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ON SOME WEIGHTED NORM INEQUALITIES FOR LITTLEWOOD–PALEY OPERATORS

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ABSTRACT. It is shown that the $L^p_w, 1<p<\infty$, operator norms of Littlewood–Paley operators are bounded by a multiple of $\|w\|_{A_p}^{\gamma_p}$, where $\gamma_p = \max\{1, p/2\}\frac{1}{p-1}$. This improves previously known bounds for all $p>2$. As a corollary, a new estimate in terms of $\|w\|_{A_p}$ is obtained for the class of Calderón–Zygmund singular integrals commuting with dilations.

1. Introduction

It is well known that many classical operators in Harmonic Analysis are bounded on the weighted space $L^p_w, 1<p<\infty$, provided a weight w satisfies the A_p condition. However, the sharp dependence of the corresponding L^p_w operator norms in terms of the A_p characteristic of w is known only for few operators. In this paper, we obtain several new estimates for Littlewood–Paley and Calderón–Zygmund operators.

Throughout the paper, all functions considered are real-valued. We recall that L^p_w denotes the space of all measurable functions f on $\mathbb{R}^n$ with norm

$$\|f\|_{L^p_w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p},$$

where a weight w is supposed to be a non-negative locally integrable function. A weight w satisfies the $A_p, 1<p<\infty$, Muckenhoupt condition [10] if

$$\|w\|_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1} < \infty,$$

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where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the axes. We call $\|w\|_{A_p}$ the A_p characteristic of w. Given a bounded operator T on a Banach space X, $\|T\|_X$ is the standard operator norm of T defined by $\sup_{\|f\|_X\le 1}\|Tf\|_X$.

In [1], Buckley proved that for the Hardy–Littlewood maximal operator M,

$$\|M\|_{L^p_w} \le c\|w\|_{A_p}^{\frac{1}{p-1}} \quad (1<p<\infty), \tag{1.1}$$

and this result is sharp in the sense that the right-hand side of (1.1) cannot be replaced by $\varphi(\|w\|_{A_p})$ for any positive nondecreasing function φ growing more slowly than $t^{1/(p-1)}$.

For $1<p<\infty$, denote $\alpha_p=\max\{1,1/(p-1)\}$. It was also shown in [1] that for the convolution Calderón–Zygmund singular integral operators T,

$$\|T\|_{L^p_w} \le c\|w\|_{A_p}^{\frac{p}{p-1}} \quad (1<p<\infty), \tag{1.2}$$

and the best power of $\|w\|_{A_p}$ is at least α_p. In the case when $p=2$ and $T=H$ is the Hilbert transform, Petermichl and Pott [13] improved the power of $\|w\|_{A_2}$ from 2 to 3/2, which in turn was improved by Petermichl [11] to the best possible linear dependence on $\|w\|_{A_2}$. Then the same dependence was obtained for the Riesz transforms [12].

In a recent paper by Dragičević et al. [5], sharp L^p_w estimates in terms of $\|w\|_{A_p}$ in the Rubio de Francia extrapolation theorem have been established. In particular, the main result of [5] shows that if a sublinear operator T is bounded on L^2_w with the linear bound for $\|T\|_{L^2_w}$ in terms of $\|w\|_{A_2}$, then T is bounded on $L^p_w, 1<p<\infty$, and $\|T\|_{L^p_w}$ is at most a multiple of $\|w\|_{A_p}^{\alpha_p}$. Therefore, the sharp L^2_w bound for the Hilbert and Riesz transforms along with extrapolation shows that for these operators the exponent $p/(p-1)$ can be improved to the best possible exponent α_p for all $p>1$. For more general singular integrals, the question about the best power of $\|w\|_{A_p}$ in (1.2) is still open.

Denote by $\mathcal{D}$, the set of all dyadic cubes in $\mathbb{R}^n$. Define the dyadic square function $S_d(f)$ by

$$S_d(f)^2(x)=\sum_{Q\in\mathcal{D}}(f_Q-f_{\widetilde{Q}})^2\chi_Q(x),$$

where $f_Q=|Q|^{-1}\int_Q f$ and $\widetilde{Q}$ denotes the smallest dyadic cube containing Q. Let $\varphi\in C^\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi\subset\{|x|\le 1\}$, and $\int\varphi=0$. Let $\mathbb{R}^{n+1}_+=\mathbb{R}^n\times\mathbb{R}_+$ and $\Gamma_\alpha(x)=\{(y,t)\in\mathbb{R}^{n+1}_+ : |y-x|<\alpha t\}, \alpha>0$. The continuous square function $S_{\varphi,\alpha}(f)$ is defined by

$$S_{\varphi,\alpha}(f)^2(x)=\iint_{\Gamma_\alpha(x)}|f*\varphi_t(y)|^2\frac{dy\,dt}{t^{n+1}} \quad (\alpha>0),$$

where $\varphi_t(y)=t^{-n}\varphi(y/t)$. We drop the subscript α if $\alpha=1$.

