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FINITE GROUPS WITH L-FREE LATTICES OF SUBGROUPS

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ABSTRACT. Balanced and strongly balanced lattices were introduced in order to generalize the uniform dimension of modular lattices. A description of finite groups with strongly balanced subgroup lattices was given by the authors in (*Colloq. Math.* **82** (1999), 65–77) and strengthened by Schmidt in (*Illinois J. Math.* **47** (2003), 515–528). In this paper, a description of finite groups with dually strongly balanced subgroup lattices is given.

Let $\mathcal{L}$ be a lattice with the least element 0 and the greatest element 1. For given $a,b \in \mathcal{L}$, $a \le b$, we denote by $[a,b]$ the interval $\{x \in \mathcal{L} : a \le x \le b\}$ in $\mathcal{L}$. If for all $x,y,z \in \mathcal{L}$

$$(x \wedge y) \vee [(x \vee y) \wedge z] = 0 \quad \Longrightarrow \quad [(y \vee z) \wedge x] \vee [(z \vee x) \wedge y] = 0,$$

then $\mathcal{L}$ is called *balanced*. It is called *strongly balanced* if all its nonempty intervals are balanced. If G is a group, then by $\mathcal{L}(G)$, we denote the subgroup lattice of G. Following [10], a lattice $\mathcal{L}$ (a group G) is called *L-free* if $\mathcal{L}$ (the subgroup lattice $\mathcal{L}(G)$) has no sublattice isomorphic to L.

Balanced lattices were introduced in [6] mainly in order to generalize the uniform dimension of modular lattices. Finite strongly balanced lattices were characterized there as L_6-free and L_7-free lattices (see Figure 1). In [1], all finite groups whose subgroup lattices are strongly balanced were determined. We call them *strongly balanced* groups. The results in [1] show that their structure is very similar to the structure of modular groups. In [10], Schmidt strengthened these results characterizing all finite groups whose subgroup lattices do not contain sublattices isomorphic to just one of the lattices from Figure 1. In [11], [12], he studies other interesting classes of groups whose

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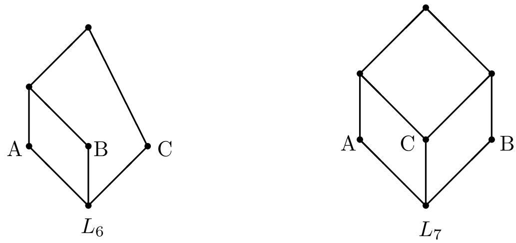

FIGURE 1.

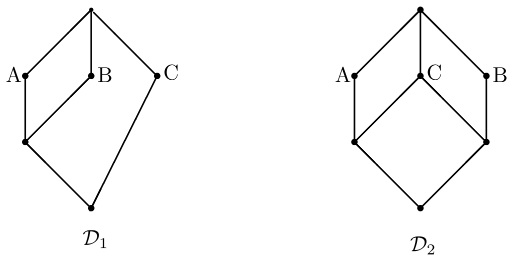

FIGURE 2.

subgroup lattices are characterized in a similar way, that is by the fact that their members do not contain certain lattices as sublattices.

The class of (strongly) balanced lattices is obviously not self-dual. So it is not strange that the class of groups whose dual subgroup lattices are (strongly) balanced substantially differs from the class of (strongly) balanced groups. In this paper, we study this class of groups. We show that when restricting attention to finite non-p-groups, a finite group from this class is very similar to L_7-free groups described in [10]. The case of finite p-groups needs further studies. A description of dually strongly balanced 2-groups seems to be the most difficult one as this depends on a description of 2-groups of rank ≤ 3 which is still not completed.

We follow standard notation which can be found in [4], [9].

1. Lattice theory preliminaries

A finite lattice $\mathcal{L}$ (a finite group G) is called *dually strongly balanced* if the lattice dual to $\mathcal{L}$ (to $\mathcal{L}(G)$) is strongly balanced.

The following lemma is an immediate consequence of the description of strongly balanced lattices given in [6].

LEMMA 1.1. *A finite lattice $\mathcal{L}$ is dually strongly balanced if and only if $\mathcal{L}$ is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free (see Figure 2).*

LEMMA 1.2. *Let $\mathcal{L}$ be a finite lattice and $x, y, z \in \mathcal{L}$ be such that $x, y, z \neq 1$, $(x \wedge y) \vee z = 1$ and $(x \wedge z) \vee y = y$. Then $x \wedge z < x \wedge y$.*

Proof. Since $(x \wedge z) \vee y = y$, we have $x \wedge z \leq y$ and so $x \wedge z \leq x \wedge y$. Now, if $x \wedge z = x \wedge y$, then $1 = (x \wedge y) \vee z = (x \wedge z) \vee z = z$. This contradiction proves that $x \wedge z < x \wedge y$. □

LEMMA 1.3. *Let $\mathcal{L}$ be a finite lattice and let $x, y, z \in \mathcal{L}$ be such that $x, y, z \neq 1$, $(x \wedge y) \vee z = 1$, $(y \wedge z) \vee x = 1$ and $(x \wedge z) \vee y = y$. If $(x \wedge y) \vee (y \wedge z) = y_1 < y$, then the sublattice of $\mathcal{L}$ generated by the set $\{x, y_1, z\}$ is isomorphic to $\mathcal{D}_2$.*

Proof. By definition of y_1, $x \wedge y \leq y_1$, so $x \wedge y \leq x \wedge y_1$. Since the opposite inequality is obvious, we obtain $x \wedge y_1 = x \wedge y$. Similarly, $y_1 \wedge z = y \wedge z$. Now, by the assumptions and easy standard calculations we get the assertion. □

A 1-*sublattice (a* 0-*sublattice)* of a lattice $\mathcal{L}$ is a sublattice of $\mathcal{L}$ containing the greatest (the smallest) element of $\mathcal{L}$.

