

APPROXIMATION PROPERTIES DEFINED BY SPACES OF OPERATORS AND APPROXIMABILITY IN OPERATOR TOPOLOGIES

ALEKSEI LISSITSIN, KRISTEL MIKKOR, AND EVE OJA

ABSTRACT. We develop a unified approach to characterize approximation properties defined by spaces of operators. Our main result describes them in terms of the approximability of weak*-weak continuous operators. In particular, we prove that if \mathcal{A} and \mathcal{B} are operator ideals satisfying $\mathcal{A} \circ \mathcal{B}^* \subset \mathcal{K}$, then the $\mathcal{A}(X, X)$ -approximation property of a Banach space X is equivalent to the following “metric” condition: for every Banach space Y and for every operator $T \in \mathcal{B}^*(Y, X)$, there exists a net $(S_\alpha) \subset \mathcal{A}(X, X)$ such that $\sup_\alpha \|S_\alpha T\| \leq \|T\|$ and $T^* S_\alpha^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(X^*, Y^*)$. As application, approximation properties of dual spaces and weak metric approximation properties are studied.

1. Introduction

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X to Y , and by $\mathcal{F}(X, Y)$, $\mathcal{K}(X, Y)$, and $\mathcal{W}(X, Y)$ its subspaces of finite-rank, compact, and weakly compact operators. If $X = Y$, then we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$, and similarly for other spaces of operators.

A Banach space X is said to have the *approximation property* if for every compact set $K \subset X$ and every $\epsilon > 0$, there exists a finite rank operator $S \in \mathcal{F}(X)$ such that $\|Sx - x\| < \epsilon$, for all $x \in K$. If one allows compact operators $S \in \mathcal{K}(X)$ in the preceding condition, then X is said to have the *compact approximation property*. As it was shown by Willis [W], these properties are not equivalent.

Received January 17, 2007; received in final form March 14, 2007.

This research was partially supported by Estonian Science Foundation Grant 5704.

2000 *Mathematics Subject Classification*. Primary 46B28. Secondary 46B20, 47B10, 47L05.

The approximation property was deeply studied by Grothendieck in his famous Memoir [G]. Among others, Grothendieck showed that the approximation property is closely related to the approximability of weakly compact operators in the strong operator topology. Namely, he proved (see [G, Chapter I, p. 141]) that if the dual space X^* of X has the approximation property, then for every Banach space Y , the closed unit ball $B_{\mathcal{F}(Y,X)}$ of $\mathcal{F}(Y,X)$ is dense in $B_{\mathcal{W}(Y,X)}$ in the strong operator topology. Grothendieck also claimed a stronger result (see [G, Chapter I, p. 184]) that the latter condition would be implied by the approximation property of X . But, his proof only goes through for the particular case when Y is complemented in Y^{**} by a norm one projection. (This proof was thoroughly analysed by Reinov [R1]; for a discussion of related questions see [R3].)

The Grothendieck's result was strengthened by Lima, Nygaard, and Oja [LNO] who proved that X has the approximation property if and only if $B_{\mathcal{F}(Y,X)}$ is dense in $B_{\mathcal{W}(Y,X)}$ in the strong operator topology. This may be considered as a “metric” characterization of the approximation property. Recently, the compact approximation property was described in terms of the approximability of weakly compact operators in the strong operator topology by Lima, Lima, and Nygaard [LLN] using similar “metric” conditions.

Let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{L}(X)$. A Banach space X is said to have the $\mathcal{A}(X)$ -approximation property if for every compact set $K \subset X$ and every $\varepsilon > 0$, there exists an operator $S \in \mathcal{A}(X)$, such that $\|Sx - x\| < \varepsilon$ for all $x \in K$. This general approximation property was studied, for instance, by Reinov [R2] and Grønboek and Willis [GW]. Its bounded version has recently been studied in [LO6] and [O1], and it has been proven useful in the studies on the duality of the distance to closed operator ideals due to Tylli [T1], [T3]. In particular, the non-self-duality was established for the essential norm of bounded linear operators on Banach spaces (see [T1]). Occasionally, we shall also use the notion of the $\mathcal{A}(X)$ -approximation property in a more general situation when $\mathcal{A}(X)$ is an arbitrary convex subset of $\mathcal{L}(X)$, extending verbatim the above definition.

The purpose of this article is to describe the $\mathcal{A}(X)$ -approximation property in terms of the approximability of weakly compact operators (or more generally, of operators from an operator ideal \mathcal{B}) with respect to the strong, weak, and norm operator topologies. The main result is Theorem 1 of the next Section 2. Section 3 contains various applications of Theorem 1, in particular, to characterize the $\mathcal{A}(X^*)$ -approximation property of the dual space X^* of X (see Section 3.3) and the weak metric $\mathcal{A}(X)$ -approximation property (see Section 3.4).

The results of this article, among others, encompass and complete similar results on the approximation and the compact approximation properties from [LNO], [LLN], [LO4], [OPe], and [Pe], yielding them a unified approach. In particular, the “metric” characterizations of the $\mathcal{A}(X)$ -approximation property via the approximability of weakly compact operators are extended from

the operator ideals $\mathcal{A} = \mathcal{F}$ and $\mathcal{A} = \mathcal{K}$ to any operator ideal \mathcal{A} such that $\mathcal{A} \circ \mathcal{W} \subset \mathcal{K}$.

Our method of proof, inspired by arguments from the recent paper [OPe] by Oja and Pelander, is rather self-contained and direct. For comparison, let us recall that in [LNO], [LLN], and [LO4], a roundabout way was used relying on characterizations of ideals [GKS] of operators and criteria of the approximation and the compact approximation properties in terms of ideals (established in [LO1], [LNO], [LO3], and [LLN]), and also on results of [LO2] describing the structure of Hahn–Banach extension operators.

Our notation is standard. A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $j_X : X \rightarrow X^{**}$. The identity operator on X is denoted by I_X . The closed unit ball of X is denoted by B_X . The closure of a set $A \subset X$ is denoted by \overline{A} . The linear span of A is denoted by $\text{span } A$ and the closed convex hull by $\overline{\text{conv}} A$.

If \mathcal{B} is an operator ideal, in the sense of Pietsch [P], then $\mathcal{B}_{w^*}(X^*, Y)$ denotes the subspace of $\mathcal{B}(X^*, Y)$ consisting of those operators which are weak*-weak continuous. Let us recall that $T \in \mathcal{L}(X^*, Y)$ is weak*-weak continuous if and only if $\text{ran } T^* \subset X$. Recall also that $\mathcal{L}_{w^*}(X^*, Y) = \mathcal{W}_{w^*}(X^*, Y)$ (if $T \in \mathcal{L}(X^*, Y)$ is weak*-weak continuous, then $T(B_{X^*})$ is weakly compact because B_{X^*} is compact in the weak* topology). The algebraic tensor product $X \otimes Y$ is always canonically identified with a linear subspace of $\mathcal{F}(X^*, Y)$. Let us recall that $X \otimes Y = \mathcal{F}_{w^*}(X^*, Y)$ and $X^* \otimes Y = \mathcal{F}(X, Y)$.

If \mathcal{B} is an operator ideal, then $\mathcal{B}(X, Y)$ will be equipped with the norm topology from $\mathcal{L}(X, Y)$, unless stated otherwise. If $x^{**} \in X^{**}$ and $y^* \in Y^*$, then the functional $y^* \otimes x^{**} \in (\mathcal{B}(X, Y))^*$ is defined by $(y^* \otimes x^{**})(T) = x^{**}(T^*y^*)$, $T \in \mathcal{B}(X, Y)$. Note that $\|y^* \otimes x^{**}\| = \|y^*\| \|x^{**}\|$. We denote by \mathcal{B}^* the dual operator ideal. Its components are $\mathcal{B}^*(X, Y) = \{T \in \mathcal{L}(X, Y) : T^* \in \mathcal{B}(Y^*, X^*)\}$. (The notation \mathcal{B}^* is reserved for another concept in [P] where the dual operator ideal is denoted by $\mathcal{B}^{\text{dual}}$.) For operator ideals \mathcal{A} and \mathcal{B} , we write $\mathcal{A} \subset \mathcal{B}$ if $\mathcal{A}(X, Y) \subset \mathcal{B}(X, Y)$ for all Banach spaces X and Y .