It was shown independently by Hukovic, Treil and Volberg [7], and Wittwer [18] that the L^2_w operator norm of $S_d(f)$ (in the case $n=1$) is bounded linearly by $\|w\|_{A_2}$, and this is best possible. The same was proved by Wittwer [19] regarding $S_\varphi(f)$ defined by means of the wavelet-type kernel φ. Therefore, setting S for any of these square functions, we obtain by the above extrapolation argument that

$$\|S\|_{L^p_w} \le c\|w\|_{A_p}^{\alpha_p} \quad (1<p<\infty). \tag{1.3}$$

Unlike singular integrals, this estimate is known to be sharp only for $1<p\le 2$ (see [5]). It was mentioned in [5] that it is unclear whether (1.3) is sharp for $p>2$. In a recent paper [9], it was observed that for $S_{\varphi,\alpha}(f)$ the linear dependence on $\|w\|_{A_p}$ in (1.3) can be improved to $\|w\|_{A_p}^{\frac{1}{2}+\frac{1}{p-1}}$ for all $p>3$.

In this paper, we improve (1.3) for all $p>2$. Our new bound improves also the exponent $\frac{1}{2}+\frac{1}{p-1}$. As in [9], instead of $S_{\varphi,\alpha}(f)$ we consider its pointwise majorant, the Littlewood–Paley function $g^*_{\varphi,\mu}(f)$ defined by

$$g^*_{\varphi,\mu}(f)^2(x) = \iint_{\mathbb{R}^{n+1}_+} |f*\varphi_t(y)|^2 \left(\frac{t}{t+|x-y|}\right)^{\mu n} \frac{dy\,dt}{t^{n+1}} \quad (\mu>0).$$

It is easy to see that $S_{\varphi,\alpha}(f)(x) \le c_{\alpha,\mu,n} g^*_{\varphi,\mu}(f)(x)$.

Denote by $S(f)$ either $S_d(f)$ or $g^*_{\varphi,\mu}(f)$. Although the linear bound for $\|S\|_{L^2_w}$ in terms of $\|w\|_{A_2}$ is best possible, we obtain a straightened form of this result through the two-weighted L^2 norm inequality. Given two weights w and v, define their A_2 characteristic by

$$\|(w,v)\|_{A_2} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q v^{-1}\right).$$

Next, by $\|w\|_{A_\infty}$ we denote the characteristic of w with the property $\|w\|_{A_\infty} \le c_{r,n}\|w\|_{A_r}$ for any $r\ge 1$ (its precise definition is given in Section 2; actually it will be important for us only that this property holds for some fixed $r>2$).

THEOREM 1.1. *Let $S(f)$ be either $S_d(f)$ or $g^*_{\varphi,\mu}(f)$ for $\mu>2$. Then for any weights w and v, and for any measurable function f,*

$$\|S(f)\|_{L^2_w} \le c\sqrt{\|v^{-1}\|_{A_\infty}\|(w,v)\|_{A_2}}\,\|f\|_{L^2_v}, \tag{1.4}$$

where $c=c_n$ if $S(f)=S_d(f)$ and $c=c_{\varphi,\mu,n}$ if $S(f)=g^*_{\varphi,\mu}(f)$.

We should mention that the proof of this theorem is based essentially on weighted Littlewood–Paley theory developed in the works by Wilson [16], Wheeden and Wilson [15].

Since $\|v^{-1}\|_{A_\infty} \le c\|v^{-1}\|_{A_2} = c\|v\|_{A_2}$, in the case of equal weights we clearly obtain from (1.4) the linear dependence on $\|w\|_{A_2}$ which yields (1.3) by extrapolation. This gives the best possible result only for $1<p\le 2$. However, inequality (1.4) contains much more information than the one with $w=v$. The following extrapolation result clarifies this point.

THEOREM 1.2. *Suppose that for two measurable functions f and g,*

$$\|f\|_{L^2_w} \le c\sqrt{\|v^{-1}\|_{A_\infty}\|(w,v)\|_{A_2}}\|g\|_{L^2_v} \tag{1.5}$$

for all weights w and v, where c is some absolute constant. Then for any $p>1$ and for any weight $w \in A_p$,

$$\|f\|_{L^p_w} \le c\|w\|_{A_p}^{\max\{1,p/2\}\frac{1}{p-1}}\|g\|_{L^p_w}, \tag{1.6}$$

where a constant c depends only on p and n.

As an immediate consequence of Theorems 1.1 and 1.2, we have the following.

COROLLARY 1.3. *Let S be as in Theorem 1.1. For any $1<p<\infty$,*

$$\|S\|_{L^p_w} \le c\|w\|_{A_p}^{\max\{1,p/2\}\frac{1}{p-1}},$$

where $c=c_{p,n}$ if $S(f)=S_d(f)$ and $c=c_{\varphi,\mu,p,n}$ if $S(f)=g^*_{\varphi,\mu}(f)$.