PROPOSITION 1.1. *If a lattice $\mathcal{L}$ contains a* 1-*sublattice isomorphic to $\mathcal{D}_1$, then $\mathcal{L}$ contains a* 1-*sublattice isomorphic to $\mathcal{D}_2$ or a* 1-*sublattice isomorphic to $\mathcal{D}_1$ containing two antiatoms of $\mathcal{L}$.*

Proof. Suppose that there is an isomorphism of $\mathcal{D}_1$ onto a 1-sublattice of $\mathcal{L}$ which maps A, B, C (see Figure 2) onto a, b, c, respectively. In particular, we have $(a \wedge b) \vee c = 1$, $(a \wedge c) \vee b = b$, $(b \wedge c) \vee a = a$. Of course, we may assume that at least one of the elements a, b, say a, is not an antiatom of $\mathcal{L}$. Now, let us consider the sublattice $\langle a_1, b, c\rangle$, where a_1 is an antiatom of $\mathcal{L}$ such that $a_1 > a$. It is clear that $(a_1 \wedge b) \vee c = 1$, and by Lemma 1.2, $b \wedge c < a_1 \wedge b$. We have also $b \wedge c \leq a_1 \wedge c$ as $(b \wedge c) \vee a_1 = a_1$ (i.e., $a_1 \wedge b \wedge c = b \wedge c$).

If $b \wedge c = a_1 \wedge c$, then $(a_1 \wedge c) \vee b = (c \wedge b) \vee b = b$, and obviously $a_1 \vee b = a_1 \vee c = b \vee c = 1$, as a_1 is an antiatom of $\mathcal{L}$. Hence, a_1, b, and c generate a sublattice isomorphic to $\mathcal{D}_1$. If b or c is an antiatom of $\mathcal{L}$, we are done. If not, we can replace a by a_1 in the sublattice generated by a, b and c and repeat the above considerations, changing the roles of a and b. Therefore, in what follows, we may assume that b is an antiatom of $\mathcal{L}$ and $b \wedge c < a_1 \wedge c$.

Now, if $a_1 \wedge c \leq b$, by the end of the first paragraph we obtain $a_1 \wedge c = a_1 \wedge c \wedge b = b \wedge c$, which contradicts the assumption $b \wedge c < a_1 \wedge c$. Therefore, $a_1 \wedge c \nleq b$, i.e., $(a_1 \wedge c) \vee b = 1$, and substituting $x = b, y = a_1, z = c$ we are in the assumptions of Lemma 1.3, which then yields that the sublattice generated by a_1, b and c is isomorphic to $\mathcal{D}_2$. This ends the proof. □

For subgroups $A_1, A_2, \ldots, A_k$ of a group G by $\mathcal{L}(A_1, A_2, \ldots, A_k)$, we denote the sublattice of $\mathcal{L}(G)$ generated by $A_1, A_2, \ldots, A_k$ as elements of $\mathcal{L}(G)$.

LEMMA 1.4. *Let A, B, C be subgroups of a group G.*

(a) *If* $\mathcal{L}(A,B,C) \simeq \mathcal{D}_1$*, where* A, B, C *are such as in Figure 2, then* $A \vee B \vee C \neq (A \wedge B)C$*; in particular* $A \wedge B$ *and* C *are not normal in* $A \vee B \vee C$*.*

(b) *If* $\mathcal{L}(A,B,C) \simeq \mathcal{D}_2$*, where* A, B, C *are such as in Figure 2, then* $A(B \wedge C) \neq A \vee B \vee C \neq (A \wedge C)B$*; in particular non of the subgroups* $A, B, A \wedge C, B \wedge C$ *is normal in* $A \vee B \vee C$*.*

Proof. For the proof of (a), suppose that $A \vee B \vee C = (A \wedge B)C$. Then by Dedekind's law ([8], 1.3.14) $A = (A \wedge B)(A \wedge C) = A \wedge B$, a contradiction. The proof of (b) is similar. □

2. Finite p-groups

Lemma 2.1. *A finite* p*-group* G *is a dually strongly balanced group if and only if* $\mathcal{L}(G)$ *is* $\mathcal{D}_2$*-free.*

Proof. By Lemma 1.1, it suffices to show that if $\mathcal{L}(G)$ is $\mathcal{D}_2$-free then it is $\mathcal{D}_1$-free. If not, then applying Proposition 1.1 we get that G contains subgroups A, B, C two of which are maximal in $A \vee B \vee C$. However, maximal subgroups of a p-group are normal which contradicts Lemma 1.4. □

Proposition 2.1. *If the commutator subgroup* G' *of a finite* p*-group* G *has order* p*, then* $\mathcal{L}(G)$ *is* D_1*-free.*

Proof. Let G be a counterexample of minimal order and let A, B, C be subgroups of G such that $\mathcal{L}(A,B,C) \simeq \mathcal{D}_1$ (see Figure 2). Then obviously $A \vee B \vee C = G = A \vee B = A \vee C = B \vee C$ and by Lemma 1.4, $G' \nleq C$ and one of the subgroups A, B, say A, does not contain G' as well. Since $|G'| = p$, a subgroup of G not containing the commutator subgroup is abelian. Thus, A and C are abelian and because of that $A \wedge C$ is central, and so, normal in $A \vee C = G$. By minimality of G, we obtain $A \wedge C = \{e\}$. If B is abelian, $A \wedge B$ is normal in $A \vee B = G$, which contradicts Lemma 1.4. So B is nonabelian, and then $G' \leq B$ which in turn means that $B \lhd G$.