We denote by \mathcal{L} , \mathcal{W} , \mathcal{K} , and \mathcal{F} the operator ideals of bounded, weakly compact, compact, and finite-rank linear operators, respectively. Occasionally, we will need the following operator ideals: \mathcal{AC} —absolutely continuous operators, \mathcal{BS} —Banach–Saks operators, \mathcal{H} —Hilbert operators, \mathcal{I} —integral operators, \mathcal{Lim} —limited operators, \mathcal{P}_p —absolutely p -summing operators (p -summing in [DJT]), \mathcal{RN} —Radon–Nikodým operators, \mathcal{V} —completely continuous operators (see [P] or [DJT]; for \mathcal{AC} , \mathcal{BS} , and \mathcal{Lim} , see [N], [DSe], and [BD]).

2. Description of the $\mathcal{A}(X)$ -approximation property in terms of pointwise convergence of weak*-weak continuous operators

The following is the main result of this article.

THEOREM 1. *Let X be a Banach space and let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{L}(X)$ containing $\mathcal{F}(X)$. Let \mathcal{B} be an operator ideal containing \mathcal{K} and satisfying the condition*

$$\{ST^* : S \in \mathcal{A}(X), T \in \mathcal{B}_{w^*}(X^*, Y)\} \subset \mathcal{K}(Y^*, X)$$

for all Banach spaces Y . The following assertions are equivalent.

(a) X has the $\mathcal{A}(X)$ -approximation property.

(b) For every Banach space Y and for every operator $T \in \mathcal{B}_{w^*}(X^*, Y)$, there exists a net $(S_\alpha) \subset \mathcal{A}(X)$, such that $\sup_\alpha \|TS_\alpha^*\| \leq \|T\|$ and $S_\alpha T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Y^*, X)$.

(c) For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}_{w^*}(X^*, Z)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $S_n T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Z^*, X)$.

(d) For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}_{w^*}(X^*, Z)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $TS_n^* \rightarrow T$ in the weak operator topology on $\mathcal{L}(X^*, Z)$.

Since $\mathcal{B}_{w^*}(X^*, Y) \subset \mathcal{W}(X^*, Y)$, Theorem 1 essentially concerns those operator ideals \mathcal{B} which are contained in \mathcal{W} . (However, there are cases when $\mathcal{B}(X^*, Y) \subset \mathcal{W}(X^*, Y)$ for any Y without assuming that $\mathcal{B} \subset \mathcal{W}$. For instance, this is the case when $\mathcal{A} = \mathcal{V}$ and X^* contains no copy of ℓ_1 , because then $\mathcal{V}(X^*, Y) = \mathcal{K}(X^*, Y)$.) Keeping this in mind, let us observe that the hypothesis of Theorem 1 could be reformulated as follows.

PROPOSITION 2. *Let X be a Banach space and let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{L}(X)$. Let \mathcal{B} be an operator ideal. If*

$$\{ST : S \in \mathcal{A}(X), T \in \mathcal{B}^*(Y, X)\} \subset \mathcal{K}(Y, X)$$

for all Banach spaces Y , then

$$\{ST^* : S \in \mathcal{A}(X), T \in \mathcal{B}_{w^*}(X^*, Y)\} \subset \mathcal{K}(Y^*, X)$$

for all Banach spaces Y . The converse holds whenever $\mathcal{B} \subset \mathcal{W}$.

Proof. Let $S \in \mathcal{A}(X)$ and $T \in \mathcal{B}_{w^*}(X^*, Y)$. Since $\text{ran } T^* \subset X$, we may write $T^* = j_X t$, where $t : Y^* \rightarrow X$ is the astriction of T^* . But then $t^* = j_Y T \in \mathcal{B}(X^*, Y^{**})$, meaning that $t \in \mathcal{B}^*(Y^*, X)$. Hence, $ST^* = St \in \mathcal{K}(Y^*, X)$.

For the converse, assume that $\mathcal{B} \subset \mathcal{W}$. Let $S \in \mathcal{A}(X)$ and $T \in \mathcal{B}^*(Y, X)$, meaning that $T^* \in \mathcal{B}(X^*, Y^*)$. Since $T^* \in \mathcal{W}(X^*, Y^*)$, we have $T \in \mathcal{W}(Y, X)$, implying that $\text{ran } T^{**} \subset X$. Hence, $T^* \in \mathcal{B}_{w^*}(X^*, Y^*)$ and by assumption, $ST^{**} \in \mathcal{K}(Y^{**}, X)$. But then $ST = ST^{**}|_Y \in \mathcal{K}(Y, X)$ as needed. \square

For the proof of Theorem 1, we shall need the following simple consequence of the description (due to [G]) of the linear functionals on $\mathcal{L}(X)$ which are continuous in the locally convex topology of uniform convergence on compact sets in X (see, e.g., [LT, pp. 31–32]).

LEMMA 3. *Let X be a Banach space and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$. Then X has the $\mathcal{A}(X)$ -approximation property if and only if, for all sequences $(x_n) \subset X$ and $(x_n^*) \subset X^*$ such that $\sum_{n=1}^\infty \|x_n^*\| \|x_n\| < \infty$, one has*

$$\inf_{S \in \mathcal{A}(X)} \left| \sum_{n=1}^\infty x_n^*(Sx_n - x_n) \right| = 0.$$

Proof. Recall the above-mentioned description. Namely, the dual space $(\mathcal{L}(X, Y), \tau)^*$, where X and Y are Banach spaces and τ is the topology of uniform convergence on compact sets in X (this is the locally convex topology generated by the seminorms of the form $\|T\|_K = \sup\{\|Tx\| : x \in K\}$, where K ranges over the compact sets in X), consists of all functionals φ of the form

$$\varphi(T) = \sum_{n=1}^\infty y_n^*(Tx_n), \quad T \in \mathcal{L}(X, Y),$$

where $(x_n) \subset X$, $(y_n^*) \subset Y^*$, and $\sum_{n=1}^\infty \|y_n^*\| \|x_n\| < \infty$.

Note that the $\mathcal{A}(X)$ -approximation property of the space X means that the identity operator I_X is in the closure of the convex set $\mathcal{A}(X)$ in the space $(\mathcal{L}(X), \tau)$. This happens if and only if, for every τ -continuous linear functional φ on $\mathcal{L}(X)$, one has

$$\mathcal{R}e\varphi(I_X) \leq \sup_{S \in \mathcal{A}(X)} \mathcal{R}e\varphi(S),$$

which is implied by

$$\inf_{S \in \mathcal{A}(X)} |\varphi(I_X - S)| = 0.$$

Since the latter condition clearly follows from the $\mathcal{A}(X)$ -approximation property of X , applying the description of $(\mathcal{L}(X), \tau)^*$ completes the proof of the lemma. □

The proof of Theorem 1 also relies on Lemma 4 which is an isometric version of the famous Davis–Figiel–Johnson–Pełczyński factorization lemma [DFJP] due to Lima, Nygaard, and Oja [LNO]. Let us recall the relevant construction.

Let a be the unique solution of the equation

$$\sum_{n=1}^\infty \frac{a^n}{(a^n + 1)^2} = 1, \quad a > 1.$$

Let X be a Banach space and let K be a closed absolutely convex subset of B_X . For each $n \in \mathbb{N}$, put $B_n = a^{n/2}K + a^{-n/2}B_X$. The gauge of B_n gives an equivalent norm $\|\cdot\|_n$ on X . Set

$$\|x\|_K = \left(\sum_{n=1}^\infty \|x\|_n^2 \right)^{1/2},$$

define $X_K = \{x \in X : \|x\|_K < \infty\}$, and let $J_K : X_K \rightarrow X$ denote the identity embedding.

LEMMA 4 (See [DFJP] and [LNO]). *With notation as above, the following holds.*

- (i) $X_K = (X_K, \|\cdot\|_K)$ is a Banach space and $\|J_K\| \leq 1$.
- (ii) $K \subset B_{X_K} \subset B_X$.
- (iii) $B_{X_K} \subset B_n$ for all $n \in \mathbb{N}$.
- (iv) $J_K^*(X^*)$ is norm dense in X_K^* .
- (v) J_K is compact if and only if K is compact; in this case X_K is separable.
- (vi) X_K is reflexive if and only if K is weakly compact.