Let Ω be homogeneous of degree zero, infinitely differentiable on the unit sphere S^{n-1}, and $\int_{S^{n-1}}\Omega=0$. Set $K(x)=\frac{\Omega(x)}{|x|^n}$ and $K_\varepsilon(x)=K(x)\chi_{\{|x|>\varepsilon\}}$. Consider the following Calderón–Zygmund singular integral operators

$$Tf(x)=\lim_{\varepsilon\to 0} f*K_\varepsilon(x) \quad \text{and} \quad T_*f(x)=\sup_{\varepsilon>0}|f*K_\varepsilon(x)|.$$

COROLLARY 1.4. *For any $1<p<\infty$, we have*

$$\|T_*\|_{L^p_w} \le c\|w\|_{A_p}^{\frac{1}{2}+\max\{1,p/2\}\frac{1}{p-1}}, \tag{1.7}$$

where a constant c depends only on p,n and Ω.

Several remarks about this result are in order. First, we do not know whether the best bounds for T and T_* are necessarily the same. As we mentioned above, for the Hilbert or Riesz transforms instead of T_* in (1.7) a better L^p_w bound is known. However, estimate (1.7) with the corresponding maximal transforms seems to be new. Next, even for T instead of T_*, Corollary 1.4 represents an improvement of (1.2) for the class of singular integrals that we consider.

The paper is organized as follows. Section 2 contains some necessary preliminary information. In Section 3, we prove Theorems 1.1, 1.2, and Corollary 1.4. Section 4 contains a further discussion of the best possible exponents for Littlewood–Paley operators and maximal singular integrals.

2. Preliminaries

We recall that the Hardy–Littlewood maximal operator is defined by

$$Mf(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes Q containing a point x.

A weight w satisfies the A_1 condition if there exists $c>0$, such that

$$Mw(x)\le cw(x) \quad \text{a.e.} \tag{2.1}$$

The smallest possible c in (2.1) is denoted by $\|w\|_{A_1}$. We shall use the well-known fact (see [4]) saying that if $Mf<\infty$ a.e., then $(Mf)^\alpha$ satisfies the A_1 condition for all $0<\alpha<1$, or, in other words, for any cube Q,

$$\frac{1}{|Q|}\int_Q (Mf)^\alpha\,dx\le c_{\alpha,n}\inf_Q (Mf)^\alpha \quad (0<\alpha<1). \tag{2.2}$$

Given a weight w and a cube Q, set $w(Q)=\int_Q w$, and define

$$\|w\|_{A_\infty}=\sup_Q\frac{1}{w(Q)}\int_Q M(w\chi_Q)(x)\,dx.$$

Using Buckley's result (1.1), one can easily show (see, e.g., [6, p. 359] or [9, Lemma 3.5]) that

$$\|w\|_{A_\infty}\le c_{r,n}\|w\|_{A_r} \quad (1\le r<\infty). \tag{2.3}$$

We say that $I\subset\mathbb{R}$ is a dyadic interval if I is of the form $(\frac{j}{2^k},\frac{j+1}{2^k})$ for some integers j and k. We say that $Q\subset\mathbb{R}^n$ is a dyadic cube if Q is a Cartesian product of n dyadic intervals of equal lengths. The side length of Q we denote by ℓ_Q. If Q is a dyadic cube with $\ell_Q=2^m$, then by $\widetilde{Q}$ we denote the unique dyadic cube containing Q such that $\ell_{\widetilde{Q}}=2^{m+1}$.

Following [15], we say that a measurable function f is in standard form if

$$f(x)=\sum_{Q\in\mathcal{D}}\lambda_Q b_Q(x),$$

where λ_Q are some constants such that $\lambda_Q\ne 0$ only for a finite number of Q, and the functions b_Q satisfy $\|\nabla b_Q\|_\infty\le \ell_Q^{-1}|Q|^{-1/2}$, $\operatorname{supp} b_Q\subset 3Q$, and $\int b_Q=0$. Here, rQ denotes the cube with the same center as Q such that $\ell_{rQ}=r\ell_Q$. For a function f in standard form, set [16, p. 666]

$$\widetilde{S}(f)(x)=\left(\sum_{Q\in\mathcal{D}}\frac{|\lambda_Q|^2}{|Q|}\chi_{3Q}(x)\right)^{1/2}.$$

The following several results are based essentially on the deep Chang–Wilson–Wolff theorem [2] saying that the boundedness of the square function $S(f)$ implies the exponential square integrability of f.

Proposition 2.1. *Suppose that f is in standard form. Then for any weight w,*

$$\|f\|_{L^p_w}\le c\|w\|_{A_\infty}^{1/2}\|\widetilde{S}(f)\|_{L^p_w} \quad (0<p<\infty), \tag{2.4}$$

where a constant c depends only on p and n.

This proposition is contained in [15, Result 3], [16, Lemma 2.3].

PROPOSITION 2.2. *For any weight w and for any $f \in L^p_w$,*

$$\|f\|_{L^p_w} \le c\|w\|^{1/2}_{A_\infty}\|S_d(f)\|_{L^p_w} \quad (0<p<\infty), \tag{2.5}$$

where a constant c depends only on p and n.

In the case $p=2$, inequality (2.5) was proved in [6, p. 358] with $\|w\|_{A_1}$ instead of $\|w\|_{A_\infty}$ on the right-hand side, and in the same paper it was mentioned on page 359 that actually $\|w\|_{A_1}$ can be replaced by any $\|w\|_{A_r}$, $r>1$. Exactly the same arguments work for $\|w\|_{A_\infty}$ and for any $p>0$. Note that the fact that inequality (2.4) holds also for the dyadic square function $S_d(f)$ was already mentioned in [16, p. 668].