Clearly, $G'C \trianglelefteq G$, and hence $G = (A \wedge B)G'C$. Applying Dedekind's law twice, we get

$$B = (A \wedge B)(B \wedge G'C) = (A \wedge B)G'(B \wedge C) = (A \wedge B)G'.$$

Since $G' \leq Z(G)$, it follows that $B = (A \wedge B) \times G'$ is abelian, a contradiction. □

In Lemma 2.1, we got that the class of $\mathcal{D}_2$-free p-groups is contained in the class of $\mathcal{D}_1$-free p-groups. The following example shows that these classes do not coincide.

Example 1. Let p be a prime and let

$$T = \langle x_1, x_2, x_3, y \mid x_1^p = x_2^p = x_3^p = y^p = e, [x_1, y] = x_2, \\ [x_2, y] = [x_3, y] = e, [x_i, x_j] = e\rangle.$$

One can easily check that for $A=\langle x_3,y\rangle$, $B=\langle x_1x_2,x_1x_3\rangle$, $C=\langle x_1,y\rangle$, we have $\mathcal{L}(A,B,C)\simeq\mathcal{D}_2$. It is easily seen that T is isomorphic to a direct product of a nonabelian group of order p^3 and exponent p and a cyclic group of order p. Therefore, $\mathcal{L}(T)$ does not contain a sublattice isomorphic to $\mathcal{D}_1$ by Proposition 2.1. It follows from the description of groups of order p^4, $p>2$, that among these groups T is the unique group whose subgroup lattice contains a sublattice isomorphic to $\mathcal{D}_2$.

In the following example, we present a p-group of minimal order whose subgroup lattice contains an isomorphic copy of $\mathcal{D}_1$.

EXAMPLE 2.

$$\begin{aligned}T_1=\langle x_1,x_2,x_3,x_4,y\mid x_1^p=x_2^p=x_3^p=x_4^p=y^p=e,[x_1,y]=x_2,\\ [x_2,y]=e,[x_3,y]=x_4,[x_4,y]=e,[x_i,x_j]=e\rangle.\end{aligned}$$

For $A=\langle x_1,y\rangle$, $B=\langle x_1x_3,y\rangle$, $C=\langle x_1x_4,x_1x_3\rangle$, the lattice $\mathcal{L}(A,B,C)$ is isomorphic to $\mathcal{D}_2$. For $A=\langle x_1,y\rangle$, $B=\langle x_3,y\rangle$, $C=\langle x_1x_4,x_2x_3\rangle$, the lattice $\mathcal{L}(A,B,C)$ is isomorphic to $\mathcal{D}_1$.

The following lemma can be easily derived from the proof of Theorem 1.11 in [7].

LEMMA 2.2. *Let G be a powerful p-group and H,K its subgroups. If $\langle H,K\rangle=G$, then $HK=G$.*

Proof. Let H and K be subgroups of G such that $H=\langle h_1,\dots,h_s\rangle$ and $K=\langle k_1,\dots,k_r\rangle$. Then obviously $G=\langle H,K\rangle=\langle h_1,\dots,h_s,k_1,\dots,k_r\rangle$. Now we choose a minimal set of generators from the above generating set, $G=\langle h_{i_1},\dots,h_{i_n},k_{j_1},\dots,k_{j_m}\rangle$. Since G is powerful, we obtain

$$G=\langle h_{i_1}\rangle\cdots\langle h_{i_n}\rangle\langle k_{j_1}\rangle\cdots\langle k_{j_m}\rangle\le\langle h_{i_1},\dots,h_{i_n}\rangle\langle k_{j_1},\dots,k_{j_m}\rangle\le HK. \qquad\square$$

The next lemma follows directly from Lemma 2.2 and Lemma 1.4(b).

LEMMA 2.3. *If G is a powerful p-group, then $\mathcal{L}(G)$ does not contain a 1-sublattice isomorphic to $\mathcal{D}_2$.*

PROPOSITION 2.2. *Let G be a p-group. If every nonpowerful subgroup of G is 2-generated, then G is dually strongly balanced.*

Proof. Let G be a counterexample of minimal order. By Lemma 2.1, there exist subgroups A,B,C of G such that $\mathcal{L}(A,B,C)\simeq\mathcal{D}_2$ as in Figure 2. By choice of G, we have $A\vee B\vee C=G$ that is $\mathcal{L}(A,B,C)$ is a 1-sublattice. Then by Lemma 2.3, G is 2-generated, that is $|G/\Phi(G)|=p^2$. So if A_1 and C_1 are maximal subgroups of G containing A and C, respectively then

$$A\wedge C\le A_1\wedge C_1=\Phi(G),$$

a contradiction since $(A\wedge C)\vee B=G$. $\square$

LEMMA 2.4. *Let G be a nonpowerful p-group, $p>2$. If $d(G)\geq 3$, then G contains a section isomorphic to the group T from Example 1.*

Proof. Without loss of generality, we may assume that $d(G)=3$. Let N be a normal subgroup of G such that $|G':N|=p$. The factor group $\bar{G}=G/G^pN$ has exponent p, $|\bar{G}|=p^4$ and $|\bar{G}'|=p$. This group is isomorphic to T (see Example 1). □

THEOREM 2.1. *Let G be a p-group, $p>2$. The following conditions are equivalent:*

(a) *G is dually strongly balanced,*
(b) *G contains no section isomorphic to T,*
(c) *every subgroup of G is either powerful or 2-generated.*

Using the description of 3-generator powerful p-groups (Theorem 3.3, [7]), we obtain the following corollary.

