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DELTA EDGE-HOMOTOPY INVARIANTS OF SPATIAL GRAPHS VIA DISK-SUMMING THE CONSTITUENT KNOTS

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ABSTRACT. In this paper, we construct some invariants of spatial graphs by disk-summing the constituent knots and show the delta edge-homotopy invariance of them. As an application, we show that there exist infinitely many slice spatial embeddings of a planar graph up to delta edge-homotopy, and there exist infinitely many boundary spatial embeddings of a planar graph up to delta edge-homotopy.

1. Introduction

Throughout this paper, we work in the piecewise linear category. Let G be a finite graph. An embedding of G into the 3-sphere is called a *spatial embedding* of G or simply a *spatial graph*. A graph G is said to be *planar* if there exists an embedding of G into the 2-sphere, and a spatial embedding of a planar graph G is said to be *trivial* if it is ambient isotopic to an embedding of G into the 2-sphere in the 3-sphere. Note that a trivial spatial embedding of a planar graph is unique up to ambient isotopy [7].

A *delta move* is a local deformation on a spatial graph as illustrated in Figure 1.1 which is known as an unknotting operation [8], [12]. A delta move is called a *self delta move* if all three strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be *delta edge-homotopic* if they are transformed into each other by self delta moves and ambient isotopies [16]. If the graph is homeomorphic to the disjoint union of 1-spheres, then this equivalence relation coincides with *self Δ-equivalence* [22] (or *delta link homotopy* [13]) on oriented links.

For self Δ-equivalence on oriented links, Shibuya proposed the following conjectures in [22] and [23].

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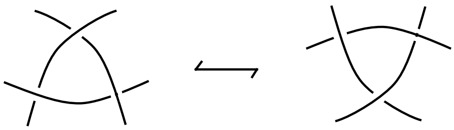

FIGURE 1.1.

CONJECTURE 1.1 ([22]). *Two cobordant oriented links are self* Δ*-equivalent.*

CONJECTURE 1.2 ([23]). *Any boundary link is self* Δ*-equivalent to the trivial link.*

He gave the partially affirmative answers to the conjectures above at the same time. He showed that any *ribbon link* is self Δ-equivalent to the trivial link [22], and any 2-component boundary link is self Δ-equivalent to the trivial link [23, Theorem 4.6]. But Nakanishi–Shibuya showed that there exists a 2-component link such that it is not self Δ-equivalent, but cobordant to the Hopf link [14, Claim 4.5], namely they gave a negative answer to Conjecture 1.1. Moreover, Nakanishi–Shibuya–Yasuhara showed that there exists a 3-component link such that it is not self Δ-equivalent, but cobordant to the Borromean rings [15, Proposition 1]. Note that both the Hopf link and the Borromean rings are not slice. On the other hand, Conjecture 1.2 was solved affirmatively by Shibuya–Yasuhara [24].

On the outcome of the results above, we investigate a more general case. A spatial embedding of a planar graph is said to be *slice* if it is cobordant[1] to the trivial spatial embedding. A spatial embedding of a graph is called a *∂-spatial embedding* if all knots in the embedding bound Seifert surfaces simultaneously such that the interiors of the surfaces are mutually disjoint and disjoint from the image of the embedding [19]. If the graph is homeomorphic to the disjoint union of 1-spheres, then this definition coincides with the definition of the boundary link. We note that any nonplanar graph does not have a ∂-spatial embedding [19, Corollary 1.3]. Then we ask the following questions.

QUESTION 1.3. (1) Is any slice spatial embedding of a planar graph delta edge-homotopic to the trivial spatial embedding?

(2) Is any ∂-spatial embedding of a graph delta edge-homotopic to the trivial spatial embedding?

In fact, for *spatial theta curves*, the affirmative answers to Question 1.3(1) and (2) have already given by the author [17, Corollary 1.3 and 1.5]. But our purpose in this paper is to give the negative answers to the Questions 1.3(1) and (2), as follows.

[1] See [25] for the precise definition of spatial graph-cobordism.

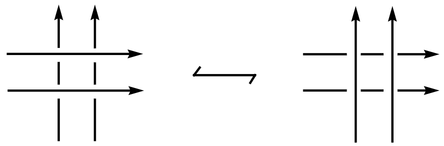

FIGURE 1.2.

THEOREM 1.4. (1) *There exist infinitely many slice spatial embeddings of a graph up to delta edge-homotopy.*

(2) *There exist infinitely many ∂-spatial embeddings of a graph up to delta edge-homotopy.*

To accomplish this, we construct some invariants of spatial graphs by considering a disk-summing operation among the constituent knots in a spatial graph in Section 2, and show the delta edge-homotopy invariance of them in Section 3 (Theorems 2.1 and 2.2). In Section 4, we give some remarkable examples which imply Theorem 1.4. Any of those examples is demonstrated by a spatial handcuff graph (see the next section) all of whose constituent links are trivial up to self Δ-equivalence. Therefore, our examples also imply that delta edge-homotopy on spatial graphs behaves quite differently than self Δ-equivalence on links.

