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AN EXACT SEQUENCE OF WEIGHTED NASH COMPLEXES

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ABSTRACT. Given a three-dimensional complex algebraic variety with isolated singular point and a sufficiently fine complete resolution of the singularity, we can make a careful choice of hyperplane that allows us to construct an exact sequence of weighted Nash complexes.

1. Introduction

Suppose (V,v) is an n-dimensional complex algebraic variety V with isolated singular point v, and $U \subset V$ is a neighborhood of v with an embedding $(U,v) \hookrightarrow (\mathbb{C}^N,0)$. The *Nash blowup* $\widehat{U}$ of U is the closure of the image of the section $\sigma : U - v \to Gr_n(T\mathbb{C}^N)$ that sends each point of $U-v$ to its tangent space, or equivalently, the blowup of the sheaf of 1-forms Ω^1_U (see [2], [4], and [6]). The *Nash bundle* $\nu : \mathfrak{N} \to \widehat{U}$ over the Nash blowup $\widehat{U}$ is the restriction of the universal subbundle of $Gr_n(T\mathbb{C}^N)$ to $\widehat{U}$, and the *Nash sheaf* $\mathcal{N}$ is the sheaf of sections of the dual of the Nash bundle. Equivalently, thinking of $\widehat{U}$ as the blowup of Ω^1_U, we can define the Nash sheaf $\mathcal{N}$ to be the locally free sheaf $\mathcal{N} := \widehat{\pi}^*\Omega^1_U / \operatorname{Torsion}(\widehat{\pi}^*\Omega^1_U) \approx \gamma^*\mathcal{Q}$, where $\mathcal{Q}$ is the universal quotient sheaf on $Gr(N-n,N)$ and $\gamma : \widehat{U} \hookrightarrow Gr(N-n,N)$ is the canonical map. A sheaf $\mathcal{N}$ on a blowup $\pi : \widetilde{U} \to U$ is a *generalized Nash sheaf* (although, we will often say simply "Nash sheaf") if $\widetilde{U}$ factors through the Nash blowup $\widehat{U}$ of U and $\mathcal{N}$ is the pullback of the Nash sheaf on $\widehat{U}$ (see the Appendix (A3) in [3]).

This paper will primarily concern the case where $n=3$, although we will suggest conjectures for the general case. Given a 3-dimensional variety V with isolated singular point v and neighborhood $U \subset V$, and a resolution $\pi : (\widetilde{U},E) \to (U,v)$ of the singularity v with exceptional divisor E, consider the following three sheaves: the sheaf-theoretic inverse image $\mathfrak{m}$ of the maximal ideal sheaf $\mathfrak{m}_v$, the generalized Nash sheaf $\mathcal{N}$ on $\widetilde{U}$, and the second Fitting

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ideal $\mathcal{F}$ of the Nash sheaf. We say that π is a *complete resolution* if $\mathfrak{m}$ and $\mathcal{F}$ are locally principal and $\mathcal{N}$ is locally free over $\widetilde{U}$. If the neighborhood U is sufficiently small, then such a complete resolution will exist (see [4]).

Given a complete resolution $\pi : (\widetilde{U}, E) \to (U, v)$, and a point $e \in E$, let W be an analytic neighborhood of e in $\widetilde{U}$. If e is a triple point of E, then we can choose coordinates $\{u, v, w\}$ on W for which the components of the exceptional divisor passing through e are $E_1 = \{u = 0\}$, $E_2 = \{v = 0\}$, and $E_3 = \{w = 0\}$. Similarly, if e is a double point, we can choose coordinates so that E_1 and E_2 are given by the vanishing of u and v, and if e is a simple point, we can choose coordinates so that E_1 is given by the vanishing of u. In each case, we will call such coordinates *divisor coordinates*.

The following theorem from [4] shows that some choice of divisor coordinates will define so-called *monomial generators* for the Nash sheaf.

THEOREM 1.1. *Given a complete resolution $\pi : (\widetilde{U}, E) \to (U, v)$ of a three-dimensional complex algebraic variety V with isolated singular point v, and a point $e \in E$ with analytic neighborhood $W \subset \widetilde{U}$, there exists a set of divisor coordinates $\{u, v, w\}$ on W so that the Nash sheaf $\mathcal{N}$ is locally generated by the differentials $d\phi$, $d\psi$, $d\rho$ of monomial functions of the form*

$$\phi = u^{m_1} v^{m_2} w^{m_3}, \qquad \psi = u^{n_1} v^{n_2} w^{n_3}, \qquad \rho = u^{p_1} v^{p_2} w^{p_3}$$

whose exponents $\{(m_1, m_2, m_3), (n_1, n_2, n_3), (p_1, p_2, p_3)\}$ are a Hsiang–Pati *ordered set in the sense that:*

(1) *If e is a double point, then either $m_3 = n_3 = 0$ and $p_3 = 1$, or $m_3 = p_3 = 0$ and $n_3 = 1$, and if e is a simple point, then $m_2 = m_3 = 0$, $n_2 = 1$, $p_2 = 0$, $n_3 = 0$, and $p_3 = 1$;*
(2) *$0 < m_l \le n_l \le p_l$ for $l = 1, 2, 3$ if e is a triple point, for $l = 1, 2$ if e is a double point, or for $l = 1$ if e is a simple point; and*
(3) $\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix} \neq 0.$

Moreover, we can assume that the functions ϕ, ψ, and ρ in Theorem 1.1 are Nash-minimal, *in the sense that:*

(4) *ϕ is a generator for $\mathfrak{m}(W)$; and*
(5) *$d\phi\, d\psi$ is a minimal element of $\Lambda^2 \mathcal{N}(W)$.*

One consequence of Theorem 1.1 is that the exponents m_i, n_i, p_i of the *Hsiang–Pati coordinates* ϕ, ψ, and ρ give rise to three divisors supported on E, denoted Z, N, and P, respectively. We will refer to these divisors (and the corresponding multiplicities) as *resolution data*, because they are invariants of the resolution used.

Given a two-dimensional complex algebraic variety with isolated singular point and a sufficiently fine resolution, Pardon and Stern constructed an exact sequence of sheaves that expresses the Nash sheaf in terms of the resolution data, and used this sequence to describe the cohomological Hodge structure on the L^2-cohomology of an algebraic surface in terms of local cohomology groups obtained from a resolution of the surface [3]. In this paper, we construct a generalization of this exact sequence to the three-dimensional case. Specifically, we show that if $\pi : \widetilde{U} \to U$ is a complete resolution, then there is a short exact sequence of the form

$$0 \to \widetilde{\Omega^1} \big/ \widetilde{\mathcal{N}}^1 \hookrightarrow \widetilde{\Omega}^2 \big/ \widetilde{\mathcal{N}}^2 \twoheadrightarrow \widetilde{\Omega}^3 \big/ \widetilde{\mathcal{N}}^3 \to 0 \tag{1.1}$$

where $\widetilde{\mathcal{N}}^k = \Lambda^k \mathcal{N} \otimes \mathcal{O}(Z - E)$ and $\widetilde{\Omega}^k = \Omega^k(\log E) \otimes \mathcal{O}(-(k-1)Z - E)$ are complexes of sheaves over $\widetilde{U}$. The maps in both complexes are given by $\wedge \frac{d\widetilde{h}}{\widetilde{h}}$, where $\widetilde{h} = h \circ \pi$ and $h : \mathbb{C}^N \to \mathbb{C}$ is a linear function defining a generic hyperplane whose proper transform satisfies certain conditions (see Section 2).

The form of this sequence suggests a possible further generalization to n dimensions, namely an exact sequence of the form

$$0 \to \widetilde{\Omega^1} \big/ \widetilde{\mathcal{N}}^1 \hookrightarrow \widetilde{\Omega}^2 \big/ \widetilde{\mathcal{N}}^2 \to \widetilde{\Omega}^3 \big/ \widetilde{\mathcal{N}}^3 \to \cdots \twoheadrightarrow \widetilde{\Omega}^n \big/ \widetilde{\mathcal{N}}^n \to 0$$

where the first three maps are defined exactly as those for sequence in the 3-dimensional case (see Section 4), and the remaining maps are induced by the map $\wedge \frac{d\widetilde{h}}{\widetilde{h}}$.