Given a natural N, let $\mathcal{A}_N$ denote the set of all functions $\eta \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\eta \subset \{|x| \le 1\}$ and $\|\sum_{|\beta|\le N}|D^\beta\eta|\|_\infty \le 1$, where $\beta=(\beta_1,\dots,\beta_n)$ is a multi-index, and $|\beta|=\sum_i \beta_i$. Consider the grand maximal function $G_N(f)$ defined by

$$G_N(f)(x) = \sup_{\eta\in\mathcal{A}_N, t>0}|f*\eta_t(x)|.$$

Next, assume that φ satisfies the same conditions as in the Introduction, and additionally, that φ is radial and rough enough. The last property means that there are positive constants c and ξ such that

$$\int_s^\infty |\widehat{\varphi}(t,0,\dots,0)|^2\frac{dt}{t} \ge c(1+s)^{-\xi} \quad (s>0). \tag{2.6}$$

PROPOSITION 2.3. *Let φ be as above. Then there exist N depending on ξ and n and α depending on n such that for any weight w and for any f with $G_N(f) \in L^p_w$,*

$$\|G_N(f)\|_{L^p_w} \le c\|w\|^{1/2}_{A_\infty}\|S_{\varphi,\alpha}(f)\|_{L^p_w} \quad (0<p<\infty), \tag{2.7}$$

where a constant c does not depend on f and w.

This result is contained in [16], although it does not appear there in such an explicit form. Inequality (2.7) with f instead of $G_N(f)$ is just an immediate combination of Lemma 2.3 and an argument on page 671 from [16]. An explanation how to replace f by $G_N(f)$ is given in [16] on pages 672–674.

3. Proofs of main results

Proof of Theorem 1.1 for $S(f)=g^*_{\varphi,\mu}(f)$. By Fubini's theorem,

$$\int_{\mathbb{R}^n} g^*_{\varphi,\mu}(f)(x)^2 w(x)\,dx = \iint_{\mathbb{R}^{n+1}_+}|f*\varphi_t(y)|^2F(y,t)\frac{dy\,dt}{t}, \tag{3.1}$$

where

$$F(y,t)=\frac{1}{t^n}\int_{\mathbb{R}^n} w(\xi)\left(\frac{t}{t+|\xi-y|}\right)^{n\mu} d\xi.$$

Now, arguing exactly as in [15, Proof of Theorem 2], for a bounded, measurable, and compactly supported function $h:\mathbb{R}^{n+1}_+\to\mathbb{R}$, such that $h(y,t)=0$ for small t independently of y, set

$$L(h)(x)=\iint_{\mathbb{R}^{n+1}_+} h(y,t)\varphi_t(y-x)F(y,t)\,dy\,dt.$$

By (3.1), and by duality, we have that

$$\|g^*_{\varphi,\mu}(f)\|_{L^2_w}\le A\|f\|_{L^2_v}$$

if and only if

$$\|L(h)\|_{L^2_{v^{-1}}}\le A\left(\iint_{\mathbb{R}^{n+1}_+} h(y,t)^2F(y,t)t\,dy\,dt\right)^{1/2} \tag{3.2}$$

for all such h. Given a dyadic cube Q, set

$$T(Q)=\{(y,t): y\in Q, \ell_Q/2\le t<\ell_Q\}.$$

Since $\mathbb{R}^{n+1}_+=\bigcup_{Q\in\mathcal{D}}T(Q)$, and the $T(Q)$'s are pairwise disjoint, we can write

$$L(h)(x)=\sum_{Q\in\mathcal{D}}\lambda_Q b_Q(x),$$

where $b_Q(x)=\frac{1}{\lambda_Q}\iint_{T(Q)}h(y,t)\varphi_t(y-x)F(y,t)\,dy\,dt$. In particular, setting

$$\lambda_Q=c\iint_{T(Q)}|h(y,t)|F(y,t)\frac{dy\,dt}{t^{n/2}}, \tag{3.3}$$

where a constant c depends only on φ and n, one can easily show (using properties of functions φ and h) that $L(h)$ is in standard form (see [15, Proof of Theorem 2] for details). Applying Proposition 2.1, yields

$$\begin{aligned}\|L(h)\|_{L^2_{v^{-1}}}&\le c\|v^{-1}\|^{1/2}_{A_\infty}\|\widetilde{S}(L(h))\|_{L^2_{v^{-1}}}\\ &=c\|v^{-1}\|^{1/2}_{A_\infty}\left(\sum_{Q\in\mathcal{D}}\frac{|\lambda_Q|^2}{|Q|}v^{-1}(3Q)\right)^{1/2}.\end{aligned} \tag{3.4}$$

By (3.3) and Hölder's inequality, we have

$$|\lambda_Q|^2\le c\left(\iint_{T(Q)}h(y,t)^2F(y,t)t\,dy\,dt\right)\left(\iint_{T(Q)}F(y,t)\frac{dy\,dt}{t^{n+1}}\right).$$

