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COMMON BOUNDED UNIVERSAL FUNCTIONS FOR COMPOSITION OPERATORS

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ABSTRACT. Let $\mathcal{A}$ be the set of automorphisms of the unit disk with 1 as attractive fixed point. We prove that there exists a *single* Blaschke product that is universal for *every* composition operator C_ϕ, $\phi \in \mathcal{A}$, acting on the unit ball of $H^\infty(\mathbb{D})$.

1. Introduction

This paper is devoted to the construction of common universal functions for some uncountable families of composition operators on the unit ball $\mathcal{B}$ of $H^\infty(\mathbb{D})$. If $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of the unit disk $\mathbb{D}$, the composition operator $C_\phi : f \mapsto f \circ \phi$ acts continuously on $\mathcal{B}$ (note that $\mathcal{B}$ will always be endowed with the topology of uniform convergence on compact sets). A function $f \in \mathcal{B}$ is said to be $\mathcal{B}$-*universal* for C_ϕ (or just *universal*, if no ambiguities arise) if $\mathcal{O}(f) = \{f \circ \phi^{[n]}; n \geq 0\}$ is dense in $\mathcal{B}$, where $\phi^{[n]} = \phi \circ \phi \circ \cdots \circ \phi$ denotes the n-th iterate of ϕ. The operator C_ϕ is $\mathcal{B}$-universal if it admits a $\mathcal{B}$-universal function, and this happens ([3]) if and only if ϕ is a hyperbolic or parabolic automorphism of the unit disk. In this case, the universal function can be chosen to be a Blaschke product. Our aim in this paper is to construct common universal Blaschke products for some uncountable families of composition operators C_ϕ acting on $\mathcal{B}$, the ϕ's being hyperbolic and parabolic automorphisms of $\mathbb{D}$.

Results on universal Blaschke products first appear in a paper by Heins [10]. A general theory of universal Blaschke products and their behaviour on the maximal ideal space of H^∞ was developed in [8] and [11]. Finally, these functions were the building blocks for studying B-universality for sequences of composition operators (C_{ϕ_n}) in [3].

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Our study of universal Blaschke products in the present paper is motivated by previous results on common hypercyclicity of [1], [4], and [5]. Indeed, the operators C_ϕ act boundedly on different spaces, such as the space $\mathcal{H}(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$, or the Hardy spaces $H^p(\mathbb{D})$, $1 \le p < +\infty$, and when ϕ is a hyperbolic or parabolic automorphism, C_ϕ is hypercyclic on $\mathcal{H}(\mathbb{D})$ (resp. $H^p(\mathbb{D})$), i.e., there exists a function $f \in \mathcal{H}(\mathbb{D})$ (resp. $f \in H^p(\mathbb{D})$) such that $\mathcal{O}(f)$ is dense in $\mathcal{H}(\mathbb{D})$ (resp. $H^p(\mathbb{D})$). It is then natural to ask about the existence of a function f which would be hypercyclic for *all* composition operators C_ϕ. Since each function in $H^p(\mathbb{D})$ has a radial limit almost everywhere on the unit circle $\mathbb{T}$ [7], such a common hypercyclic function cannot exist on $H^p(\mathbb{D})$: if $\mathcal{A}$ is a family of hyperbolic or parabolic automorphisms of $\mathbb{D}$, the fact that the family $(C_\phi)_{\phi\in\mathcal{A}}$ has a common hypercyclic vector necessarily implies that the subset B of $\mathbb{T}$ consisting of all the attractive fixed points of the automorphisms $\phi \in \mathcal{A}$ has Lebesgue measure zero. Hence, a natural family to consider is $(C_\phi)_{\phi\in\mathcal{A}_0}$, where $\mathcal{A}_0$ is the class of hyperbolic or parabolic automorphisms of $\mathbb{D}$ with 1 as attractive fixed point. Then this restricted family of composition operators acting on $H^p(\mathbb{D})$ admits a common hypercyclic vector ([4] or [5]).

We deal here with the same question, but our underlying space is now the unit ball $\mathcal{B}$ of $H^\infty(\mathbb{D})$. The main difficulty in this new setting lies in the fact that all the techniques of [1], [4], or [5] are “additive” and strongly use the linearity of the space, making it difficult to control the H^∞-norm of the functions which are constructed. We have to use “multiplicative” techniques instead to prove the following theorem, which is the main result of this paper.

THEOREM 1. *There exists a Blaschke product B which is universal for all composition operators C_ϕ associated to hyperbolic or parabolic automorphisms of $\mathbb{D}$ with 1 as attractive fixed point.*

The proof of this result uses an argument of Costakis and Sambarino [6]. The hyperbolic and parabolic cases will be treated separately in Sections 2 and 3, respectively, the hyperbolic case being as usual, easier than the parabolic one, since we have a better control of the rate of convergence of the iterates to the attractive fixed point.