In [4], it was shown that a complete resolution always exists in the $n = 3$ case. The proof of this fact in the general case is nontrivial and is an open problem. The definition of *complete* in the general case is similar to the $n = 3$ case, but with the requirement that the Fitting invariants $Fitt_j(\alpha_1)\mathcal{O}_{\widetilde{U}} = Fitt_{n-1}(\alpha_{n-j})$ are locally principal ideal sheaves on $\widetilde{U}$ for $1 \le j \le n-2$, where $\alpha_{n-j} : \Lambda^{n-j}\mathcal{N}_{\widetilde{U}} \hookrightarrow \Omega^{n-j}_{\widetilde{U}}(\log E)$. There exists an analogue of Hsiang–Pati coordinates (and thus, monomial generators for the Nash sheaf), that Hironaka's resolution theorem can be used to make a careful choice of generic hyperplane, and that the conjectured sequence in the n-dimensional case is well defined and exact are results in progress that may appear in a future paper.

In Section 2 of this paper, we use genericity and a theorem from Hironaka [1] to make a careful choice of transverse hyperplane that will define the maps of our exact sequence. In Section 3, we establish some further notation and use the properties of the monomial generators of the Nash sheaf to construct a local basis for a certain sheaf of logarithmic 1-forms. Finally, in Section 4, we prove that the sequence given in (1.1) that relates the Nash sheaf to the resolution data is well defined and exact.

2. A careful choice of hyperplane

The two lemmas in Sections 2.1 and 2.2 will show that in a sufficiently fine resolution, we can find a generic hyperplane passing through v in $\mathbb{C}^N$

whose proper transform intersects the exceptional divisor transversely at simple points. In Section 2.3, we will show that such a hyperplane will help us make certain choices for the monomial generators ϕ, ψ, and ρ referred to in Theorem 1.1. This careful choice of hyperplane will enable us to construct the exact sequence in Section 4.

2.1. Finding a nice hyperplane. Given a resolution $\pi : (\widetilde{U}, E) \to (U, v)$, and a hyperplane $H \in \mathbb{C}^n$, the *proper transform* $\widetilde{H}$ of H is the closure in $\widetilde{U}$ of $\pi^{-1}(H \cap (U - v))$, and the *total transform* of H is simply $\pi^{-1}(H \cap U)$. The following theorem allows us to generically choose a hyperplane with nice properties.

LEMMA 2.1. *Suppose $\pi : (\widetilde{U}, E) \to (U, v)$ is a complete resolution. A generic hyperplane $H \subset \mathbb{C}^n$ is nice, in the sense that:*

(1) *$H \cap (U - v)$ is smooth;*
(2) *$H \cap U$ is reduced; and*
(3) *the total transform of H vanishes to minimum order along E.*

Proof. Parts (1) and (2) follow from Lemma 1.1 in Teissier's paper [5], which states that in a small enough neighborhood of v, there exists an open, Zariski dense set $\mathcal{G} \subset Gr(N-1, N)$ of hyperplanes in $\mathbb{C}^N$ passing through v such that for each $H \in \mathcal{G}$ we have $(H \cap U)_{\text{sing}} = H \cap U_{\text{sing}}$ (and thus the singular set of $H \cap (U - v)$ is empty). In fact, the proof of Lemma 1.1 from [5] shows that a generic H will meet $U - v$ transversely.

Now, let $h : \mathbb{C}^n \to \mathbb{C}$ be the linear function defining H. To prove part (3), we must show that the total transform $\pi^{-1}(H \cup U)$ of $H \cup U$ in $\widetilde{U}$ vanishes to minimum order along E, i.e., that the linear function $h \circ \pi$ vanishes to minimum order along E. It suffices to show that there is some perturbation h' of h so that $h' \circ \pi$ vanishes to the minimum order along E. Since $\widetilde{U}$ is a complete resolution, $\pi^*(\mathfrak{m}_v)$ is a locally principal sheaf of ideals on $\widetilde{U}$; let ϕ be the local generator. If h vanishes to more than the order of ϕ, we can write $h \circ \pi = \lambda\phi$ for some holomorphic function λ. Since ϕ is an element of $\pi^*\mathfrak{m}_v$, there is an $f \in \mathfrak{m}_v$ with $\phi = \pi^* f = f \circ \pi$. Note that since f is an element of the maximal ideal for v, it defines a hyperplane passing through v. Now, let $h' := h + \epsilon f$; then

$$h' \circ \pi = (h + \epsilon f) \circ \pi = (h \circ \pi) + \epsilon(f \circ \pi) = \lambda\phi + \epsilon\phi = (\lambda + \epsilon)\phi.$$

Since $\lambda + \epsilon$ is a local unit, h' vanishes to minimum order along E. □

2.2. Finding a transverse hyperplane. Given a complete resolution π from $(\widetilde{U}, E)$ to (U, v) and a "nice" hyperplane H with proper transform $\widetilde{H}$, we would like to be able to say that $E \cup \widetilde{H}$ is a divisor with normal crossings in $\widetilde{U}$, but this is not in general the case. However, we can find a finer resolution over which this is true, with the following lemma.

LEMMA 2.2. *Suppose* $\pi : (\widetilde{U}, E) \to (U, v)$ *is a complete resolution, and* $H \subset \mathbb{C}^n$ *is a "nice" hyperplane in the sense of Lemma 2.1. Then there exists a further resolution* $\bar{U}$ *of* $\widetilde{U}$ *in which the proper transform* $\bar{H}$ *is reduced and meets* $\bar{E}$ *transversely at smooth points of* $\bar{H}$*.*

We will prove Lemma 2.2 by putting our notation in the context of Hironaka's paper [1] and applying his Theorem $I_2^{N,n}$. This theorem involves permissible resolutions of resolution datum with open restriction; we will present these concepts here only in the cases that we need. We start with the definition of a resolution datum (i.e., an object that we wish to resolve in some fashion) on $\widetilde{U}$ (following Definition 3(I) from [1]).

DEFINITION 2.3. A *resolution datum* on a dimension n space X is a triple $\mathfrak{R}_I^{n,m} = (D; V; W)$ where

(1) D is reduced and codimension 1 in X with normal crossings;
(2) V is a subvariety of X with $V \supset W$; and
(3) W is a reduced subvariety of X of dimension m.

We will also call a pair $\mathfrak{R}_I^{n,m}(D; W)$ a *resolution datum* if it satisfies conditions (1) and (3) above.

Clearly, the pair $(E; \widetilde{H})$ is a resolution datum of type $\mathfrak{R}_I^{n,n-1}$ on $\widetilde{U}$ because E is reduced and codimension 1 in $\widetilde{U}$ with normal crossings, and $\widetilde{H}$ is reduced and dimension $n-1$. We will denote $\mathfrak{R}_I^{n,m}$ simply by $\mathfrak{R}$ when convenient.

We now state what it means for such a datum to be resolved at a point of W (see Definition 4(I) in [1]).

DEFINITION 2.4. The datum $\mathfrak{R} = (D; V; W)$ (and similarly, the datum $\mathfrak{R} = (D; W)$) is said to be *resolved* at $x \in W$ if:

(1) x is a smooth point of W; and
(2) D has only normal crossings with W at x.

We define a *datum with open restriction* to be a resolution datum that is resolved on a dense open subset (see Definition 5(I.2) of [1]) as follows.

DEFINITION 2.5. Given a resolution datum $\mathfrak{R} = (D; V; W)$ (similarly, a datum $(D; W)$), a pair $(\mathfrak{R}, Y)$ is a *resolution datum with open restriction* on X if

(1) Y is a dense open subset of W; and
(2) $\mathfrak{R}$ is resolved at every point of Y.

The pair $((E; \widetilde{H}), \widetilde{H} - E)$ is a resolution datum with open restriction: the subset $\widetilde{H} - E = \widetilde{H} - (\widetilde{H} \cap E)$ is open and dense in $\widetilde{H}$ since $\widetilde{H}$ is the Zariski closure of $\widetilde{H} - E$. The datum $(E; \widetilde{H})$ is resolved along all of $\widetilde{H} - E$ because $\widetilde{H}$

is smooth away from E (by our careful choice of H), and E vacuously has only normal crossings with $\widetilde{H}$ along $\widetilde{H}-E$ since $E\cap(\widetilde{H}-E)=\emptyset$.

Given a smooth, irreducible subset $B\subset X$, we say that a map $f:X'\to X$ is the *monoidal transformation with center* B if it is the blowup of X along the sheaf of ideals defining B. We now define (as in Definition 6 of [1]) what is means for such a transformation to be *permissible* with respect to some resolution datum.