REMARK 1.5. (1) Recently, Question 1.3(1) for oriented links was solved affirmatively by Yasuhara [26, Corollary 1.9].

(2) A *sharp move* is a local deformation on a spatial *oriented* graph as illustrated in Figure 1.2 which is also known as an unknotting operation [11]. A sharp move is called a *self sharp move* if all four strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be *sharp edge-homotopic* (or *self sharp-equivalent* [19]) if they are transformed into each other by self sharp moves and ambient isotopies [18].[2] It is known that two delta edge-homotopic spatial embeddings of a graph are sharp edge-homotopic [18, Lemma 2.1(2)]. The author showed that two cobordant spatial embeddings of a graph are sharp edge-homotopic [18, Lemma 2.2], and the author and Shinjo showed that any ∂-spatial embedding of a graph is sharp edge-homotopic to the trivial spatial embedding [19, Theorem 1.5(1)].

2. Invariants

In this section, we introduce the invariants of spatial graphs needed later. Let H_n ($n \geq 2$) be the graph as illustrated in Figure 2.1. We give the label to each of the edges and give an orientation to each of the loops as presented in Figure 2.1. A spatial embedding of H_n is called a *spatial n-handcuff graph*, or

[2] This equivalence relation does not depend on the edge orientations.

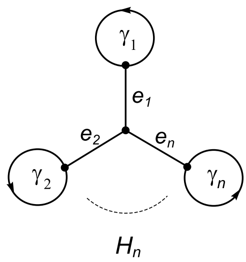

FIGURE 2.1.

simply a *spatial handcuff graph* if $n = 2$. On that occasion, we regard $e_1 \cup e_2$ as an edge of H_2 and denote by e.

Let $L = J_1 \cup J_2 \cup \cdots \cup J_n$ be an ordered and oriented n-component link. Let D be an oriented 2-disk and $x_1, x_2, \ldots, x_n$ are mutually disjoint arcs in ∂D, where ∂D has the orientation induced by the one of D, and these arcs appear along the orientation of ∂D in order and each arc has an orientation induced by the one of ∂D. We assume that D is embedded in the 3-sphere so that $D \cap L = x_1 \cup x_2 \cup \cdots \cup x_n$ and $x_i \subset J_i$ with opposite orientations for any i. Then we call a knot $K_D^{12\cdots n} = L \cup \partial D - \bigcup_{i=1}^{n} \operatorname{int} x_i$ a *D-sum* of L. For a spatial n-handcuff graph f, we denote $f(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)$ by L_f and consider a D-sum of L_f so that $f(e_1 \cup e_2 \cup \cdots \cup e_n) \subset D$ and $f(e_i) \cap \partial D = f(e_i \cap \gamma_i) \subset \operatorname{int} x_i$ for any i. We call such a D-sum of L_f a *D-sum of L_f with respect to f* and denote it by $K_D^{12\cdots n}(f)$.

For a spatial handcuff graph f, we define that

$$n_{12}(f, D) = a_2(K_D^{12}(f)) - a_2(f(\gamma_1)) - a_2(f(\gamma_2))$$

and denote the modulo $\operatorname{lk}(L_f)$ reduction of $n_{12}(f, D)$ by $\bar{n}_{12}(f)$, where lk denotes the *linking number* in the 3-sphere. Then we have the following.

THEOREM 2.1. *If two spatial handcuff graphs f and g are delta edge-homotopic, then $\bar{n}_{12}(f) = \bar{n}_{12}(g)$.*

On the other hand, let f be a spatial 3-handcuff graph and $K_D^{123}(f)$ a D-sum of L_f with respect to f. Then by using the same disk D, we can obtain three knots $K_D^{12}(f)$, $K_D^{23}(f)$ and $K_D^{13}(f)$ by forgetting the components $f(\gamma_3)$, $f(\gamma_1)$ and $f(\gamma_2)$, respectively, namely by the D-sums of sublinks $f(\gamma_1) \cup f(\gamma_2)$, $f(\gamma_2) \cup f(\gamma_3)$ and $f(\gamma_1) \cup f(\gamma_3)$ of L_f. Then we define that

$$n_{123}(f, D) = -v_3(K_D^{123}(f)) + \sum_{1 \le i < j \le 3} v_3(K_D^{ij}(f)) - \sum_{i=1}^{3} v_3(f(\gamma_i)),$$

where $v_3(J)=(1/36)V_J^{(3)}(1)$ and $V_J^{(3)}(1)$ denotes the third derivative at 1 of the *Jones polynomial*[3] of a knot J. Assume that L_f is *algebraically split*, namely all of the pairwise linking numbers of L_f are zero. Then we denote the modulo $\mu_{123}(L_f)$ reduction of $n_{123}(f,D)$ by $\bar{n}_{123}(f)$, where μ_{123} denotes the *triple linking number*, namely *Milnor's μ-invariant* of length 3 of a 3-component algebraically split link [9], [10]. Then we have the following.