2. The hyperbolic case

We first consider for $\lambda > 1$ the family of hyperbolic automorphisms

$$z \mapsto \frac{z + \frac{\lambda-1}{\lambda+1}}{1 + z\frac{\lambda-1}{\lambda+1}}$$

of $\mathbb{D}$ with 1 as attractive fixed point and -1 as repulsive fixed point. The action of such an automorphism is best understood when considered on the right half-plane $\mathbb{C}_+ = \{w \in \mathbb{C}; \mathrm{Re} w > 0\}$: if $\sigma : \mathbb{D} \to \mathbb{C}_+$ is the Cayley map defined by $\sigma(z) = \frac{1+z}{1-z}$, such an automorphism is conjugated via σ to a dilation

$\varphi_\lambda : w \mapsto \lambda w$, where $\lambda > 1$, We will denote by ϕ_λ the hyperbolic automorphism of $\mathbb{D}$ such that $\phi_\lambda = \sigma^{-1} \circ \varphi_\lambda \circ \sigma$. A general hyperbolic automorphism with 1 as attractive fixed point has the form $\phi_{\lambda,\beta} = \sigma^{-1} \circ \varphi_{\lambda,\beta} \circ \sigma$, where $\varphi_{\lambda,\beta}$ acts on $\mathbb{C}_+$ as $\varphi_{\lambda,\beta}(w) = \lambda(w - i\beta) + i\beta$, $\lambda > 1$, $\beta \in \mathbb{R}$. We first show that the parameters β play essentially no role in this problem.

LEMMA 2. *Let B be a Blaschke product which is universal for C_{ϕ_λ}, $\lambda > 1$. For any $\beta \in \mathbb{R}$, B is universal for $C_{\phi_{\lambda,\beta}}$.*

Proof. Let $f \in \mathcal{B}$ and let K be a compact subset of $\mathbb{D}$. For $z \in K$, we define

$$z_1(n) = \sigma^{-1}\big(\lambda^n\big(\sigma(z) - i\beta\big) + i\beta\big) = \phi_{\lambda,\beta}^{[n]}(z),$$
$$z_2(n) = \sigma^{-1}\big(\lambda^n\big(\sigma(z) - i\beta\big)\big) = \phi_\lambda^{[n]}\big(\sigma^{-1}\big(\sigma(z) - i\beta\big)\big).$$

It is easy to show that there exists a constant C_1 which depends only on K and β, such that

$$|z_1(n) - z_2(n)| \le \frac{C_1}{\lambda^{2n}}.$$

In fact, if $w_1(n) = \lambda^n(\sigma(z) - i\beta) + i\beta$ and $w_2(n) = \lambda^n(\sigma(z) - i\beta)$, then

$$|z_1(n) - z_2(n)| = \left| \int_{[w_1(n), w_2(n)]} \frac{2}{(1+w)^2}\, dw \right| \le |\beta| \max_{w \in [w_1(n), w_2(n)]} \frac{2}{|1+w|^2}$$
$$\le |\beta| \frac{2}{\lambda^{2n}[\min\{\mathrm{Re}\sigma(z) : z \in K\}]^2} \le \frac{C_1}{\lambda^{2n}}.$$

On the other hand, there is another constant C_2, depending only on K and β, such that

$$|z_1(n)| \le 1 - \frac{C_2}{\lambda^n} \quad \text{and} \quad |z_2(n)| \le 1 - \frac{C_2}{\lambda^n}.$$

This can be seen in the following way:

$$|1 - z_j(n)| = 1 - \left| \frac{w_j(n) - 1}{w_j(n) + 1} \right| = 2/|w_j(n) + 1| \ge \frac{C_2}{\lambda^n}.$$

Since B belongs to $H^\infty(\mathbb{D})$, Cauchy's inequalities show that $B(z_1(n)) - B(z_2(n))$ converges uniformly on K to 0. In fact,

$$|B(z_1(n)) - B(z_2(n))| \le \int_{[z_1(n), z_2(n)]} \frac{|B'(\xi)|(1 - |\xi|^2)}{1 - |\xi|^2} |d\xi|$$
$$\le |z_1(n) - z_2(n)| \frac{1}{\min\{1 - |z_1(n)|^2, 1 - |z_2(n)|^2\}}$$
$$\le \frac{C_3}{\lambda^n}.$$