DEFINITION 2.6. A monoidal transformation $f:X'\to X$ with center B is *permissible* for the resolution datum $\mathfrak{R}=(D;V;W)$ (respectively, $(D;W)$) if

(1) $(D;V\cap W;B)$ (respectively, $(D;W;B)$) is a resolution datum on X; and
(2) the datum $(D;V\cap W;B)$ (respectively, $(D;W;B)$) is resolved everywhere, i.e., on all of B.

Such a monoidal transformation is *permissible* for a resolution datum with open restriction $(\mathfrak{R},Y)$ if it is permissible for $\mathfrak{R}$ as defined above with $B\subset Y$.

In our case where $\mathfrak{R}=(E;\widetilde{H})$, a monoidal transformation $f:\widetilde{U}'\to\widetilde{U}$ with center B is permissible if the triple $(E;\widetilde{H};B)$ is a resolution datum (and thus B is reduced and contained in $\widetilde{H}$) and E has only normal crossings with B. If f with center B is permissible for the datum with open restriction $((E;\widetilde{H});\widetilde{H}-E)$, then in addition we have $B\subset\widetilde{H}-(\widetilde{H}-E)$, i.e., $B\subset\widetilde{H}\cap E$.

We now define what it means to *pull back* a resolution datum by a permissible monoidal transformation f (as in Definition 7 of [1]). Given such an f, define

$$D'=\mathrm{pt}_{X'}(D),$$
$$V'=\mathrm{pt}_{X'}(V),$$
$$W'=\mathrm{pt}_{X'}(W),$$

where $\mathrm{pt}_{X'}(D)$ denotes the proper transform of D in X', et cetera, and

$$B'=\mathrm{tt}_{X'}(B),$$
$$Y'=\mathrm{tt}_{X'}(Y),$$

where $\mathrm{tt}_{X'}(B)$ denotes the total transform (i.e., the inverse image $f^{-1}(B)$) of B in X'. We can now define the pullback of a resolution datum $\mathfrak{R}$ by f as follows.

DEFINITION 2.7. Given a resolution datum $\mathfrak{R}$ and a monoidal transformation f as above (permissible with respect to $\mathfrak{R}$), the *pullback* of $\mathfrak{R}$ by f is defined to be the triple

$$f^*(\mathfrak{R}):=(D'\cup B';V';W')$$

(simply omit the V' if $\mathfrak{R}$ is a pair rather than a triple). The pullback of the resolution datum with open restriction $(\mathfrak{R},Y)$ by such an f is defined to be the pair

$$f^*(\mathfrak{R}, Y) := (f^*(\mathfrak{R}), Y').$$

By the discussion following Definition 7 in [1], the pullback $f^*(\mathfrak{R})$ is itself a resolution datum (of the same type, i.e., the same dimensions) on X (as long as B does not contain any irreducible components of W; in that case the dimension m may be smaller). Let us investigate what this means in our case, where $(\mathfrak{R}, Y) = ((E; \widetilde{H}), \widetilde{H} - E)$. In this case, we have

$$f^*\big((E; \widetilde{H}), \widetilde{H} - E\big) = \big((E' \cup B'; \widetilde{H}'), \widetilde{H}' - (E' \cup B')\big),$$

since $Y' = (\widetilde{H} - E)' = f^{-1}(\widetilde{H} - E) = \widetilde{H}' - (E' \cup B')$. The fact that this is a resolution datum (with open restriction) means that $E' \cup B'$ is reduced, codimension 1 in $\widetilde{U}'$, and has normal crossings; and moreover, that $\widetilde{H}'$ is reduced and dimension $n-1$ (note that B cannot contain any irreducible components of H because $B \subset \widetilde{H} \cap E$).

Our final definition describes what it means for a series of monoidal transformations to be *permissible* (following Definition 8 from [1]).

Definition 2.8. Given a resolution datum $\mathfrak{R}$ on X, a series of monoidal transformations $f = \{f_i : X_{i+1} \to X_i\}_{0 \le i < s}$ with centers B_i on X_i (where $X_0 = X$) is *permissible* if there exists, for $0 \le i < s$, a resolution datum $\mathfrak{R}_i$ (with $\mathfrak{R}_0 = \mathfrak{R}$) for X_i such that:

(1) f_i is permissible with respect to $\mathfrak{R}_i$; and
(2) $\mathfrak{R}_{i+1} = f_i^*(\mathfrak{R}_i)$.

Given such a permissible series $f : X' \to X$ of monoidal transformations (with $X' = X_s$), we will define the *pullback* $f^*(\mathfrak{R})$ of $\mathcal{R}$ under f to be the final resolution datum $\mathfrak{R}_s$.

We can now state the theorem of Hironaka that we wish to apply (Theorem $I_2^{N,n}$ in [1]).

Theorem 2.9. *There exists a finite succession of monoidal transformations $f : X' \to X$ which is permissible for the resolution datum with open restriction $(\mathfrak{R}, Y)$ such that the resolution datum $f^*(\mathcal{R})$ is resolved everywhere.*

Now, we can finally prove Lemma 2.2.

Proof of Lemma 2.2. If $(\mathfrak{R}, Y) = ((E; \widetilde{H}), \widetilde{H} - E)$, then Theorem 2.9 says that we can find a series of monoidal transformations $f : \bar{U} \to \widetilde{U}$ (here $\bar{U} = \widetilde{U}'$ from the above), with centers B_i contained in $E_1 \cap \widetilde{H}_i$ at each level, so that in $\bar{U}$, $\bar{H} \cup \bar{E}$ is a divisor with normal crossings and $\bar{H}$ is smooth (where $\bar{H}$ is $\widetilde{H}' = \widetilde{H}_s$ in the notation above, and $\bar{E}$ is the union of the proper transform of E with the total transforms of the centers B_i). ☐

2.3. Using a generic hyperplane to choose a monomial generator. The following theorem will allow us to use a generic hyperplane to define one of the monomial functions ϕ, ψ, or ρ that appear in Theorem 1.1.

THEOREM 2.10. *Suppose $\pi : (\widetilde{U}, E) \to (U, v)$ is a sufficiently fine complete resolution and $H \subset \mathbb{C}^n$ is a "nice" hyperplane in the sense of Lemmas 2.1 and 2.2. Let $Z := \sum m_i E_i$ be the divisor on E corresponding to the pull-back $\pi^*(\mathfrak{m}_v)$. If h is the linear function that defines H, and $\widetilde{H}$ is the proper transform of H, then:*

(1) $\operatorname{div}(h \circ \pi) = Z + \widetilde{H}$;
(2) *$\widetilde{H}$ meets E only at double or simple points of E;*
(3) *near a point $e \notin \widetilde{H}$ we can choose ϕ to be $h \circ \pi$;*
(4) *near a double point $e \in \widetilde{H} \cap E_1 \cap E_2$ we have $m_i = n_i$ and $m_j = n_j$, and we can choose ψ to be $h \circ \pi$.*
(5) *near a simple point $e \in \widetilde{H} \cap E_1$ we have $m_i = n_i = p_i$ and we can choose either ψ or ρ to be $h \circ \pi$.*

Proof. Part (1) follows directly from Lemma 2.2, which ensures that $\widetilde{H} \cup E$ is a divisor with normal crossings in $\widetilde{U}$, and the fact that H is a "nice" hyperplane, and thus that $h \circ \pi$ vanishes to minimum order along E.

Part (2) follows from Theorem 2.9, which guarantees that $\widetilde{H} \cup E$ is a divisor with normal crossings, and thus that we can choose h, so that $\widetilde{H}$ misses the triple points of E.

To prove part (3), suppose e is a point that is not contained in $\widetilde{H}$, and let W be an analytic neighborhood of e in $\widetilde{U}$. By part (1), we have $h \circ \pi = u^{m_1} v^{m_2} w^{m_3}$ near e (at a triple point; at double or simple points simply set $m_2 = 0$ or $m_2 = m_3 = 0$, respectively), and thus, $h \circ \pi = \phi$.

