* = p$, it is clear that $s_h^* = h$ $(1 \leqq h < p)$, and hence $\sigma_s(b_h) = h$ $(1 \leqq h \leqq p)$. It is clear from 5.1.5 that $\gamma_0^j = 0$ in this case. If $s_p^* \neq p$, then $\delta_{s_p^*, p} = 0$ and $A(\sigma_s(b_1), \cdots, \sigma_s(b_p)) = 0$ because there is necessarily a repetition among the numbers $\sigma_s(b_1), \cdots, \sigma_s(b_p)$.

6.3 We now define $B_0 = 0$ and $C_0 = 0$. (This choice is of course dictated by 5.1.8 and 5.1.7.) Equalities 6.2.1 show that condition 6.1.1 is satisfied if $i = 0$ or $j = 0$. The matrices $B_1, \cdots, B_u$ and $C_1, \cdots, C_v$ are now to satisfy the condition

6.3.1 $$\begin{pmatrix} B_1C_1 & \cdots & B_1C_v \\ \cdots & \cdots & \cdots \\ B_uC_1 & \cdots & B_uC_v \end{pmatrix} = \begin{pmatrix} \gamma_1^1 & \cdots & \gamma_v^1 \\ \cdots & \cdots & \cdots \\ \gamma_1^u & \cdots & \gamma_v^u \end{pmatrix}.$$

We write Γ for the matrix on the right side of 6.3.1. It is a (ku, kv) matrix.

6.4 Let r be the rank of Γ. Let α and β be any positive integers greater than or equal to r. (Note that r is positive.) Let $J(\alpha, \beta)$ be the (α, β) matrix

6.4.1
$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & 1_{(r)} & \\ & & 0 & & 0 \end{pmatrix},$$

having 1's in the first r places of the main diagonal and 0's elsewhere. We write $J(ku, kv)$ as J, $J(ku, r)$ as J_1, and $J(r, kv)$ as J_2. Obviously

6.4.2
$$J = J_1 J_2 .$$

It is a familiar fact that there exist a nonsingular (ku, ku) matrix P and a nonsingular (kv, kv) matrix Q such that

6.4.3
$$P\Gamma Q = J.$$

If we define the (k, r) matrices B_j by

6.4.4
$$\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_u \end{pmatrix} = P^{-1}J_1$$

and the (r, k) matrices C_i by

6.4.5
$$(C_1 C_2 \cdots C_v) = J_2 Q^{-1},$$

we see that

6.4.6
$$\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_u \end{pmatrix} (C_1 C_2 \cdots C_v) = P^{-1}J_1 J_2 Q^{-1} = P^{-1}JQ^{-1} = \Gamma.$$

Condition 6.3.1 is then obviously satisfied. By Theorem 5.7, we have obtained a representation of $\mathfrak{B}_p \cup \{z\}$ for which $l = r$.

6.5 For use in §7, we need two facts. Let Y be an arbitrary (k, r) matrix. Then there are (k, k) matrices $M_1, \cdots, M_u$ such that $Y = \sum_{j=1}^{u} M_j B_j$. To see this, we note that

6.5.1
$$YJ(r, ku)P \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_u \end{pmatrix} = Y,$$

and that the left side of 6.5.1 has the form $\sum_{j=1}^{u} M_j B_j$. Similarly, let Z be an arbitrary (r, k) matrix. Then we have

6.5.2
$$(C_1 C_2 \cdots C_v) QJ(kv, r)Z = Z,$$

and it follows that every (r, k) matrix can be written in the form $\sum_{i=1}^{v} C_i N_i$, where the N_i are (k, k) matrices.

7. The irreducible representations

Condition 5.1.2 implies that the matrices A appearing in 5.1.1 yield a representation of $\mathfrak{S}_p$. We will establish in this section a 1-to-1 correspondence between the irreducible representations of $\mathfrak{S}_p$ and those representations 5.1.1 of $\mathfrak{B}_p \cup \{z\}$ that are irreducible.

