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ON 2-KNOTS WITH TOTAL WIDTH EIGHT

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Abstract. A 2-knot is (the isotopy class of) a 2-sphere smoothly embedded in 4-space. The apparent contour of a generic planar projection of a 2-knot divides the plane into several regions, and to each such region, we associate the number of sheets covering it. The total width of a 2-knot is defined to be the minimum of the sum of these numbers, where we take the minimum among all generic planar projections of the given 2-knot. In this paper, we show that a 2-knot has total width eight if and only if it is an n-twist spun 2-bridge knot for some $n \neq \pm 1$.

1. Introduction

By a *surface knot,* we mean (the isotopy class of) a closed connected (possibly nonorientable) surface smoothly embedded in $\mathbf{R}^4$. A surface knot is called a 2-*knot* if it is homeomorphic to the 2-dimensional sphere S^2.

In this paper, we study 2-knots by using generic planar projections. Usually, for the study of surface knots in $\mathbf{R}^4$, generic projections into $\mathbf{R}^3$ are used. For example, many important invariants have been constructed by using such projections into $\mathbf{R}^3$ (for example, see [4, 5]). Generic planar projections of surface knots have also been studied and certain interesting results have been obtained (for example, see [2, 3, 11, 13, 15]).

For a surface knot, its generic planar projections have fold points and cusps as their singularities. Cusps appear as discrete points and fold points appear as a 1-dimensional submanifold of the surface. Let us call the set of cusps and fold points in the surface the *singular set* and its image the *apparent contour.*

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Recently, in [12], the authors have developed a theory of generic planar projections of surface knots. In classical knot theory, from a knot diagram drawn on the plane, we can recover the original knot in 3-space. In a similar fashion, to a surface knot, we associate a planar object, called a *braided diagram*, which consists of the apparent contour of a generic projection into the plane and some "banded braids" attached to the arcs of the apparent contour. A *banded braid* is a braid together with a band spanned by two of the strings and disjoint from the other strings. Note that the bands correspond to the set of fold points. We have shown that a braided diagram can recover the original surface knot (for details, see Section 3 of the present paper).

On the other hand, in [13], the second author defined a numerical invariant of a surface knot using generic planar projections as follows. For a given surface knot, the apparent contour of a generic planar projection divides the plane into several regions, and to each such region, we associate the number of sheets covering it. The *total width* of a surface knot F is defined to be the minimum of the sum of these numbers, where we take the minimum among all generic planar projections of surface knots isotopic to F. Note that the total width is always a positive even integer. This was considered as an analogy of the widths for classical knots defined by Gabai [7].

In [13], the second author showed that a surface knot is trivial if and only if its total width is equal to two. Furthermore, if a 2-knot has total width less than or equal to six, then it must be trivial.

In this paper, we completely determine those 2-knots which have total width eight. More precisely, a 2-knot has total width eight if and only if it is an n-twist spun 2-bridge knot for some $n \neq \pm 1$. For the proof, we use the theory of braided diagrams developed in [12] by the authors. As a corollary of the proof, we also get a result concerning connected sums of twist spun 2-bridge knots.

The paper is organized as follows. In Section 2, we recall some materials from singularity theory necessary for our purpose. We also give a precise definition of the width of a surface knot and state our main theorem. In Section 3, we recall the definition and basic properties of braided diagrams of surface knots in order to make the paper self-contained. Most of the materials are taken from [12]. In Section 4, we prove our main theorem using the notion of a braided diagram. We also give a result which characterizes the connected sum of twist spun 2-bridges knots.

Throughout the paper, we work in the smooth category.

2. Preliminaries

In this section, we prepare several notions from singularity theory and recall the definition and some properties of the total width of a surface knot in $\mathbf{R}^4$. For singularity theory, for example, the reader is referred to [8].

Let F be a closed connected surface. Denote by $C^\infty(F, \mathbf{R}^2)$ the space of all smooth maps of F into $\mathbf{R}^2$, endowed with the Whitney C^∞ topology. Two maps g and $h \in C^\infty(F, \mathbf{R}^2)$ are said to be *equivalent* if there exist diffeomorphisms $\varphi : F \to F$ and $\psi : \mathbf{R}^2 \to \mathbf{R}^2$, such that $\psi \circ g = h \circ \varphi$. A map $g \in C^\infty(F, \mathbf{R}^2)$ is said to be C^∞ *stable* if there exists a neighborhood N_g of g in $C^\infty(F, \mathbf{R}^2)$ such that each h in N_g is equivalent to g.