Proof of Theorem 1. (a) \Rightarrow (b). The standard scheme (see, e.g., [LNO, Theorem 1.2 and Corollary 1.5] or [OPe, Theorem 3]) would be to show first the claim for reflexive Banach spaces Y and then deduce from this the general case. However, this scheme does not work if $\mathcal{A}(X)$ is not contained in $\mathcal{K}(X)$. Thus, let Y be an arbitrary Banach space and let $T \in \mathcal{B}_{w^*}(X^*, Y)$. We clearly may assume that $\|T\| = 1$.

Denote $K = \overline{T^*(B_{Y^*})}$. Then K is a closed absolutely convex subset of B_X . Since K is also weakly compact, the space X_K is reflexive (see Lemma 4). Define $t \in \mathcal{L}(Y^*, X_K)$ by $ty^* = T^*y^*, y^* \in Y^*$. Then T^* factorizes through X_K as $T^* = J_K t$. Moreover, it is known (see [LNO, Theorem 2.2]) and straightforward to verify that $\|t\| = \|T^*\| = 1$ and $\|J_K\| = 1$.

Let $S \in \mathcal{A}(X)$. Then $SJ_K \in \mathcal{L}(X_K, X)$. We show that $SJ_K \in \mathcal{K}(X_K, X)$.

By assumption, $ST^* \in \mathcal{K}(Y^*, X)$. We know (see Lemma 4) that

$$J_K(B_{X_K}) \subset B_n = a^{n/2} \overline{T^*(B_{Y^*})} + a^{-n/2} B_X$$

for all $n \in \mathbb{N}$. Hence,

$$(SJ_K)(B_{X_K}) \subset a^{n/2} \overline{(ST^*)(B_{Y^*})} + a^{-n/2} \|S\| B_X$$

for all $n \in \mathbb{N}$. This implies that $(SJ_K)(B_{X_K})$ has for all $\varepsilon > 0$, a compact ε -net, and therefore it is relatively compact in X . Hence, $SJ_K \in \mathcal{K}(X_K, X)$ as needed.

By (a), there is a net $(S_\beta) \subset \mathcal{A}(X)$ such that $S_\beta \rightarrow I_X$ uniformly on compact subsets of X .

Consider the linear subspace

$$\mathcal{Z} := \{SJ_K : S \in \mathcal{A}(X)\} \subset \mathcal{K}(X_K, X).$$

We shall construct an operator

$$\Phi : \mathcal{Z}^* \rightarrow (\text{span}\{J_K\})^*$$

using the net (S_β) and relying on the description of $(\mathcal{K}(Z, X))^*$ due to Feder and Saphar [FS, Theorem 1] which holds whenever Z is a reflexive Banach space. According to this description, the trace mapping τ from the projective tensor product $X^* \widehat{\otimes} X_K$ to $(\mathcal{K}(X_K, X))^*$, defined by

$$(\tau u)(S) = \text{trace}(Su), \quad u \in X^* \widehat{\otimes} X_K, S \in \mathcal{K}(X_K, X),$$

is a quotient mapping and, moreover, for all $g \in (\mathcal{K}(X_K, X))^*$, there exists $u_g \in X^* \widehat{\otimes} X_K$ such that $g = \tau u_g$ and $\|g\| = \|u_g\|_\pi$.

Let $g \in \mathcal{Z}^*$. By passing to a norm-preserving extension, we may assume that $g \in (\mathcal{K}(X_K, X))^*$. Then

$$u_g = \sum_{n=1}^\infty x_n^* \otimes z_n,$$

where $x_n^* \in X^*$ and $z_n \in X_K$ satisfy $\sum_{n=1}^\infty \|x_n^*\| = 1$ and $z_n \rightarrow 0$. Define

$$(\Phi g)(\lambda J_K) = \lambda(\tau u_g)(J_K), \quad \lambda \in \mathbb{K}.$$

Similarly to [LNO, proof of Theorem 1.2], we can show that

$$(\Phi g)(J_K) = \lim_{\beta} g(S_{\beta} J_K).$$

Indeed,

$$\begin{aligned} |(\tau u_g)(J_K) - g(S_{\beta} J_K)| &= |(\tau u_g)(J_K - S_{\beta} J_K)| = \left| \sum_{n=1}^\infty x_n^* ((I_X - S_{\beta}) J_K z_n) \right| \\ &\leq \sup_n \|(I_X - S_{\beta})(J_K z_n)\| \rightarrow_n 0 \end{aligned}$$

because $\{0, J_K z_1, J_K z_2, \dots\}$ is a compact subset of X . Hence, Φ is correctly defined, linear, and $\|\Phi\| \leq 1$.

Since $\Phi^*(J_K) \in B_{\mathcal{Z}^{**}}$, by Goldstine’s theorem, there exists a net $(S_{\alpha}) \subset \mathcal{A}(X)$ such that $\sup_{\alpha} \|S_{\alpha} J_K\| \leq 1$ and $S_{\alpha} J_K \rightarrow_{\alpha} \Phi^*(J_K)$ in the weak*-topology of \mathcal{Z}^{**} . In particular, for all $x^* \in X^*$ and $z \in X_K$, consider the functional $x^* \otimes z \in (\mathcal{K}(X_K, X))^*$. Then $\|x^* \otimes z\| = \|x^*\| \|z\|$. Let $g = (x^* \otimes z)|_{\mathcal{Z}} \in \mathcal{Z}^*$. Since $\overline{X^* \otimes X} = \mathcal{F}(X) \subset \mathcal{A}(X)$, we clearly have $J_K^*(X^*) \otimes X \subset \mathcal{Z}$. But $X_K^* = \overline{J_K^*(X^*)}$ (see Lemma 4). Hence $X_K^* \otimes X \subset \overline{\mathcal{Z}}$ in $\mathcal{K}(X_K, X)$. This implies that

$$\|g\| = \|(x^* \otimes z)|_{\overline{\mathcal{Z}}}\| \geq \|(x^* \otimes z)|_{X_K^* \otimes X}\| = \|x^*\| \|z\|.$$

Consequently, $x^* \otimes z$ is a norm-preserving extension of g , and we may take $u_g = x^* \otimes z \in X^* \widehat{\otimes} X_K$. But then

$$\begin{aligned} x^*(S_{\alpha} J_K z) &= g(S_{\alpha} J_K) \\ \rightarrow_{\alpha} (\Phi^*(J_K))(g) &= (\Phi g)(J_K) = (\tau u_g)(J_K) = x^*(J_K z). \end{aligned}$$

This means that $(S_{\alpha} J_K)$ converges to J_K in the weak operator topology on $\mathcal{L}(X_K, X)$. Since the weak and strong operator topologies yield the same dual space [DS, Theorem VI.1.4], we may, by passing to convex combinations, assume that $(S_{\alpha} J_K)$ converges to J_K in the strong operator topology on $\mathcal{L}(X_K, X)$. Recalling that $T^* = J_K t$, this implies that $(S_{\alpha} T^*)$ converges to T^* in the strong operator topology on $\mathcal{L}(Y^*, X)$. Moreover, since $T S_{\alpha}^* = (S_{\alpha} T^*)^*$, we also have $\sup_{\alpha} \|T S_{\alpha}^*\| = \sup_{\alpha} \|S_{\alpha} J_K t\| \leq \|t\| = 1$.

(b) \Rightarrow (c). Let Z be a separable reflexive Banach space and let $T \in \mathcal{K}_{w^*}(X^*, Z)$. As $\mathcal{K} \subset \mathcal{B}$, there exists a net $(S_{\alpha}) \subset \mathcal{A}(X)$ such that

$\sup_{\alpha} \|TS_{\alpha}^*\| \leq \|T\|$ and $S_{\alpha}T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Z^*, X)$. Since $\text{ran } T^* \subset X$, we have $T^* \in \mathcal{K}(Z^*, X)$ and $ST^* \in \mathcal{K}(Z^*, X)$ for every $S \in \mathcal{A}(X)$. Further, as Z is reflexive, we have $(ST^*)^* = TS^*$ for every $S \in \mathcal{A}(X)$. Therefore, $\sup_{\alpha} \|S_{\alpha}T^*\| < \infty$.