COROLLARY 2.1. *Let G be a powerful p-group. If $d(G)\leq 3$, then G is dually strongly balanced.*

Note that it follows from [2] that if G is a finite p-group in which every nonabelian subgroup is 2-generated, then G is metabelian.

3. Simple groups

In this section, we show that the lattice of subgroups of an arbitrary finite simple group contains isomorphic copies of the both lattices $\mathcal{D}_1$ and $\mathcal{D}_2$. So in particular, we prove that if a finite group G is dually strongly balanced then G is solvable. It is clear that it suffices to consider the minimal simple groups only. So we list first these groups and some known results concerning them.

THEOREM 3.1 ([13]). *Every minimal simple group is isomorphic to one of the following groups:*

(a) *$PSL(2,2^p)$, p any prime;*
(b) *$PSL(2,3^p)$, p any odd prime;*
(c) *$PSL(2,p)$, p any prime exceeding 3 such that $p^2+1\equiv 0 \pmod 5$;*
(d) *$Sz(2^p)$, p any odd prime;*
(e) *$PSL(3,3)$.*

The next two lemmas are immediate consequences of II.8.27 of [4].

LEMMA 3.1. *Let p, $p>2$, be a prime and let k be a positive integer (if $p=3$ then we assume that $k>1$). The group $PSL(2,p^k)$ contains a subgroup isomorphic to S_4 if and only if $p^{2k}-1\equiv 0 \pmod{16}$.*

LEMMA 3.2. *Let p be a prime, $p>3$, or $p=3^k$ where k is a prime. Moreover, let $G=PSL(2,p)$. Every maximal subgroup M of G is isomorphic to one of the following groups:*

(a) *the dihedral group of order* $p-1$*;*
(b) *the dihedral group of order* $p+1$*;*
(c) A_4 *– the alternating group of degree* 4*;*
(d) *a semidirect product of a Sylow* p*-subgroup of* G *and a cyclic group of order* $\frac{p-1}{2}$*.*

Corollary 3.1. *Let* G *be a group as in the previous lemma with order not divisible by* 8 *(i.e.* $16 \nmid p^2-1$*). If* H *is a proper subgroup of* G *with order divisible by* 4 *and* $P_1 \neq P_2$ *are Sylow* 2*-subgroups of* H*, then* $P_1 \cap P_2 \neq \{e\}$*.*

We need also some basic information on the Suzuki groups (see for instance [5], pp. 182–194). Let G be a Suzuki group $Sz(2^p)$, with $p = 2m+1$. Let

$$F = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & a^\theta & 1 & 0 \\ a^{2+\theta}+ab+b^\theta & a^\theta + b & a & 1 \end{pmatrix} \;\middle|\; a, b \in GF(2^p) \right\rangle,$$

where θ is the automorphism of $GF(2^p)$ such that $\theta^2 = id$. Then F is a Sylow 2-subgroup of G. Let

$$H = \left\langle \begin{pmatrix} c^{1+\theta} & 0 & 0 & 0 \\ 0 & c^\theta & 0 & 0 \\ 0 & 0 & c^\theta & 0 \\ 0 & 0 & 0 & c^{-1-\theta} \end{pmatrix} \;\middle|\; 0 \neq c \in GF(2^p) \right\rangle \quad \text{and}$$

$$t = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $Sz(q) = \langle F, H, t \rangle$.

Lemma 3.3. *A maximal subgroup of a Suzuki group* $Sz(q)$ *is conjugate to one of the following subgroups:*
(a) $FH = N_G(F)$*;*
(b) $N_G(H)$ *which is isomorphic to the dihedral group of order* $2(q-1)$*;*
(c) $B_i = \langle U_i, t_i \rangle$*, where* U_i*,* $i = 1, 2$*, is a cyclic group of order* $q \pm 2^{m+1} + 1$*; moreover for every* $u \in U_i$*,* $u_i^{t_i} = u^q$ *and* $|B_i : U_i| = 4$*;*
(d) $Sz(s)$*, where* q *is a power of* s*.*

We need also the following, rather easy observation.

Lemma 3.4. *The lattices* $\mathcal{L}(S_4)$ *of all subgroups of the symmetric group of degree* 4 *and* $\mathcal{L}(A_5)$ *of all subgroups of the alternating group of degree* 5 *contain sublattices isomorphic to* $\mathcal{D}_1$ *and sublattices isomorphic to* $\mathcal{D}_2$*.*

Proof. One can easily check that for $A = \langle (1\ 2), (1\ 2\ 3) \rangle$, $B = \langle (1\ 2\ 3\ 4) \rangle$ and $C = \langle (1\ 2), (1\ 3\ 2\ 4) \rangle$, the sublattice $\mathcal{L}(A, B, C)$ of $\mathcal{L}(S_4)$ generated by A, B and C is isomorphic to $\mathcal{D}_2$.

For a construction of a sublattice isomorphic to $\mathcal{D}_1$, we take A as in the previous case, $B = \langle (1\,2), (3,4)\rangle$ a noncyclic group of order 4 and $C = \langle (1\,2\,3\,4)\rangle$—a cyclic group of order 4.

In the group A_5, let $A = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, $B = \{e, (1\ 2)(3\ 5), (1\ 3)(2\ 5), (1\ 5)(2\ 3)\}$, and $C = \langle (1\ 2)(3\ 4), (3\ 4\ 5)\rangle$. Simple computations show that A, B, and C generate a sublattice isomorphic to $\mathcal{D}_2$.