7.1 A glance at 5.1.1 shows that if the representation M of $\mathfrak{B}_p \cup \{z\}$ is irreducible, then the representation A of $\mathfrak{S}_p$ must be irreducible. Conversely, suppose that A is an irreducible representation of $\mathfrak{S}_p$. From 5.1.1, we have

7.1.1
$$M\begin{pmatrix}\{1\} & \cdots & \{p-1\}\{p, \cdots, n\} \\ 1 & \cdots & p-1 \quad\quad p\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}.$$

If X is any (k, k) matrix, then X can be written as a linear combination $\sum \beta_{(a_1, \cdots, a_p)} A(a_1, \cdots, a_p)$. Then

7.1.2
$$\begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix} \sum \beta_{(a_1, \cdots, a_p)} M\begin{pmatrix}\{1\} & \cdots & \{p-1\}\{p, \cdots, n\} \\ a_1 & \cdots & a_{p-1} \quad\quad a_p\end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & 0\end{pmatrix}.$$

Since M is a representation, the left side of 7.1.2 is a linear combination of matrices 5.1.1.

Next, consider an arbitrary $b_1, \cdots, b_p$, and let φ be a 1-to-1 mapping of $\{1, \cdots, p\}$ onto itself. Then 5.1.1 shows that

7.1.3
$$M\begin{pmatrix}\{1\} & \cdots & \{p-1\}\{p, \cdots, n\} \\ b_{\varphi(1)} & \cdots & b_{\varphi(p-1)} \quad\quad b_{\varphi(p)}\end{pmatrix} = \begin{pmatrix} H & 0 \\ C(b_1, \cdots, b_p)A(\varphi(1), \cdots, \varphi(p)) & 0\end{pmatrix},$$

where H is some (k, k) matrix. Since A is irreducible, we can, for every (k, k) matrix N, find a linear combination of matrices 7.1.3 that has the form

7.1.4
$$\begin{pmatrix} H' & 0 \\ C(b_1, \cdots, b_p)N & 0\end{pmatrix}.$$

Then 6.5 shows that for an arbitrary (r, k) matrix Z, there is a linear combination of matrices 5.1.1 that has the form

7.1.5
$$\begin{pmatrix} H'' & 0 \\ Z & 0\end{pmatrix}.$$

Next consider an arbitrary $s_1, \cdots, s_p$, and let φ be as above. Then 5.1.1 and 5.1.7 show that

7.1.6 $$M\begin{pmatrix} s_1 & \cdots & s_p \\ \varphi(1) & \cdots & \varphi(p)\end{pmatrix} = \begin{pmatrix} H''' & A(\varphi(1), \cdots, \varphi(p))B(s_1, \cdots, s_p) \\ 0 & 0 \end{pmatrix}.$$

As before, we apply 6.5 and see that, for an arbitrary (k, r) matrix Y, there is a linear combination of matrices 5.1.1 having the form

7.1.7 $$\begin{pmatrix} H'''' & Y \\ 0 & 0 \end{pmatrix}.$$

From 7.1.2, 7.1.5, and 7.1.7, it is clear that linear combinations of the matrices 5.1.1 give arbitrary matrices

7.1.8 $$\begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix}.$$

Let $E_{ij}(\alpha, \beta)$ be an (α, β) matrix with 1 in the i^{th} row and j^{th} column and 0's elsewhere. Then

7.1.9 $$\begin{pmatrix} 0 & 0 \\ E_{i1}(r, k) & 0 \end{pmatrix}\begin{pmatrix} 0 & E_{1j}(k, r) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & E_{ij}(r, r) \end{pmatrix}.$$

From 7.1.8, 7.1.9, and the fact that M is a representation, we now see that M is irreducible.

7.2 We next show that equivalent irreducible representations of $\mathfrak{S}_p$ produce equivalent representations of $\mathfrak{B}_p \cup \{z\}$. If A and $\bar{A}$ are equivalent irreducible representations of $\mathfrak{S}_p$ by (k, k) matrices, then there is a nonsingular (k, k) matrix R such that $\bar{A}(\varphi(1), \cdots, \varphi(p)) = RA(\varphi(1), \cdots, \varphi(p))R^{-1}$ for all φ as in 5.1.2. Let M and $\bar{M}$ be the irreducible representations of $\mathfrak{B}_p \cup \{z\}$ obtained from A and $\bar{A}$ respectively by applying 5.1.1 and 5.1.3. Writing A_{ij} for the entry in the i^{th} row and j^{th} column of A, and similarly for $\bar{A}$, M, and $\bar{M}$, we now have

7.2.1 $$\bar{A}_{11}(\varphi(1), \cdots, \varphi(p)) = \textstyle\sum_{i,j} \tau_{ij} A_{ij}(\varphi(1), \cdots, \varphi(p))$$

for all 1-to-1 mappings φ of $\{1, \cdots, p\}$ onto itself. Condition 5.1.3 shows that