Since Z^* is separable, the strong operator topology is metrizable on bounded subsets of $\mathcal{L}(Z^*, X)$ (see, e.g., [SW, III.4.7]). Consequently, there exists a sequence $(S_n) \subset \mathcal{A}(X)$ such that $S_nT^* \rightarrow T^*$ in the strong operator topology.

(c) \Rightarrow (d). The convergence $S_nT^* \rightarrow T^*$ in the strong operator topology clearly implies the convergence $TS_n^* \rightarrow T$ in the weak operator topology.

(d) \Rightarrow (a). We shall apply Lemma 3 to show that X has the $\mathcal{A}(X)$ -approximation property.

Let $(x_n) \subset X$ and $(x_n^*) \subset X^*$ satisfy $\sum_{n=1}^{\infty} \|x_n^*\| \|x_n\| < \infty$. One needs to prove that

$$\inf_{S \in \mathcal{A}(X)} \left| \sum_{n=1}^{\infty} x_n^*(Sx_n - x_n) \right| = 0.$$

We shall make use of an idea from the proof of [OPe, Theorem 3, (c₃) \Rightarrow (a)].

We may (and shall) assume that $x_n \rightarrow 0$, $\sup_n \|x_n\| \leq 1$ and $\sum_{n=1}^{\infty} \|x_n^*\| < \infty$. Denote by K the closed absolutely convex hull of (x_n) in X . Since $K \subset B_X$ and K is compact, by Lemma 4, the Banach space $Z := X_K$ is reflexive and separable, the identity embedding $J := J_K : Z \rightarrow X$ is compact and $\|J\| \leq 1$. Moreover, since $(x_n) \subset K \subset B_Z$, denoting by z_n the element x_n considered as an element of the Banach space Z , we have $x_n = Jz_n$ for every $n \in \mathbb{N}$, and $\sup_n \|z_n\| \leq 1$.

Since $J^* \in \mathcal{K}(X^*, Z^*)$ and Z is reflexive, we have $\text{ran } J^{**} \subset X$, so that $J^* \in \mathcal{K}_{w^*}(X^*, Z^*)$. Since Z^* is separable and reflexive, there is a sequence $(S_k) \subset \mathcal{A}(X)$ such that $J^*S_k^* \rightarrow J^*$ in the weak operator topology on $\mathcal{L}(X^*, Z^*)$. By the uniform boundedness principle, the sequence $(J^*S_k^*)$ is bounded in the norm operator topology. Put $M := \sup_k \|J^*S_k^*\|$.

Now, let us fix an arbitrary $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that

$$\sum_{n > N} \|x_n^*\| < \frac{\epsilon}{2(M+1)}.$$

Let us also fix S_k , so that

$$|(J^*S_k^*x_n^* - J^*x_n^*)(z_n)| < \frac{\epsilon}{2N}, \quad n = 1, \dots, N.$$

Then

$$\begin{aligned} \left| \sum_{n=1}^{\infty} x_n^*(S_kx_n - x_n) \right| &= \left| \sum_{n=1}^{\infty} x_n^*(S_kJz_n - Jz_n) \right| \\ &= \left| \sum_{n=1}^{\infty} (J^*S_k^*x_n^* - J^*x_n^*)(z_n) \right| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n=1}^N |(J^* S_k^* x_n^* - J^* x_n^*)(z_n)| \\ &\quad + \sum_{n>N} (\|J^* S_k^*\| + \|J^*\|) \|z_n\| \|x_n^*\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence,

$$\inf_{S \in \mathcal{A}(X)} \left| \sum_{n=1}^{\infty} x_n^*(Sx_n - x_n) \right| = 0,$$

and X has the $\mathcal{A}(X)$ -approximation property. □

REMARK 1. Let W be a Banach space. If X is a subspace of W^* , then the hypothesis $\mathcal{F}(X) \subset \mathcal{A}(X)$, that is $X^* \otimes X \subset \mathcal{A}(X)$ of Theorem 1 may be relaxed to $W \otimes X \subset \mathcal{A}(X)$, where the operator $w \otimes x$ with $w \in W$ and $x \in X$ is defined by $(w \otimes x)(w^*) = w^*(w)x$ for all $w^* \in X \subset W^*$. In fact, let $X_K, j_K,$ and \mathcal{Z} be as in the proof of Theorem 1, (a) \Rightarrow (b). Moreover, let $j : X \rightarrow W^*$ denote the identity embedding, and consider $J_K^* j^* : W^{**} \rightarrow X_K^*$. Since j^* is surjective and $X_K^* = \overline{J_K^*(X^*)}$ (see Lemma 4), we have

$$X_K^* = \overline{J_K^*(j^*(W^{**}))}.$$

For any $w^{**} \in W^{**}$, there is a bounded net $(w_\alpha) \subset W$ such that $w_\alpha \rightarrow w^{**}$ weak* in W^{**} . Hence, $J_K^* j^* w_\alpha \rightarrow J_K^* j^* w^{**}$ weak* in X_K^* . The convergence being weak because X_K^* is reflexive we have

$$J_K^* j^* w^{**} \in \overline{(J_K^* j^*)(W)}^w = \overline{(J_K^* j^*)(W)}.$$

Hence,

$$X_K^* = \overline{(J_K^* j^*)(W)}.$$

Since $W \otimes X \subset \mathcal{A}(X)$, we clearly have $(J_K^* j^*)(W) \otimes X \subset \mathcal{Z}$, and therefore $X_K^* \otimes X \subset \overline{\mathcal{Z}}$ as it is needed in the proof of Theorem 1, (a) \Rightarrow (b).

3. Applications of Theorem 1

3.1. The $\mathcal{A}(X)$ -approximation property for X . Let us recall that a Banach space X has the approximation property if and only if $B_{\mathcal{F}(Y,X)}$ is dense in $B_{\mathcal{W}(Y,X)}$ in the strong operator topology for every Banach space Y (see [LNO, Corollary 1.5]). A version of this result for the $\mathcal{A}(X)$ -approximation property is as follows.

THEOREM 5. *Let X be a Banach space and let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{L}(X)$ containing $\mathcal{F}(X)$. Let \mathcal{B} be an operator ideal such that $\mathcal{K} \subset \mathcal{B} \subset \mathcal{W}$ and*

$$\{ST : S \in \mathcal{A}(X), T \in \mathcal{B}^*(Y, X)\} \subset \mathcal{K}(Y, X)$$

for all Banach spaces Y . The following assertions are equivalent.

- (a) X has the $\mathcal{A}(X)$ -approximation property.

(b) For every Banach space Y and for every operator $T \in \mathcal{B}^*(Y, X)$, there exists a net $(S_\alpha) \subset \mathcal{A}(X)$, such that $\sup_\alpha \|S_\alpha T\| \leq \|T\|$ and $T^* S_\alpha^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(X^*, Y^*)$.

(c) For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(Z, X)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $S_n T \rightarrow T$ in the strong operator topology on $\mathcal{L}(Z, X)$.

(d) For every compact subset K of X , there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $S_n x \rightarrow x$ for all $x \in K$.

Proof. Notice that, by Proposition 2, the hypothesis of Theorem 1 is satisfied.

(a) \Rightarrow (b). It suffices to show that (b) is implied by condition (b) of Theorem 1. Let $T \in \mathcal{B}^*(Y, X)$. Since $T \in \mathcal{W}(Y, X)$, we have $T^* \in \mathcal{W}_{w^*}(X^*, Y^*)$. Therefore $T^* \in \mathcal{B}_{w^*}(X^*, Y^*)$. Hence, there is a net $(S_\alpha) \subset \mathcal{A}(X)$ such that $\sup_\alpha \|T^* S_\alpha^*\| \leq \|T^*\|$ and $S_\alpha T^{**} \rightarrow T^{**}$ in the strong operator topology on $\mathcal{L}(Y^{**}, X)$. This implies $\sup_\alpha \|S_\alpha T\| \leq \|T\|$ and $T^* S_\alpha^* \rightarrow T^*$ in the weak operator topology on $\mathcal{L}(X^*, Y^*)$. By passing to convex combinations we can acquire the needed net.

