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# DEFINABLE SMOOTHING OF LIPSCHITZ CONTINUOUS FUNCTIONS

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ABSTRACT. Let $\mathcal{M}$ be an $o$-minimal expansion of a real closed field. We prove the definable smoothing of definable Lipschitz continuous functions. In the case of Lipschitz functions of one variable, we are even able to preserve the Lipschitz constant.

## 1. Introduction

The present paper is motivated by the recently studied smoothing of Lipschitz continuous functions defined on separable Riemannian manifolds, cf. [1], of which we prove an $o$-minimal version.

Let $R$ denote a real closed field and $\mathcal{M}$ an $o$-minimal expansion of $R$. In the sequel, "definable" means "definable with parameters in $\mathcal{M}$." We assume the reader to be familiar with basic properties of $o$-minimal structures, cf. [9] or [3]. For examples of $o$-minimal structures, we refer to [2], Chapter 2, [10], [4], [5], and [12].

We endow $R^n$ with the Euclidean $R$-norm $\|\cdot\|$ (note that an $R$-norm has the same definition as norm just taking its values in $R$) and the corresponding topology. Moreover, $\mathcal{C}^m$ is short for "$m$ times continuously differentiable" where $m \in \mathbb{N}$.

The aim of this paper is to prove the following theorem.

THEOREM 1. *Let $U \subset R^n$ be open, let $f : U \to R$ be a definable Lipschitz continuous function, and let $\varepsilon : U \to (0,\infty)$ be a definable continuous function. Then there is a definable Lipschitz continuous $\mathcal{C}^m$ function $g : U \to R$ such that*

$$|g(u) - f(u)| < \varepsilon(u), \quad u \in U.$$

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Related approximation theorems for definable differentiable functions can be found in [6] and [11].

The classical methods for smoothing functions use integration which is not applicable to $o$-minimal structures. We bypass integration by using a consequence of the concept of $\Lambda^m$-regular stratification of definable sets which was developed in [7]. Our method does not allow us to control the Lipschitz constant while smoothing the function. To be more precise, if the definable function depends on at least two variables, the Lipschitz constant of the approximating function may be much bigger than that of the original function.

As indicated above, we obtain a stronger result for definable functions of one variable, cf. Proposition 2.

REMARK 1. The method of definable smoothing has some further property which may be of interest for some applications. As a definable function, $f$ is continuously differentiable outside a definable set $A \subset U$ of lower dimension. If $V$ is an open definable neighborhood of $\mathrm{cl}(A) \cap U$, where $\mathrm{cl}(A)$ denotes the topological closure of $A$, we may assume that $g$ coincides with $f$ outside $V$.

## 2. One-dimensional functions

If we consider functions of one variable, we can preserve the Lipschitz constant during the smoothing process.

PROPOSITION 2. *Let $f : I \to R$ be a definable Lipschitz continuous function with constant $L$ defined on an open interval $I$, and let $\varepsilon : I \to (0,\infty)$ be a definable continuous function. Then there is a definable Lipschitz continuous $\mathcal{C}^1$ function $g : I \to R$, such that $|g(t) - f(t)| < \varepsilon(t)$ and $|g'(t)| \leq L$, $t \in I$.*

*Proof.* As a definable function of one variable, $f$ is continuously differentiable outside of a finite set $\{a_1, \dots, a_k\}$, cf. [3], Chapter 7, Theorem 3.2. The Lipschitz continuity implies the existence of

$$\lim_{t \nearrow 0} f(a_i + t)/t = c \quad \text{and} \quad \lim_{t \searrow 0} f(a_i + t)/t = d$$

in $R$, cf. [3], Chapter 3, Corollary 1.6. By definability of $f$, there is a pointed definable neighborhood $U_i$ of $a_i$, such that $f$ is continuously differentiable in $U_i$. Without loss of generality, we may assume that $a_i = 0$ and $c < d$. For $0 < \sigma < 1$, let $h_\sigma : (-1,1) \to R$ be defined by