To prove part (4), suppose $e \in \widetilde{H} \cap E_1 \cap E_2$ is a double point contained in H. By Lemma 2.2 and part (1), we can choose coordinates $\{u, v, w\}$ on $\widetilde{U}$ so that $E_1 = \{u = 0\}$, $E_2 = \{v = 0\}$, and $\widetilde{H} = \{w = 0\}$; then by the definition of m_1 and m_2 we have (after possibly rechoosing coordinates by multiplying w by a local unit) $h \circ \pi = u^{m_1} v^{m_2} w$. There exists a perturbation g of h so that $g \circ \pi = \delta u^{m_1} v^{m_2}$ near e, where δ is a local unit (this corresponds to a hyperplane $\widetilde{G} \subset \widetilde{U}$ that is shifted away from e, off of $\{w = 0\}$, but still transverse to E). The exponents $\{n_1, n_2\}$ and $\{p_1, p_2\}$ are minimal in the sense that we have either $m_1 = p_1$ and $m_2 = p_2$, or $m_1 = n_1$ and $m_2 = n_2$. Suppose first that we have $m_1 = p_1$ and $m_2 = p_2$. Then since $m_1 \leq n_1 \leq p_1$ and $m_2 \leq n_2 \leq p_2$, we must have $m_1 = n_1$ and $m_2 = n_2$. But, we must also have $m_1 n_2 - m_2 n_1 \neq 0$, and thus we have a contradiction. Therefore, we must have $m_1 = n_1$ and $m_2 = n_2$, and thus $h \circ \pi = u^{m_1} v^{m_2} w = u^{n_1} v^{n_2} w = \psi$.

Finally, we prove part (5). Given a simple point $e \in \widetilde{H} \cap E_1$, and an analytic neighborhood W of e in $\widetilde{U}$, we can choose coordinates $\{u, v, w\}$ on W so that

$E_1 = \{u = 0\}$ and $\widetilde{H} = \{v = 0\}$ (by Lemma 2.2; then by part (1) we have $h \circ \pi = u^{m_1} v$ near e. There exists a perturbation g of h so that $g \circ \pi = \delta u^{m_1}$ near e, where δ is a local unit (this corresponds to a hyperplane $\widetilde{G} \subset \widetilde{U}$ that is shifted away from e, off of $\{v = 0\}$, but still transverse to E). There also exists a perturbation f of h so that $f \circ \pi = \tau u^{m_1}$ near e, where τ is a coordinate independent of u and v (this corresponds to a hyperplane $\widetilde{F} \subset \widetilde{U}$ that is rotated off of $\{v = 0\}$, but still transverse to E). Rechoose coordinates by

$$\begin{cases} u \mapsto u\delta^{-1/m_1}, \\ v \mapsto v\delta, \\ w \mapsto w; \end{cases}$$

with these coordinates we have $h \circ \pi = u^{m_1} v$, $g \circ \pi = u^{m_1}$, and $f \circ \pi = \tau' u^{m_1}$, where τ' is some coordinate independent of u and v. Finally, redefine $w = \tau'$; then $f \circ \pi = u^{m_1} w$. By minimality, we now have $m_1 = n_1 = p_1$ on this component E_1, and we can choose $\phi = g \circ \pi$, $\psi = h \circ \pi$, and $\rho = f \circ \pi$. We clearly could have also changed coordinates to have $\rho = h \circ \pi$. □

3. The logarithmic Nash frame

We first collect and extend our notation. Let $\pi : (\widetilde{U}, E) \to (U, v)$ be a sufficiently fine complete resolution, and let $H \subset \mathbb{C}^n$ be a "nice" hyperplane in the sense of Theorem 2.10. Let W be an analytic neighborhood of e in $\widetilde{U}$, and choose divisor coordinates $\{u, v, w\}$ on W so that ϕ, ψ, and ρ are Hsiang–Pati coordinates as in Theorem 1.1. Let $Z = \sum m_i E_1$, $N = \sum n_i E_1$, and $P = \sum p_i E_1$ be the divisors that represent the resolution data.

At a triple point $e \in E_1 \cap E_2 \cap E_3$, we have

$$\begin{aligned} \phi &= u^{m_1} v^{m_2} w^{m_3}, \\ \psi &= u^{n_1} v^{n_2} w^{n_3}, \\ \rho &= u^{p_1} v^{p_2} w^{p_3}, \end{aligned}$$

where $m_l \le n_l \le p_l$ for $l = 1, 2, 3$. Similarly, at a double point $e \in E_1 \cap E_2$, we have either

$$\begin{aligned} \phi &= u^{m_1} v^{m_2}, \\ \psi &= u^{n_1} v^{n_2}, \\ \rho &= u^{p_1} v^{p_2} w \end{aligned} \qquad \text{or} \qquad \begin{aligned} \phi &= u^{m_1} v^{m_2}, \\ \psi &= u^{n_1} v^{n_2} w, \\ \rho &= u^{p_1} v^{p_2}, \end{aligned}$$

where $m_l \le n_l \le p_l$ for $l = 1, 2$. When we have the situation on the above left, we say that e is a *case I* double point, and when we have the situation on the above right, we say that e is a *case II* double point. Finally, at a simple point $e \in E_1$, we have

$$\begin{aligned}\phi &= u^{m_1},\\ \psi &= u^{n_1} v,\\ \rho &= u^{p_1} w\end{aligned}$$

where $m_1 \leq n_1 \leq p_1$.

Suppose h is the linear function that defines the hyperplane H. We will denote the composition $h \circ \pi$ by $\widetilde{h}$. By Theorem 2.10, we can choose ϕ, ψ, and ρ such that $\widetilde{h} = \phi$ near any triple point (since $\widetilde{H}$ cannot pass through such points), and $\widetilde{h} = \psi$ near any double or simple point. In addition, Theorem 2.10 tells us that near a double point $e \in E_1 \cap E_2 \cap \widetilde{H}$ we have $m_1 = n_1$ and $m_2 = n_2$, and thus in an analytic neighborhood of e we have $Z = N$. Similarly, near a simple point $e \in E_1 \cap \widetilde{H}$ we have $Z = N = P$. Moreover, since by Theorem 2.10 we have $\operatorname{div}(\widetilde{h}) = Z + \widetilde{H}$, multiplication by $\widetilde{h}$ gives us an isomorphism $\mathcal{O}(\widetilde{H}) \approx \mathcal{O}(-Z)$.

By Theorem 1.1, $\{d\phi, d\psi, d\rho\}$ is a basis for the Nash sheaf $\mathcal{N}_{\widetilde{U}}(W)$. The sheaf $\Omega^1_W(\log E)$ has as its standard basis over W the logarithmic frame

$$\begin{aligned}&\left\{\frac{du}{u}, \frac{dv}{v}, \frac{dw}{w}\right\}, \quad \text{if } e \text{ is a triple point;}\\ &\left\{\frac{du}{u}, \frac{dv}{v}, dw\right\}, \quad \text{if } e \text{ is a double point;}\\ &\left\{\frac{du}{u}, dv, dw\right\}, \quad \text{if } e \text{ is a simple point.}\end{aligned}$$

To clarify the relationship between $\mathcal{N}_{\widetilde{U}}(W)$ and $\Omega^1_W(\log E)(W)$ we will define a *logarithmic Nash frame* for $\Omega^1_W(\log E)(W)$. We begin by defining

$$\psi' = \begin{cases} \psi, & \text{if } e \text{ is a triple point,}\\ \psi, & \text{if } e \text{ is a ``case I'' double point,}\\ \psi w^{-1}, & \text{if } e \text{ is a ``case II'' double point,}\\ \psi v^{-1}, & \text{if } e \text{ is a simple point;}\end{cases}$$

$$\rho' = \begin{cases} \rho, & \text{if } e \text{ is a triple point,}\\ \rho w^{-1}, & \text{if } e \text{ is a ``case I'' double point,}\\ \rho, & \text{if } e \text{ is a ``case II'' double point,}\\ \rho w^{-1}, & \text{if } e \text{ is a simple point.}\end{cases}$$

Note that under these definitions, ϕ, ψ', and ρ' are local defining functions for the divisors Z, N, and P, respectively, regardless of whether the chosen point $e \in E$ is a simple, double, or triple point. Now, define the logarithmic

Nash frame to be

$$\left\{\frac{d\phi}{\phi}, \frac{d\psi}{\psi'}, \frac{d\rho}{\rho'}\right\}.$$

THEOREM 3.1. *The logarithmic Nash frame is a basis for $\Omega^1_W(\log E)(W)$.*

Proof. We must show that every element of $\Omega^1_{\widetilde{U}}(\log E)(W)$ (written in the standard logarithmic frame) can be written in the logarithmic Nash frame. In each case (triple point, double point, and simple point), we will do this by calculating the transformation from the logarithmic frame to the logarithmic Nash frame and then showing that this transformation has an inverse. As usual, all computations here take place over the analytic neighborhood W of our chosen point e.