On the other hand, since B is universal, there exists a sequence (n_k) such that $B \circ \phi_\lambda^{[n_k]}(\sigma^{-1}(\sigma - i\beta))$ converges uniformly to f on K (the map $z \mapsto \sigma^{-1}(\sigma(z) - i\beta)$ is an automorphism of $\mathbb{D}$). We conclude that $B \circ \phi_{\lambda,\beta}^{[n_k]}$ converges uniformly on K to f. □

In order to construct a common universal Blaschke product for all the C_{ϕ_λ}'s, we will decompose $]1,+\infty[$ as an increasing union of compact sub-intervals $[a_k,b_k]$. Following [6], we then decompose each interval $[a,b]$ as $[a,b]=\bigcup_{j=1}^{q-1}[\lambda_j,\lambda_{j+1}]$ where $\lambda_1=a$, $\lambda_2=\lambda_1+\frac{\delta}{2N}$, ..., $\lambda_{j+1}=\lambda_j+\frac{\delta}{(j+1)N}$ if $\lambda_j+\frac{\delta}{(j+1)N}\le b$ and $\lambda_{j+1}=b$ if $\lambda_j+\frac{\delta}{(j+1)N}>b$. Here, N is a positive integer which will be chosen very large in the sequel, and δ is a positive real number which will be chosen very small. The interval $[a,b]$ has been divided into q successive sub-intervals (q depending on δ and N, of course). The interest of such a decomposition of $[a,b]$ in our context is explained in the following lemma. Recall that $\|f\|_K$ denotes the supremum of the function f on the compact set K.

LEMMA 3. *Let f be a finite Blaschke product such that $f(1)=f(-1)=1$. For every compact subset K of $\mathbb{D}$ and each interval $[a,b]\subseteq]1,\infty[$, there exists a positive constant M depending on K, f and a such that for every $j=1,\dots,q$ and every $\lambda\in[\lambda_j,\lambda_{j+1}[$ the following assertions are true:*

(1) *for every $l<j$, $\|C_{\phi_\lambda}^{jN}C_{\phi_{\lambda_l}}^{-lN}(f)-1\|_K\le Ma^{-(j-l)N}$;*
(2) *for every $l>j$, $\|C_{\phi_\lambda}^{jN}C_{\phi_{\lambda_l}}^{-lN}(f)-1\|_K\le Ma^{-(l-j)N}$;*
(3) *$\|C_{\phi_\lambda}^{jN}C_{\phi_{\lambda_j}}^{-jN}(f)-f\|_K\le M\delta$.*

We will use repeatedly the following fact, which is a consequence of the Schwarz–Pick estimates.

LEMMA 4. *Let $u\in\mathcal{B}$. Then for every $z\in\mathbb{D}$,*

$$|u(z)-1|\le\frac{1+|z|}{1-|z|}|u(0)-1|.$$

Proof. We obviously have that $\frac{u(z)-u(0)}{1-\overline{u(0)}u(z)}=zg(z)$ for some $g\in\mathcal{B}$. Hence,

$$\begin{aligned}|u(z)-1|&\le|u(z)-u(0)|+|u(0)-1|\le|z||1-\overline{u(0)}u(z)|+|u(0)-1|\\&\le|z|\big|\big(1-\overline{u(0)}\big)+\overline{u(0)}\big(1-u(z)\big)\big|+|u(0)-1|.\end{aligned}$$

Therefore, $|u(z)-1|(1-|z|)\le|1-u(0)|(1+|z|)$ for every $z\in\mathbb{D}$. □

Thus, in order to prove Assertions 1 and 2 above, for instance, it suffices to control in a suitable way the quantities $f(\phi_{\lambda_l}^{-[lN]}(\phi_\lambda^{[jN]}(0)))$.

Proof of Lemma 3. For every $z\in\mathbb{D}$ we have

$$C_{\phi_\lambda}^{jN}C_{\phi_{\lambda_l}}^{-lN}(f)(z)=f\left(\sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_l^{lN}}\sigma(z)\right)\right).$$

Since $f(1)=1$ and f is Lipschitz with constant C up to the boundary of $\mathbb{D}$, we have

$$\left|f\left(\phi_{\lambda_l}^{-[lN]}\left(\phi_{\lambda}^{[jN]}(0)\right)\right)-1\right| \leq C\left|\sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_l^{lN}}\right)-1\right| = \frac{2C}{\frac{\lambda^{jN}}{\lambda_l^{lN}}+1}.$$

Assertion 1 follows from this estimate: since $l<j$,

$$\frac{\lambda^{jN}}{\lambda_l^{lN}} \geq \frac{\lambda^{jN}}{\lambda_{j-1}^{lN}} \geq \lambda_{j-1}^{(j-l)N}\left(1+\frac{\delta}{\lambda_{j-1}Nj}\right)^{Nj} \geq \lambda_{j-1}^{(j-l)N} \geq a^{(j-l)N}.$$