$$h_\sigma(t) = \begin{cases} -\frac{c-d}{4\sigma}(\sigma + t)^2, & -\sigma < t \leq 0, \\ -\frac{c-d}{4\sigma}(\sigma - t)^2, & 0 < t < \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $h_\sigma$ is $\mathcal{C}^1$ outside 0 and that

$$\lim_{t \nearrow 0} h'_\sigma(t) = -(c-d)/2 \quad \text{and} \quad \lim_{t \searrow 0} h'_\sigma(t) = (c-d)/2.$$

Let $g_\sigma := f + h_\sigma$. Then $g_\sigma$ is $\mathcal{C}^1$ at 0 and $g_\sigma(t) = f(t)$, $|t| > \sigma$. Since $h_\sigma$ is bounded by $|c-d|\sigma$, we may also assume that $|g_\sigma(t) - f(t)| \le \varepsilon(t)$ for $\sigma$ being sufficiently small. The derivative of $h_\sigma$ (outside 0) is bounded by $|c-d|/2$, nonpositive for $t<0$, and nonnegative for $t>0$. So, if we choose $\sigma$ sufficiently small we obtain the estimates $|f'(t)-c| < |c-d|/4$ for $-\sigma < t < 0$ and $|f'(t)-d| < |c-d|/4$ for $0 < t < \sigma$. Hence, $g_\sigma$ has the same Lipschitz constant as $f$. Applying this method to all $a_i$, we obtain a $g$ with the desired properties. □

## 3. Preliminaries

The proof of Theorem 1 is prepared by several lemmas. We recall the well-known fact that a definable Lipschitz continuous function $f : U \to R$ can always be extended to a definable Lipschitz continuous function $\overline{f}$ defined on $\mathrm{cl}(U)$. This extended function is unique.

The next lemma names conditions to assume Lipschitz continuity for definable differentiable functions with bounded derivative. Note that in general a continuously differentiable function of several variables with bounded derivative is not Lipschitz continuous. In the sequel, the symbol $\nabla$ is used to denote the gradient operator.

Lemma 3. *Let $U \subset R^n$ be open and $f : U \to R$ be definable and Lipschitz continuous. Let $V \subset U$ be open and $g : V \to R$ be definable and continuously differentiable with bounded gradient, such that*

$$F(\xi) := \begin{cases} g(\xi), & \xi \in V, \\ \overline{f}(\xi), & \xi \in \mathrm{cl}(U) \setminus V, \end{cases}$$

*is continuous. Then $F$ is Lipschitz continuous.*

*Proof.* We select $L > 0$ large enough such that $f$ is Lipschitz continuous with constant $L$ and $\|\nabla g\|$ is bounded by $L$. For $x, y \in \mathrm{cl}(U)$, we set

$$[x,y] := \{x + t(y-x) : 0 \le t \le 1\}.$$

The set $[x,y]$ is not necessarily contained in $\mathrm{cl}(U)$. But according to $o$-minimality, there exist $0 = t_1 \le \cdots \le t_{2N} = 1$ such that

$$[x,y] \cap \mathrm{cl}(U) = \bigcup_{i=1}^{N} [\xi_{2i-1}, \xi_{2i}],$$

where $\xi_i := x + t_i(y-x)$, $i = 1, \ldots, 2N$. We may further assume that for $1 \le i \le N$ either $[\xi_{2i-1}, \xi_{2i}] \subset \mathrm{cl}(U) \setminus V$ or $[\xi_{2i-1}, \xi_{2i}] \setminus \{\xi_{2i-1}, \xi_{2i}\} \subset V$ applies.