7.2.2 $$\bar{A}_{11}(c_1, \cdots, c_p) = \textstyle\sum_{i,j} \tau_{ij} A_{ij}(c_1, \cdots, c_p)$$

for all integers $c_1, \cdots, c_p$ lying between 1 and p. Now 5.1.1 and 7.2.2 show that

7.2.3 $$\begin{aligned}\bar{M}_{11}\begin{pmatrix} s_1 & \cdots & s_p \\ a_1 & \cdots & a_p\end{pmatrix} &= \delta_{s_p^*, p} \textstyle\sum_{i,j} \tau_{ij} A_{ij}(a_1', \cdots, a_p') \\ &= \textstyle\sum_{i,j} \tau_{ij} M_{ij}\begin{pmatrix} s_1 & \cdots & s_p \\ a_1 & \cdots & a_p\end{pmatrix}.\end{aligned}$$

Also

7.2.4 $$\bar{M}_{11}(z) = 0 = \sum_{i,j} \tau_{ij} M_{ij}(z).$$

Consequently the function $\bar{M}_{11}$ is a linear combination of the functions M_{ij} $(1 \leqq i \leqq k,\ 1 \leqq j \leqq k)$. Theorem 5.18 of [3] implies that the representations M and $\bar{M}$ are equivalent.

7.3 We will now show that inequivalent irreducible representations of $\mathfrak{S}_p$ produce inequivalent representations of $\mathfrak{B}_p \cup \{z\}$. Suppose that A and $\bar{A}$ are irreducible representations of $\mathfrak{S}_p$ by (k, k) and $(\bar{k}, \bar{k})$ matrices, respectively, and that M and $\bar{M}$ are the corresponding representations of $\mathfrak{B}_p \cup \{z\}$ obtained by 5.1.3 and 5.1.1. We may obviously suppose that $\bar{k} \geqq k$. Let I_s denote the (s, s) identity matrix $(s = 1, 2, 3, \cdots)$. Now suppose that M and $\bar{M}$ are equivalent. There exist $(k + r, k + r)$ matrices

$$\begin{pmatrix} S & T \\ U & V \end{pmatrix} \text{ and } \begin{pmatrix} S' & T' \\ U' & V' \end{pmatrix}$$

(written in (k, k), (k, r), (r, k) and (r, r) blocks) that are inverses of each other and have the property that

7.3.1 $$\begin{pmatrix} S & T \\ U & V \end{pmatrix} M \begin{pmatrix} s_1 & \cdots & s_p \\ a_1 & \cdots & a_p \end{pmatrix} \begin{pmatrix} S' & T' \\ U' & V' \end{pmatrix} = \bar{M} \begin{pmatrix} s_1 & \cdots & s_p \\ a_1 & \cdots & a_p \end{pmatrix}$$

for all

$$\begin{pmatrix} s_1 & \cdots & s_p \\ a_1 & \cdots & a_p \end{pmatrix} \in \mathfrak{B}_p .$$

Putting

$$\begin{pmatrix} s_1 & \cdots & s_p \\ a_1 & \cdots & a_p \end{pmatrix} = \begin{pmatrix} \{1\} & \cdots & \{p-1\} & \{p, \cdots, n\} \\ 1 & \cdots & p-1 & p \end{pmatrix}$$

in 7.3.1, and using 5.1.1, we have

7.3.2 $$\begin{pmatrix} S & T \\ U & V \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S' & T' \\ U' & V' \end{pmatrix} = \begin{pmatrix} I_{\bar{k}} & 0 \\ 0 & 0 \end{pmatrix}.$$

We also have

7.3.3 $$\begin{pmatrix} S & T \\ U & V \end{pmatrix} \begin{pmatrix} S' & T' \\ U' & V' \end{pmatrix} = I_{k+r} .$$

From 7.3.2, we have

7.3.4 $$SS' = I_k , \qquad US' = 0, \qquad ST' = 0.$$

Hence

7.3.5 $$U = 0, \qquad T' = 0.$$

From 7.3.5 and 7.3.3, we infer in turn

7.3.6 $$VV' = I_r, \qquad U' = 0, \qquad T = 0.$$

The left side of 7.3.2 is therefore equal to

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

and this implies that $k = \bar{k}$. Let φ be a 1-to-1 mapping of $\{1, \cdots, p\}$ onto itself. Consider 7.3.1 for $s_1, \cdots, s_p = \{1\}, \cdots, \{p-1\}, \{p, \cdots, n\}$ and $a_1, \cdots, a_p = \varphi(1), \cdots, \varphi(p)$. We obtain