Next, it is easy to see that for $(y,t)\in T(Q)$,

$$F(y,t)\le c\sum_{k=0}^{\infty}\frac{w(2^kQ)}{2^{kn\mu}|Q|}.$$

Therefore, since $\mu > 2$,

$$\frac{v^{-1}(3Q)}{|Q|}\iint_{T(Q)} F(y,t)\frac{dy\,dt}{t^{n+1}} \le c\sum_{k=0}^{\infty}\frac{w(2^kQ)v^{-1}(3Q)}{2^{kn\mu}|Q|^2}$$
$$\le c\sum_{k=0}^{\infty}\frac{\|(w,v)\|_{A_2}}{2^{kn(\mu-2)}} \le c\|(w,v)\|_{A_2},$$

and thus,

$$\sum_{Q\in\mathcal{D}}\frac{|\lambda_Q|^2}{|Q|}v^{-1}(3Q) \le c\|(w,v)\|_{A_2}\sum_{Q\in\mathcal{D}}\iint_{T(Q)} h(y,t)^2F(y,t)t\,dy\,dt$$
$$= c\|(w,v)\|_{A_2}\iint_{\mathbb{R}^{n+1}_+} h(y,t)^2F(y,t)t\,dy\,dt.$$

Combining this with (3.4), we get (3.2) with $A = c\|v^{-1}\|_{A_\infty}^{1/2}\|(w,v)\|_{A_2}^{1/2}$, which completes the proof. $\square$

Proof of Theorem 1.1 for $S(f) = S_d(f)$. We follow similar ideas as in the previous proof. For a sequence $\mu = \{\mu_Q\}_{Q\in\mathcal{D}}$, set

$$L(\mu)(x) = \sum_{Q\in\mathcal{D}}\mu_Q w(Q)h_Q(x),$$

where $h_Q(x) = |Q|^{-1}\chi_Q(x) - |\widetilde{Q}|^{-1}\chi_{\widetilde{Q}}(x)$. By duality, we have that

$$\|S_d(f)\|_{L^2_w} \le A\|f\|_{L^2_v}$$

if and only if

$$\|L(\mu)\|_{L^2_{v^{-1}}} \le A\Big(\sum_{Q\in\mathcal{D}}\mu_Q^2 w(Q)\Big)^{1/2}. \tag{3.5}$$

It suffices to check (3.5) for all $\{\mu_Q\}$ such that $\mu_Q \ne 0$ for a finite number of Q. In this case $L(\mu)\in L^2_{v^{-1}}$, and by Proposition 2.2, we obtain

$$\|L(\mu)\|_{L^2_{v^{-1}}} \le c\|v^{-1}\|_{A_\infty}^{1/2}\|S_d(L(\mu))\|_{L^2_{v^{-1}}} \tag{3.6}$$
$$= c\|v^{-1}\|_{A_\infty}^{1/2}\Big(\sum_{Q\in\mathcal{D}}\big(L(\mu)_Q - L(\mu)_{\widetilde{Q}}\big)^2 v^{-1}(Q)\Big)^{1/2}.$$

Let $Q, Q' \in \mathcal{D}$. We have

$$\int_Q h_{Q'} = \frac{|Q\cap Q'|}{|Q'|} - \frac{|Q\cap\widetilde{Q}'|}{|\widetilde{Q}'|}.$$

Suppose that $Q'\cap Q\neq\emptyset$. If $Q'\subset Q$, then $\widetilde{Q}'\subseteq Q$, and we clearly get $(h_{Q'})_Q=0$. If $Q\subseteq Q'$, then $(h_{Q'})_Q=(1-1/2^n)/|Q'|$. Assume now that $Q'\cap Q=\emptyset$. If $Q\cap\widetilde{Q}'=\emptyset$, then $(h_{Q'})_Q=0$. It remains only the case when $Q\subset\widetilde{Q}'$. Then we get $(h_{Q'})_Q=-1/2^n|Q'|$. Therefore, setting $\mathcal{D}_1(Q)=\{Q'\in\mathcal{D}:Q\subseteq Q'\}$ and

$$\mathcal{D}_2(Q)=\{Q'\in\mathcal{D}:Q\cap Q'=\emptyset \text{ and } Q\subset\widetilde{Q}'\},$$

we get

$$\begin{aligned}L(\mu)_Q&=\sum_{Q'\in\mathcal{D}}\mu_{Q'}w(Q')(h_{Q'})_Q\\&=\left(1-\frac{1}{2^n}\right)\sum_{Q'\in\mathcal{D}_1(Q)}\frac{\mu_{Q'}w(Q')}{|Q'|}-\frac{1}{2^n}\sum_{Q'\in\mathcal{D}_2(Q)}\frac{\mu_{Q'}w(Q')}{|Q'|}.\end{aligned}$$