By Lemma 4, there exists a positive constant M_1 such that

$$\|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_l}}^{-lN}(f)-1\|_K \leq \frac{M_1}{a^{(j-l)N}} \quad \text{for } l<j.$$

Assertion 2 is proved in the same fashion, using this time the fact that $f(-1)=1$, so that

$$\left|f\left(\phi_{\lambda_l}^{-[lN]}\circ\phi_{\lambda}^{[jN]}(0)\right)-1\right| \leq 2C\frac{\frac{\lambda^{jN}}{\lambda_l^{lN}}}{\frac{\lambda^{jN}}{\lambda_l^{lN}}+1}$$

and that for $l>j$,

$$\frac{\lambda^{jN}}{\lambda_l^{lN}} \leq \lambda_{j+1}^{(j-l)N} \leq a^{(j-l)N}.$$

As to Assertion 3, we have for every $z\in\mathbb{D}$

$$\begin{aligned}|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_j}}^{-jN}(f)(z)-f(z)| &\leq C\left|\sigma^{-1}\left(\frac{\lambda^{jN}}{\lambda_j^{jN}}\sigma(z)\right)-z\right| \\ &\leq C\left|\frac{\lambda^{jN}}{\lambda_j^{jN}}-1\right|\cdot\frac{2|\sigma(z)|}{|\frac{\lambda^{jN}}{\lambda_j^{jN}}\sigma(z)+1|^2}.\end{aligned}$$

Since $|\frac{\lambda^{jN}}{\lambda_j^{jN}}\sigma(z)+1|$ is bigger than its real part, which is bigger than 1, and since $0\leq(\frac{\lambda}{\lambda_j})^{jN}-1\leq(1+\frac{\delta}{aNj})^{Nj}-1\leq 2\delta/a$ when δ is small enough, we have

$$\|C_{\phi_\lambda}^{jN} C_{\phi_{\lambda_j}}^{-jN}(f)-f\|_K \leq M_3\delta$$

for some positive constant M_3. □

We need a last lemma.

LEMMA 5. *The finite Blaschke products f such that $f(1)=f(-1)=1$ are dense in $\mathcal{B}$ (for the topology of uniform convergence on compact sets).*

Proof. We use Carathéodory's theorem that the set of finite Blaschke products is dense in $\mathcal{B}$, as well as a special case of an interpolation result given in [9, Lemma 2.10]: for every $\varepsilon>0$, every compact subset $K\subseteq\mathbb{D}$ and $\alpha,\beta\in\mathbb{T}$ there exists a finite Blaschke product B_1 satisfying $B_1(1)=\alpha, B_1(-1)=\beta$

and $\|B_1 - 1\|_K < \varepsilon$. Thus, given $f \in \mathcal{B}$ and a finite Blaschke product B_0 that is close to f on K, we solve the interpolation problem with $\alpha = \overline{B_0(1)}$ and $\beta = \overline{B_0(-1)}$ and set $B = B_0 B_1$, in order to get the desired Blaschke product. □

With these two lemmas in hand, we prove the following proposition.

PROPOSITION 6. *Let $(f_k)_{k\geq 1}$ be a dense sequence of finite Blaschke products with $f_k(1) = f_k(-1) = 1$. Let (K_k) be an exhaustive sequence of compact subsets of $\mathbb{D}$, and $([a_k, b_k])_{k\geq 1}$ an increasing sequence of compact intervals such that*

$$\bigcup_{k\geq 1} [a_k, b_k] =]1, +\infty[.$$

There exist

- *a sequence $(B_n)_{\geq 1}$ of finite Blaschke products;*
- *an increasing sequence $(p_n)_{n\geq 1}$ of positive integers*

such that the following properties are satisfied for every $k \geq 1$:

(1) $B_k(1) = 1$;

(2) $\|B_k - 1\|_{K_k} < 2^{-k}$;

(3) *for every $\lambda \in [a_k, b_k]$, there exists an integer $n_k(\lambda) \leq p_k$ such that for every $i \geq k$,*

$$\left\|C_{\phi_\lambda}^{n_k(\lambda)}(B_1 \cdots B_i) - f_k\right\|_{K_k} < 2^{-k}. \tag{1}$$

As a corollary, we obtain the corollary below.