The function $F$ restricted to $[\xi_{2i-1}, \xi_{2i}]$ is Lipschitz continuous with constant $L$. By the properties of $g$ and $f$, we conclude that

$$|F(\xi_j) - F(\xi_{j+1})| \le L\|\xi_{j+1} - \xi_j\| = L\|y-x\|(t_{j+1} - t_j),$$

for $j=1,\dots,2N-1$. So,

$$|F(y)-F(x)| = \left|\sum_{j=1}^{2N-1} F(\xi_{j+1})-F(\xi_j)\right| \le \sum_{j=1}^{2N-1} L(t_{j+1}-t_j)\|y-x\|$$
$$= L\|y-x\|. \qquad \square$$

For a definable open set $U\subset R^n$, we denote by $\mathcal{C}_b^m(U,R^k)$ the definable $\mathcal{C}^m$ functions from $U$ to $R^k$ with bounded (first) derivative. $\pm\infty$ are regarded as constant functions.

Definition 1. A $\mathcal{C}_b^m$ *cell* of $R$ is either an open interval or a singleton. Suppose that we know all $\mathcal{C}_b^m$ cells of $R^\ell$, $1\le \ell\le n$. Then a $\mathcal{C}_b^m$ *cell* $M$ of $R^{n+1}$ is either a set of the form $\{(x,y): x\in X, y=h(x)\}$ where $X\subset R^d$ is an open $\mathcal{C}_b^m$ cell in $R^d$ and $h: X\to R^{n+1-d}$ is a definable $\mathcal{C}_b^m$ function; or $M$ is of the form $\{(x,y): x\in X, f(x)<y<g(x)\}$ where $X\subset R^n$ is an open $\mathcal{C}_b^m$ cell and $f,g\in \mathcal{C}_b^m(X,R)\cup\{\pm\infty\}$ such that for all $x\in X$, $f(x)<g(x)$; or $M$ is a singleton.

By construction, all $\mathcal{C}_b^m$ cells are definable. Moreover, $\mathcal{C}_b^1$ functions defined on a $\mathcal{C}_b^1$ cell are Lipschitz continuous, cf. [7], Corollary 9.9.

A definable function $f: A\to R^d$, where $A$ is not necessarily open, is called $\mathcal{C}^m$ if there exists an open definable set $B$ containing $A$ and a definable $\mathcal{C}^m$ function $g: B\to R^d$ which coincides with $f$ on $A$.

The *dimension* of a definable set is the maximal integer $d$, such that $A$ contains a definable set which is definably homeomorphic to $R^d$. This definition is well defined, cf. [3], and we refer the reader to [3], Chapter 4, for a detailed description of dimension. Moreover, it is straight forward to verify that a $\mathcal{C}_b^m$ cell is definably homeomorphic to some $R^d$.

For a differentiable function $f$, the symbol $\nabla_x f$ is used to denote its gradient with respect to the variables $x$.

Lemma 4. *Let $M\subset R^n$ be a $\mathcal{C}_b^m$ cell of dimension $d<n$ and $M\subset V\subset U$ definable open neighborhoods of $M$. Let $f: U\to R$ be definable and Lipschitz continuous, such that both $f|_{U\setminus M}$ and $f|_M$ are $\mathcal{C}^m$. Then for every definable continuous $\varepsilon: U\to(0,\infty)$ there is a definable Lipschitz continuous $\mathcal{C}^m$ function $g: U\to R$ such that $f=g$ outside $V$ and*

$$|g(u)-f(u)|<\varepsilon(u), \quad u\in U.$$

*Proof.* The dimension of $M$ is less than $n$. So, $M$ is the graph of a definable $\mathcal{C}_b^m$ function $h: X\to R^{n-d}$ where $X\subset R^d$ is an open $\mathcal{C}_b^m$ cell. Let $U' := U\cap X\times R^{n-d}$. For each $\xi\in M$, $\varepsilon(\xi)>0$. So, the continuity of $f$ implies that there is an open definable neighborhood $V'$ of $M$ such that $|f(\xi)-f(\xi+\eta)|<\varepsilon(\xi+\eta)$ whenever $\xi\in M$ and $\eta\in\{0\}\times R^{n-d}$ with $\xi+\eta\in V'$. We may further choose $V'$ in such a way that $M\subset V'\subset (V\cap(X\times R^{n-d}))$.