Denote $\mathcal{D}_3(Q)=\{Q_i\in\mathcal{D}:Q_i\subset\widetilde{Q},|Q_i|=|Q|,i=1,\dots,2^n\}$. It is easy to see that $\mathcal{D}_1(Q)\setminus\mathcal{D}_1(\widetilde{Q})=\{Q\}$, and $\mathcal{D}_2(Q)\setminus\mathcal{D}_2(\widetilde{Q})=\mathcal{D}_3(Q)\setminus\{Q\}$. Hence,

$$\begin{aligned}|L(\mu)_Q-L(\mu)_{\widetilde{Q}}|&=\left|\frac{\mu_Q w(Q)}{|Q|}-\frac{1}{2^n|Q|}\sum_{Q_i\in\mathcal{D}_3(Q)}\mu_{Q_i}w(Q_i)\right|\\&\le\frac{2}{|Q|}\max_{Q_i\in\mathcal{D}_3(Q)}|\mu_{Q_i}w(Q_i)|,\end{aligned}$$

which along with (3.6) yields

$$\begin{aligned}\|L(\mu)\|_{L^2_{v^{-1}}}&\le c\|v^{-1}\|_{A_\infty}^{1/2}\left(\sum_{Q\in\mathcal{D}}\Big(\max_{Q_i\in\mathcal{D}_3(Q)}\mu_{Q_i}^2w^2(Q_i)\Big)v^{-1}(Q)/|Q|^2\right)^{1/2}\\&\le 2^nc\|v^{-1}\|_{A_\infty}^{1/2}\|(w,v)\|_{A_2}^{1/2}\left(\sum_{Q\in\mathcal{D}}\max_{Q_i\in\mathcal{D}_3(Q)}\mu_{Q_i}^2w(Q_i)\right)^{1/2}\\&\le 2^nc\|v^{-1}\|_{A_\infty}^{1/2}\|(w,v)\|_{A_2}^{1/2}\left(2^n\sum_{Q\in\mathcal{D}}\mu_Q^2w(Q)\right)^{1/2}.\end{aligned}$$

Thus, we have obtained (3.5) with $A=c\|v^{-1}\|_{A_\infty}^{1/2}\|(w,v)\|_{A_2}^{1/2}$, which finishes the proof. □

Proof of Theorem 1.2. Consider first the case $1<p\le 2$. As we mentioned in the Introduction, in this case one can simply reduce (1.5) to the one weighted inequality and then use the sharp form of the Rubio de Francia extrapolation theorem proved in [5, Theorem 1]. Namely, setting in (1.5)

$v = w$, and using (2.3), we get

$$\|f\|_{L^2_w} \le c\|w\|_{A_2}\|g\|_{L^2_w}.$$

Applying to this inequality Theorem 1 from [5], we obtain (1.6) when $1 < p \le 2$.

Suppose that $p > 2$. Take an arbitrary function u with $\|u\|_{L_w^{(p/2)'}} = 1$ (here, as usual, $q' = q/(q-1), q > 1$), and set

$$\mathcal{S}(u) = \left\{w^{-1}M\left(|u|^{\frac{p-1}{p-2}}w\right)\right\}^{\frac{p-2}{p-1}}.$$

Applying (1.5) and Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|f|^2uw &\le c\|(w\mathcal{S}(u))^{-1}\|_{A_\infty}\|(uw,\mathcal{S}(u)w)\|_{A_2}\int_{\mathbb{R}^n}|g|^2\mathcal{S}(u)w\\ &\le c\|(w\mathcal{S}(u))^{-1}\|_{A_\infty}\|(uw,\mathcal{S}(u)w)\|_{A_2}\|\mathcal{S}(u)\|_{L_w^{(p/2)'}}\|g\|^2_{L^p_w}.\end{aligned}$$

To estimate $\|(w\mathcal{S}(u))^{-1}\|_{A_\infty}$, we use (2.3) with some $r > 2$. We have trivially

$$\left(\frac{1}{|Q|}\int_Q (w\mathcal{S}(u))^{-1}\right) \le \left(\inf_Q M\left(|u|^{\frac{p-1}{p-2}}w\right)\right)^{-\frac{p-2}{p-1}}\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}.$$

Next, by Hölder's inequality,

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q (wS(u))^{\frac{1}{r-1}}\right)^{r-1}\\ &\quad\le \left(\frac{1}{|Q|}\int_Q w\right)^{\frac{1}{p-1}}\left(\frac{1}{|Q|}\int_Q M\left(|u|^{\frac{p-1}{p-2}}w\right)^{\frac{p-2}{(p-1)(r-1)-1}}\right)^{\frac{(p-1)(r-1)-1}{p-1}}.\end{aligned}$$

If $r > 2$, then $p-2 < (p-1)(r-1)-1$, and hence, by (2.2),

$$\frac{1}{|Q|}\int_Q M\left(|u|^{\frac{p-1}{p-2}}w\right)^{\frac{p-2}{(p-1)(r-1)-1}} \le c_{p,r,n}\inf_Q M\left(|u|^{\frac{p-1}{p-2}}w\right)^{\frac{p-2}{(p-1)(r-1)-1}}.$$