Near a triple point e, we have

$$\begin{aligned}
\frac{d\phi}{\phi} &= \frac{d(u^{m_1}v^{m_2}w^{m_3})}{u^{m_1}v^{m_2}w^{m_3}} = m_1\frac{du}{u} + m_2\frac{dv}{v} + m_3\frac{dw}{w},\\
\frac{d\psi}{\psi'} &= \frac{d(u^{n_1}v^{n_2}w^{n_3})}{u^{n_1}v^{n_2}w^{n_3}} = n_1\frac{du}{u} + n_2\frac{dv}{v} + n_3\frac{dw}{w},\\
\frac{d\rho}{\rho'} &= \frac{d(u^{p_1}v^{p_2}w^{p_3})}{u^{p_1}v^{p_2}w^{p_3}} = p_1\frac{du}{u} + p_2\frac{dv}{v} + p_3\frac{dw}{w}.
\end{aligned}$$

In other words, the change of basis from the logarithmic to the logarithmic Nash frame of $\Omega^1_W(\log E)(W)$ is given by

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ p_1 & p_2 & p_3 \end{pmatrix}\begin{pmatrix} du/u \\ dv/v \\ dw/w \end{pmatrix} = \begin{pmatrix} d\phi/\phi \\ d\psi/\psi' \\ d\rho/\rho' \end{pmatrix}.$$

By Theorem 1.1, we have

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ p_1 & p_2 & p_3 \end{vmatrix} \neq 0,$$

and thus the change of basis matrix is invertible. Therefore, the logarithmic Nash frame is a local basis for $\Omega^1_W(\log E)(W)$.

The double point case is similar. The change of basis matrix for "case I" double points is

$$\begin{pmatrix} m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \\ wp_1 & wp_2 & 1 \end{pmatrix},$$

while for "case II" double points we have

$$\begin{pmatrix} m_1 & m_2 & 0 \\ wn_1 & wn_2 & 1 \\ p_1 & p_2 & 0 \end{pmatrix}.$$

In either case, by Theorem 1.1 the matrix has nonzero determinant (since we have either $\begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix} \neq 0$ or $\begin{vmatrix} m_1 & m_2 \\ p_1 & p_2 \end{vmatrix} \neq 0$, respectively), and thus is invertible.

In the simple point case the change of basis matrix is

$$\begin{pmatrix} m_1 & 0 & 0 \\ vn_1 & 1 & 0 \\ wp_1 & 0 & 1 \end{pmatrix}.$$

Since by Theorem 1.1 we have $m_1 \neq 0$, this matrix has nonzero determinant and is invertible. □

4. An exact sequence of weighted Nash complexes

In this paper, we are considering resolutions of three-dimensional complex algebraic varieties with isolated singular points. In the two-dimensional case, Pardon and Stern construct an exact sequence of sheaves over $\widetilde{U}$ that expresses the Nash sheaf in terms of the resolution data (see [3]). In this section, we develop a generalization of that exact sequence. The sequence here only partially describes the Nash sheaf in terms of the resolution data Z, N, and P (the problem is that the exact sequence also involves the second exterior power of the Nash sheaf and is thus self-referential regarding the Nash sheaf).

4.1. The exact sequence. Suppose $\pi : (\widetilde{U}, E) \to (U, v)$ and $\widetilde{H}$ are as in Theorem 2.10. We now define two weighted complexes of sheaves that will enable us to build the short exact sequence that is the focus of this paper.

Definition 4.1. The *weighted Nash complex* is the complex of sheaves over $\widetilde{U}$ whose kth level is given by

$$\widetilde{\mathcal{N}}^k := \Lambda^k \mathcal{N} \otimes \mathcal{O}(Z - E),$$

with maps $\widetilde{\mathcal{N}}^k \to \widetilde{\mathcal{N}}^{k+1}$ given by $\wedge \frac{d\widetilde{h}}{\widetilde{h}}$.

Definition 4.2. The *weighted log forms complex* is the complex of sheaves over $\widetilde{U}$ with kth level

$$\widetilde{\Omega}^k := \Omega^k(\log E) \otimes \mathcal{O}\big(-(k-1)Z - E\big),$$

with maps $\widetilde{\Omega}^k \to \widetilde{\Omega}^{k+1}$ given by $\wedge \frac{d\widetilde{h}}{\widetilde{h}}$.

Notice that we can utilize the isomorphism $\mathcal{O}(-Z) \approx \mathcal{O}(\widetilde{H})$ to rewrite $\widetilde{\mathcal{N}}^k$ and $\widetilde{\Omega}^k$ as

$$\widetilde{\mathcal{N}}^k = \Lambda^k \mathcal{N} \otimes \mathcal{O}\big(kZ + (k-1)\widetilde{H} - E\big)$$

and

$$\widetilde{\Omega}^k = \Omega^k(\log E) \otimes \mathcal{O}\big((k-1)\widetilde{H} - E\big).$$

In this form, it is more apparent that the maps $\wedge\frac{d\widetilde{h}}{\widetilde{h}}$ are well defined for these complexes.

THEOREM 4.3. *There is a short exact sequence of the form*

$$0 \quad\to\quad \widetilde{\Omega^1}\big/\widetilde{\mathcal{N}}^1 \quad\hookrightarrow\quad \widetilde{\Omega}^2\big/\widetilde{\mathcal{N}}^2 \quad\twoheadrightarrow\quad \widetilde{\Omega}^3\big/\widetilde{\mathcal{N}}^3 \quad\to\quad 0.$$

We will prove Theorem 4.3 in Section 4.2. The existence of a short exact sequence as in Theorem 4.3 is equivalent to the existence of an exact sequence of the form given in Theorem 4.4, which is a 3-dimensional generalization of the 2-dimensional sequence that appears in Proposition 3.20 of [3].

THEOREM 4.4. *The exact sequence in Theorem 4.3 is equivalent to an exact sequence of sheaves on $\widetilde{U}$ of the form*

$$\begin{aligned} 0 &\to \mathcal{N}(Z-E) \overset{\alpha}{\hookrightarrow} \mathcal{I}_E\Omega^1(\log E) \\ &\overset{\beta}{\to} \big(\Omega^2(\log E)\big/\Lambda^2\mathcal{N}(2Z)\big)\otimes\mathcal{O}(-Z-E) \\ &\overset{\gamma}{\twoheadrightarrow} \Omega^3\otimes\mathcal{O}_{P+N-2Z}(-2Z)\to 0. \end{aligned}$$

Proof. We first show that the sequence in Theorem 4.4 is equivalent to an exact sequence that will enable us to use the generic hyperplane H discussed in Section 2. Since $\mathcal{O}(\widetilde{H})\approx\mathcal{O}(-Z)$ (by multiplication by $\widetilde{h}$), we have

$$\begin{aligned} &\big(\Omega^2(\log E)\big/\Lambda^2\mathcal{N}(2Z)\big)\otimes\mathcal{O}(-Z-E) \\ &\quad\approx \Omega^2(\log E)\otimes\mathcal{O}(-Z-E)\big/\Lambda^2\mathcal{N}(2Z)\otimes\mathcal{O}(-Z-E) \\ &\quad\approx \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})\big/\Lambda^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(\widetilde{H}). \end{aligned}$$

The last term in the sequence above can be rewritten using the fact that there is an isomorphism

$$\Lambda^3\mathcal{N}\approx\Omega^3\otimes\mathcal{O}(-Z-N-P+E).$$

The proof that there is such an isomorphism is as follows. Let $e\in E$ be a point with analytic neighborhood $W\subset\widetilde{U}$. By Lemma 4 in [4] and the definition of ϕ, ψ, and ρ, near a triple point, we can write the generator of $\Lambda^3\mathcal{N}(W)$ as

$$d\phi\wedge d\psi\wedge d\rho = u^{d_i}v^{d_j}w^{d_k}(\mu du\wedge dv\wedge dw)$$

where $d_l=m_l+n_l+p_l-1$ for $l=i,j,k$. The arguments for double and simple points are similar.