Theorem 2.2. *Let f and g be two spatial 3-handcuff graphs which are delta edge-homotopic. Assume that both L_f and L_g are algebraically split. Then it holds that $\bar{n}_{123}(f)=\bar{n}_{123}(g)$.*

For example, if a spatial handcuff (resp. 3-handcuff) graph f contains a Hopf link (resp. Borromean rings), then our invariants are no use. But if L_f is *link-homotopic* [9] to the trivial link, then our invariants take effect on our purpose. Because any slice link is link-homotopic to the trivial link [3], [4], and any boundary link is also link-homotopic to the trivial link [1], [2]. We prove Theorems 2.1 and 2.2 in the next section.

3. Proofs of Theorems 2.1 and 2.2

To prove Theorems 2.1 and 2.2, we first recall some results and show a lemma needed later.

Lemma 3.1. *Let J_+ and J_- be two oriented knots and $J_0=K_1\cup K_2$ an oriented 2-component link which are identical except inside the depicted regions as illustrated in Figure 3.1. Then we have that:*

(1) ([5, Lemma 5.6]) $a_2(J_+)-a_2(J_-)=\mathrm{lk}(J_0)$.
(2) ([18, Proposition 4.2])

$$V_{J_+}^{(3)}(1)-V_{J_-}^{(3)}(1)=36a_2(J_+)+18\{\mathrm{lk}(J_0)\}^2-36\{a_2(K_1)+a_2(K_2)\}.$$

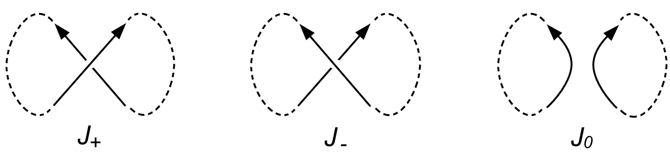

Figure 3.1.

[3] We calculate the Jones polynomial of a knot by the skein relation

$$tV_{J_+}(t)-t^{-1}V_{J_-}(t)=(t^{-\frac{1}{2}}-t^{\frac{1}{2}})V_{J_0}(t),$$

where J_+ and J_- are two oriented knots and J_0 an oriented 2-component link which are identical except inside the depicted regions as illustrated in Figure 3.1.

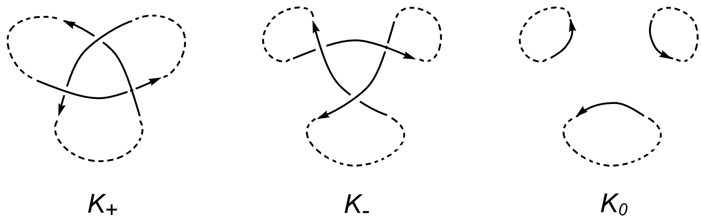

FIGURE 3.2.

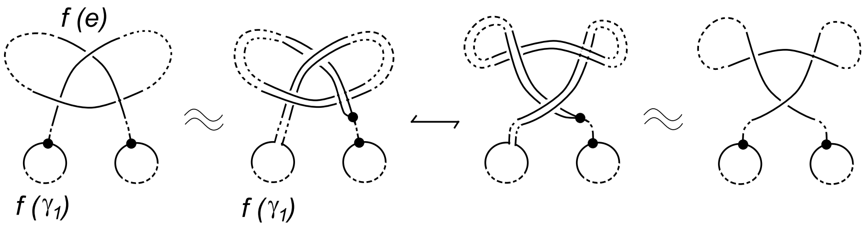

FIGURE 3.3.

LEMMA 3.2. *Let K_+ and K_- be two oriented knots and K_0 an oriented 3-component link which are identical except inside the depicted regions as illustrated in Figure 3.2. Then we have that*

(1) ([20, Theorem 1.1]) $a_2(K_+) - a_2(K_-) = 1$.
(2) ([16, Theorem 3.2]) $V^{(3)}_{K_+}(1) - V^{(3)}_{K_-}(1) = 36\,\mathrm{Lk}(K_0) - 18$, *where* $\mathrm{Lk}(L)$ *denotes the total linking number of an oriented link L.*

LEMMA 3.3. (1) *Let f be a spatial handcuff graph. Then any of the self delta moves on $f(e)$ is realized by self delta moves on $f(\gamma_1)$ and ambient isotopies.*

(2) *Let f be a spatial 3-handcuff graph. Then any of the self delta moves on $f(e_i)$ is realized by self delta moves on $f(\gamma_i)$ and ambient isotopies ($i = 1, 2, 3$).*

Proof. (1) We can see that a self delta move on $f(e)$ is realized by a "doubled-delta move" on $f(\gamma_1)$, see Figure 3.3. It is easy to see that a doubled-delta move is realized by eight delta moves on the strings in the move and ambient isotopies. Thus, we have the result.