COROLLARY 7. *There exists a Blaschke product B which is universal for all the composition operators $C_{\phi_{\lambda,\beta}}$, $\lambda > 1$, $\beta \in \mathbb{R}$.*

Proof. Consider $B = \prod_{n=1}^{\infty} B_n$: this is a convergent Blaschke product by Assertion 2 of Proposition 6, and going to the limit as i goes to infinity in equation (1) implies that for every $\lambda \in [a, b]$ and k large enough ($[a, b] \subseteq [a_k, b_k]$),

$$\left\|C_{\phi_\lambda}^{n_k(\lambda)}(B) - f_k\right\|_{K_k} \leq 2^{-k}.$$

Since the family $(f_k)_{k\geq 1}$ is locally uniformly dense in $\mathcal{B}$, this proves the universality of B for C_{ϕ_λ}, hence for $C_{\phi_{\lambda,\beta}}$. □

We turn now to the proof of Proposition 6.

Proof of Proposition 6. The proof is done by induction on k. We consider a first partition $a_1 = \lambda_1 < \lambda_2 < \cdots < \lambda_{q_1} = b_1$ of $[a_1, b_1]$ with parameters N_1 and δ_1, and the finite Blaschke product

$$B_1 = \prod_{l=1}^{q_1} C_{\phi_{\lambda_l}}^{-lN_1}(f_1).$$

We have $B_1(1)=1$. Since $f_1(-1)=1$, $C_{\phi_{\lambda_l}}^{-lN_1}(f_1)$ tends to 1 uniformly on compact sets as N_1 tends to infinity, and if N_1 is large enough,

$$\|B_1-1\|_{K_1}<2^{-1}. \tag{2}$$

Since $|\prod_{j=1}^s a_j-\prod_{j=1}^s b_j|\le\sum_{j=1}^s|a_j-b_j|$ whenever $a_j,b_j\in\mathbb{D}$, we have for every $j=1,\dots,q_1$, every $\lambda\in[\lambda_j,\lambda_{j+1}[$, and any compact subset K of $\mathbb{D}$

$$\|C_{\phi_\lambda}^{jN_1}(B_1)-f_1\|_K\le\sum_{l=1,l\neq j}^{q_1}\|C_{\phi_\lambda}^{jN_1}C_{\phi_{\lambda_l}}^{-lN_1}(f_1)-1\|_K \\ +\|C_{\phi_\lambda}^{jN_1}C_{\phi_{\lambda_j}}^{-jN_1}(f_1)-f_1\|_K.$$

But, by Lemma 3, the quantity on the right-hand side is less than

$$\sum_{l=1,l\neq j}^{q_1}\frac{M}{a_1^{|j-l|N_1}}+M\delta_1\le 2M\sum_{k=1}^{\infty}\frac{1}{a_1^{kN_1}}+M\delta_1.$$

Thus, if N_1 is large enough and δ_1 small enough

$$\|C_{\phi_\lambda}^{jN_1}(B_1)-f_1\|_{K_1}<2^{-1}. \tag{3}$$

We now fix N_1 large enough and δ_1 small enough so that inequalities (2) and (3) are satisfied. It is easy to check that Assertions 2 and 3 of Proposition 6 are satisfied with $p_1=q_1N_1$ and $n_1(\lambda)=jN_1$ for $\lambda\in[\lambda_j,\lambda_{j+1}[$. This terminates the first step of the construction.

If now the construction has been carried out until step $k-1$, we consider again a partition $a_k=\lambda_1<\cdots<\lambda_{q_k}=b_k$ of $[a_k,b_k]$ with parameters δ_k and N_k, and set

$$B_k=\prod_{l=1}^{q_k}C_{\phi_{\lambda_l}}^{-lN_k}(f_k),$$

so that B_k is a finite Blaschke product with $B_k(1)=1$. Just as above if N_k is large enough and δ_k small enough, we have for every $j\le q_k$ and every $\lambda\in[\lambda_j,\lambda_{j+1}[$

$$\|C_{\phi_\lambda}^{jN_k}(B_k)-f_k\|_{K_k}<2^{-(k+1)}$$

and

$$\|B_k-1\|_{K_k}<2^{-k}.$$

Because $B_1(1)=\cdots=B_{k-1}(1)=1$, we can also choose simultaneously N_k large enough so that $C_{\phi_\lambda}^{jN_k}(B_1\cdots B_{k-1})$ is very close to 1 on K_k. This gives (1) for $i=k$.