7.3.7 $$\begin{pmatrix} S & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} A(\varphi(1), \cdots, \varphi(p)) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} = \begin{pmatrix} \bar{A}(\varphi(1), \cdots, \varphi(p)) & 0 \\ 0 & 0 \end{pmatrix},$$

so that

7.3.8 $$SA(\varphi(1), \cdots, \varphi(p))S^{-1} = \bar{A}(\varphi(1), \cdots, \varphi(p)).$$

Hence the representation A of $\mathfrak{S}_p$ is equivalent to the representation $\bar{A}$ of $\mathfrak{S}_p$. We have therefore proved the following.

7.4 THEOREM. *Let the representation A of $\mathfrak{S}_p$, as described in* 5.1.2, *run through a complete set of inequivalent irreducible representations of $\mathfrak{S}_p$. The corresponding representations M of $\mathfrak{B}_p \cup \{z\}$ defined by* 5.1.3 *and* 5.1.1 *are all irreducible and inequivalent. Furthermore, every irreducible representation of $\mathfrak{B}_p \cup \{z\}$ is obtained in this way.*

7.5 Theorems 7.4 and 3.7 show that we have a method for obtaining all irreducible representations of $\mathfrak{T}_n$. To write down any of these representations, begin with an irreducible representation of $\mathfrak{S}_p$. These representations are well known, and a method for their construction can be found, for example, in Ch. IV of [1]. The construction in §6 gives the matrices B_i and C_j. Formula 5.1.1 gives the associated irreducible representation of $\mathfrak{B}_p \cup \{z\}$. Theorem 3.5 shows how to extend this representation over all of $\mathfrak{T}_n$. A numerical example is given in 8.6.

8. Special results

We here give the special forms of the irreducible representations of $\mathfrak{T}_n$ that correspond to certain special values of p and A. We also work out some numerical examples.

8.1 *The case $p = 1$.* The semigroup $\mathfrak{B}_1$ has the simple multiplication rule $fg = f$. The only irreducible representation of $\mathfrak{B}_1$ is the 1-dimensional identity representation. By Theorem 3.5, the only irreducible representation of $\mathfrak{T}_n$ not zero on $\mathfrak{B}_1$ is the 1-dimensional identity representation. This also fits into the general theory of §§5–7, since $u + 1 = 1$ if $p = 1$, and the matrix Γ does not appear at all. Note also that z is an adjoined zero in the semigroup $\mathfrak{B}_1 \cup \{z\}$.

8.2 *The case* $p = n$. It is clear that an irreducible representation of $\mathfrak{T}_n$ that is zero on $\bigcup_{j=1}^{n-1} \mathfrak{B}_j$ must be an irreducible representation on the group $\mathfrak{B}_n = \mathfrak{S}_n$. Conversely, every irreducible representation of $\mathfrak{B}_n$ can be extended to an irreducible representation of $\mathfrak{T}_n$ by being defined as 0 on $\bigcup_{j=1}^{n-1} \mathfrak{B}_j$. Thus we know all irreducible representations of $\mathfrak{T}_n$ that vanish on $\bigcup_{j=1}^{n-1} \mathfrak{B}_j$ in terms of the irreducible representations of the symmetric group $\mathfrak{B}_n$. This fits into the general theory of §§5–7: for $p = n$, we have $u + 1 = v + 1 = 1$, and the matrix Γ does not appear.

8.3 *The case in which A is the identity.* If the representation A of $\mathfrak{S}_p$ appearing in 5.1.2 is the 1-dimensional identity representation, then the corresponding irreducible representation of $\mathfrak{T}_n$ can be written in a simple form. Suppose that $1 < p < n$. Consider the semigroup algebra $\mathfrak{L}_1(\mathfrak{B}_p \cup \{z\})$ as defined in [3]. We may think of $\mathfrak{L}_1(\mathfrak{B}_p \cup \{z\})$ as consisting of all formal complex linear combinations $\sum \alpha_f f$, the sum being taken over all $f \in \mathfrak{B}_p \cup \{z\}$, with $(\sum_f \alpha_f f)(\sum_g \beta_g g) = \sum_f \sum_g \alpha_f \beta_g fg$. For every sequence $b_1, \cdots, b_p$ (recall that $1 \leqq b_1 < \cdots < b_p \leqq n$), let $F_{b_1 \cdots b_p}$ be the element of $\mathfrak{L}_1(\mathfrak{B}_p \cup \{z\})$

$$8.3.1 \quad F_{b_1 \cdots b_p} = \sum_{\varphi \in \mathfrak{S}_p} \begin{pmatrix} \{1\} & \{2\} & \cdots & \{p-1\} & \{p, \cdots, n\} \\ b_{\varphi(1)} & b_{\varphi(2)} & \cdots & b_{\varphi(p-1)} & b_{\varphi(p)} \end{pmatrix} - p!z.$$