Let $g : F \to \mathbf{R}^2$ be a smooth map. Then $q \in F$ is called a *fold point* if we can choose local coordinates (x, y) centered at q and (X, Y) centered at $g(q)$ such that g, in a neighborhood of q, is of the form $(X, Y) = (x, y^2)$. Moreover, $q \in F$ is called a *cusp* if we can choose local coordinates as above such that g, in a neighborhood of q, is of the form $(X, Y) = (x, xy + y^3)$. We denote by $S_1(g)$ the set of fold points and cusps, and by $S_1^2(g)$ the set of cusps. Note that $S_1(g)$ is a regular 1-dimensional submanifold of F while $S_1^2(g)$ is a finite set of points.

For a smooth map $g : F \to \mathbf{R}^2$, we denote by $S(g)$ the set of its singular points. It is known that a smooth map $g : F \to \mathbf{R}^2$ is C^∞ stable if and only if $S(g) = S_1(g)$, the map $f|_{S_1(g)\setminus S_1^2(g)}$ is an immersion with normal crossings, and for each cusp q, we have:

$$g^{-1}(g(q)) \cap S_1(g) = \{q\}.$$

The singular value set $g(S(g))$ is often called the *apparent contour* of g.

Let $g : F \to \mathbf{R}^2$ be a C^∞ stable map. For a point $q \in S(g) \setminus S_1^2(g)$, we give a local orientation to $S(g)$ at q as follows. First, we locally orient $g(S(g))$ near $g(q)$ so that the points in the left hand side region of $\mathbf{R}^2 \setminus g(S(g))$ have a larger number of inverse image points. Then we locally orient $S(g)$ at q so that $g|_{S(g)}$ preserves the orientations near q. It is easy to see that the local orientations vary continuously and that they define a globally well-defined orientation on $S(g)$.

By considering the "line" $dg_q(T_qS(g))$ for each $q \in S(g) \setminus S_1^2(g)$, we obtain a smooth map $S(g) \setminus S_1^2(g) \to \mathbf{R}P^1$. It is not difficult to see that this map extends to a smooth map $\tau_g : S(g) \to \mathbf{R}P^1$. We orient $\mathbf{R}P^1$ so that the rotation in the counter-clockwise direction corresponds to the positive direction of $\mathbf{R}P^1$. Then we define $\mathrm{rot}(g)$ to be the mapping degree of $\tau_g : S(g) \to \mathbf{R}P^1$. Then the following is proved in [9].

Proposition 2.1. *The Euler characteristic $\chi(F)$ of F coincides with $\mathrm{rot}(g)$.*

Let us now recall the notion of a total width of a surface knot.

Definition 2.2. Let $f : F \to \mathbf{R}^4$ be an embedding of a closed connected surface F. Then an orthogonal projection $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ is said to be *generic* with respect to f (or with respect to $f(F)$) if $\pi \circ f$ is C^∞ stable.

By [10], almost every orthogonal projection is generic with respect to f.

Definition 2.3. Let $f : F \to \mathbf{R}^4$ be an embedding of a closed connected surface F and $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ an orthogonal projection which is generic with respect to f. Set $g = \pi \circ f$. The apparent contour $g(S(g))$ divides the plane $\mathbf{R}^2$ into several regions. For a point x in a given region, we call the number of elements in the set $g^{-1}(x)$ the *local width*, which does not depend on a choice of x and is always even. Let $tw(f, \pi)$ (or $tw(f(F), \pi)$) be the total of the local widths over all the regions. The *total width* $tw(f(F))$ of a surface knot $f(F)$ is the minimum of $tw(\widetilde{f}, \widetilde{\pi})$, where $\widetilde{f}$ runs over all embeddings isotopic to f and $\widetilde{\pi}$ runs over all orthogonal projections which are generic with respect to $\widetilde{f}$.