Now, using the isomorphism above, and the fact that $\Omega^3(\log E)\approx\Omega^3\otimes\mathcal{O}(E)$, we have

$$\begin{aligned} &\Omega^3\otimes\mathcal{O}_{N+P-2Z}(-2Z) \\ &\quad\approx\Omega^3\otimes\mathcal{O}(2\widetilde{H})\otimes\mathcal{O}\big/\mathcal{O}(-N-P-2Z) \\ &\quad\approx\Omega^3\otimes\mathcal{O}(2\widetilde{H})\big/\Omega^3\otimes\mathcal{O}(2\widetilde{H}-N-P+2Z) \\ &\quad\approx\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})\big/\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H}). \end{aligned}$$

Therefore, the sequence in Theorem 4.4 is equivalent to the sequence

$$
\begin{aligned}
(4.1) \qquad 0 &\to \mathcal{N}(Z-E)\\
&\overset{\alpha}{\hookrightarrow} \mathcal{I}_E\Omega^1(\log E)\\
&\xrightarrow{\beta} \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})\big/\Lambda^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(\widetilde{H})\\
&\overset{\gamma}{\twoheadrightarrow} \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})\big/\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H})\to 0,
\end{aligned}
$$

which is clearly equivalent to the sequence of weighted Nash complexes in Theorem 4.3. □

4.2. Proof of exactness. To prove that the sequence in Theorem 4.3 is exact, we will prove that the equivalent sequence in expression (4.1) at the end of the proof of Theorem 4.4 is exact.

Proof. The first parts of the proof are similar to the proof of the 2-dimensional version that appears as Proposition 3.20 in [3]. We first show that we have an injection

$$\alpha : \mathcal{N}(Z-E)\hookrightarrow \mathcal{I}_E\Omega^1(\log E).$$

The following computation assumes we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively. Since the Nash sheaf $\mathcal{N}$ is generated by $\{d\phi, d\psi, d\rho\}$, we have

$$
\begin{aligned}
(4.2) \quad \mathcal{N}(Z-E) &= \{(a\,d\phi+b\,d\psi+c\,d\rho)\cdot f \mid a,b,c\in\mathcal{O}, f\in\mathcal{O}(Z-E)\}\\
&= \left\{k_1\frac{d\phi}{\phi}+k_2\frac{d\psi}{\psi'}+k_2\frac{d\rho}{\rho'}\,\Big|\, k_1\in\mathcal{O}(-E),\right.\\
&\qquad\left. k_2\in\mathcal{O}(Z-N-E), k_3\in\mathcal{O}(Z-P-E)\right\}.
\end{aligned}
$$

Since $\mathcal{O}(Z-P-E)\subset\mathcal{O}(Z-N-E)\subset\mathcal{O}(-E)\approx\mathcal{I}_E$ (recall that $P>N$ since $p_i\geq n_i$ for all i, by Theorem 1.1), we have the desired injection α.

To define β, we first define the map

$$\widetilde{\beta}:\mathcal{I}_E\Omega^1(\log E)\longrightarrow\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})$$

by $\widetilde{\beta}(\omega)=\omega\wedge\frac{d\widetilde{h}}{\widetilde{h}}$. Take $\omega\in\mathcal{I}_E\Omega^1(\log E)$. Then $\omega=k_1\frac{d\phi}{\phi}+k_2\frac{d\psi}{\psi'}+k_3\frac{d\rho}{\rho'}$, with $k_i\in\mathcal{O}(-E)$. We need to show that $\widetilde{\beta}(\omega)$ is actually in $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})$. We do this locally, examining the three possible cases: $e\in E$ away from $\widetilde{H}$, $e\in E_1\cap\widetilde{H}$ is a simple point of E on $\widetilde{H}$, and $e\in E_1\cap E_2\cap\widetilde{H}$ is a double point of E On $\widetilde{H}$ (and necessarily a "case II" double point). By Theorem 2.10, we know that $\operatorname{div}(\widetilde{h})=Z+\widetilde{H}$. Therefore, away from $\widetilde{H}$, $\widetilde{h}=\phi$, in this case, we have

$$\widetilde{\beta}(\omega)=\omega\wedge\frac{d\widetilde{h}}{\widetilde{h}}=-k_2\frac{d\phi\,d\psi}{\phi\psi'}+k_3\frac{d\rho\,d\phi}{\rho'\phi}.$$

This is clearly in $\mathcal{I}_E\Omega^2(\log E)$ since $\frac{d\phi\, d\psi}{\phi\psi'}$ and $\frac{d\rho\, d\phi}{\rho'\phi}$ are each nontrivial linear combinations of $\frac{du\, dv}{uv}$, $\frac{dv\, dw}{vw}$, $\frac{dw\, du}{wu}$ (because the logarithmic Nash frame serves as a basis for $\Omega^1_{\widetilde{U}}(\log E)$; see Section 3).

At a simple point of E contained in $\widetilde{H}$, say $e \in E_1 \cap \widetilde{H}$, we can choose coordinates $\{u, v, w\}$ so that $E_1 = \{u = 0\}$ and $\widetilde{H} = \{v = 0\}$. Since $m_i = n_i = p_i$ on components E_1 that intersect $\widetilde{H}$, up to unit we have

$$\widetilde{h} = u^{m_i} v = u^{n_i} v = \psi' v = \psi.$$

In such a case, we have

$$\widetilde{\beta}(\omega) = \omega \wedge \frac{d\widetilde{h}}{\widetilde{h}} = \frac{k_1}{v}\frac{d\phi\, d\psi}{\phi\psi'} - \frac{k_3}{v}\frac{d\psi\, d\rho}{\psi'\rho'},$$

which is clearly in $\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H})$.

Finally, at a double point of E contained in $\widetilde{H}$, $e \in E_1 \cap E_2 \cap \widetilde{H}$, we can choose coordinates $\{u, v, w\}$ centered at e so that $E_1 = \{u = 0\}$, $E_2 = \{v = 0\}$, and $\widetilde{H} = \{w = 0\}$. Since $m_i = n_i$ and $m_j = n_j$, in such a case (see Theorem 2.10), and $\operatorname{div}(\widetilde{h}) = Z + \widetilde{H}$, we have (up to unit)

$$\widetilde{h} = u^{m_i} v^{m_j} w = u^{n_i} v^{n_j} w = \psi = \psi' w.$$

Thus, $\widetilde{\beta}(\omega)$ is given in this case by

$$\widetilde{\beta}(\omega) = \omega \wedge \frac{d\widetilde{h}}{\widetilde{h}} = \frac{k_1}{w}\frac{d\phi\, d\psi}{\phi\psi'} - \frac{k_3}{w}\frac{d\psi\, d\rho}{\psi'\rho'},$$

which as above, is clearly an element of $\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H})$.

Since $\mathcal{O}(\widetilde{H}) \approx \mathcal{O}$, away from $\widetilde{H}$, and $\mathcal{O}(\widetilde{H})$ is generated by v^{-1} (respectively, w^{-1}) near a point e in the simple (respectively, double) point case near $\widetilde{H}$, the computations above show that $\omega \wedge \frac{d\widetilde{h}}{\widetilde{h}}$ is always in $\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H})$.

We will define β to be the composition of the map $\widetilde{\beta}$ with the projection

$$\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H}) \overset{p}{\twoheadrightarrow} \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H}) \big/ \Lambda^2\mathcal{N}(2Z - E) \otimes \mathcal{O}(\widetilde{H});$$

however, first we must show that this projection is well defined; i.e., we must show that $\Lambda^2\mathcal{N}(2Z - E) \otimes \mathcal{O}(\widetilde{H})$ is a subset of $\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H})$. (The following computation assumes we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively.)

$$\begin{aligned}(4.3)\qquad &\Lambda^2\mathcal{N}(2Z - E) \otimes \mathcal{O}(\widetilde{H}) \\ &\quad = \{(a\, d\phi\, d\psi + b\, d\psi\, d\rho + c\, d\rho\, d\phi) \cdot g \cdot r \mid a, b, c \in \mathcal{O}, \\ &\qquad g \in \mathcal{O}(2Z - E), r \in \mathcal{O}(\widetilde{H})\}\end{aligned}$$

$$= \left\{ Ar\frac{d\phi\, d\psi}{\phi\psi'} + Br\frac{d\psi\, d\rho}{\psi'\rho'} + Cr\frac{d\rho\, d\phi}{\rho'\phi} \,\middle|\, A \in \mathcal{O}(Z-N-E), \right.$$
$$\left. B \in \mathcal{O}(2Z-N-P-E), C \in \mathcal{O}(Z-P-E), r \in \mathcal{O}(\widetilde{H}) \right\}.$$

Since $\mathcal{O}(2Z-N-P-E) \subset \mathcal{O}(Z-P-E) \subset \mathcal{O}(Z-N-E) \approx \mathcal{I}_E\mathcal{O}(Z-N)$, we see from the above computation that

$$\Lambda^2\mathcal{N}(2Z-E) \otimes \mathcal{O}(\widetilde{H}) \subset \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(Z-N+\widetilde{H}).$$

Moreover, since $\mathcal{O}(Z-N) \subset \mathcal{O}$, we have shown that $\Lambda^2\mathcal{N}(2Z-E) \otimes \mathcal{O}(\widetilde{H})$ is contained in $\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H})$. Thus, the projection p is well defined, and we can define $\beta := p \circ \widetilde{\beta}$.