Finally, if in the above set of subgroups we replace B by the subgroup $B_1 = \langle (1\ 2\ 3\ 4\ 5)\rangle$ then the set obtained in this way generate a sublattice isomorphic to $\mathcal{D}_1$. □

Theorem 3.2. *If G is a finite simple group, then $\mathcal{L}(G)$ contains isomorphic copies of the lattices $\mathcal{D}_1$ and $\mathcal{D}_2$.*

Proof. As it was mentioned earlier, we consider only the minimal simple groups. We begin with studying the easiest cases. So first let us consider the group $PSL(3,3)$. Let $G = T(3,3)$ be the group of upper triangular matrices from $SL(3,3)$ and let

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Straightforward calculations show that for $A = \langle a, d\rangle$, $B = \langle b, d\rangle$, $C = \langle ab\rangle$ we get $\mathcal{L}(A, B, C) \simeq \mathcal{D}_1$. Obviously, the image of G in $PSL(3,3)$ is isomorphic to G.

Now let

$$a_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$
$$b_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easily seen that the subgroups $H_1 = \langle a_1, b_1\rangle$, $H_2 = \langle a_2, b_2\rangle$ are isomorphic to S_3, the subgroup $\langle d\rangle = [H_1, H_2]$ is normal in G and $\langle H_1, H_2\rangle/\langle d\rangle$ is isomorphic to $S_3 \times S_3$. Let $\bar{G} = G/\langle d\rangle$ and consider the subgroups $\bar{A} = \langle \bar{a}_1, \bar{a}_2\rangle$, $\bar{B} = \langle \bar{b}_1\bar{a}_1, \bar{b}_2\bar{a}_2\rangle$, $\bar{C} = \langle \bar{a}_1, \bar{b}_1\rangle$. Again easy calculations show that $\mathcal{L}(\bar{A}, \bar{B}, \bar{C}) \simeq \mathcal{D}_2$.

To get a proof for the group $G = PSL(2, 2^p)$ where p is an odd prime, for a sublattice $\mathcal{L}(A, B, C)$ isomorphic to $\mathcal{D}_2$ one can take

$$A = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in F \right\}, \qquad B = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, c \in F \right\},$$
$$C = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle.$$

For the sublattice $\mathcal{L}(A,B,C)$ isomorphic to $\mathcal{D}_1$, put

$$A = \left\{ \begin{pmatrix} d & a \\ 0 & d^{-1} \end{pmatrix}, a \in F, d \in F^* \right\}, \qquad B = \left\{ \begin{pmatrix} d & 0 \\ a & d^{-1} \end{pmatrix}, a \in F, d \in F^* \right\},$$
$$C = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Not difficult standard details of the proof we leave to the reader.

Now let p be a prime, $p > 3$, or $p = 3^k$ where k is a prime and $G = PSL(2,p)$. By Lemmas 3.1 and 3.4, we may assume that $16 \nmid p^2 - 1$, i.e., $|PSL(2,p)|$ is not divisible by 8 (that is the Sylow 2-subgroups of G are isomorphic to the four group). Since p is odd, exactly one of the numbers $\frac{p-1}{2}$, $\frac{p+1}{2}$ is an odd integer. Let us denote this number by q. Let Q be a fixed cyclic subgroup of order q. It follows from II.8.27 of [4] also that the subgroup $C = N_G(Q)$ is a dihedral group of order $2q$. By Lemma 3.2, it is a maximal subgroup of G. Let $x, y \in C$ be elements of order 2 such that $Q = \langle xy \rangle$. Now let A be a Sylow 2-subgroup of G containing x and let B be a Sylow 2-subgroup of G containing y. Note that $A \wedge B = \{e\}$. Otherwise, there exists an element t of order 2, such that $A = \langle x, t \rangle$ and $B = \langle y, t \rangle$. This element t would centralize C, i.e., the group $\langle C, t \rangle = C \times \langle t \rangle$ would be a proper subgroup of G which is not possible as C is maximal in G.

Now by maximality of C in G and by Corollary 3.1, we obtain $A \vee B = A \vee C = B \vee C$. Moreover, we have $A \wedge C = \langle x \rangle$ i $B \wedge C = \langle y \rangle$. All these yield that $\mathcal{L}(A,B,C) \simeq \mathcal{D}_2$.

Since $PSL(2,5)$ is isomorphic to A_5, by Lemma 3.4, we may additionally assume that $p > 5$. This and the fact that $8 \nmid |G|$ yield that there exists an odd integer dividing $|G|$ and relatively prime to pq. Let r be the biggest among such numbers (it follows from the assumptions that if $q = \frac{p-1}{2}$ then $r = \frac{p+1}{4}$ and if $q = \frac{p+1}{2}$ then $r = \frac{p-1}{4}$). Now let A be a dihedral subgroup of order $4r$ (such a subgroup exists by Lemma 3.2) and let $B = N_G(Q)$ be such that $|A \wedge B| = 2$. Moreover, let C be a cyclic subgroup of order r such that $A \wedge C = \{e\}$. Now, it is not difficult to check that $\mathcal{L}(A,B,C) \simeq \mathcal{D}_1$.

Finally, let $G = Sz(q)$, where $q = 2^{2m+1}$. Then $|G| = (q^2+1)q^2(q-1)$. Let A be a Sylow 2-subgroup containing the element t. Let B be a Sylow 2-subgroup containing t^x, where $x \in G \setminus A$ and let $C = \langle t, t^x \rangle$. Since two distinct Sylow 2-subgroups have trivial intersection, $A \wedge B = \{e\}$. It follows from Lemma 3.3 that the subgroups $\langle A, t^x \rangle$ and $\langle B, t \rangle$ are not contained in a maximal subgroup of $Sz(q)$, so $A \vee (B \wedge C) = G$ and $B \vee (A \wedge C) = G$. Thus, $\mathcal{L}(A,B,C) \simeq \mathcal{D}_2$.