It is easy to show that the total width of a trivial surface knot is equal to two, where a surface knot is *trivial* if it is the boundary of a (possibly nonorientable) 3-dimensional handlebody embedded in $\mathbf{R}^4$. In [13], the second author proved that a 2-knot has total width ≤ 6 if and only if it is trivial and that n-twist spun 2-bridge knots with $n \neq \pm 1$ have total width eight. In this paper, we prove the following.

Theorem 2.4. *A* 2*-knot has total width eight if and only if it is an n-twist spun* 2*-bridge knot for some $n \neq \pm 1$.*

3. Braided diagram

In order to prove Theorem 2.4, we need the notion of a braided diagram introduced in [12]. As this is new and is not widely known, in this section we review its definition and properties necessary for our purpose. Many of the materials in this section are thus taken from [12].

Let us first recall the notion of a banded braid. A usual *braid* is a finite disjoint union of arcs, called strings, embedded in $[0,1] \times \mathbf{R}^2$ such that the projection $pr_1 : [0,1] \times \mathbf{R}^2 \to [0,1]$ to the first factor restricted to each component is a diffeomorphism. We adopt the convention that the end points of the arcs lie on the lines $\{0,1\} \times (\mathbf{R} \times \{0\})$.

A *banded braid* b is a braid b_E together with a band, diffeomorphic to $[0,1] \times [-1,1]$, spanned by a pair of two strings and disjoint from the other strings, where the projection pr_1 restricted to the band is equivalent to the projection $[0,1] \times [-1,1] \to [0,1]$ to the first factor. We assume that the ends of the arcs and bands lie on the lines $\{0,1\} \times (\mathbf{R} \times \{0\})$. The braid b_E is called the *edge braid* of b, and the *hollow braid* of b is the braid obtained from b_E by taking off the two strings spanning the band. The number of strings of a banded braid b is, by convention, equal to that of its hollow braid. Two banded braids are considered to be the same if there exists a smooth 1-parameter family of banded braids connecting them.

Let F be a closed connected surface embedded in $\mathbf{R}^4$ and $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ an orthogonal projection which is generic with respect to F. We may assume that π is given by $\pi(x_1, x_2, x_3, x_4) = (x_1, x_2)$ for $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$. We set

$g = \pi|_F$, which is a C^∞ stable map. For simplicity, we assume that g has no cusps and $g|_{S(g)}$ is an embedding, which is enough for our purpose.

Let R be a region of $g(F) \setminus g(S(g))$. If R is not an open disk, then we take disjointly embedded arcs $a_1, a_2, \ldots, a_k$ in $\widetilde{R} = R \cup g(S(g))$ such that $(a_1 \cup a_2 \cup \cdots \cup a_k) \cap g(S(g)) = \partial(a_1 \cup a_2 \cup \cdots \cup a_k)$ and each component of $R \setminus (a_1 \cup a_2 \cup \cdots \cup a_k)$ is an open disk. We call the arcs $a_1, a_2, \ldots, a_k$ *additional arcs*. For each nondisk region of $g(F) \setminus g(S(g))$ we take such additional arcs and orient them arbitrarily. (In [12], for simplicity it is assumed that additional arcs in distinct regions do not intersect with each other, while in this paper, they may intersect. This will make essentially no difference.)

For each component of $g(S(g))$ that is disjoint from the additional arcs, we take a point as a vertex. The end points of the additional arcs are also considered to be vertices. The edges in $g(S(g))$ are oriented as in Section 2. The oriented planar graph thus obtained from $g(S(g))$ together with the additional arcs is denoted by $\Gamma_{F,\pi}$.

In order to construct a (banded) braid for each edge of $\Gamma_{F,\pi}$, we need to arrange F over the vertices of $\Gamma_{F,\pi}$ as follows. Let us begin by introducing two notions.

Definition 3.1. An isotopy $h_t : \mathbf{R}^4 \to \mathbf{R}^4$, $t \in [0,1]$, is said to be *vertical* if for all $t \in [0,1]$, $h_t(\{x\} \times \mathbf{R}^2) \subset \{x\} \times \mathbf{R}^2$ holds for all $x \in \mathbf{R}^2$, i.e. $\pi \circ h_t = \pi$.