Now, we show that the sequence is exact at $\mathcal{I}_E\Omega^1(\log E)$, in other words, that $\ker(\beta) = \mathrm{im}(\alpha)$. Let $\omega = k_1\frac{d\phi}{\phi} + k_2\frac{d\psi}{\psi'} + k_3\frac{d\rho}{\rho'}$ be any element of $\mathcal{I}_E\Omega^1(\log E)$. We have

$$\begin{aligned} \omega \in \ker(\beta) \quad &\Longleftrightarrow \quad \widetilde{\beta}(\omega) \in \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H}), \\ \omega \in \mathrm{im}(\alpha) \quad &\Longleftrightarrow \quad \omega \in \mathcal{N}(Z-E). \end{aligned}$$

Let us first handle the case where we are away from $\widetilde{H}$. Looking back on our computation of $\Lambda^2\mathcal{N}(2Z-E) \otimes \mathcal{O}(\widetilde{H})$ in (4.3), where r is now equal to 1, we see that $\omega \in \ker(\beta)$ if and only if $-k_2 = A \in \mathcal{O}(Z-N-E)$ and $k_3 = C \in \mathcal{O}(Z-P-E)$. Comparing this with our computation of $\mathcal{N}(Z-E)$ in (4.2), it is clear that this is precisely the condition we need in order to have $\omega \in \mathcal{N}(Z-E)$, i.e., $\omega \in \mathrm{im}(\alpha)$.

Near $\widetilde{H}$, say at a simple point $e \in E_1 \cap \widetilde{H}$, we have $Z = N = P$. We see that $\omega \in \ker(\beta)$ if and only if $k_1 = A \in \mathcal{O}(Z-N-E) \approx \mathcal{O}(-E)$ and $-k_3 = B \in \mathcal{O}(2Z-N-P-E) \approx \mathcal{O}(-E)$. Note that since $Z = N = P$, k_2 and k_3 are a priori in $\mathcal{O}(Z-N-E) \approx \mathcal{O}(-E)$; thus we have exactly the conditions we need in order to have $\omega \in \mathrm{im}(\alpha)$.

Likewise, at a double point of E contained in $\widetilde{H}$, we have $Z = N$. Now $\omega \in \ker(\beta)$ if and only if $k_1 = A \in \mathcal{O}(Z-N-E) \approx \mathcal{O}(-E)$ and $-k_3 = B \in \mathcal{O}(2Z-N-P-E) \approx \mathcal{O}(Z-P-E)$. Looking back at (4.2) we see that these conditions imply that $\omega \in \mathcal{N}(Z-E)$ We have now shown that in all cases, the sequence is exact at $\mathcal{I}_E\Omega^1(\log E)$.

As a first step towards defining γ, we define the map

$$\widetilde{\gamma} : \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H}) \longrightarrow \mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2\widetilde{H})$$

by $\widetilde{\gamma}(\tau) = \tau \wedge \frac{d\widetilde{h}}{\widetilde{h}}$. Take $\tau \in \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(\widetilde{H})$. Then $\tau = Ar\frac{d\phi\, d\psi}{\phi\psi'} + Br\frac{d\psi\, d\rho}{\psi'\rho'} + Cr\frac{d\rho\, d\phi}{\rho'\phi}$, with $A, B, C \in \mathcal{O}(-E)$, and $r \in \mathcal{O}(\widetilde{H})$. We will first show that the map $\widetilde{\gamma}$ is well defined, i.e., that $\widetilde{\gamma}(\tau)$ is in fact an element of $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})$. Away from $\widetilde{H}$ (so $r=1$), we have $\widetilde{h}=\phi$, and thus

$$\widetilde{\gamma}(\tau)=\tau\wedge\frac{d\widetilde{h}}{\widetilde{h}}=B\frac{d\phi\,d\psi\,d\rho}{\phi\psi'\rho'};$$

which is in $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})$ since $\frac{d\phi\,d\psi\,d\rho}{\phi\psi'\rho'}$ is a nowhere-vanishing multiple of $\frac{du\,dv\,dw}{uvw}$ and $\mathcal{O}(\widetilde{H})\approx\mathcal{O}$ away from $\widetilde{H}$.

Near a simple point $e\in E_1$ contained in $\widetilde{H}$, we have (up to unit) $\widetilde{h}=\psi=\psi'v$ (where as above we have chosen coordinates $\{u,v,w\}$ for $\widetilde{U}$, so that $E_1=\{u=0\}$ and $\widetilde{H}=\{v=0\}$), and thus

$$\widetilde{\gamma}(\tau)=\tau\wedge\frac{d\widetilde{h}}{\widetilde{h}}=\frac{Cr}{v}\frac{d\phi\,d\psi\,d\rho}{\phi\psi'\rho'},$$

which is clearly in $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})$ since r and $\frac{1}{v}$ are in $\mathcal{O}(\widetilde{H})$.

Near a double point $e\in E_1\cap E_2\cap\widetilde{H}$, up to unit we have $\widetilde{h}=\psi=\psi'w$ (in appropriate coordinates). In this case, we have

$$\widetilde{\gamma}(\tau)=\tau\wedge\frac{d\widetilde{h}}{\widetilde{h}}=\frac{Cr}{w}\frac{d\phi\,d\psi\,d\rho}{\phi\psi'\rho'},$$

which is an element of $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})$. Thus, in all cases we have shown that $\widetilde{\gamma}$ is well defined.

As a further step towards defining γ, we will show that we have a well-defined projection

$$\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})\overset{\widetilde{p}}{\twoheadrightarrow}\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})/\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H}).$$

It suffices to prove that we have an injection of $\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H})$ into $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})$. (Once again, we assume we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively.)

$$\begin{aligned}
&\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H})\\
&\quad=\{(a\,d\phi\,d\psi\,d\rho)\cdot f\cdot r^2\,|\,a\in\mathcal{O}(3Z-E),r\in\mathcal{O}(\widetilde{H})\}\\
&\quad=\left\{Kr^2\frac{d\phi\,d\psi\,d\rho}{\phi\psi'\rho'}\;\middle|\;K\in\mathcal{O}(2Z-P-N-E),r\in\mathcal{O}(\widetilde{H})\right\}\\
&\quad=\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2Z-P-N+2\widetilde{H}).
\end{aligned}$$

Since $\mathcal{O}(2Z-N-P)\approx\mathcal{O}(Z-N)\otimes\mathcal{O}(Z-P)\subset\mathcal{O}$, we have $\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H})$ as a subsheaf of $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})$, and the projection $\widetilde{p}$ is well defined.

We will define the map γ using the maps $\widetilde{\gamma}$, p, and $\widetilde{p}$ via the diagram

$$\begin{array}{ccc}
\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H}) & \overset{p}{\twoheadrightarrow} & \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})\big/\Lambda^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(\widetilde{H}) \\
{\scriptstyle\widetilde{\gamma}}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow \\
\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H}) & \overset{\widetilde{p}}{\twoheadrightarrow} & \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})\big/\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H}).
\end{array}$$

In other words, given

$$\bar{\tau}\in\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})/\Lambda^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(\widetilde{H}),$$

with representative $\tau\in\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})$, $p(\tau)=\bar{\tau}$, we define $\gamma(\bar{\tau})=\widetilde{p}(\widetilde{\gamma}(\tau))$. This is well defined because the restriction of $\widetilde{\gamma}$ to $\Lambda^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(\widetilde{H})$ maps into $\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H})$; if

$$\tau=Ar\frac{d\phi\,d\psi}{\phi\psi'}+Br\frac{d\psi\,d\rho}{\psi'\rho'}+Cr\frac{d\rho\,d\phi}{\rho'\phi}$$

is an element of $\Lambda^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(\widetilde{H})$, then we have $A\in\mathcal{O}(Z-N-E)$, $B\in\mathcal{O}(2Z-N-P-E)$, $C\in\mathcal{O}(Z-P-E)$, and $r\in\mathcal{O}(\widetilde{H})$. Looking at the computations above, it is clear that in this case we have $\widetilde{\gamma}(\tau)\in\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H})$.