The elements $F_{b_1 \cdots b_p}$ are linearly independent, and span an $\binom{n}{p}$-dimensional subspace $\mathcal{S}$ of $\mathfrak{L}_1(\mathfrak{B}_p \cup \{z\})$. For every $f \in \mathfrak{T}_n$, let

$$8.3.2 \quad T_f F_{b_1 \cdots b_p} = \begin{cases} \sum_{\varphi \in \mathfrak{S}_p} f \cdot \begin{pmatrix} \{1\} & \cdots & \{p-1\} & \{p, \cdots, n\} \\ b_{\varphi(1)} & \cdots & b_{\varphi(p-1)} & b_{\varphi(p)} \end{pmatrix} - p!z & \text{if } f(b_1), \cdots, f(b_p) \text{ are all distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $T_f F_{b_1 \cdots b_p} = F_{c_1 \cdots c_p}$, where $c_1, \cdots, c_p$ is the sequence $f(b_1), \cdots, f(b_p)$ arranged in increasing order, if $f(b_1), \cdots, f(b_p)$ are all distinct. Extend the transformations T_f over $\mathcal{S}$ by linearity. It is easy to see that they form a representation of $\mathfrak{T}_n$ by linear transformations on $\mathcal{S}$. The set $\{T_f\}_{f \in \mathfrak{B}_p}$ of linear transformations can be shown to be irreducible on $\mathcal{S}$. Choose a new basis for $\mathcal{S}$:

$$\{F_{12 \cdots p}\} \cup \{F_{12 \cdots p-1x} - F_{12 \cdots p}\}_{p < x \leqq n} \cup \{F_{b_1 \cdots b_p}\}_{b_{p-1} > p-1}.$$

Consider the matrices $N(f)$ corresponding to the linear transformations T_f in this particular basis ($f \in \mathfrak{B}_p$). It is easy to see that the upper left corners of these matrices are the same as the upper left corners of the matrices 5.1.1 for A the identity. Hence the irreducible representation of $\mathfrak{B}_p \cup \{z\}$ defined by

$$f \to N(f) \quad \text{for} \quad f \in \mathfrak{B}_p,$$
$$z \to 0,$$

is equivalent to the representation 5.1.1 with A the identity. This follows from Theorem 5.18 of [3]. Formula 8.3.2 thus defines in one step the irreducible representation of $\mathfrak{T}_n$ corresponding to A the identity and any fixed value of p, $1 < p < n$. We see that the representation is by means of $\left(\binom{n}{p}, \binom{n}{p}\right)$ matrices. Furthermore it is easy to show that the rank of the matrix corresponding to T_f for $f \in \mathfrak{B}_{p+j}$ is $\binom{p+j}{p}$ $(j = 0, 1, \cdots, n - p)$.

8.4 *The case* $p = 2$. In view of 8.1, 8.2, and the general theory, we have on y one more irreducible representation of $\mathfrak{T}_n$ not vanishing on $\mathfrak{B}_2$: the representation 5.1.1 for $p = 2$ and A the alternating representation of $\mathfrak{S}_2$. Consider the semigroup algebra $\mathfrak{L}_1(\mathfrak{T}_n)$, and let $H_b \in \mathfrak{L}_1(\mathfrak{T}_n)$ be defined by

8.4.1 $$H_b = \begin{pmatrix} \{1, 2, \cdots, n\} \\ 1 \end{pmatrix} - \begin{pmatrix} \{1, 2, \cdots, n\} \\ b \end{pmatrix}, \qquad b = 2, 3, \cdots, n.$$