Definition 3.2. For $q \in S(g)$, the line

$$\mathrm{Ker}\big(dg_q : T_qF \to T_{g(q)}\mathbf{R}^2\big) \subset T_qF \subset \mathbf{R}^4$$

passing through q is called the *kernel line* at $q \in F$, where we identify the tangent plane T_qF to F at q as a plane in $\mathbf{R}^4$ passing through $q \in \mathbf{R}^4$ with the origin being identified with q.

Let v be a vertex of $\Gamma_{F,\pi}$. Note that $\pi^{-1}(v)$ contains a unique fold point. Then we arrange F by a vertical isotopy whose support lies in a small neighborhood of $\pi^{-1}(v)$ so that the points $\pi^{-1}(v) \cap F$ and the kernel line at the fold point all lie in $\{v\} \times (\mathbf{R} \times \{0\}) \subset \{v\} \times \mathbf{R}^2 = \pi^{-1}(v)$.

Remark 3.3. In the general case where g has cusps and/or $g|_{S(g)}$ has self-intersection points, we need to arrange F over the corresponding points as well.

Now, we are ready to construct a (banded) braid for each edge of $\Gamma_{F,\pi}$. Take an edge $e \subset g(S(g))$. Let $N(\partial e)$ be a small open neighborhood of the end point(s) of e in $\mathbf{R}^2$ and set $e' = e \setminus N(\partial e)$. If we identify e' with $[\varepsilon, 1-\varepsilon]$ and $\pi^{-1}(e')$ with $[\varepsilon, 1-\varepsilon] \times \mathbf{R}^2$, where $\varepsilon > 0$ is a sufficiently small positive real number, then $\pi^{-1}(e') \cap F$ can be regarded as a braid with an odd number of strings. Note that exactly one of the strings consist of fold points of g. Then we replace this string with a band which corresponds to the union of small kernel line segments at the fold points. In this way, we get a "banded braid" in $\pi^{-1}(e')$. Since F has been arranged over the end point(s) ∂e of e appropriately, we can canonically extend this "banded braid" to a genuine banded braid.

Similarly, to each additional edge of $\Gamma_{F,\pi}$ we associate a usual braid. The oriented graph $\Gamma_{F,\pi}$ together with the (banded) braids associated with the edges is called the *braided diagram* associated with F.

PROPOSITION 3.4. *Let F and F' be surface knots in $\mathbf{R}^4$ such that an orthogonal projection $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ is generic with respect to both F and F'. If they have the same braided diagram with respect to π, then the surface knots F and F' are vertically isotopic.*

Proof. Set $\Gamma = \Gamma_{F,\pi} = \Gamma_{F',\pi}$. By construction, for each edge e of Γ, the associated (banded) braids for F and F' are vertically isotopic. So we may assume that they are the same for each e. Then for each region R of $\pi(F) \setminus \Gamma = \pi(F') \setminus \Gamma$, the disjoint union of 2-disks $\pi^{-1}(R) \cap F$ and $\pi^{-1}(R) \cap F'$ have common boundaries. Since the configuration space of a fixed number of distinct points on the plane has vanishing second homotopy group (see, for example, [6]), they are vertically isotopic relative to their boundaries. Therefore, we have the desired conclusion. □

Note that a braided diagram is not unique for a surface knot. However, a surface knot having a given braided diagram is unique up to vertical isotopy.

4. Proof

In this section, we prove Theorem 2.4.

Proof of Theorem 2.4. In [13], it has been shown that an n-twist spun 2-bridge knot with $n \neq \pm 1$ has total width equal to eight.

Conversely, suppose that F is a 2-knot whose total width is equal to eight. Then there exists an orthogonal projection $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ which is generic with respect to F such that $tw(F, \pi) = 8$. Set $g = \pi|_F : F \to \mathbf{R}^2$, which is a C^∞ stable map.

Let us first list up all the possibilities for the apparent contour of g by using Proposition 2.1. By an argument similar to that in [13, Section 5], we can show that the apparent contour of a stable map of a surface of Euler characteristic 2 whose total width is equal to 8 is equivalent to one of the figures as depicted in Figure 1 up to a diffeomorphism of $\mathbf{R}^2$. In the figures, the integers (without parentheses) indicate the local widths of the regions.

It is known that if a C^∞ stable map has no cusp and the apparent contour has an odd number of self-intersection points, then the source surface is nonorientable [1]. Therefore, figures (2), (3), and (4) of Figure 1 do not occur, since our source surface is the 2-sphere, which is orientable.