Combining three latter estimates yields

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q (w\mathcal{S}(u))^{-1}\right)\left(\frac{1}{|Q|}\int_Q (w\mathcal{S}(u))^{\frac{1}{r-1}}\right)^{r-1}\\ &\quad\le c\left(\frac{1}{|Q|}\int_Q w\right)^{\frac{1}{p-1}}\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right) \le c\|w\|_{A_p}^{\frac{1}{p-1}}.\end{aligned}$$

Therefore,

$$\|(w\mathcal{S}(u))^{-1}\|_{A_\infty} \le c\|(w\mathcal{S}(u))^{-1}\|_{A_r} \le c\|w\|_{A_p}^{\frac{1}{p-1}}. \tag{3.7}$$

To estimate $\|(uw,\mathcal{S}(u)w)\|_{A_2}$ and $\|\mathcal{S}(u)\|_{L_w^{(p/2)'}}$, we follow the same argument as in [5, Lemma 1]. By Hölder's inequality,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q uw\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{\mathcal{S}(u)w}\right)\\
&\quad\le\left(\frac{1}{|Q|}\int_Q |u|^{\frac{p-1}{p-2}}w\right)^{\frac{p-2}{p-1}}\left(\frac{1}{|Q|}\int_Q w\right)^{\frac{1}{p-1}}\left(\frac{1}{|Q|}\int_Q \frac{1}{\mathcal{S}(u)w}\right)\\
&\quad\le\left(\frac{1}{|Q|}\int_Q w\right)^{\frac{1}{p-1}}\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)\le\|w\|_{A_p}^{\frac{1}{p-1}}.
\end{aligned}$$

Therefore,

$$\|(uw,\mathcal{S}(u)w)\|_{A_2}\le\|w\|_{A_p}^{\frac{1}{p-1}}. \tag{3.8}$$

Finally, using Buckley's inequality (1.1), and the fact that $\|w^{-1/(p-1)}\|_{A_{p'}}=\|w\|_{A_p}^{p'-1}$, we have

$$\begin{aligned}
\|\mathcal{S}(u)\|_{L_w^{(p/2)'}}&=\left(\int_{\mathbb{R}^n} M\big(|u|^{\frac{p-1}{p-2}}w\big)^{p'}w^{-\frac{1}{p-1}}\right)^{\frac{p-2}{p}}\\
&\le c\big\|w^{-\frac{1}{p-1}}\big\|_{A_{p'}}^{\frac{1}{p'-1}\frac{p-2}{p-1}}=c\|w\|_{A_p}^{\frac{p-2}{p-1}}.
\end{aligned}$$

Unifying the latter estimate with (3.7) and (3.8) gives

$$\int_{\mathbb{R}^n}|f|^2uw\le c\|w\|_{A_p}^{\frac{p}{p-1}}\|g\|_{L_w^p}^2.$$

Taking the supremum over all u with $\|u\|_{L_w^{(p/2)'}}=1$, we get

$$\|f\|_{L_w^p}^2\le c\|w\|_{A_p}^{\frac{p}{p-1}}\|g\|_{L_w^p}^2,$$

which proves (1.6) for $p>2$. □

Proof of Corollary 1.4. We just combine arguments from [16]. Let $\varphi\in C^\infty$ be radial, $\operatorname{supp}\varphi\subset\{|x|\le 1/2\}$, φ satisfies (2.6) and $\int\varphi(x)P(x)\,dx=0$ for every polynomial P of degree $\le 2n$. Let $\psi=\varphi*\varphi$. For singular integrals T as in the introduction we have

$$S_{\psi,\alpha}(Tf)(x)\le c_{\alpha,\varphi,\Omega}g^*_{\varphi,3}(f)(x)\quad(\alpha>0). \tag{3.9}$$

The proof of (3.9) can be found in [16, p. 677]. Observe that inequalities of such type were known long ago (see, e.g. [14, p. 233]). Next, it is well known [14, pp. 67–68] that for all $x\in\mathbb{R}^n$,

$$T_*f(x)\le c\big(G_N(Tf)+Mf(x)\big). \tag{3.10}$$

Note that ψ satisfies conditions of Proposition 2.3. Also, $f \in L^p_w$ implies $G_N(Tf) \in L^p_w$ for $w \in A_p$. Hence, combining Proposition 2.3 and Corollary 1.3 with inequalities (3.9), (3.10), and with Buckley's result (1.1), we get

$$\begin{aligned}\|T_*f\|_{L^p_w} &\le c\big(\|G_N(Tf)\|_{L^p_w} + \|Mf\|_{L^p_w}\big)\\ &\le c\|w\|_{A_p}^{\frac{1}{2}}\|S_{\psi,\alpha}(Tf)\|_{L^p_w} + c\|w\|_{A_p}^{\frac{1}{p-1}}\|f\|_{A_p}\\ &\le \|w\|_{A_p}^{\frac{1}{2}}\|g^*_{\varphi,3}(f)\|_{L^p_w} + c\|w\|_{A_p}^{\frac{1}{p-1}}\|f\|_{A_p}\\ &\le c\|w\|_{A_p}^{\frac{1}{2}+\max\{1,p/2\}\frac{1}{p-1}}\|f\|_{L^p_w},\end{aligned}$$