(2) We can show in the same way as (1). □

Proof of Theorem 2.1. We first show that $\bar{n}_{12}(f)$ is an ambient isotopy invariant. Let $K^{12}_D(f)$ be a D-sum of L_f with respect to f and $K^{12}_{D'}(f)$

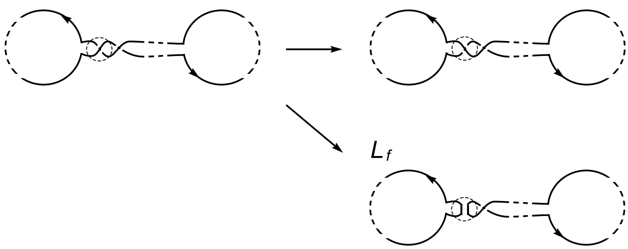

FIGURE 3.4.

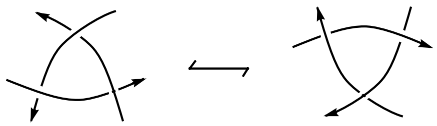

FIGURE 3.5.

another D'-sum of L_f with respect to f. We may assume that $K^{12}_{D'}(f)$ is obtained from $K^{12}_D(f)$ by a positive full twist of the band corresponding to $f(e)$. Then by Lemma 3.1(1), we have that

$$n_{12}(f, D') - n_{12}(f, D) = a_2(K^{12}_{D'}(f)) - a_2(K^{12}_D(f)) = \mathrm{lk}(L_f),$$

see Figure 3.4. This implies that $\bar{n}_{12}(f)$ is an ambient isotopy invariant.

Next, we show that $\bar{n}_{12}(f)$ is a delta edge-homotopy invariant. Let f and g be delta edge-homotopic two spatial handcuff graphs. Then by Lemma 3.3(1), g is obtained from f by self delta moves on $f(\gamma_i)$ $(i = 1, 2)$ and ambient isotopies. Moreover, it is known that each of the oriented delta moves can be realized by the one as illustrated in Figure 3.5 [12, Figure 1.1]. Hence, we may assume that g is obtained from f by a self delta move on $f(\gamma_1)$ as illustrated in Figure 3.6 without loss of generality. Let $K^{12}_D(f)$ be the D-sum of L_f with respect to f as illustrated in Figure 3.6 and $K^{12}_D(g)$ the D-sum of L_g with respect to g by using the same D as illustrated in Figure 3.6. Namely, $K^{12}_D(f)$ and $K^{12}_D(g)$ are identical except the depicted parts which represents the delta move. Note that $f(\gamma_2)$ and $g(\gamma_2)$ are ambient isotopic. Then by Lemma 3.2(1) we have that

$$\begin{aligned} &n_{12}(f, D) - n_{12}(g, D) \\ &\quad = a_2(K^{12}_D(f)) - a_2(K^{12}_D(g)) - \{a_2(f(\gamma_1)) - a_2(g(\gamma_1))\} \\ &\quad = 1 - 1 = 0. \end{aligned}$$

Since a delta move preserves the linking number, we have that $\bar{n}_{12}(f) = \bar{n}_{12}(g)$. This completes the proof. □

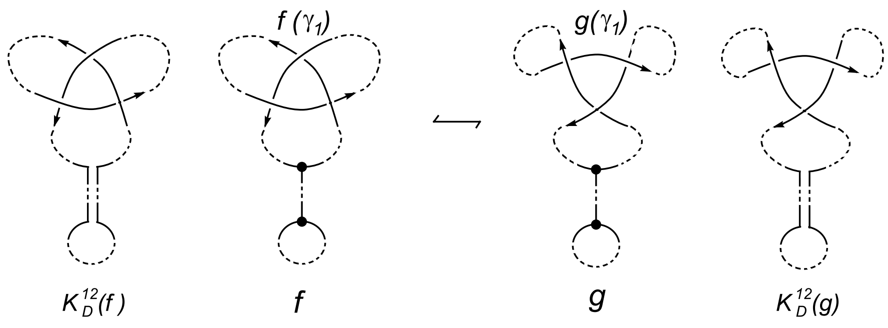

FIGURE 3.6.

REMARK 3.4. (1) By the first half of the proof of Theorem 2.1, we can also see that the modulo $\mathrm{lk}(L_f)$ reduction of $a_2(K_D^{12}(f))$ is an ambient isotopy invariant of a spatial handcuff graph f.

(2) For a spatial n-handcuff graph f and a D-sum $K_D^{12\cdots n}(f)$ of L_f with respect to f, we can generalize Theorem 2.1 as follows. Let l be the greatest common divisor of $\mathrm{lk}(f(\gamma_i), L_f - f(\gamma_i))$ $(i=1,2,\ldots,n)$. Then it can be shown that the modulo l reduction of

$$a_2(K_D^{12\cdots n}(f)) - \sum_{i=1}^{n} a_2(f(\gamma_i))$$

is a delta edge-homotopy invariant of f in the same way as the proof of Theorem 2.1. But this generalized version for $n \geq 3$ is not so strong as we will see in Examples 4.2 and 4.4.

To prove Theorem 2.2, we recall another result. By Polyak's formula of the triple linking number [21], we have the following.