If the apparent contour of g is given by figure (5) or (6), then for an orthogonal projection $\pi_1^2 : \mathbf{R}^2 \to \mathbf{R}^1$, the composition $\pi_1^2 \circ g = \pi_1^2 \circ \pi|_F$ is a

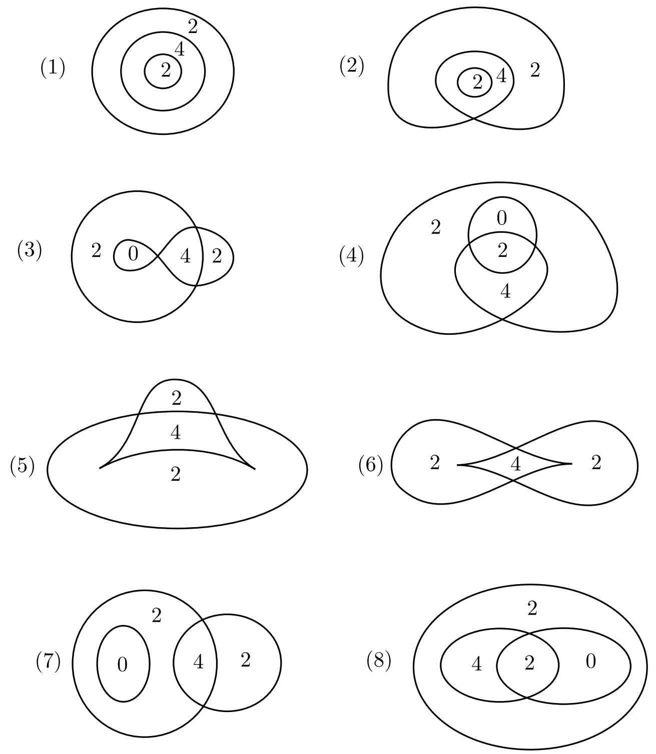

FIGURE 1. List of possible apparent contours.

Morse function with exactly two critical points. Hence, the 2-knot F is trivial (for example, see [3]).

In the case of figure (7), by [13, Lemma 3.1], the 2-knot F is a connected sum of two surface knots one of which has zero Euler characteristic. This is a contradiction and this case does not occur.

In the case of figure (8), by using an argument similar to that in [13, Lemma 3.1], we can cut open the 2-knot along the inverse image of a line segment, which is depicted by a dotted line in Figure 2, and we attach two disks along the boundary circles. Then the resulting surface is the disjoint union of two 2-spheres (for example, see [3]). Furthermore, we see that the above surgery was performed only on one of the two components. This implies that the original surface is not connected. This is a contradiction.

Therefore, we may assume that the apparent contour is of the form as depicted in Figure 1(1). We may assume that it forms a disjoint union of three concentric circles, whose common center we denote by c.

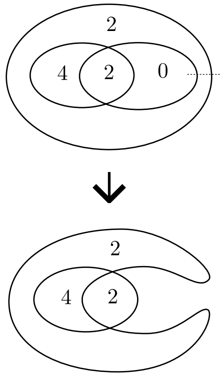

FIGURE 2. A surgery in case (8).

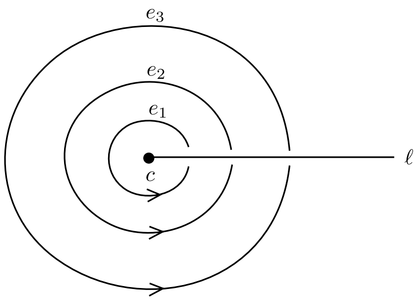

FIGURE 3. Oriented arcs e_1, e_2, e_3 and a half line ℓ.

Let ℓ be a half line having c as the end point. We take two additional arcs contained in ℓ to get the graph $\Gamma_{F,\pi}$ from $g(S(g))$. Furthermore, let e_1, e_2, and e_3 be oriented arcs contained in the apparent contour $g(S(g))$ and disjoint from ℓ as depicted in Figure 3. (Note that the orientation of the arc e_1 is not consistent with that of the apparent contour introduced in Section 2). More precisely, each of e_1, e_2, and e_3 is obtained from a component of $g(S(g))$ by removing a small open neighborhood of the intersection point with ℓ. In the notation of Section 3, e_i corresponds to e' for an appropriate edge e of $\Gamma_{F,\pi}$. Furthermore, let e_{i+} (resp. e_{i-}) be an arc parallel to e_i which is very close to e_i and lies on its outer side (resp. on its inner side), $i = 1, 2, 3$.