(b) \Rightarrow (c). Take $T \in \mathcal{K}(Z, X)$. Since $\mathcal{K} = \mathcal{K}^* \subset \mathcal{B}^*$, there is a net $(S_\alpha) \subset \mathcal{A}(X)$ such that $\sup_\alpha \|S_\alpha T\| \leq \|T\|$ and $T^* S_\alpha^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(X^*, Z^*)$. This implies the convergence $S_\alpha T \rightarrow T$ in the weak operator topology on $\mathcal{L}(Z, X)$. By passing to convex combinations, we may assume that $S_\alpha T \rightarrow T$ in the strong operator topology on $\mathcal{L}(Z, X)$. Since Z is separable, the strong operator topology is metrizable on bounded subsets of $\mathcal{L}(Z, X)$. Hence, there is a needed sequence.

(c) \Rightarrow (a). It suffices to observe that (c) implies condition (c) of Theorem 1. Let $T \in \mathcal{K}_{w^*}(X^*, Z)$. Then $T^* \in \mathcal{K}(Z^*, X)$. Hence, there is a sequence $(S_n) \subset \mathcal{A}(X)$ such that $S_n T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Z^*, X)$, as needed.

(a) \Rightarrow (d). This is obvious from the definition of the $\mathcal{A}(X)$ -approximation property.

(d) \Rightarrow (c). This is immediate if one takes $K = \overline{T(B_Z)}$. □

REMARK 2. Observe that the proofs of the implications (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a) of Theorem 1 and of the implications (b) \Rightarrow (c) \Rightarrow (a) \Rightarrow (d) \Rightarrow (c) of Theorem 5 remain valid in the more general case when $\mathcal{A}(X)$ is an arbitrary convex subset of $\mathcal{L}(X)$.

REMARK 3. In the special case when $\mathcal{A}(X) = \mathcal{F}(X)$, the equivalence (a) \Leftrightarrow (d) has been pointed out in [FJPP]. In the same special case when $\mathcal{A}(X) = \mathcal{F}(X)$, the net $(S_\alpha T)$ in condition (b) of Theorem 5 can be replaced by a net $(T_\alpha) \subset \mathcal{F}(Y, X)$ (see [LO4, Theorem 3.1]). Similarly, the sequence $(S_n T)$ in condition (c) of Theorem 5 can be replaced by a sequence $(T_n) \subset \mathcal{F}(Z, X)$ (see [Pe, Corollary 2]). However, already in the case when $\mathcal{A}(X) = \mathcal{K}(X)$,

this is no longer possible. In fact, by [LO4, Corollary 2.4], the condition “for every Banach space Y and for every operator $T \in \mathcal{W}(Y, X)$, there exists a net $(T_\alpha) \subset \mathcal{K}(Y, X)$ such that $\sup_\alpha \|T_\alpha\| \leq \|T\|$ and $T_\alpha^* \rightarrow T^*$ in the strong operator topology” is equivalent to the condition “ $\mathcal{K}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for all Banach spaces Y ”. But the latter condition may be satisfied even when X does not have the compact approximation property (see [LNO, p. 340]).

REMARK 4. In the special case when $\mathcal{A}(X) = \mathcal{K}(X)$, equivalence (a) \Leftrightarrow (b) of Theorem 5 has been established in [LLN, Theorem 2.3]. The proof in [LLN] uses a roundabout way that relies on criteria of the compact approximation property in terms of ideals (see [LLN, Theorem 2.2]). Concerning this special case, let us notice that condition (c) of Theorem 5 improves condition (iv) in [LLN, Theorem 2.3] removing the boundedness condition in the sequential version of (iv).

REMARK 5. In the case when \mathcal{A} and \mathcal{B} are operator ideals such that $\mathcal{A} = \mathcal{B}^*$, condition (b) of Theorem 5 represents a weakening of the *outer \mathcal{A} -approximation property*. This notion was introduced in [T3] and studied in [T1], [T2], and [T3] (cf. Remark 11 below).

To conclude this subsection, let us discuss some situations when Theorems 1 and 5 can be applied. A general case is when \mathcal{A} and \mathcal{B} are operator ideals satisfying

$$(\circ) \qquad \mathcal{A} \circ \mathcal{B}^* \subset \mathcal{K}.$$

Condition (\circ) holds always if $\mathcal{A} \subset \mathcal{K}$; in particular, if $\mathcal{A} = \mathcal{F}$ (the classical approximation property) or $\mathcal{A} = \mathcal{K}$ (cf. Remark 4 above).

If we take $\mathcal{B} = \mathcal{W}$ (recall that $\mathcal{W}^* = \mathcal{W}$), then condition (\circ) is satisfied for several important operator ideals \mathcal{A} which are *not* contained in \mathcal{K} , for instance, if \mathcal{A} equals \mathcal{AC} , \mathcal{J} , $\mathcal{L}im^*$, \mathcal{P}_p with $1 \leq p < \infty$, or \mathcal{V} . (Indeed, all of them are contained in \mathcal{V} and $\mathcal{V} \circ \mathcal{W} \subset \mathcal{K}$, which is well known and an easy-to-see fact.) Notice that \mathcal{AC} , $\mathcal{L}im^*$, and \mathcal{V} are even larger than \mathcal{K} . Let us recall that before in the literature (as was discussed in the Introduction), the only operator ideals \mathcal{A} , for which one had been able to characterize the $\mathcal{A}(X)$ -approximation property through a “metric” condition like (b) in Theorems 1 and 5 (with \mathcal{B} larger than \mathcal{K}), were \mathcal{F} and \mathcal{K} .

Moreover, there are some other interesting pairs of operator ideals \mathcal{A} and \mathcal{B} that satisfy (\circ) , and Theorems 1 and 5 apply. For instance, take $\mathcal{B} = \mathcal{J}$. Then $\mathcal{J}^* = \mathcal{J}$ and (\circ) is satisfied for any $\mathcal{A} \subset \mathcal{RN}^{**}$ (in fact, if $T \in \mathcal{RN}^{**} \circ \mathcal{J}$, then T^* is a nuclear operator). Here, important cases are \mathcal{RN}^{**} , \mathcal{W} , and, of course, any operator ideal contained in \mathcal{W} , like \mathcal{AC} , \mathcal{BS} , \mathcal{H} , \mathcal{J} , \mathcal{P}_p , etc.

3.2. On Grothendieck’s classics. Let us recall the following result from Grothendieck’s classics [G, Chapter I, p. 165].

THEOREM 6 (Grothendieck). *Let X be a Banach space. Then the following assertions are equivalent.*

- (a) X has the approximation property.
- (b) For every Banach space Y , one has $\mathcal{K}(Y, X) = \overline{\mathcal{F}(Y, X)}$.
- (c) For every Banach space Y , one has $\mathcal{K}_{w^*}(X^*, Y) = \overline{\mathcal{F}_{w^*}(X^*, Y)}$.

It is more than obvious that in Theorem 6, one cannot replace \mathcal{F} by \mathcal{K} . However, a version of Theorem 6 which holds for all $\mathcal{A}(X)$ -approximation properties can easily be obtained using Theorem 1.

THEOREM 7. *Let X be a Banach space and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$. The following assertions are equivalent.*

- (a) X has the $\mathcal{A}(X)$ -approximation property.
- (b) For every Banach space Y and for every operator $T \in \mathcal{K}(Y, X)$, one has $T \in \overline{\{ST : S \in \mathcal{A}(X)\}}$.
- (b') For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(Z, X)$, one has $T \in \overline{\{ST : S \in \mathcal{A}(X)\}}$.
- (c) For every Banach space Y and for every operator $T \in \mathcal{K}_{w^*}(X^*, Y)$, one has $T \in \overline{\{TS^* : S \in \mathcal{A}(X)\}}$.
- (c') For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}_{w^*}(X^*, Z)$, one has $T \in \overline{\{TS^* : S \in \mathcal{A}(X)\}}$.