LEMMA 3.5 ([21]). *Let $L = J_1 \cup J_2 \cup J_3$ be an ordered and oriented algebraically split 3-component link. Let K_D be a D-sum of L and K_D^{23}, K_D^{13} and K_D^{12} three knots obtained from K_D^{123} by forgetting the components J_1, J_2 and J_3, respectively. Then it holds that*

$$\mu_{123}(L) = -a_2(K_D^{123}) + \sum_{1 \leq i < j \leq 3} a_2(K_D^{ij}) - \sum_{i=1}^{3} a_2(J_i).$$

Proof of Theorem 2.2. We first show that $\bar{n}_{123}(f)$ is an ambient isotopy invariant. Let $K_D^{123}(f)$ be a D-sum of L_f with respect to f and $K_{D'}^{123}(f)$ another D'-sum of L_f with respect to f. We may assume that $K_{D'}^{123}(f)$ is obtained from $K_D^{123}(f)$ by a positive full twist of the band corresponding to $f(e_1)$, see Figure 3.7. Then by the skein relation as illustrated in Figure 3.8,

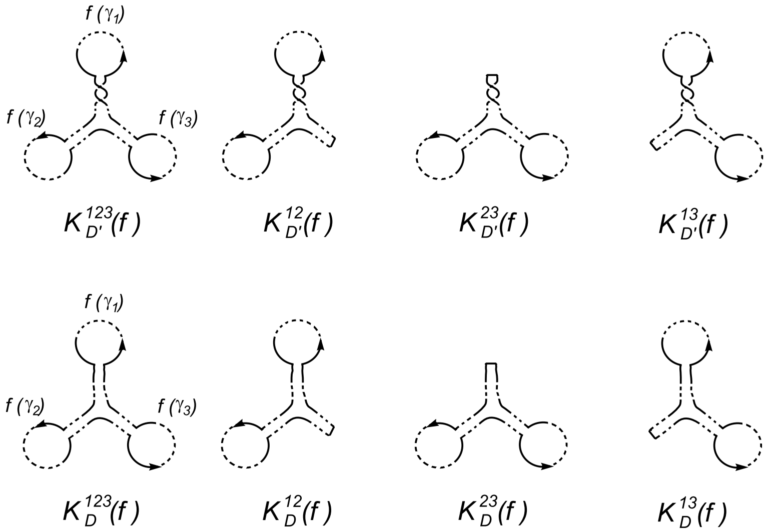

FIGURE 3.7.

Lemmas 3.1(2) and 3.5, we have that

$$\begin{aligned}
&n_{123}(f,D') - n_{123}(f,D)\\
&\quad = -\{v_3(K^{123}_{D'}(f)) - v_3(K^{123}_{D}(f))\} + \{v_3(K^{12}_{D'}(f)) - v_3(K^{12}_{D}(f))\}\\
&\quad\quad + \{v_3(K^{13}_{D'}(f)) - v_3(K^{13}_{D}(f))\}\\
&\quad = -a_2(K^{123}_{D'}(f)) - \frac{1}{2}\operatorname{lk}\big(f(\gamma_1), f(\gamma_2)\cup f(\gamma_3)\big)^2\\
&\quad\quad + \{a_2(f(\gamma_1)) + a_2(K^{23}_{D'}(f))\}\\
&\quad\quad + a_2(K^{12}_{D'}(f)) + \frac{1}{2}\operatorname{lk}(f(\gamma_1), f(\gamma_2))^2 - \{a_2(f(\gamma_1)) + a_2(f(\gamma_2))\}\\
&\quad\quad + a_2(K^{13}_{D'}(f)) + \frac{1}{2}\operatorname{lk}(f(\gamma_1), f(\gamma_3))^2 - \{a_2(f(\gamma_1)) + a_2(f(\gamma_3))\}\\
&\quad = -a_2(K^{123}_{D'}(f)) + \sum_{1\le i<j\le 3} a_2(K^{ij}_{D'}(f)) - \sum_{i=1}^{3} a_2(f(\gamma_i))\\
&\quad = \mu_{123}(L_f).
\end{aligned}$$

Hence, we have that $\bar{n}_{123}(f)$ is an ambient isotopy invariant.

Next, we show that $\bar{n}_{123}(f)$ is a delta edge-homotopy invariant. Let f and g be delta edge-homotopic two spatial 3-handcuff graphs. Then by Lemma 3.3(2), g is obtained from f by self delta moves on $f(\gamma_i)$ $(i=1,2,3)$ and ambient isotopies. Hence, we may assume that g is obtained from f by

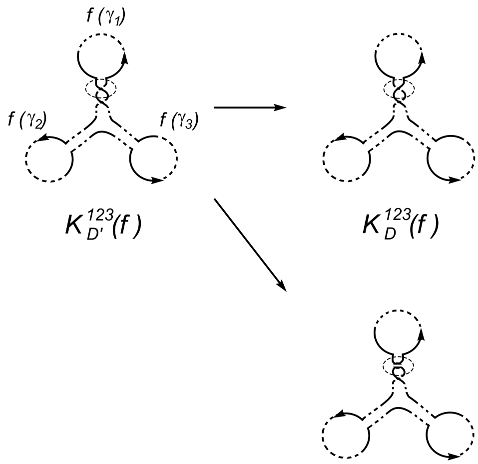

FIGURE 3.8.