For every $f \in \mathfrak{T}_n$, let

8.4.2 $$U_f H_b = fH_b.$$

Clearly

8.4.3 $$U_f H_b = H_{f(b)} - H_{f(1)}.$$

Just as in 8.3, one can show that the U_f produce linear transformations (also written as U_f) on the linear subspace of $\mathfrak{L}_1(\mathfrak{T}_n)$ spanned by $H_2, \cdots, H_n$. These linear transformations yield an irreducible representation of $\mathfrak{T}_n$ which on $\mathfrak{B}_2 \cup \{z\}$ is equivalent to the representation 5.1.1 with $p = 2$ and A the alternating representation of $\mathfrak{S}_2$. Hence the matrices M of 5.1.1 are in this case $(n - 1, n - 1)$ matrices. It is not hard to see that the rank of the matrix corresponding to U_f is $p - 1$ for $f \in \mathfrak{B}_p$ $(p = 1, 2, \cdots, n)$.

8.5 *The case* $p = n - 1$. Carefully chosen transformations of the matrix Γ lead to the following results for $p = n - 1$. Let the degree k of the representation A of $\mathfrak{S}_{n-1}$ be greater than 1. Then the rank of Γ is $k(n - 1)$, and hence the degree of the corresponding representation of $\mathfrak{T}_n$ is kn. If $k = 1$ and A is the alternating representation of $\mathfrak{S}_{n-1}$, then the rank of Γ is $n - 2$. Thus the degree of the corresponding representation of $\mathfrak{T}_n$ is $n - 1$. If A is the identity representation of $\mathfrak{S}_{n-1}$, then the rank of Γ is $n - 1$, and the degree of the corresponding representation of $\mathfrak{T}_n$ is n. (This last follows also from 8.3.) The calculations are long, and we omit them.

8.6 As an example of the general theory, we consider the case $n = 4$, $p = 3$.

We order the s_1, s_2, s_3 and the b_1, b_2, b_3:

8.6.1
$$\begin{matrix} \{1\} & \{2\} & \{34\} \\ \{1\} & \{23\} & \{4\} \\ \{12\} & \{3\} & \{4\}, \\ \{1\} & \{24\} & \{3\} \\ \{14\} & \{2\} & \{3\} \\ \{13\} & \{2\} & \{4\} \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 4 \\ 2 & 3 & 4 \end{matrix}.$$

Using 5.1.5 and the definition of Γ in 6.3, we find

$$\Gamma = \begin{Bmatrix} I & I & 0 \\ 0 & I & I \\ -I & A(1,3,2) & 0 \\ -I & 0 & A(2,3,1) \\ I & 0 & A(2,1,3) \end{Bmatrix}.$$

where the A's form an irreducible (k, k) matrix representation of $\mathfrak{S}_3$ and I is the (k, k) identity matrix. We have used the equalities $A(1, 2, 3) = I$ and $A(c_1, c_2, c_3) = 0$ if there is a duplication among the c's.

Now if we take

$$P_1 = \begin{Bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ I & -I - A(1,3,2) & I & 0 & 0 \\ I & -I & 0 & I & 0 \\ A(2,1,3) & -A(2,1,3) - A(2,3,1) & A(2,1,3) & I & I \end{Bmatrix}$$

and

$$Q_1 = \begin{pmatrix} I & -I & -\tfrac{1}{2}I \\ 0 & I & \tfrac{1}{2}I \\ 0 & 0 & -\tfrac{1}{2}I \end{pmatrix},$$

we obtain

$$P_1\Gamma Q_1 = \begin{Bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tfrac{1}{2}(I + A(1,3,2)) \\ 0 & 0 & \tfrac{1}{2}(I - A(2,3,1)) \\ 0 & 0 & 0 \end{Bmatrix},$$

where we have used the equality $A(2, 1, 3)A(1, 3, 2) = A(2, 3, 1)$.

There are two nonequivalent irreducible representations of $\mathfrak{S}_3$ by (1, 1) matrices and one by (2, 2) matrices. Now $\mathfrak{S}_3$ is generated by the elements corresponding to $A(1, 3, 2)$ and $A(2, 1, 3)$, so we need list only these two matrices. We have the three cases

(i) $A(1, 3, 2) = A(2, 1, 3) = (1)$,

(ii) $A(1, 3, 2) = A(2, 1, 3) = (-1)$,

(iii) $A(1, 3, 2) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad A(2, 1, 3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

In case (i), we have $k = 1$ and

$$P_1 \Gamma Q_1 = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \qquad P = P_1, \qquad Q = Q_1, \qquad r = 3.$$