Now let $A = FH$ i $B = F'H$ (see the paragraph before Lemma 3.1), where F' is the subgroup consisting of all matrices which are transpositions of matrices from F. Moreover, let $C = S_t$, where S_t is a Sylow 2-subgroup of G

containing the element t. Again by the argument already used in the previous paragraph, $A \wedge C = B \wedge C = \{e\}$. Additionally, $A \wedge B = H$. Since the elements t and t^h, where $h \in H$, are of order 2 and tt^h has odd order, t and t^h belong to distinct Sylow 2-subgroups of G. Thus, $h \notin N_G(S_t)$, that is H is not contained in $N_G(S_t)$. Since $\langle H, S_t\rangle$ is not contained in a maximal subgroup of G, $(A \wedge B) \vee C = G$. Thus, $\mathcal{L}(A,B,C) \simeq \mathcal{D}_1$. □

4. Solvable groups

LEMMA 4.1. *Let p be a prime and let $G = H \ltimes P$ be a semidirect product of an elementary abelian p-group P and a cyclic p'-group H and suppose that G is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free. Then:*

(a) *If $|H| = q$, where q is a prime, then H acts trivially on P or H acts irreducibly on P or H acts by power automorphisms on P.*

(b) *If H acts nontrivially on P, then $|G : C_G(P)| = q$, with q a prime.*

Proof. (a) Assume that H acts nontrivially on P and H does not induce power automorphisms on P. By way of contradiction, suppose that there exist H-invariant nontrivial subgroups P_1 and P_2 of P such that $P = P_1 \times P_2$. We may suppose also that P_1 and P_2 are H-irreducible.

If there exists $t \in P$ such that $\langle t\rangle^H = P$, then the subgroups $A = HP_1$, $B = HP_2$ and $C = \langle t\rangle$ generate a sublattice of $\mathcal{L}(G)$ isomorphic to $\mathcal{D}_1$. So we may assume that $\langle t\rangle^H < P$, for any $t \in P \setminus \{e\}$. Put $H = \langle y\rangle$, $P_1 = \langle x_1\rangle^H$ and $P_2 = \langle x_2\rangle^H$. None of the subgroups P_1 and P_2 is cyclic as otherwise we would have $\langle t\rangle^H = P$ for $t = x_1x_2$. In particular, we may assume that $q > 2$.

Let $C = \langle x_1^y x_2, x_1 x_2^y\rangle$ be a noncyclic subgroup of P of order p^2. It is seen that for $i = 1, 2$, $P_i \wedge C = \{e\}$. In fact, if $P_1 \wedge C \neq \{e\}$, then for some integers k, l we have $e \neq (x_1^y x_2)^k (x_1 x_2^y)^l \in P_1$. By easy calculations and the fact that $P_1 \wedge P_2 = \{e\}$, we obtain that $x_2^k (x_2^y)^l = e$. Therefore $x_2^y = x_2^{-kl'}$, where l' is the inversion of l modulo p. Then P_2 is a cyclic group, a contradiction. Furthermore, we have $C^H = P$. Actually $((x_1^y x_2)^{-1})^{y^{-1}} x_1 x_2^y = (x_2^{-1})^{y^{-1}} x_2^y$, and if $(x_2^{-1})^{y^{-1}} x_2^y = e$, we obtain $x_2^{y^2} = x_2$, a contradiction. Therefore, $(x_2^{-1})^{y^{-1}} x_2^y = t \neq e$ and then $P = \langle t\rangle^H < C^H$. This means that $C^H = P$. Now, it is easily seen that $\mathcal{L}(A,B,C) \simeq \mathcal{D}_1$. A contradiction.

(b) Obviously, we may assume that $H \cap C_G(P) = Z(G) = \{e\}$. Suppose that there exist distinct primes q and r dividing $|H|$ and let Q and R be subgroups such that $|Q| = q$ and $|R| = r$. In view of (a), we may assume that Q and R act on P irreducibly or induce power automorphisms on P. Moreover, if Q and R induce power automorphisms on P, we may assume that P is a cyclic group. Clearly, $H \wedge H^x = \{e\}$ for any $x \notin N_G(H)$. Let $A = H$, $B = H^x$ for some $x \in P$ and $x \neq e$. Let also $C = \langle Q, Q^x\rangle$, if Q acts irreducibly on P or $C = \langle R, R^x\rangle$, if R acts irreducibly on P. Now it is easy to check that a sublattice of $\mathcal{L}(G)$ generated by A, B and C is isomorphic to $\mathcal{D}_2$.

This contradicts the assumption, so $|H|$ is not divisible by distinct primes, say H is a q-group.