a self delta move on $f(\gamma_1)$ as illustrated in Figure 3.9 without loss of generality. Let $K_D^{123}(f)$ be the D-sum of L_f with respect to f as illustrated in Figure 3.9 and $K_D^{123}(g)$ the D-sum of L_g with respect to g by using the same D as illustrated in Figure 3.9. Namely, $K_D^{123}(f)$ and $K_D^{123}(g)$ are identical except the depicted parts which represents the delta move. Let h be the spatial 3-handcuff graph and k_1 and k_2 two oriented knots as illustrated in Figure 3.9, where $f(H_3)$, $g(H_3)$ and $h(H_3) \cup k_1 \cup k_2$ are identical except the depicted parts. Let $K_D^{123}(h)$ be the D-sum of L_h with respect to h by using the same D as illustrated in Figure 3.9. Then by Lemma 3.2(2) and the homological invariance of the linking number, we have that

$$\begin{aligned} &n_{123}(f,D) - n_{123}(g,D)\\ &= -\operatorname{lk}(k_1, K_D^{123}(h)) - \operatorname{lk}(k_2, K_D^{123}(h)) - \operatorname{lk}(k_1,k_2) + \frac{1}{2}\\ &\quad + \operatorname{lk}(k_1, K_D^{12}(h)) + \operatorname{lk}(k_2, K_D^{12}(h)) + \operatorname{lk}(k_1,k_2) - \frac{1}{2}\\ &\quad + \operatorname{lk}(k_1, K_D^{13}(h)) + \operatorname{lk}(k_2, K_D^{13}(h)) + \operatorname{lk}(k_1,k_2) - \frac{1}{2}\\ &\quad - \operatorname{lk}(k_1, h(\gamma_1)) - \operatorname{lk}(k_2, h(\gamma_2)) - \operatorname{lk}(k_1,k_2) + \frac{1}{2}\\ &= 0. \end{aligned}$$

Note that L_f and L_g are self Δ-equivalent. Thus, they are also link-homotopic, namely $\mu_{123}(L_f) = \mu_{123}(L_g)$. Thus, we have that $\bar{n}_{123}(f) = \bar{n}_{123}(g)$. This completes the proof. $\square$

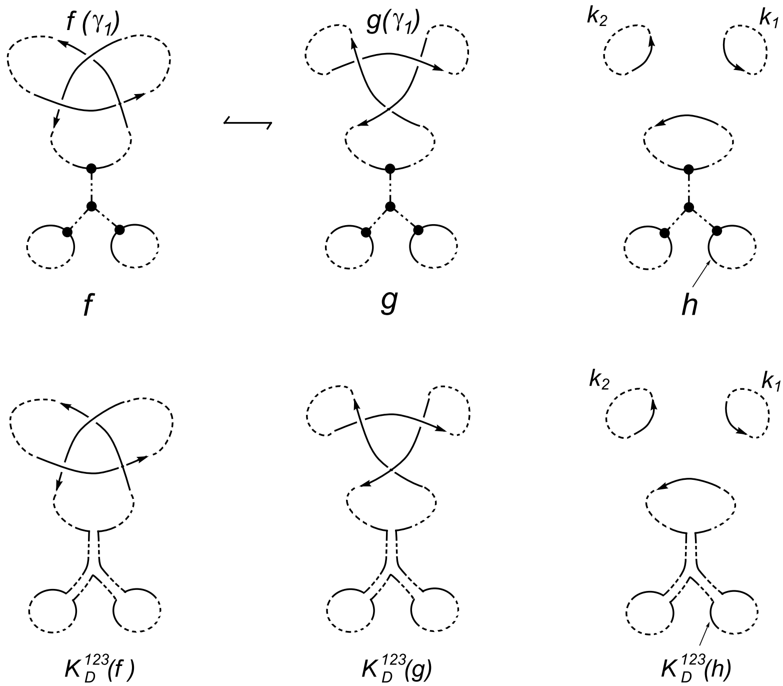

FIGURE 3.9.

4. Examples

EXAMPLE 4.1. Let f_m be the spatial handcuff graph for $m \in \mathbb{N}$ as illustrated in Figure 4.1. We can see that L_{f_m} is the trivial 2-component link for any $m \in \mathbb{N}$, namely $\mathrm{lk}(L_{f_m}) = 0$. We can also see that f_m is slice by the hyperbolic transformation on $f_m(\gamma_2)$ along the band B shown in Figure 4.1.

Now, we consider the D-sum of L_{f_m} with respect to f_m as illustrated in Figure 4.1. Then by a calculation we have that $a_2(K_D^{12}(f_m)) = 2m$ and therefore $\bar{n}_{12}(f_m) = 2m$. Thus, by Theorem 2.1, we have that f_m is not delta edge-homotopic to the trivial spatial handcuff graph for any $m \in \mathbb{N}$, and f_i and f_j are not delta edge-homotopic for $i \neq j$.