Proof. (a) \Rightarrow (b). Let $T \in \mathcal{K}(Y, X)$. Then $K := \overline{T(B_Y)}$ is compact. Therefore, for every $\epsilon > 0$, there exists $S \in \mathcal{A}(X)$ such that $\|Sx - x\| < \epsilon$ for all $x \in K$. Hence, $\|ST - T\| = \sup_{y \in B_Y} \|STy - Ty\| \leq \epsilon$. This means that $T \in \overline{\{ST : S \in \mathcal{A}(X)\}}$.

(b) \Rightarrow (c). Let $T \in \mathcal{K}_{w^*}(X^*, Y)$. Then $T^* \in \mathcal{K}(Y^*, X)$. Therefore, $T^* \in \overline{\{ST^* : S \in \mathcal{A}(X)\}}$. Since $TS^* - T = (ST^* - T^*)^*$ for any $S \in \mathcal{A}(X)$, this is equivalent to $T \in \overline{\{TS^* : S \in \mathcal{A}(X)\}}$.

Note that the above proof is also suitable for the implication (b') \Rightarrow (c'), while the implications (b) \Rightarrow (b') and (c) \Rightarrow (c') are obvious.

(c') \Rightarrow (a). Condition (c') clearly implies condition (d) of Theorem 1, which is sufficient by Remark 2. \square

REMARK 6. In the special case when $\mathcal{A}(X) = \mathcal{K}(X)$, equivalence (a) \Leftrightarrow (b) has been established in [LLN, Theorem 2.1]. The proof in [LLN] relies on a special case of Lemma 3 and on the Davis–Figiel–Johnson–Pełczyński factorization procedure [DFJP].

The following characterization of the $\mathcal{A}(X)$ -approximation property, that holds for convex subsets $\mathcal{A}(X)$ of $\mathcal{L}(X)$, easily follows from Theorems 1, 5, and 7.

COROLLARY 8. *Let X be a Banach space and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$ containing 0. The following assertions are equivalent.*

(a) X has the $\mathcal{A}(X)$ -approximation property.

(b) For every Banach space Y and for every operator $T \in \mathcal{K}(Y, X)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $\sup_n \|S_n T\| \leq \|T\|$ and $S_n T \rightarrow T$ in the norm operator topology.

(b') For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(Z, X)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $S_n T \rightarrow T$ in the strong operator topology.

(c) For every Banach space Y and for every operator $T \in \mathcal{K}_{w^*}(X^*, Y)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $\sup_n \|TS_n^*\| \leq \|T\|$ and $TS_n^* \rightarrow T$ in the norm operator topology.

(c') For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}_{w^*}(X^*, Z)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $TS_n^* \rightarrow T$ in the weak operator topology.

Proof. It is an easy task to check that conditions (b) and (c) of Theorem 7 imply conditions (b) and (c) of Corollary 8, respectively. Indeed, for instance, let $T_n \in \mathcal{A}(X)$ be such that $T_n T \rightarrow T$ in the norm topology (see (b) of Theorem 7). Then $\|T_n T\| \rightarrow \|T\|$. Let $\varepsilon_n > 0$, $\varepsilon_n \rightarrow 0$, be such that $\|T_n T\| \leq \|T\| + \varepsilon_n$ for all n . Then

$$S_n := \frac{\|T\|}{\|T\| + \varepsilon_n} T_n \in \mathcal{A}(X),$$

$\|S_n T\| \leq \|T\|$, and $S_n T \rightarrow T$ in the norm topology. By Remark 2, the rest of the proof is immediate from Theorems 5 and 1. □

Corollary 8 applies, for instance, when X is a Banach lattice to characterize positive approximation properties. This is the case when $\mathcal{A}(X)$ is the subset of all positive operators contained in some fixed subspace of $\mathcal{L}(X)$. If the positive finite-rank operators are considered, then one speaks about the *positive approximation property*. It is not known whether the approximation property implies the positive approximation property (see [C, Problem 2.18]).

3.3. The $\mathcal{A}(X^*)$ -approximation property for X^* . We begin with an immediate application of Theorem 7 and Corollary 8.

THEOREM 9. *Let X be a Banach space and let $\mathcal{A}(X^*)$ be a convex subset of $\mathcal{L}(X^*)$ containing 0. The following assertions are equivalent.*

(a) X^* has the $\mathcal{A}(X^*)$ -approximation property.

(b) For every Banach space Y and for every operator $T \in \mathcal{K}(X, Y)$, one has $T^* \in \overline{\{ST^* : S \in \mathcal{A}(X^*)\}}$.

(b') For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$, one has $T^* \in \{ST^* : S \in \mathcal{A}(X^*)\}$.

(c) For every Banach space Y and for every operator $T \in \mathcal{K}(X, Y)$, there exists a sequence $(S_n) \subset \mathcal{A}(X^*)$, such that $\sup_n \|S_n T^*\| \leq \|T\|$ and $S_n T^* \rightarrow T^*$ in the norm operator topology.

(c') For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$, there exists a sequence $(S_n) \subset \mathcal{A}(X^*)$, such that $S_n T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Z^*, X^*)$.

Proof. (a) \Rightarrow (c). Condition (c) is immediate from condition (b) of Corollary 8 applied to X^* .

(c) \Rightarrow (b) \Rightarrow (b') and (c) \Rightarrow (c') \Rightarrow (b') are obvious.

(b') \Rightarrow (a). It suffices to show that (b') implies condition (b') of Theorem 7 applied to X^* . Let $T \in \mathcal{K}(Z, X^*)$. Then $T^*|_X \in \mathcal{K}(X, Z^*)$ and $(T^*|_X)^* = T$. Hence, $T \in \overline{\{ST : S \in \mathcal{A}(X^*)\}}$ as needed. \square

REMARK 7. In the special case when $\mathcal{A}(X^*) = \mathcal{K}(X^*)$, in [LLN, Theorem 3.1], (a) has been proven to be equivalent to the condition “for every Banach space Y and for every operator $T \in \mathcal{K}(X, Y)$, one has $T \in \overline{\{T^{**}S : S \in \mathcal{K}(X, X^{**})\}}$ ” which can be easily seen to be equivalent to (b). The proof in [LLN] relies on a special case of Lemma 3 and on the Davis–Figiel–Johnson–Pelczyński factorization procedure [DFJP]. Concerning this special case, let us notice that condition (c') of Theorem 9 improves condition (v) in [LLN, Theorem 3.2].

The following is an easy application of our main Theorem 1 (see also Proposition 2 and Remark 1).

THEOREM 10. Let X be a Banach space and let $\mathcal{A}(X^*)$ be a linear subspace of $\mathcal{L}(X^*)$ containing $\mathcal{F}_{w^*}(X^*) = X \otimes X^*$. Let \mathcal{B} be an operator ideal such that $\mathcal{K} \subset \mathcal{B} \subset \mathcal{W}$ and

$$\{ST : S \in \mathcal{A}(X^*), T \in \mathcal{B}^*(Y, X^*)\} \subset \mathcal{K}(Y, X^*)$$

for all Banach spaces Y . The following assertions are equivalent.

(a) X^* has the $\mathcal{A}(X^*)$ -approximation property.

(b) For every Banach space Y and for every operator $T \in \mathcal{B}^{**}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{A}(X^*)$, such that $\sup_\alpha \|S_\alpha T^*\| \leq \|T\|$ and $S_\alpha T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Y^*, X^*)$.

Proof. (a) \Leftrightarrow (b). By Remark 1 and Proposition 2, it suffices to prove that (b) is equivalent to condition (b) of Theorem 1 applied to X^* .

For the necessity part, let $T \in \mathcal{B}_{w^*}(X^{**}, Y)$. Observe that $(T|_X)^{**} = j_Y T \in \mathcal{B}(X^{**}, Y^{**})$, where $j_Y : Y \rightarrow Y^{**}$ is the canonical embedding. Hence, $T|_X \in \mathcal{B}^{**}(X, Y)$. Also, note that $(T|_X)^* = T^* \in \mathcal{L}(Y^*, X^*)$. Hence, there is a net $(S_\alpha) \subset \mathcal{A}(X^*)$ such that $\sup_\alpha \|S_\alpha T^*\| \leq \|T|_X\| \leq \|T\|$ and $S_\alpha T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Y^*, X^*)$, as needed.