The map γ is surjective because the map $\widetilde{\gamma}$ is: given $\tau\in\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})$ as above, we can choose B (if away from $\widetilde{H}$) or C (if near $\widetilde{H}$) in the coefficients of τ, so that $\widetilde{\gamma}(\tau)$ hits any specified element of $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2\widetilde{H})$.

It now remains only to prove that $\ker(\gamma)=\operatorname{im}(\beta)$. It is easy to show that $\operatorname{im}(\beta)\subseteq\ker(\gamma)$; given ω in $\mathcal{I}_E\Omega^1(\log E)$ we must show that $\gamma(\beta(\omega))=0$, i.e., that $\widetilde{p}(\widetilde{\gamma}(\widetilde{\beta}(\omega)))=0$. We have

$$\widetilde{p}(\widetilde{\gamma}(\widetilde{\beta}(\omega)))=\widetilde{p}\biggl(\omega\wedge\frac{d\widetilde{h}}{\widetilde{h}}\wedge\frac{d\widetilde{h}}{\widetilde{h}}\biggr)=\widetilde{p}(0)=0.$$

To show that $\ker(\gamma)\subseteq\operatorname{im}(\beta)$, take $\bar{\tau}=[\tau]$ with τ in $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(\widetilde{H})$. If $\bar{\tau}\in\ker(\gamma)$, then τ must be in $\ker(\widetilde{p}\circ\widetilde{\gamma})$; that is to say, $\widetilde{\gamma}(\tau)$ is contained in $\Lambda^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2\widetilde{H})$. Suppose

$$\tau=Ar\frac{d\phi\,d\psi}{\phi\psi'}+Br\frac{d\psi\,d\rho}{\psi'\rho'}+Cr\frac{d\rho\,d\phi}{\rho'\phi}$$

(a priori A, B, and C are in $\mathcal{O}(-E)$, and $r\in\mathcal{O}(\widetilde{H})$). Away from $\widetilde{H}$ we have $\widetilde{h}=\phi$ (and $r=1$ in τ), and we see that if $\tau\in\ker(\widetilde{p}\circ\widetilde{\gamma})$, then $B\in\mathcal{O}(2Z-N-P-E)$. To show that $\bar{\tau}\in\operatorname{im}(\beta)$, we must show that there exists an $\omega\in\mathcal{I}_E\Omega^1(\log E)$ so that $\bar{\tau}=\beta(\omega)=p(\widetilde{\beta}(\omega))$, i.e. $p(\tau)=p(\widetilde{\beta}(\omega))$. Choose $\omega=k_1\frac{d\phi}{\phi}+k_2\frac{d\psi}{\psi'}+k_3\frac{d\rho}{\rho'}$ with $k_2=-A$ and $k_3=C$; then we have

$$p(\widetilde{\beta}(\omega))=p\biggl(A\frac{d\phi\,d\psi}{\phi\psi'}+C\frac{d\rho\,d\phi}{\rho'\phi}\biggr).$$

On the other hand, since $B \in \mathcal{O}(2Z - N - P - E)$, we have

$$p(\tau) = p\left(A\frac{d\phi\, d\psi}{\phi\psi'} + B\frac{d\psi\, d\rho}{\psi'\rho'} + C\frac{d\rho\, d\phi}{\rho'\phi}\right) = p\left(A\frac{d\phi\, d\psi}{\phi\psi'} + C\frac{d\rho\, d\phi}{\rho'\phi}\right).$$

Thus, we have shown that away from $\widetilde{H}$, $\ker(\gamma) \subseteq \operatorname{im}(\beta)$.

At a simple point $e \in E_1 \cap \widetilde{H}$ near $\widetilde{H}$, we have coordinates $\{u, v, w\}$ in an analytic neighborhood of e so that $E_1 = \{u = 0\}$, $\widetilde{H} = \{v = 0\}$, and $\widetilde{h} = \psi = \psi' v$ (recall that $Z = N = P$ on components E_1 that intersect $\widetilde{H}$). We see that if $\tau \in \ker(\widetilde{p} \circ \widetilde{\gamma})$, then $C \in \mathcal{O}(Z - P - E) \approx \mathcal{O}(-E)$. Again, we must find an $\omega \in \mathcal{I}_E \Omega^1(\log E)$ so that $p(\tau) = p(\widetilde{\beta}(\omega))$; choose $\omega = k_1 \frac{d\phi}{\phi} + k_2 \frac{d\psi}{\psi'} + k_3 \frac{d\rho}{\rho'}$ with $k_1 = A$ and $k_3 = -B$. Then we have

$$p(\widetilde{\beta}(\omega)) = p\left(\frac{A}{v}\frac{d\phi\, d\psi}{\phi\psi'} + \frac{B}{v}\frac{d\psi\, d\rho}{\psi'\rho'}\right).$$

On the other hand, since $C \in \mathcal{O}(Z - P - E) \approx \mathcal{O}(-E)$ we have

$$p(\tau) = p\left(Ar\frac{d\phi\, d\psi}{\phi\psi'} + Br\frac{d\psi\, d\rho}{\psi'\rho'} + Cr\frac{d\rho\, d\phi}{\rho'\phi}\right) = p\left(Ar\frac{d\phi\, d\psi}{\phi\psi'} + Br\frac{d\psi\, d\rho}{\psi'\rho'}\right).$$

Since $r = \frac{1}{v}$, this shows that $p(\tau) = p(\widetilde{\beta}(\omega))$, and thus we have shown that near a simple point of E contained in $\widetilde{H}$, $\ker(\gamma) \subseteq \operatorname{im}(\beta)$.

Finally, let $e \in E_1 \cap E_2 \cap \widetilde{H}$ be a double point of E that is contained in $\widetilde{H}$. With coordinates $\{u, v, w\}$ about e so that $E_1 = \{u = 0\}$, $E_2 = \{v = 0\}$, and $\widetilde{H} = \{w = 0\}$, we have $\widetilde{h} = \psi = \psi' w$. Moreover, $Z = N$ on this analytic neighborhood of e. It is evident that if $\tau \in \ker(\widetilde{p} \circ \widetilde{\gamma})$, then $C \in \mathcal{O}(Z - P - E)$. Once more, we wish to find an element ω of $\mathcal{I}_E \Omega^1(\log E)$ with the property that $p(\tau) = p(\widetilde{\beta}(\omega))$. As above, choose $\omega = k_1 \frac{d\phi}{\phi} + k_2 \frac{d\psi}{\psi'} + k_3 \frac{d\rho}{\rho'}$ with $k_1 = A$ and $k_3 = -B$. Then we have

$$p(\widetilde{\beta}(\omega)) = p\left(\frac{A}{v}\frac{d\phi\, d\psi}{\phi\psi'} + \frac{B}{v}\frac{d\psi\, d\rho}{\psi'\rho'}\right).$$

Moreover, since p mods out by $\Lambda^2 \mathcal{N}(2Z - E) \otimes \mathcal{O}(\widetilde{H})$, expression (4.3) shows that again we have

$$p(\tau) = p\left(Ar\frac{d\phi\, d\psi}{\phi\psi'} + Br\frac{d\psi\, d\rho}{\psi'\rho'} + Cr\frac{d\rho\, d\phi}{\rho'\phi}\right) = p\left(Ar\frac{d\phi\, d\psi}{\phi\psi'} + Br\frac{d\psi\, d\rho}{\psi'\rho'}\right).$$

Thus, in each of the three possible cases, we have $\ker(\gamma) \subseteq \operatorname{im}(\beta)$. This completes the proof. □

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References

[1] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero I*, Ann. of Math. **79** (1964), 109–203. MR 0199184

[2] G. Gonzalez-Sprinberg, *L'obstruction locale d'Euler et le théorème de MacPherson*, Astérisque **82–83** (1981), 7–32. MR 0629121

[3] W. Pardon and M. Stern, *Pure Hodge structure on the L_2-cohomology of varieties with isolated singularities*, J. Reine Angew. Math. **533** (2001), 55–80. MR 1823864

[4] L. Taalman, *The Nash sheaf of a complete resolution*, Manuscripta Math. **106** (2001), 249–270. MR 1865567

[5] B. Tessier, *Cycles evanescents, sections planes et conditions de Whitney*, Astérisque **7–8** (Singularités à Cargèse) (1973), 285–362.

[6] L. D. Trang and B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulières*, Ann. of Math. **114** (1981), 457–491. MR 0634426

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