Now we let $H = \langle y \rangle$, where y is an element of order q^2 and suppose $y^q \notin C_G(P)$. Since G is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free, y^q induces a power automorphism on P or $\langle y^q \rangle$ acts irreducibly on P. Let $x \in P$ and consider the subgroups $A = \langle y \rangle$, $B = \langle y \rangle^x$ and $C = \langle y^q, (y^q)^x \rangle$. Obviously, $\langle y \rangle \cap \langle y \rangle^x = \{e\}$. Hence, the lattice $\mathcal{L}(A, B, C)$ is isomorphic to $\mathcal{D}_2$. A contradiction. □

Lemma 4.2. *Let p and q be distinct primes and let $G = Q \ltimes P$ be a semidirect product of an elementary abelian p-group P and a noncyclic q-group Q. If Q acts nontrivially on P, then $\mathcal{L}(G)$ contains a sublattice isomorphic to $\mathcal{D}_1$ or $\mathcal{D}_2$.*

Proof. Let $H = C_G(P) \cap Q$. Of course H is normal in G, so we can consider the factor group $\bar{G} = G/H$. By Lemma 4.1(a), every element of $\bar{Q}$ acts irreducibly on $\bar{P}$ or induces a power automorphism on $\bar{P}$. Therefore, $\bar{Q}$ is a regular group of automorphisms of $\bar{P}$ and by [3], 5.4.11, $\bar{Q}$ is a cyclic group or a generalized quaternion group. On the other hand, by Lemma 4.1(b), for every $\bar{y} \in \bar{Q}$, we have $\bar{y}^q = 1$. Hence, $\bar{Q}$ is cyclic and by Lemma 4.1(b), $|Q/H| = q$. Since Q is not cyclic, we have $|Q/\Phi(Q)| \geq q^2$ and $\Phi(Q) < H$. Thus, $\Phi(Q)$ is a normal subgroup of G and we may assume that $\Phi(Q) = \{e\}$. Now, take elements y, z in G such that $o(y) = o(z) = q$, $y \in Q \setminus H$ and $z \in Z(G)$. Let also $x \in P$ be such that $xy \neq yx$. It is easily seen that for $A = \langle x, y \rangle$, $B = \langle y, z \rangle$, and $C = \langle xyz \rangle$, the subgroups A, B, C generate a sublattice of $\mathcal{L}(G)$ isomorphic to $\mathcal{D}_1$. This completes the proof. □

We call a group G *L-indecomposable* if the lattice $L(G)$ is not a direct product of its nontrivial sublattices. This means that G is not a direct product of its nontrivial subgroups with coprime orders. Otherwise, the group G will be called *L-decomposable.*

Lemma 4.3. *Let $G = H \rtimes Q$ be a semidirect product of a normal nilpotent group H and a q-group Q, with $(|H|, q) = 1$. If G is L-indecomposable and G is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free, then H is a p-group for a prime p.*

Proof. Let $\pi(H) = \{p_1, \ldots, p_r\}$. By assumption Q acts nontrivially on all Sylow p_i-subgroups of G and then Q induces nontrivial action on every primary component of $H/\Phi(H)$. Obviously, we may assume that $\Phi(H) = \{e\}$. By Lemma 4.2, Q is cyclic, that is $Q = \langle y \rangle$ for some $y \in G$. In view of Lemma 4.1, $y^q \in C_G(P_i)$ for every $i \in \{1, \ldots, r\}$. Hence, $\langle y^q \rangle$ is normal in G and we may replace G by the factor group $G/\langle y^q \rangle$. Now, let $a \in P_1$, $b \in P_2$ be such that $a^y \neq a$ and $b^y \neq b$. Then as it can be easily checked the subgroups $A = \langle a, y \rangle$, $B = \langle b, y \rangle$, and $C = \langle aby \rangle$ generate a sublattice of $\mathcal{L}(G)$ isomorphic to $\mathcal{D}_1$. A contradiction. □

LEMMA 4.4. *Let G be a solvable group. If $\pi(G)=\{p,q\}$, $p\neq q$, and G is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free, then either a Sylow p-subgroup or a Sylow q-subgroup of G is normal in G.*

Proof. Suppose that no Sylow subgroup of G is normal in G. Let $H=K\times M$ be the Fitting subgroup of G, where $K=O_p(G)$ and $M=O_q(G)$. Let also H_1/H be a minimal normal subgroup of G/H. Obviously, H_1 is not nilpotent and since H_1/H is a primary group we may assume that H_1/H is a p-group. Since the Fitting subgroup of a solvable group is not trivial, we have $M\neq\{e\}$. Otherwise, we get a contradiction because H_1 would be a nilpotent normal subgroup of G greater than its Fitting subgroup. Furthermore, suppose that M is an elementary abelian q-group and $K=\{e\}$ (if it is not the case we can replace G by the factor group $G/(K\times\Phi(M))$.

Let P be a Sylow p-subgroup of G. Since H_1 is not nilpotent, P acts nontrivially on M. By Lemma 4.2, P is cyclic and by Lemma 4.1(b), $|P:C_P(M)|=p$. It follows from well-known properties of the Fitting subgroup that $C_G(H)\leq H$, so $C_P(M)=C_P(H)\leq H=M$. Hence, $C_P(M)=\{e\}$ and then $|P|=p$. Therefore $P\leq H_1$, and G/H_1 is a cyclic q-group by Lemma 4.2.

Now, observe that $|G/H_1|=q$. In fact, if $C_{G/M}(H_1/M)$ contains q-elements then $O_q(G/M)\neq\{e\}$ and it means that there exists a normal q-subgroup of G greater than M, which is not possible. So all q-elements of G/M act nontrivially on H_1/M and in view of Lemma 4.1 we get $|G/H_1|=q$.