We define $\psi: X\times R^{n-d}\to X\times R^{n-d}$ by $\psi(x,y)=(x,y-h(x))$. The function $\psi$ is $\mathcal{C}_b^m$, and so since $M\times R^{n-d}$ is a $\mathcal{C}_b^1$ cell, $\psi$ is Lipschitz continuous. Hence, we can extend $\psi$ to a Lipschitz continuous function $\overline{\psi}$ defined on $\mathrm{cl}(U')$. In addition, $\overline{\psi}$ is bijective with Lipschitz continuous inverse.

The function $F:=f\circ\psi^{-1}$ is, as composition of Lipschitz continuous functions, also Lipschitz continuous. In addition, $F$ is $\mathcal{C}^m$ in $\psi(U')\setminus(X\times\{0\})$ and $X\times\{0\}$.

*Step* 1: We construct a $\mathcal{C}_b^m$ function $\sigma: X\to R$ which tends to 0 as $x$ tends to the boundary of $X$ or infinity, such that $\psi(V')$ contains the set $W:=\{(x,y)\in X\times R^{n-d}:\|y\|<\sigma(x)\}$.

Let the semi-algebraic function $\phi: R^d\to(-1,1)^d$ be given by $\phi(x_1,\dots,x_d)=(x_1/\sqrt{1+x_1^2},\dots,x_d/\sqrt{1+x_d^2})$. This map is obviously $\mathcal{C}_b^m$, and the set $\phi(X)$ is bounded and open. We select a definable $\mathcal{C}^m$ function $\theta: R^d\to R$ which vanishes outside $\phi(X)$ and is positive on $\phi(X)$. The support of $\theta$ is bounded, so $\theta$ is $\mathcal{C}_b^m$. Note that the zero-set of $D: R^d\to R$, $x\mapsto\mathrm{dist}((x,0),R^d\setminus\psi(V'))$, is contained in the zero-set of $\theta$. This allows us to apply the generalized Łojasiewicz inequality, cf. [9], Theorem C14, to $\theta$ and $D$. So, we obtain a bijective definable continuous map $\rho: R\to R$ with $\rho(0)=0$ such that $\rho\circ\theta(x)\le D(x)$ for $x\in R^d$. By definability, $\rho$ is $\mathcal{C}^m$ in $(0,\delta)$ for some $0<\delta<1$. We define $\tilde{\rho}: R\to R$ by

$$t\mapsto\frac{t^{2m}}{1+t^{2m}}\rho\left(\frac{\delta t^{2m}}{1+t^{2m}}\right).$$

Hence, $\tilde{\rho}$ is $m$ times Peano differentiable at 0 and by [7], Proposition 7.2, the function $\tilde{\rho}$ is even $\mathcal{C}^m$ at 0. So, $\sigma=\tilde{\rho}\circ\theta\circ\phi$ is the desired function. We may further assume that $\|\nabla\sigma\|\le 1$.

*Step* 2: Let $\varphi:[0,\infty)\to[0,1]$ be a definable $\mathcal{C}^m$ function with $\varphi|_{[0,1/2]}=1$ and $\varphi|_{[1,\infty)}=0$. Then the derivative $\varphi'$ is bounded by some constant $K>0$. Note that $x\mapsto F(x,0)$ is a definable Lipschitz continuous $\mathcal{C}^m$ function on $X$. We define $G:\psi(U')\to R$ by

$$G(x,y):=F(x,0)\varphi\left(\frac{\|y\|}{\sigma(x)}\right)+F(x,y)\left(1-\varphi\left(\frac{\|y\|}{\sigma(x)}\right)\right).$$

The function $G$ is definable and $\mathcal{C}^m$ in $\psi(U')$. Since, for $(x,y)\in W$ the value $G(x,y)$ lies between $F(x,0)$ and $F(x,y)$, we obtain the inequality $|G(x,y)-F(x,y)|<\varepsilon(\psi^{-1}(x,y))$. We now prove the Lipschitz continuity of $G$.