The inverse image $g^{-1}(\ell)$ is in general a disjoint union of several arcs. Since our source surface is the 2-sphere and is connected, $g^{-1}(\ell)$ must be connected. Furthermore, the function $g|_{g^{-1}(\ell)} : g^{-1}(\ell) \to \ell$ is a Morse function with exactly three critical points. Therefore, we may assume that $g^{-1}(\ell) \subset$

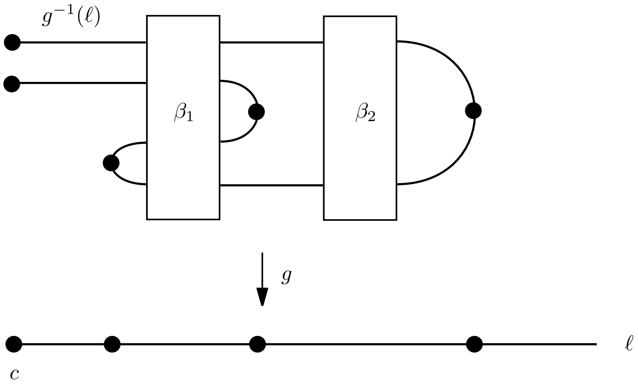

FIGURE 4. The arc $g^{-1}(\ell)$ embedded in $\pi^{-1}(\ell) = \ell \times \mathbf{R}^2$.

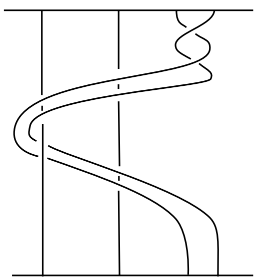

FIGURE 5. An example of the 4-string braid $g^{-1}(e_{1+})$.

$\pi^{-1}(\ell) = \ell \times \mathbf{R}^2$ is in a form as depicted in Figure 4, where β_1 is a 4-string pure braid and β_2 is a 2-string pure braid.

Now, $g^{-1}(e_{1-}) \subset \pi^{-1}(e_{1-}) = e_{1-} \times \mathbf{R}^2$ can be regarded as a 2-string pure braid. A pure braid can be identified with an element of the fundamental group of a certain configuration space, and $g^{-1}(e_{1-})$ corresponds to the neutral element in the fundamental group, since the inverse image of the disk bounded by the circle containing e_{1-} gives the null homotopy (see the proof of Proposition 3.4). This means that $g^{-1}(e_{1-})$ is the trivial 2-string braid.

Let us now consider the 4-string braid $g^{-1}(e_{1+}) \subset \pi^{-1}(e_{1+}) = e_{1+} \times \mathbf{R}^2$. As has been explained in Section 3, to each fold point of $g^{-1}(e_1)$ is associated a small kernel line segment, and this gives rise to a band whose center coincides with the set of fold points contained in $g^{-1}(e_1)$. Hence, the 4-string braid $g^{-1}(e_{1+})$ is the edge braid of a 2-string banded braid whose hollow braid is trivial (see Figure 5 for an example).

Let a and a' be oriented line segments parallel to ℓ connecting end points of e_{1+} and e_{2-} as depicted in Figure 6. Note that e_{2-} is homotopic to the

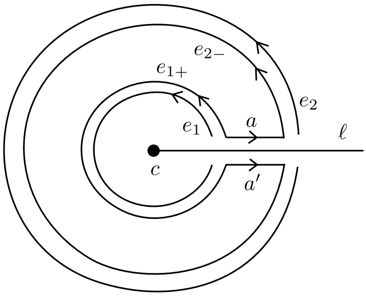

FIGURE 6. e_{2-} is homotopic to $\overline{a} * e_{1+} * a'$ relative to end points.