It remains to check that if $r\le k-1$, $\lambda\in[a_r,b_r]$,

$$\left\|C_{\phi_\lambda}^{n_r(\lambda)}(B_1\cdots B_{k-1}B_k)-f_r\right\|_{K_r}<2^{-r}.$$

We already know that

$$\left\|C_{\phi_\lambda}^{n_r(\lambda)}(B_1\cdots B_{k-1})-f_r\right\|_{K_r}<2^{-r},$$

and since B_k can be made arbitrarily close to 1 on any compact set if N_k is large enough, we also choose N_k so that $\|B_k-1\|_K$ is small enough, where

$$K=\bigcup_{r\leq k-1,\lambda\in[a_r,b_r]}\phi_\lambda^{n_r(\lambda)}(K_r),$$

and then Assertions 2 and 3 are satisfied at step k. □

3. The parabolic case

We consider now the family of parabolic automorphisms of $\mathbb{D}$ with 1 as attractive fixed point. If $T_\lambda:\mathbb{C}_+\to\mathbb{C}_+$ is the translation defined by $w\mapsto w+i\lambda$, $\lambda\in\mathbb{R}\setminus\{0\}$, then such parabolic automorphisms have the form $\psi_\lambda(z)=\sigma^{-1}\circ T_\lambda\circ\sigma$. Our aim in this section is to construct a Blaschke product which is universal for all composition operators (C_{ψ_λ}), $\lambda>0$. This is more difficult than the hyperbolic case because we have no suitable analog of Lemma 3: the estimate we get has the form

$$\|C_{\psi_\lambda}^{jN}C_{\psi_{\lambda_l}}^{-lN}(f)-1\|_K\leq\frac{M}{|j-l|N}\quad\text{for } l\neq j,$$

and the series on the right-hand side is not convergent when we sum over all $l\neq j$.

In other words, if K is any compact set, the sets $\psi_\lambda^{[n]}(K)$ go towards the point 1 at a rate of $1/n$, which is too slow. This difficulty was tackled for the study of common hypercyclicity on the Hardy space $H^2(\mathbb{D})$ by using either a fine analysis of properties of disjointness in [4] or probabilistic ideas in [5]. Here, we use in a crucial way the tangential convergence of the sequence $(\psi_\lambda^{[n]}(0))$ to the boundary. Indeed, the series $\sum_n(1-|\psi_n(0)|)$ is summable, whereas the series $\sum_n|1-\psi_n(0)|$ is not. The following lemma will play the same role as Lemma 3 in the hyperbolic case. We keep the notation of Section 2 and use the same kind of decomposition $a=\lambda_1,\dots,\lambda_q=b$ of a compact sub-interval $[a,b]$ of $]0,+\infty[$.

LEMMA 8. *Let f be a finite Blaschke product such that $f(1)=1$. For every compact subset K of $\mathbb{D}$, there exists a positive constant M depending on K, f and a such that for every $j=1,\dots,q$ and every $\lambda\in[\lambda_j,\lambda_{j+1}[$ the following assertions are true:*

(1) *for every $l<j$, $\||C_{\psi_\lambda}^{jN}C_{\psi_{\lambda_l}}^{-lN}(f)|-1\|_K\leq\frac{M}{(j-l)^2N^2}$;*
(2) *for every $l>j$, $\||C_{\psi_\lambda}^{jN}C_{\psi_{\lambda_l}}^{-lN}(f)|-1\|_K\leq\frac{M}{(l-j)^2N^2}$;*
(3) *$\|C_{\psi_\lambda}^{jN}C_{\psi_{\lambda_j}}^{-jN}(f)-f\|_K\leq M\delta$.*

Proof. In order to prove Assertions 1 and 2, it suffices to work at the point 0. Since the modulus of f is equal to 1 on $\mathbb{T}$, and since the operators commute, we have

$$\big||f(\psi_{\lambda_l}^{-[lN]}\circ\psi_\lambda^{[jN]}(0))|-1\big| = \bigg||f(\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0))| - \left|f\left(\frac{\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0)}{|\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0)|}\right)\right|\bigg|.$$

Since f is C-Lipschitz on $\overline{\mathbb{D}}$ for some positive constant C, this quantity is less than

$$C\left|\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0) - \frac{\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0)}{|\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0)|}\right| = C\big(1-|\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0)|\big). \tag{4}$$

An easy computation shows that

$$\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{[-lN]}(0) = \frac{iN(j\lambda-l\lambda_l)}{2+iN(j\lambda-l\lambda_l)}\cdot$$

Observe that $|j\lambda-l\lambda_l|\geq|j-l|a$. This gives

$$\begin{aligned}1-|\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0)| &\leq 1-|\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0)|^2\\ &\leq 1-\frac{(j\lambda-l\lambda_l)^2N^2}{4+(j\lambda-l\lambda_l)^2N^2}\\ &\leq \frac{C_1}{N^2(j-l)^2}\end{aligned}$$

for some positive constant C_1 which does not depend on λ. Now, equation (4) implies that for $l\neq j$,

$$\big||f(\psi_\lambda^{[jN]}\circ\psi_{\lambda_l}^{-[lN]}(0))|-1\big| \leq \frac{C_2}{N^2(j-l)^2}\cdot$$