In case (ii), we have $k = 1$ and

$$P_1 \Gamma Q_1 = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \qquad P = P_1, \qquad Q = Q_1, \qquad r = 2.$$

In case (iii), we have $k = 2$ and

$$P_1 \Gamma Q_1 = \begin{Bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Bmatrix};$$

To bring this matrix to our standard form, we multiply on the left by

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}.$$

$$P_2 P_1 \Gamma Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P = P_2 P_1, \qquad Q = Q_1, \qquad r = 6.$$

In all three cases, we have

$$Q^{-1} = Q_1^{-1} = \begin{pmatrix} I & I & 0 \\ 0 & I & I \\ 0 & 0 & -2I \end{pmatrix},$$

$$P_1^{-1} = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ -I & I + A(1,3,2) & I & 0 & 0 \\ -I & I & 0 & I & 0 \\ I & -I & -A(2,1,3) & -I & I \end{pmatrix},$$

$$P_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In cases (i) and (ii), we have $P^{-1} = P_1^{-1}$, and in case (iii) we have $P^{-1} = P_1^{-1}P_2^{-1}$. The irreducible representations of $\mathfrak{B}_3 \cup \{z\}$ can be obtained from the matrices P^{-1} and Q^{-1}, and they can be then extended over $\mathfrak{T}_4$. We will not do this, but we will carry one case a little further.

In case (ii), we have $k = 1$, $r = 2$,

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix},$$

and

$$Q^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}.$$

From this we find

$$\begin{aligned} B_1 &= (1 \quad 0) \\ B_2 &= (0 \quad 1) \\ B_3 &= (-1 \quad 0), \\ B_4 &= (-1 \quad 1) \\ B_5 &= (1 \quad -1) \end{aligned} \qquad C_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

using the ordering 8.6.1. These matrices can be used in 5.1.1 to find the corresponding irreducible representations of $\mathfrak{B}_3 \cup \{z\}$. For example, we find

$$M\begin{pmatrix}\{1\} & \{23\} & \{4\} \\ 1 & 3 & 4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 \\ 0 & & \\ 0 & & C_2B_1 \end{pmatrix} = \begin{pmatrix}0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0\end{pmatrix},$$

$$M\begin{pmatrix}\{1\} & \{23\} & \{4\} \\ 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 \\ 0 & & \\ 0 & & C_3B_1 \end{pmatrix} = \begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix},$$

$$M\begin{pmatrix}\{12\} & \{3\} & \{4\} \\ 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 \\ 0 & & \\ 0 & & C_3B_2 \end{pmatrix} = \begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}.$$

We also have

$$M\begin{pmatrix}\{1\} & \{2\} & \{34\} \\ 1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix},$$

and hence

$$M\begin{pmatrix}\{1\} & \{2\} & \{34\} \\ 1 & 2 & 3\end{pmatrix} + M\begin{pmatrix}\{1\} & \{23\} & \{4\} \\ 1 & 3 & 4\end{pmatrix}$$
$$- M\begin{pmatrix}\{1\} & \{23\} & \{4\} \\ 2 & 3 & 4\end{pmatrix} + M\begin{pmatrix}\{12\} & \{3\} & \{4\} \\ 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix},$$

from which we can read off the values of the α_f for use in Theorem 3.5.

8.7 The matrix Γ has ku rows and kv columns. The number v is obviously

$$v = \binom{n}{p} - 1.$$

The number u is not as easy to find. We write $u = u(n, p) - 1$. Consider the set of all $s_1, \cdots, s_p$ counted by $u(n - 1, p)$. If we replace any s_i by $s_i \cup \{n\}$, we obtain an $s_1, \cdots, s_p$ counted by $u(n, p)$. We will also get an $s_1, \cdots, s_p$ counted by $u(n, p)$ if we take an $s_1, \cdots, s_{p-1}$ counted by $u(n - 1, p - 1)$ and change it to $s_1, \cdots, s_{p-1}, s_p$ with $s_p = \{n\}$. It is easy to see that there are no duplicates and that this enumeration is exhaustive. Thus we have

$$u(n, p) = pu(n - 1, p) + u(n - 1, p - 1), \qquad 2 \leqq p \leqq n - 1.$$

Since $u(n, 1) = u(n, n) = 1$, we obtain the following table.

n	p					
	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

It can be shown that

$$u(n, p) = \sum_{j=1}^{p} \frac{(-1)^{p-j} j^n}{j!\,(p - j)!}$$

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