For the sufficiency part, let $T \in \mathcal{B}^{**}(X, Y)$. Then $T^{**} \in \mathcal{B}(X^{**}, Y^{**})$. In fact, $T^{**} \in \mathcal{B}_{w^*}(X^{**}, Y^{**})$, because $T \in \mathcal{W}(X, Y)$. So, there is a net $(S_\alpha) \subset \mathcal{A}(X^*)$, such that $\sup_\alpha \|T^{**} S_\alpha\| \leq \|T^{**}\|$ and $S_\alpha T^{***} \rightarrow T^{***}$ in the strong operator topology on $\mathcal{L}(Y^{***}, X^*)$. Hence, $\sup_\alpha \|S_\alpha T^*\| \leq \|T\|$ and $S_\alpha T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Y^*, X^*)$, as needed. \square

REMARK 8. In the special case when $\mathcal{A}(X^*) = \mathcal{K}(X^*)$, equivalence (a) \Leftrightarrow (b) of Theorem 10 has been established in [LLN, Theorem 3.2] using a round-about way involving ideals and relying on a factorization lemma from [LO3, Lemma 4.1].

Let X be a Banach space and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$. We say that X^* has the $\mathcal{A}(X)$ -approximation property with conjugate operators if, for every compact set $K \subset X^*$ and every $\varepsilon > 0$, there exists an operator $S \in \mathcal{A}(X)$, such that $\|S^*x^* - x^*\| < \varepsilon$ for all $x^* \in K$. This approximation property of X^* is clearly equivalent to the $\mathcal{A}(X^*)$ -approximation property where $\mathcal{A}(X^*) = \{S^* : S \in \mathcal{A}(X)\}$. Therefore, the following characterizations are immediate from Theorems 9 and 10, respectively.

COROLLARY 11. *Let X be a Banach space and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$ containing 0. The following assertions are equivalent.*

- (a) X^* has the $\mathcal{A}(X)$ -approximation property with conjugate operators.
- (b) For every Banach space Y and for every operator $T \in \mathcal{K}(X, Y)$, one has $T \in \overline{\{TS : S \in \mathcal{A}(X)\}}$.
- (b') For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$, one has $T \in \overline{\{TS : S \in \mathcal{A}(X)\}}$.
- (c) For every Banach space Y and for every operator $T \in \mathcal{K}(X, Y)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $\sup_n \|TS_n\| \leq \|T\|$ and $TS_n \rightarrow T$ in the norm operator topology.
- (c') For every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$, such that $S_n^*T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Z^*, X^*)$.

Proof. Note that all conditions from (b) till (c') are exactly conditions from (b) till (c') of Theorem 9 with $\mathcal{A}(X^*) = \{S^* : S \in \mathcal{A}(X)\}$. Indeed, for instance, the condition $T^* \in \overline{\{ST^* : S \in \mathcal{A}(X^*)\}} = \overline{\{S^*T^* : S \in \mathcal{A}(X)\}}$ is equivalent to $T \in \overline{\{TS : S \in \mathcal{A}(X)\}}$. □

REMARK 9. In the special case when $\mathcal{A}(X) = \mathcal{K}(X)$, equivalence (a) \Leftrightarrow (b) has been established in [LLN, Theorem 3.3] relying on a special case of Lemma 3 and on the Davis–Figiel–Johnson–Pełczyński factorization procedure [DFJP]. Again, concerning this special case, let us notice that condition (c') of Corollary 11 improves condition (v) in [LLN, Theorem 3.4].

COROLLARY 12. *Let X be a Banach space and let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{L}(X)$ containing $\mathcal{F}(X)$. Let \mathcal{B} be an operator ideal such that $\mathcal{K} \subset \mathcal{B} \subset \mathcal{W}$ and*

$$\{TS : S \in \mathcal{A}(X), T \in \mathcal{B}(X, Y)\} \subset \mathcal{K}(X, Y)$$

for all Banach spaces Y . The following assertions are equivalent.

- (a) X^* has the $\mathcal{A}(X)$ -approximation property with conjugate operators.

(b) For every Banach space Y and for every operator $T \in \mathcal{B}^{**}(X, Y)$, there exists a net $(S_\alpha) \subset \mathcal{A}(X)$, such that $\sup_\alpha \|TS_\alpha\| \leq \|T\|$ and $S_\alpha^*T^* \rightarrow T^*$ in the strong operator topology on $\mathcal{L}(Y^*, X^*)$.

Proof. It suffices to show that the hypotheses of Corollary 12 imply the hypotheses of Theorem 10 with $\mathcal{A}(X^*) = \{S^* : S \in \mathcal{A}(X)\}$. First, notice that $X \otimes X^* = \{S^* : S \in \mathcal{F}(X)\} \subset \mathcal{A}(X^*)$. Secondly, let $S \in \mathcal{A}(X)$ and let $T \in \mathcal{B}^*(Y, X^*)$. Then $T^* \in \mathcal{B}(X^{**}, Y^*)$ and we have $T^*|_X = T^*j_X \in \mathcal{B}(X, Y^*)$. Hence, $T^*|_X S \in \mathcal{K}(X, Y^*)$. Note that $(T^*|_X)^*|_Y = T$. Therefore, $S^*T = (T^*|_X S)^*|_Y \in \mathcal{K}(Y, X^*)$, as needed. \square

REMARK 10. In the special case when $\mathcal{A}(X) = \mathcal{K}(X)$, equivalence (a) \Leftrightarrow (b) has been established in [LLN, Theorem 3.4] by the method described in Remark 8.

REMARK 11. In the case when \mathcal{A} and \mathcal{B} are operator ideals such that $\mathcal{A} = \mathcal{B}^{**}$, condition (b) of Corollary 12 represents a weakening of the *inner \mathcal{A} -approximation property*. This notion was introduced in [T3] and studied in [T1], [T2], and [T3] (cf. Remark 5 above).

Theorem 10 can be applied in the situations discussed at the end of Section 3.1. A general case when Corollary 12 can be applied is if \mathcal{A} and \mathcal{B} are operator ideals satisfying

$$\mathcal{B} \circ \mathcal{A} \subset \mathcal{K}.$$

Before in the literature (see the Introduction for references), the only operator ideals \mathcal{A} , for which one had been able to characterize the $\mathcal{A}(X)$ -approximation property with conjugate operators through a “metric” condition like (b) in Corollary 12 (with $\mathcal{K} \subset \mathcal{B} \subset \mathcal{W}$), were \mathcal{F} and \mathcal{K} . Here, already the inclusion

$$\mathcal{J} \circ \mathcal{RN}^* \subset \mathcal{K}$$

(see, e.g., [P, 24.6.1]) enables us to take \mathcal{A} to be any operator ideal contained in \mathcal{RN}^* , for instance, \mathcal{RN}^* , \mathcal{W} , \mathcal{AC} , \mathcal{BS} , \mathcal{H} , \mathcal{J} , \mathcal{P}_p , etc. (recall that $\mathcal{J}^{**} = \mathcal{J}$). Other important cases are given by $\mathcal{W} \circ \mathcal{J} \subset \mathcal{K}$ (recall that $\mathcal{W}^{**} = \mathcal{W}$) and $\mathcal{V} \circ \mathcal{W} \subset \mathcal{K}$.

3.4. The weak metric $\mathcal{A}(X)$ -approximation property. Recently, the weak metric approximation property was introduced and studied by Lima and Oja [LO5]. Several similar results for its compact version have already been obtained by Lima and Lima [LL]. In the context of the present paper, it is natural to extend this notion as follows.

Let X be a Banach space and let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{L}(X)$. We say that X has the *weak metric $\mathcal{A}(X)$ -approximation property* if for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$ there exists a net $(S_\alpha) \subset \mathcal{A}(X)$, such that $\sup_\alpha \|TS_\alpha\| \leq \|T\|$ and $S_\alpha \rightarrow I_X$ uniformly on compact subsets of X . In the special case when $\mathcal{A}(X) = \mathcal{F}(X)$,

we have the *weak metric approximation property* (see [LO5, Theorem 2.4]), and when $\mathcal{A}(X) = \mathcal{K}(X)$, we have the *weak metric compact approximation property* studied in [LL].