This proves Assertions 1 and 2 of Lemma 8. Assertion 3 is proved in the same way as in Lemma 3, writing for $\lambda\in[\lambda_j,\lambda_{j+1}[$ and $z\in\mathbb{D}$

$$|C_{\psi_\lambda}^{jN}C_{\psi_{\lambda_j}}^{-jN}f(z)-f(z)| \leq CjN|\lambda_j-\lambda| \leq CjN\frac{\delta}{(j+1)N}\leq C'\delta. \qquad \square$$

The following proposition is the main ingredient of the proof.

Proposition 9. *Let f be a finite Blaschke product, $[a,b]\subset]0,+\infty[$, m_0 a positive integer, K a compact subset of $\mathbb{D}$ and $\varepsilon>0$. There exist an integer m, integers $(n(\lambda))_{\lambda\in[a,b]}$ with $n(\lambda)\in[m_0,m]$ and a finite Blaschke product B such that*

(1) $B(1)=1$;
(2) $\|B-1\|_K<\varepsilon$;
(3) $C_{\psi_\lambda}^{n(\lambda)}(B)=u_\lambda v_\lambda$ *where u_λ and v_λ belong to $\mathcal{B}$, $\|u_\lambda-f\|_K<\varepsilon$ and $|v_\lambda(0)|>1-\varepsilon$.*

Proof. We use again the decomposition $\lambda_1 = a, \lambda_2 = a + \frac{\delta}{2N}, \ldots, \lambda_q = b$, where $\delta > 0$ and $N \geq m_0$ will be fixed during the proof. Consider the Blaschke product

$$B_1 = \prod_{l=1}^{q} C_{\psi_{\lambda_l}}^{-lN}(f).$$

For $\lambda \in [\lambda_j, \lambda_{j+1}[$, we have

$$C_{\psi_\lambda}^{jN}(B_1) = C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_j}}^{-jN}(f) \left(\prod_{l \neq j} C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f) \right) := u_{1,\lambda} v_{1,\lambda}$$

with $u_{1,\lambda} = C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_j}}^{-jN}(f)$ and $v_{1,\lambda} = \prod_{l \neq j} C_{\psi_\lambda}^{jN} C_{\psi_{\lambda_l}}^{-lN}(f)$. Set $n(\lambda) = jN$ for $\lambda \in [\lambda_j, \lambda_{j+1}[$. By Assertion 3 of Lemma 8, $\|u_{1,\lambda} - f\|_K \leq M\delta < \varepsilon$ if δ is small enough. Moreover, still by Lemma 8,

$$\begin{aligned} 1 - |v_{1,\lambda}(0)| &= 1 - \prod_{l \neq j} \left| f\left(\psi_\lambda^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)\right)\right| \\ &\leq \sum_{l \neq j} \left(1 - \left| f\left(\psi_\lambda^{[jN]} \circ \psi_{\lambda_l}^{-[lN]}(0)\right)\right|\right) \\ &\leq \frac{C'}{N^2} \end{aligned}$$

for some positive constant C'. Thus, if N is large enough, $|v_{1,\lambda}(0)| > 1 - \varepsilon$. To conclude, it remains to observe that the same proof leads to

$$|B_1(0)| \geq 1 - \frac{C''}{N^2}$$

for some positive constant C'', so that using Lemma 4 and adjusting N large enough, there exists a real number θ such that $\|e^{i\theta} B_1 - 1\|_K < \varepsilon$. If we set $B_2 = e^{i\theta} B_1$, then B_2 satisfies the conclusions of the proposition (setting $u_{2,\lambda} = u_{1,\lambda}$ and $v_{2,\lambda} = e^{i\theta} v_{1,\lambda}$), except that we are not sure that $B_2(1) = 1$. To conclude, let F be a finite Blaschke product such that F is very close to 1 on a big compact set $L \subset \mathbb{D}$ and $F(1) = \overline{B_2(1)}$. Then $B = FB_2$ is the finite Blaschke product we are looking for. Indeed, setting $u_\lambda = u_{2,\lambda}$ and $v_\lambda = C_{\psi_\lambda}^{n(\lambda)}(F) v_{2,\lambda}$, Assertion 3 is satisfied, provided L is big enough to contain all the points $\psi_\lambda^{n(\lambda)}(0)$, $\lambda \in [a, b]$. □

We can now proceed with the construction.