Let $P=\langle x\rangle$ be a fixed Sylow p-subgroup and let $A=N_G(P)$. If $y\notin M$ is a q-element of G, then $x^y=x^im$ for $0<i<p$ and some $m\in M$. But every element of the form x^iu, $u\in M$, is conjugated to x^i by some element $t\in M$. Let $t\in M$ be such that $(x^i)^t=x^im^{-1}$. Then

$$x^{yt}=(x^im)^t=x^im^{-1}m=x^i.$$

Therefore, $yt\in A$ and of course yt is a q-element. It is also clear that $A\cap M=\{e\}$ and so $|A|=pq$. Let $A=N_G(P)=\langle x,y\rangle$, where $\langle x\rangle=P$ and $o(y)=q$. Let also $t\in A$ be such that t does not normalize some p-subgroup of G and let $z\in M$ be such that $tz=zt$. Now, the subgroup $B=\langle t,z\rangle$ is abelian of order q^2 and $A\wedge B=\langle t\rangle$. Take a p-element v, $v\notin P$, such that $t\notin N_G(\langle v\rangle)$ and set $C=\langle v\rangle$. Then $(A\wedge B)\vee C=\langle t,C\rangle=G$ because $\langle C,C^t\rangle=PM$ and $t\notin PM$. Moreover, $A\vee C=B\vee C=G$. Hence, the subgroups A, B, and C generate a sublattice of $\mathcal{L}(G)$ isomorphic to $\mathcal{D}_1$. This contradiction ends the proof. □

As an immediate consequence of the above lemmas we obtain the following theorem.

THEOREM 4.1. *Let G be a nonnilpotent group with $|\pi(G)|=2$. If G is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free, then $G=P\rtimes Q$, where P is a Sylow p-subgroup, and Q is a Sylow q-subgroup for some different primes p and q. Moreover:*

(a) *P is powerful or 2-generated;*
(b) *$Q = \langle y\rangle$ is cyclic with $y^q \in C_G(P)$;*
(c) *Q acts irreducibly on $P/\Phi(P)$ or it acts on $P/\Phi(P)$ by power automorphisms.*

LEMMA 4.5. *Let G be a solvable group and let $|\pi(G)| = 3$. If G is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free, then G is L-decomposable.*

Proof. Assume that G is L-indecomposable and $\pi(G) = \{p, q, r\}$, where p, q, r are distinct primes. Since G is solvable, there exist a Sylow p-subgroup P, a Sylow q-subgroup Q and a Sylow r-subgroup R such that $G = PQR$ and PQ, PR and QR are subgroups of G.

Suppose first that one of the subgroups P, Q, R, say P, is normal in G. We assume also that $\Phi(P) = \{e\}$, that is P is elementary abelian. If it is not the case, we replace G by the factor group $G/\Phi(P)$. By assumption QR is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free, so in view of Lemma 4.4, we may assume that $Q \lhd QR$. Since G is L-indecomposable, one of the subgroups Q or R is not contained in $C_G(P)$. If $Q \leq C_G(P)$, then $G = (P \times Q) \rtimes R$ and we obtain a contradiction by Lemma 4.3. Hence $Q \nleq C_G(P)$. Since the subgroup PQ is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free, Q is cyclic by Lemma 4.2. We set $Q = \langle a\rangle$. Observe that $\langle a^q\rangle$ is a normal subgroup in $QR = Q \rtimes R$ and by Lemma 4.1(b) $a^q \in C_G(P)$. Hence $\langle a^q\rangle$ is normal in G and we replace G by $G/\langle a^q\rangle$ that is we assume $a^q = e$.

If $R \nleq C_G(P)$, then by Lemma 4.2 R is also cyclic; set $R = \langle b\rangle$. If $QR = Q \times R$, then we get a contradiction by Lemma 4.1. Thus, $QR = Q \rtimes R$. Since the subgroups PR and QR are $\mathcal{D}_1$-free or $\mathcal{D}_2$-free, we have $b^r \in C_G(P) \cap C_G(Q)$. Hence, $\langle b^r\rangle$ is normal in G and then we can suppose $b^r = e$. Therefore, QR is a regular group of automorphisms of P of order qr. But by [3], Theorem 5.3.14, QR is cyclic which again contradicts to Lemma 4.1. Hence, $R \subseteq C_G(P)$ and since $|Q| = q$ it follows that $R = C_{QR}(P) \lhd QR$, a contradiction.

Suppose now that none of the subgroups P, Q, R is normal in G. By 10.1.10 of [8] at least one of them, say P, is not cyclic. Then it follows from Lemma 4.2 that $P \lhd PR$ and $P \lhd PQ$; hence $P \lhd G$, the final contradiction. □

THEOREM 4.2. *If G is $\mathcal{D}_1$-free and $\mathcal{D}_2$-free, then G is a direct product of p-groups and $\{p, q\}$-groups with pairwise relatively prime orders.*

Proof. By Lemma 4.5, the theorem is true when $|\pi(G)| \leq 3$. So assume that $|\pi(G)| \geq 4$ and let G be a counterexample of minimal order. Since G is solvable, there exist Sylow p_i-subgroups P_i of the group G such that $G = P_1 \cdots P_n$ and $P_iP_j = P_jP_i$, for $i, j \in \{1, \ldots, n\}$. Let $H = P_2 \cdots P_n$. By choice of G, H is a direct product of p-groups and $\{p, q\}$-groups with pairwise relatively prime orders. Since P_1 is not a direct factor of G, H has a direct factor which does not centralize P_1. If for some $i, j \in \{2, \ldots, n\}$, P_iP_j is not a direct product of its Sylow subgroups, then by minimality of G and by Lemma 4.5 we have

$P_1P_iP_j = P_1 \times P_iP_j$. Hence, there exists $i \in \{2, \ldots, n\}$ such that P_i is a direct factor of H and $P_1 \nleq C_G(P_i)$. Therefore, the group P_1P_i is L-indecomposable. Let us consider the subgroup P_j of H, where $j \neq i$. Since $P_1P_iP_j < G$, by choice of G we have $P_1P_iP_j = P_1P_i \times P_j$. Hence, P_1P_i is a direct factor in G. This contradiction ends the proof. $\square$

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