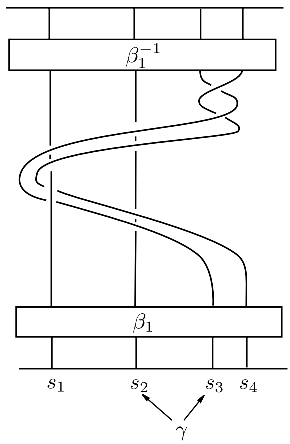

FIGURE 7. 4-string braid $g^{-1}(e_{2-})$.

oriented arc $\overline{a} * e_{1+} * a'$ relative to end points, where $\overline{a}$ denotes the arc a with the reversed orientation and "$*$" means the product of arcs. More precisely, the simple closed curve $\overline{e_{2-}} * (\overline{a} * e_{1+} * a')$ bounds a disk in $\mathbf{R}^2$ which does not intersect $g(S(g))$. Then again by using an argument of the fundamental group of a certain configuration space, we see that $g^{-1}(e_{2-})$ is a 4-string braid as depicted in Figure 7, where β_1 is the 4-string pure braid appearing in Figure 4 and corresponds to $g^{-1}(a) \subset a \times \mathbf{R}^2$ (or $g^{-1}(a') \subset a' \times \mathbf{R}^2$).

Note that the 2-string braid γ in Figure 7 corresponds to the edges of the band contained in the banded braid associated with $g^{-1}(e_2)$, according to Figure 4. Therefore, the two strings s_2 and s_3 must span a band disjoint from the other two strings. Note that the band is not linked with the left-most string s_1, since s_2 is not linked with s_1.

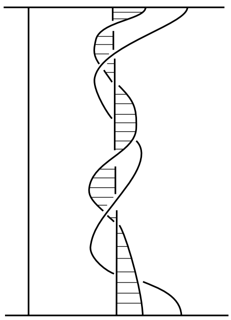

FIGURE 8. A conjugate of the 4-string braid $g^{-1}(e_{2-})$ and a band spanned by s_2 and s_3.

Since s_3 and s_4 span a band which is disjoint from the other two strings, we see that, up to a conjugate by a pure braid, the 4-string braid $g^{-1}(e_{2-})$ is of the form $(\sigma_2^{\varepsilon}\sigma_3^{\varepsilon})^{3n}$ for some integer n, where $\varepsilon = \pm 1$, and σ_1, σ_2, and σ_3 are the standard generators of the 4-string braid group. Therefore, the closure of the 4-string braid $g^{-1}(e_{2-})$ is equivalent to the closure of a 4-string pure braid as depicted in Figure 8, where the case $n = 2$ is shown.

Now, the 2-string braid $g^{-1}(e_{2+})$ is obtained from $g^{-1}(e_{2-})$ by removing the strings s_2 and s_3. Therefore, $g^{-1}(e_{2+})$ is the trivial 2-string braid. Hence, the 2-string braid $g^{-1}(e_{3-})$ is also trivial.

Summarizing, we see that the braided diagram associated with F is equivalent to that associated with the 2-knot F' constructed by rotating a properly embedded arc in $\pi^{-1}(\ell)$ as depicted in Figure 9 around $\pi^{-1}(c) \cong \mathbf{R}^2$ in such a way that the box A rotates n times during the rotation.

Now, by Proposition 3.4 (see also [12]), F and F' are vertically isotopic. On the other hand, the 2-knot F' is nothing but the n-twist spin of the knot K as depicted in Figure 10. Since F has total width different from two, it is nontrivial, and hence K is nontrivial. This implies that K is a 2-bridge knot. If $n = \pm 1$, then the n-twist spin of K is trivial (for example, see [16]). Therefore, we have $n \neq \pm 1$. This completes the proof. □

REMARK 4.1. In [14], it has been shown that a fibered 2-knot whose fiber is a punctured lens space is actually a 2-twist spun 2-bridge knot. So, this forms a subclass of 2-knots with total width equal to eight.

By an argument similar to that in the proof of Theorem 2.4, we can also prove the following.