By the assumption, $|F(\xi)-F(\eta)|\le L\|\xi-\eta\|$, and $\|\nabla F(x,y)\|$ is bounded by $L$ outside $X\times\{0\}$ as well as $\|\nabla(F(x,0))\|$ on $X\times R^{n-d}$.

We first show that $\|\nabla\varphi(\|y\|/\sigma(x))\|$ is bounded by $2K/\sigma(x)$.

$$\begin{aligned}\left\|\nabla\varphi\left(\frac{\|y\|}{\sigma(x)}\right)\right\| &\le \left|\varphi'\left(\frac{\|y\|}{\sigma(x)}\right)\right|\cdot\left\|\nabla\left(\frac{\|y\|}{\sigma(x)}\right)\right\|\\ &\le K\left\|\left(\frac{\|y\|}{\sigma^2(x)}\nabla_x\sigma(x), \frac{y}{\|y\|\sigma(x)}\right)\right\|\\ &= \frac{K}{\sigma(x)}\left\|\left(\frac{\nabla_x\sigma(x)\|y\|}{\sigma(x)}, \frac{y}{\|y\|}\right)\right\|\\ &\le \frac{2K}{\sigma(x)}.\end{aligned}$$

So, for $0<\|y\|\le\sigma(x)$

$$\begin{aligned}\|\nabla G(x,y)\| = \Bigg\|&\big(\nabla\big(F(x,0)-F(x,y)\big)\big)\varphi\left(\frac{\|y\|}{\sigma(x)}\right)\\ &+\big(F(x,0)-F(x,y)\big)\nabla\varphi\left(\frac{\|y\|}{\sigma(x)}\right)+\nabla F(x,y)\Bigg\|\\ \le L+L\|y\|&\frac{2K}{\sigma(x)}+L\le 2L(1+K).\end{aligned}$$

As a consequence, we see that $G$ is $\mathcal{C}_b^m$ in $W$, and that $G=F$ outside $W$. For the Lipschitz continuity of $G$, we use Lemma 3; i.e., we have to show that $G$ extends continuously to $\mathrm{cl}(\psi(U'))$, and that $G=F$ on the boundary $\partial\psi(U')$ of $\psi(U')$. This is evident for the points outside $\partial X\times\{0\}$ since there $G=F$, and $F$ is Lipschitz continuous by construction. We further note that

$$|G(x,y)-F(x,y)|\le\big|\big(F(x,0)-F(x,y)\big)\big|\le L\sigma(x).$$

So, for $\xi\in\partial X$ and $(x,y)\in\psi(U')$

$$\begin{aligned}|G(x,y)-F(\xi,0)| &\le |G(x,y)-G(x,0)|+|G(x,0)-F(\xi,0)|\\ &\le L\sigma(x)+L\|x-\xi\|.\end{aligned}$$

Therefore, the difference $G(u)-F(u)$ tends to 0 as $u\in\psi(U')$ tends to $\partial\psi(U')$.

*Step* 3: Now, we define $g: U\to R$ by

$$g(\xi):=\begin{cases}G(\psi(\xi)), & \text{if } \xi\in V',\\ f(\xi), & \text{otherwise.}\end{cases}$$

By using Lemma 3, we obtain the desired properties for $g$. □

## 4. Proof of Theorem 1

For the proof of Theorem 1, we use a consequence of $\Lambda^m$-regular stratification.