EXAMPLE 4.2. Let f_m be the spatial 3-handcuff graph for $m \in \mathbb{N}$ as illustrated in Figure 4.2. We can see that L_{f_m} is the trivial 3-component link for any $m \in \mathbb{N}$, namely $\mu_{123}(L_{f_m}) = 0$. We can also see that f_m is slice by the hyperbolic transformation on $f_m(\gamma_3)$ along the band B shown in Figure 4.2.

Now, we consider the D-sum of L_{f_m} with respect to f_m as illustrated in Figure 4.2 and a skein tree as illustrated in Figure 4.3. Then we have

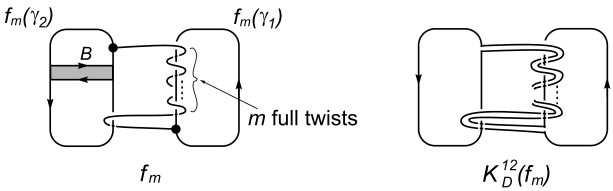

FIGURE 4.1.

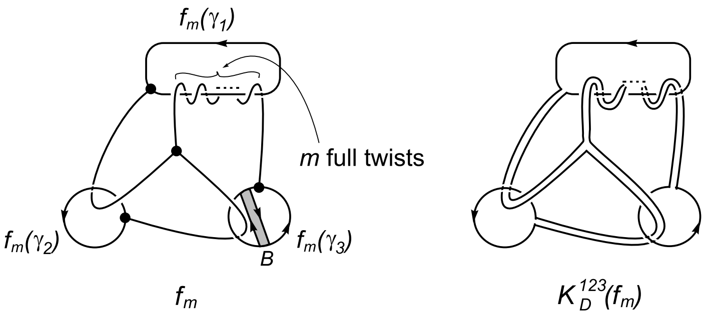

FIGURE 4.2.

that

$$a_2(K_D^{123}(f_m)) = a_2(J_m) = a_2(J) - m - 1 \tag{4.1}$$
$$= a_2(J_{m-1}) + m + 1 - m - 1$$
$$= a_2(J_{m-1}) = \cdots = a_2(J_0) = 0.$$

Then by Lemma 3.1 and (4.1) we have that

$$\begin{aligned} v_3(K_D^{123}(f)) &= v_3(J_m) = v_3(J) + a_2(J_m) + \frac{1}{2}(m+1)^2 + 1 \\ &= \left\{ v_3(J_{m-1}) - a_2(J_{m-1}) - \frac{1}{2}(m+1)^2 \right\} \\ &\quad + a_2(J_m) + \frac{1}{2}(m+1)^2 + 1 \\ &= v_3(J_{m-1}) + 1 = \cdots = v_3(J_0) + m \\ &= m. \end{aligned}$$

Since $K_D^{ij}(f_m)$ is a trivial knot for any $1 \leq i < j \leq 3$, we have that $\bar{n}_{123}(f_m) = m$. Thus, by Theorem 2.2, we have that f_m is not delta edge-homotopic to the trivial spatial handcuff graph for any $m \in \mathbb{N}$, and f_i and f_j are not delta edge-homotopic for $i \neq j$. Note that the generalized version of $\bar{n}_{12}$ for $n = 3$ as mentioned in Remark 3.4(2) vanishes for f_m by (4.1).

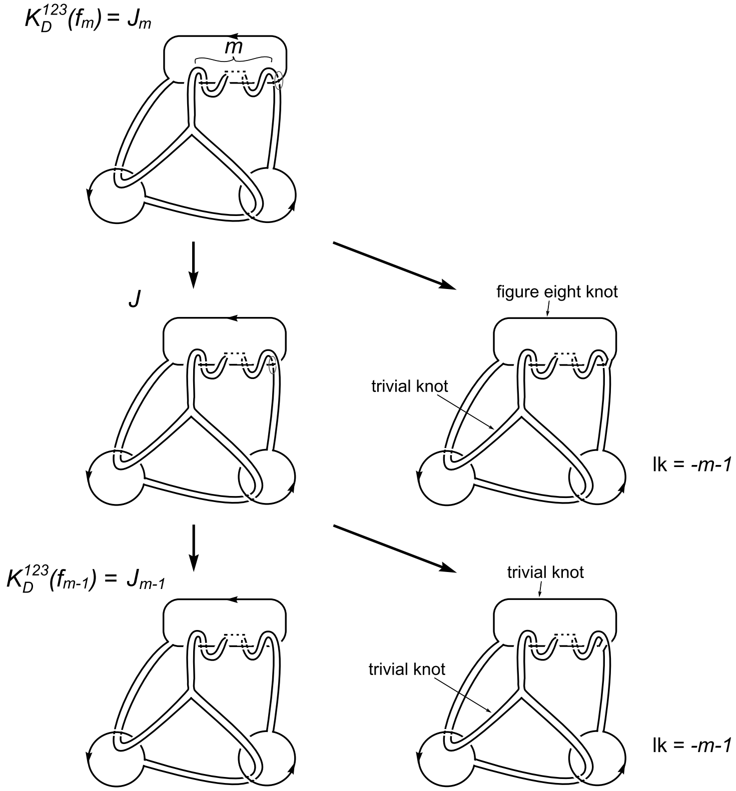

FIGURE 4.3.