proving Corollary 1.4. □

4. Concluding remarks

REMARK 4.1. Let S denote any one of Littlewood–Paley operators considered in this paper. Denote by ν_p the best possible exponent in the inequality

$$\|S\|_{L^p_w} \le c\|w\|_{A_p}^{\nu_p}. \tag{4.1}$$

Let $\beta_p = \max\{\frac{1}{2}, \frac{1}{p-1}\}$. It was conjectured in [9] that $\nu_p = \beta_p$. This was motivated by the fact that

$$\|M(g^*_{\varphi,\mu}(f))\|_{L^p_w} \le c\|w\|_{A_p}^{\beta_p}\|Mf\|_{L^p_w} \quad (\mu > 3, 1 < p < \infty), \tag{4.2}$$

and the exponent β_p is sharp for all $p > 1$. We observe here that essentially the same argument establishing the lower bound in (4.2) proves the same lower bound in (4.1). This along with the main results of this paper shows that $\nu_p = \frac{1}{p-1}$ for $1 < p \le 2$ (for the dyadic square function this was proved in [5]) and

$$\beta_p \le \nu_p \le \frac{p}{2(p-1)} \quad (p > 2).$$

So, the question about the exact value of ν_p for $p > 2$ is still open.

REMARK 4.2. Given a measurable function f, define the local maximal operator $m_{1/2}f$ by

$$m_{1/2}f(x) = \sup_{Q \ni x}(f\chi_Q)^*(|Q|/2).$$

Here f^* denotes the non-increasing rearrangement of f. It was proved in [8] that for Calderón–Zygmund operators we have

$$T_*(f)(x) \le cMf(x) + m_{1/2}(Tf)(x)$$

and

$$m_{1/2}(Tf)(x) \le cMf(x) + T_*(f)(x).$$

This shows that the best bounds for T_* and $m_{1/2}(T)$ coincide. However, using this idea and assuming that the best bound for T is equal to $\alpha_p = \max\{1, \frac{1}{p-1}\}$ (which, as we mentioned in the Introduction, is currently known only for the

Hilbert and Riesz transforms), we can recover estimate (1.7) only for $p = 2$. Indeed, it is easy to show that

$$\|m_{1/2}f\|_{L^p_w} \le c\|w\|_{A_p}^{1/p}\|f\|_{L^p_w} \quad (1 \le p < \infty), \tag{4.3}$$

and the exponent $1/p$ is best possible. Hence,

$$\|m_{1/2}(Tf)\|_{L^p_w} \le c\|w\|_{A_p}^{\alpha_p + \frac{1}{p}}\|f\|_{L^p_w} \quad (1 < p < \infty),$$

proving the same bound for T_*. But $\alpha_p + 1/p > 1/2 + \max\{1, p/2\}\frac{1}{p-1}$ except the case $p = 2$. These observations lead naturally to a question whether the estimate

$$\|H_*\|_{L^2_w} \le c\|w\|_{A_2}^{3/2}$$

can be improved (H_* denotes the maximal Hilbert transform).

We outline briefly the proof of (4.3). First, we observe that (4.3) is equivalent to

$$w\{M\chi_E > 1/2\} \le c\|w\|_{A_p} w(E) \quad (p \ge 1) \tag{4.4}$$

for any measurable set E. Indeed, setting in (4.3) $f = \chi_E$, we get (4.4). Next, applying (4.4) to $E_\alpha = \{|f| > \alpha\}, \alpha > 0$, and using the fact that $\{M\chi_{E_\alpha} > 1/2\} = \{m_{1/2}f > \alpha\}$, we easily obtain (4.3).

Now, (4.4) follows immediately from the inequality

$$w\{Mf > \alpha\} \le c\|w\|_{A_p}(\|f\|_{L^p_w}/\alpha)^p$$

proved in [1, p. 256]. To show the sharpness of (4.4), take $w(x) = |x|^{\delta - 1}$ and $E = [1/2, 2]$. Then $(0, 2) \subset \{M\chi_E > 1/2\}$, and hence

$$w\{M\chi_E > 1/2\}/w(E) \ge c/\delta \ge c\|w\|_{A_p}.$$

Remark 4.3. Note that Theorem 1.1 implies easily that for any locally integrable function w,

$$\|S(f)\|_{L^2_w} \le c\|f\|_{L^2_{Mw}}. \tag{4.5}$$

For different S, this result was proved in [2, 3, 17].

To prove (4.5), set in (1.4) $v = Mw$. Then $\|(w, Mw)\|_{A_2} \le 1$. Now (4.5) follows immediately from the fact that $\|(Mw)^{-1}\|_{A_\infty} \le c$, where c does not depend on w. To prove the latter fact, we use (2.3) with any $r > 2$. Applying (2.2), we easily have

$$\left(\frac{1}{|Q|}\int_Q (Mw)^{-1}\right)\left(\frac{1}{|Q|}\int_Q (Mw)^{\frac{1}{r-1}}\right)^{r-1} \le c_{r,n},$$

which yields

$$\|(Mw)^{-1}\|_{A_\infty} \le c\|(Mw)^{-1}\|_{A_r} \le c.$$

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