A $\mathcal{C}_b^m$ (resp. $\Lambda^m$-regular) *stratification* is a finite partition of $R^n$ into disjoint definable sets $X_1,\dots,X_r$; for $i=1,\dots,r$ there is a linear orthogonal isomorphism $\phi_i : R^n \to R^n$ such that $\phi_i(X_i)$ is a $\mathcal{C}_b^m$ (resp. standard $\Lambda^m$-regular) cell; in addition, the frontier $\partial X_i$ is the union of some of the $X_j$. We call a stratification *compatible* with the definable sets $A_1, \dots, A_s \subset R^n$ if each $A_j$ is the union of some $X_i$. Both [7], Theorem 4.5 and [8], Theorem 1.4, imply that for any definable sets $A_1, \dots, A_k \subset R^n$ there is $\mathcal{C}_b^m$ stratification of $R^n$ compatible with the $A_i$, $i = 1, \dots, k$.

*Proof of Theorem* 1. By [3], Chapter 7, Section 3, we can partition $U$ into finitely many definable sets $X_1, \dots, X_r$ such that the restrictions of $f$ to $X_i$ are $\mathcal{C}^m$. We select a $\mathcal{C}_b^m$ stratification of $\mathrm{cl}(U)$ compatible with the $X_1, \dots, X_r$. We use $N$ to denote the number of $\mathcal{C}_b^m$ cells $Z_i$ of dimension less than $n$ which are contained in $U$. Moreover, we may assume that $\dim(Z_i) \geq \dim(Z_{i+1})$ for $i = 1, \dots, N-1$. We choose for each $Z_i$ a definable open neighborhood $U_i$. Since, we deal with a stratification, we may assume that $U_i \cap U_j = \emptyset$ if $j > i$. For each $i = 1, \dots, N$, we choose a further definable open neighborhood $V_i$ of $Z_i$ by

$$Z_i \subset V_i \subset \left\{ x : \mathrm{dist}(x, Z_i) < \frac{1}{2} \mathrm{dist}(x, R^n \setminus U_i) \right\}. \tag{1}$$

Obviously, $Z_j \subset V_j \subset U_j$.

We define a sequence of functions $f_0, \dots, f_N$ in the following way. Set $f_0 = f$. Let $f_{j-1}$ be given such that $f_{j-1}$ is Lipschitz continuous,

$$|f(u) - f_{j-1}(u)| < (j-1)\varepsilon(u)/N, \quad u \in U,$$

and $f_{j-1}$ is $\mathcal{C}^m$ outside of $\bigcup_{i \geq j} Z_i$. By Lemma 4, there is a definable Lipschitz continuous $\mathcal{C}^m$ function $F_j : U_j \to R$ with

$$|f_{j-1}(u) - F_j(u)| < \varepsilon(u)/N, \quad u \in U_j,$$

such that $F_j = f_{j-1}$ outside $V_j$. This implies also that $\overline{F}_j(u) = \overline{f}_{j-1}(u)$ for $u \in \partial U_j$. We define

$$f_j(u) := \begin{cases} f_{j-1}(u), & u \in U \setminus U_j, \\ F_j(u), & u \in U_j. \end{cases}$$

By construction, $f_j$ is $\mathcal{C}^m$ outside $\bigcup_{i > j} Z_i$, $f_j$ is Lipschitz in $U$ by Lemma 3, and

$$|f_j(u) - f_{j-1}(u)| < \varepsilon(u)/N, \quad u \in U.$$

So, $f_N$ is a definable Lipschitz continuous $\mathcal{C}^m$ function, and

$$|f_N(u) - f(u)| < N\varepsilon(u)/N = \varepsilon(u), \quad u \in U. \qquad \square$$

In the proof of Theorem 1, we modified the values of the function $f : U \to R$ only in a special open neighborhood of $\mathrm{cl}(D) \cap U$ where $D$ is the set of points at which $f$ is not continuously differentiable. We can choose the $V_i$ arbitrarily small as long as they satisfy (1). Lemma 3 requires no special neighborhoods as long as they are definable and open. So, we may assume that there is an approximating $g : U \to R$ which coincides with $f$ outside an arbitrarily small definable open neighborhood of $\mathrm{cl}(D) \cap U$.

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