EXAMPLE 4.3. Let f_m be the spatial handcuff graph for $m \in \mathbb{N}$ as illustrated in Figure 4.4. It is easy to see that f_m is a ∂-spatial handcuff graph for any $m \in \mathbb{N}$. Since L_{f_m} is a 2-component boundary link, we have that L_{f_m} is self Δ-equivalent to the 2-component trivial link for any $m \in \mathbb{N}$.

Now, we consider the D-sum of L_{f_m} with respect to f_m as illustrated in Figure 4.4. Then by a calculation, we have that $a_2(K_D^{12}(f_m)) = -2m$. Since $f_m(\gamma_1)$ and $f_m(\gamma_2)$ are trivial knots, we have that $\bar{n}_{12}(f_m) = -2m$. Thus, by Theorem 2.1, we have that f_m is not delta edge-homotopic to the trivial spatial handcuff graph for any $m \in \mathbb{N}$, and f_i and f_j are not delta edge-homotopic for $i \neq j$.

EXAMPLE 4.4. Let f be the spatial 3-handcuff graph as illustrated in Figure 4.5. Note that $f|_{\gamma_i \cup \gamma_j \cup e_i \cup e_j}$ is the trivial spatial handcuff graph for any $1 \leq i < j \leq 3$. It is easy to see that f is a ∂-spatial 3-handcuff graph. Since L_f is a 3-component boundary link, we have that $\mu_{123}(L_{f_m}) = 0$ and L_f is

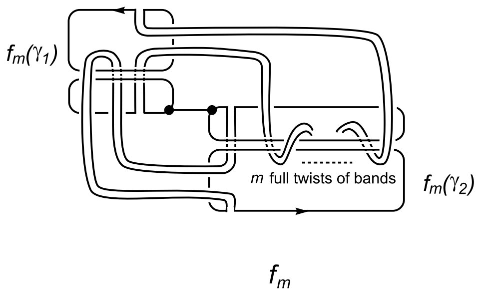

FIGURE 4.4.

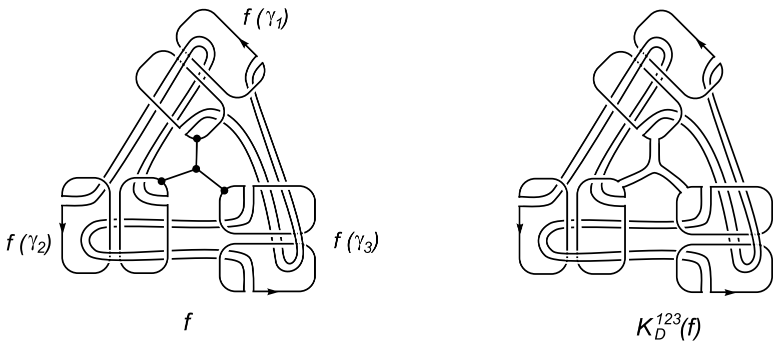

FIGURE 4.5.

self Δ-equivalent to the 3-component trivial link. Note that L_f is *Brunnian*, namely any 2-component sublink of L_f is trivial.

Now, we consider the D-sum of L_f with respect to f as illustrated in Figure 4.5. By a calculation, we have that $a_2(K_D^{123}(f)) = 0$. Since $f(\gamma_i)$ is a trivial knot for $i = 1, 2, 3$, we have that the generalized version of $\bar{n}_{12}$ for $n = 3$ as mentioned in Remark 3.4(2) vanishes for f. On the other hand, by a calculation we have that

$$\begin{aligned} V_{K_D^{123}(f)}(t) &= -t^{-12} + 6t^{-11} - 11t^{-10} + t^{-9} + 28t^{-8} - 52t^{-7} \\ &\quad + 36t^{-6} + 17t^{-5} - 61t^{-4} + 67t^{-3} - 43t^{-2} + 11t^{-1} \\ &\quad + 22 - 57t + 84t^2 - 78t^3 + 32t^4 + 23t^5 - 43t^6 \\ &\quad + 24t^7 - 4t^9 - 4t^{10} + 5t^{11} + t^{12} - 3t^{13} + t^{14}, \\ V^{(3)}_{K_D^{123}(f)}(1) &= 36. \end{aligned}$$

Since $K_D^{ij}(f)$ is also a trivial knot for any $1 \leq i < j \leq 3$, we have that $\bar{n}_{123}(f) = -1$. Thus, by Theorem 2.2, we have that f is not delta edge-homotopic to the trivial spatial 3-handcuff graph.

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