PROPOSITION 10. *Let $(f_k)_{k \geq 1}$ be a dense sequence of finite Blaschke products with $f_k(1) = 1$. Let (K_k) be an exhaustive sequence of compact subsets of $\mathbb{D}$, and $([a_k, b_k])_{k \geq 1}$ an increasing sequence of compact intervals such that*

$$\bigcup_{k \geq 1} [a_k, b_k] =]1, +\infty[.$$

There exist finite Blaschke products (B_k)*, integers* (m_k)*, and other integers* $(n_k(\lambda))_{\lambda\in[a_k,b_k]}$ *with* $n_k(\lambda)\le m_k$ *such that*

(1) $B_k(1)=1$;
(2) $\|B_k-1\|_{K_k}<2^{-k}$;
(3) *for every* $j<k$*, every* $\lambda\in[a_k,b_k]$*,* $|B_j\circ\psi_\lambda^{[n_k(\lambda)]}(0)-1|<2^{-k}$;
(4) *for every* $j<k$*, every* $\lambda\in[a_j,b_j]$*,* $|B_k\circ\psi_\lambda^{[n_j(\lambda)]}(0)-1|<2^{-k}$;
(5) *for every* $\lambda\in[a_k,b_k]$*,* $C_{\psi_\lambda}^{n_k(\lambda)}(B_k)=u_{k,\lambda}v_{k,\lambda}$ *where*

$$\|u_{k,\lambda}-f_k\|_{K_k}<2^{-k}\quad\text{and}\quad|v_{k,\lambda}(0)|>1-2^{-k}.$$

Proof. The first step of the construction follows directly from Proposition 9. Now, we assume that the construction has been done until step $k-1$ and show how to complete step k. By continuity at the point 1 of the functions $(B_j)_{j<k}$, we choose an integer m such that for every $\lambda\in[a_k,b_k]$, for any $n\ge m$, $|B_j\circ\psi_\lambda^{[n]}(0)-1|<2^{-k}$. We then set

$$K=K_k\cup\bigcup_{j<k,\lambda\in[a_j,b_j],n\le m_j}\{\psi_\lambda^{[n]}(0)\}.$$

The function B_k is then given immediately by Proposition 9. □

Corollary 11. *There exists a Blaschke product* B *which is universal for all the composition operators* C_{ψ_λ}*,* $\lambda>0$.

Proof. Set

$$B=\prod_{l\ge1}B_l,$$

which is a convergent Blaschke product by Assertion 2 of Proposition 10. We claim that B is $\mathcal{B}$-universal with respect to every composition operator C_{ψ_λ}. Indeed, fix $\lambda>0$ and k_0 such that $\lambda\in[a_{k_0},b_{k_0}]$. Let g be a universal function for *this* particular operator C_{ψ_λ}. Using the notation of Proposition 10, let (p_k) be an increasing sequence of integers such that f_{p_k} converges uniformly to g on compact subsets of $\mathbb{D}$. Now, we decompose

$$C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B)=C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B_{p_k})\left(\prod_{j\ne p_k}B_j\circ\psi_\lambda^{[n_{p_k}(\lambda)]}\right):=u_{p_k,\lambda}v_{p_k,\lambda}w_{p_k,\lambda}$$

where $C_{\psi_\lambda}^{[n_{p_k}(\lambda)]}(B_{p_k})=u_{p_k,\lambda}v_{p_k,\lambda}$ is the decomposition of Proposition 9. From Assertions 3 and 4 of Proposition 10, we get that $w_{p_k,\lambda}(0)$ tends to 1 (see [2] for details), so that (cf. Fact 4) $w_{p_k,\lambda}$ converges uniformly on compacta to 1. Taking a subsequence if necessary, we can assume that $v_{p_k,\lambda}(0)$ converges to some unimodular number $e^{i\theta}$, and by Fact 4, again we have uniform convergence on compacta. Thus, $C_{\psi_\lambda}^{n_{p_k}(\lambda)}(B)$ converges uniformly to the function $e^{i\theta}g$ on compacta. Since the function $e^{i\theta}g$ is universal for C_{ψ_λ}, this implies that B is universal for C_{ψ_λ} too, and this terminates the proof of Corollary 11. □

The proof of Theorem 1 is now concluded by "intertwining" the two proofs of the hyperbolic and parabolic cases: the common universal Blaschke product has the form

$$B = \prod_{l \geq 1} B_l$$

where the B_l's are finite Blaschke products satisfying a number of properties: B_1 is constructed using Proposition 6, then B_2 using Proposition 10, then B_3 using Proposition 6 again, etc... taking care at each step not to destroy what has been done previously. Details are left to the reader.

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