THEOREM 4.2. *Let F be a 2-knot in $\mathbf{R}^4$ which admits an orthogonal projection $\pi : \mathbf{R}^4 \to \mathbf{R}^2$, such that*

(1) *π is generic with respect to F,*

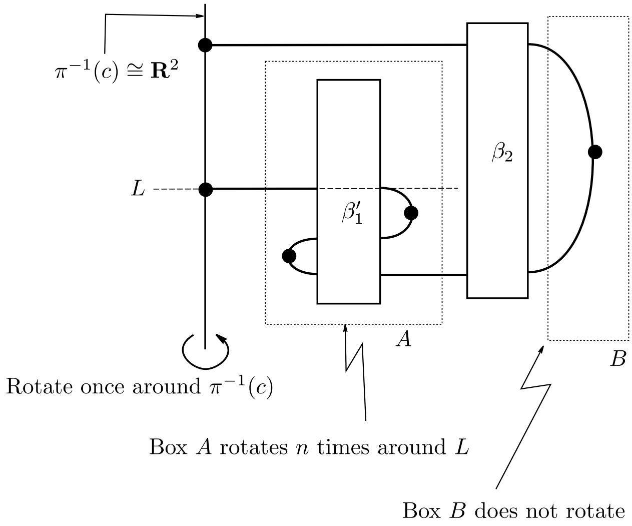

FIGURE 9. Construction of the 2-knot F'.

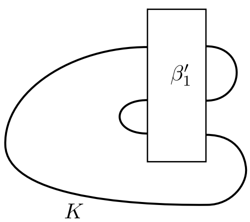

FIGURE 10. The 2-bridge knot K.

(2) *the apparent contour of* $g = \pi|_F$ *consists of* $2k+1$ *concentric circles for some* $k \geq 1$, *and*
(3) *the local width of the innermost region is equal to* 2 *and the widths of the other bounded regions are equal to* 2 *or* 4 (*see Figure* 11).

Then F *is the connected sum of* 2*-knots* $F_1, F_2, \dots, F_k$, *where each* F_i *is an* n_i*-twist spin of a knot* K_i *for some* $n_i \in \mathbf{Z}$, *and* K_i *is a knot with bridge index at most* 2.

Proof. By an argument as in the proof of Theorem 2.4, we see that F is isotopic to a 2-knot F' which is obtained by rotating an arc as in Figure 12 around $\mathbf{R}^2 \subset \mathbf{R}^4$, where each box A_i rotates n_i times during the rotation for some $n_i \in \mathbf{Z}$, $i = 1, 2, \dots, k$, and each $\bar{K}_i$ is an arc corresponding to a knot K_i with bridge index being equal to 1 or 2.

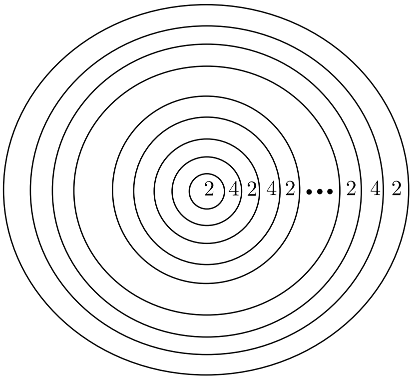

FIGURE 11. Apparent contour consisting of concentric circles.

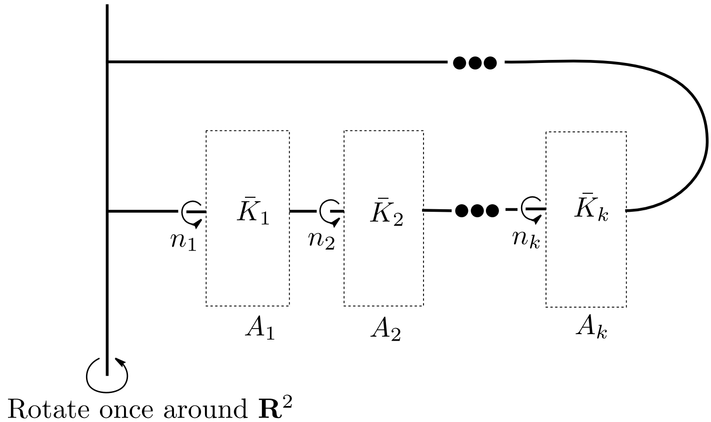

FIGURE 12. The 2-knot F'.

It is not difficult to see that this 2-knot F' is isotopic to the connected sum $F_1 \sharp F_2 \sharp \cdots \sharp F_k$, where F_i is the n_i-twist spin of the knot K_i, $i = 1, 2, \ldots, k$. This completes the proof. $\square$

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