The weak metric $\mathcal{A}(X)$ -approximation property clearly implies the $\mathcal{A}(X)$ -approximation property (take $T = 0$ in the definition). The converse is not true in general (see [LO5] or [O2] for examples in the case when $\mathcal{A}(X) = \mathcal{F}(X)$; it can be easily seen that in all these examples X , which has the approximation property and does not have the weak metric approximation property, actually does not have the weak metric compact approximation property).

Corollary 11 enables us to extend [LO5, Theorem 4.2, (a) \Leftrightarrow (c)] from the case when $\mathcal{A}(X) = \mathcal{F}(X)$ to the general case.

THEOREM 13. *Let X be a Banach space and let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{L}(X)$. The following assertions are equivalent.*

- (a) X^* has the $\mathcal{A}(X)$ -approximation property with conjugate operators.
- (b) X has the weak metric $\mathcal{A}(X)$ -approximation property in every equivalent norm.

Proof. (a) \Rightarrow (b). Since the $\mathcal{A}(X)$ -approximation property with conjugate operators is preserved under changes to equivalent norms, it suffices to prove that X has the weak metric $\mathcal{A}(X)$ -approximation property. By Corollary 11, for every separable reflexive Banach space Z and for every operator $T \in \mathcal{K}(X, Z)$, there exists a sequence $(S_n) \subset \mathcal{A}(X)$ such that $\sup_n \|TS_n\| \leq \|T\|$ and $TS_n \rightarrow T$ in the norm operator topology. Repeating verbatim the proof of (c) \Rightarrow (d') \Rightarrow (a') in [LO5, Theorem 2.4] yields the desired property for X .

(b) \Rightarrow (a). Here, one verifies the condition of Lemma 3. This can be done repeating verbatim the proof of (a) \Rightarrow (c) in [LO5, Theorem 4.2]. □

REMARK 12. In the special case when $\mathcal{A}(X) = \mathcal{K}(X)$, Theorem 13 has been proven in [LL, Theorem 4.9]. The proof in [LL] uses [LLN, Theorem 3.4]) (see Remark 10 above) and a characterization of the weak metric compact approximation property that involves Hahn–Banach extension operators (see [LL, Theorem 4.3]).

In [O2, Theorem 2 and Corollary 1], it is proven that the weak metric and the metric approximation properties are equivalent for a Banach space X whenever X^* or X^{**} has the Radon–Nikodým property. By the same proof, the following extension holds.

THEOREM 14. *Let X be a Banach space and let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{K}(X)$. If X^* or X^{**} has the Radon–Nikodým property, then the weak metric and the metric $\mathcal{A}(X)$ -approximation properties are equivalent for X .*

We do not know whether Theorem 14 holds for $\mathcal{A}(X) = \mathcal{W}(X)$, for instance.

Corollary 15 below is well known when $\mathcal{A}(X) = \mathcal{F}(X)$ or $\mathcal{A}(X) = \mathcal{K}(X)$. The case of $\mathcal{A}(X) = \mathcal{F}(X)$ goes back to Grothendieck's classics [G]. There have been many different proofs of it (see [LO6, Remark 3.4] for comments and references). The most recent one from [O2] can be repeated nearly verbatim to obtain the following conclusion.

COROLLARY 15. *Let X be a Banach space and let $\mathcal{A}(X)$ be a linear subspace of $\mathcal{K}(X)$. If X^* or X^{**} has the Radon–Nikodým property, then the $\mathcal{A}(X)$ -approximation property with conjugate operators and the metric $\mathcal{A}(X)$ -approximation property with conjugate operators are equivalent for X^* .*

Proof. Assume that X^* has the $\mathcal{A}(X)$ -approximation property with conjugate operators. By Theorem 13, X has the weak metric $\mathcal{A}(X)$ -approximation property in every equivalent norm. Since the Radon–Nikodým property is preserved under changes to equivalent norms, by Theorem 14, X has the metric $\mathcal{A}(X)$ -approximation property in every equivalent norm. Using the characterization of bounded $\mathcal{A}(X^*)$ -approximation properties of Reinov [R2, Lemma 1.2] and repeating verbatim the proof of Johnson's theorem on lifting of the metric approximation property from Banach spaces to their dual spaces [J, Theorem 4], one obtains the metric $\mathcal{A}(X)$ -approximation property with conjugate operators for X^* . \square

Concerning the hypothesis of Theorem 14 and Corollary 15 that $\mathcal{A}(X)$ is a linear subspace of $\mathcal{K}(X)$, let us notice that besides the obvious examples like $\mathcal{F}(X)$, nuclear operators, and $\mathcal{K}(X)$, one can take $\mathcal{A}(X) = \mathcal{K}(X) \cap \mathcal{B}(X)$, where \mathcal{B} is an operator ideal which is not comparable with \mathcal{K} (e.g., $\mathcal{P}_p, 1 \leq p < \infty$).

REFERENCES

- [BD] J. Bourgain and J. Diestel, *Limited operators and strict cosingularity*, Math. Nachr. **119** (1984), 55–58. MR 0774176
- [C] P. G. Casazza, *Approximation properties*, Handbook of the geometry of Banach spaces, vol. 1 (W. B. Johnson and J. Lindenstrauss, eds.), Elsevier, Amsterdam, 2001, pp. 271–316. MR 1863695
- [DFJP] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327. MR 0355536
- [DJT] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge Univ. Press, Cambridge, 1995. MR 1342297
- [DSe] J. Diestel and C. J. Seifert, *The Banach–Saks ideal, I. Operators acting on $C(\Omega)$* , Comment. Math. **tom. spec.** (1979), 109–118, 343–344. MR 0504156
- [DS] N. Dunford and J. T. Schwartz, *Linear operators. Part 1: General theory*, Wiley Interscience, New York–London, 1958. MR 1009162
- [FS] M. Feder and P. D. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. **21** (1975), 38–49. MR 0377591
- [FJPP] V. P. Fonf, W. B. Johnson, G. Pisier and D. Preiss, *Stochastic approximation properties in Banach spaces*, Studia Math. **159** (2003), 103–119. MR 2030905
- [GKS] G. Godefroy, N. J. Kalton and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), 13–59. MR 1208038

- [GW] N. Grønbæk and G. A. Willis, *Approximate identities in Banach algebras of compact operators*, Canad. Math. Bull. **36** (1993), 45–53. MR 1205894
- [G] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955), 1–196; 1–140. MR 0075539
- [J] W. B. Johnson, *A complementary universal conjugate Banach space and its relation to the approximation problem*, Israel J. Math. **13** (1972), 301–310. MR 0326356
- [LNO] Á. Lima, O. Nygaard and E. Oja, *Isometric factorization of weakly compact operators and the approximation property*, Israel J. Math. **119** (2000), 325–348. MR 1802659
- [LO1] Á. Lima and E. Oja, *Ideals of finite rank operators, intersection properties of balls, and the approximation property*, Studia Math. **133** (1999), 175–186. MR 1686696
- [LO2] Á. Lima and E. Oja, *Hahn–Banach extension operators and spaces of operators*, Proc. Amer. Math. Soc. **130** (2002), 3631–3640. MR 1920043
- [LO3] Á. Lima and E. Oja, *Ideals of compact operators*, J. Aust. Math. Soc. **77** (2004), 91–110. MR 2069027
- [LO4] Á. Lima and E. Oja, *Ideals of operators, approximability in the strong operator topology, and the approximation property*, Michigan Math. J. **52** (2004), 253–265. MR 2069799
- [LO5] Á. Lima and E. Oja, *The weak metric approximation property*, Math. Ann. **333** (2005), 471–484. MR 2198796
- [LO6] Á. Lima and E. Oja, *Metric approximation properties and trace mappings*, Math. Nachr. **280** (2007), 571–580. MR 2308485
- [LL] Á. Lima and V. Lima, *Geometry of spaces of compact operators*, Ark. Mat. **46** (2008), 113–142.
- [LLN] V. Lima, Á. Lima and O. Nygaard, *On the compact approximation property*, Studia Math. **160** (2004), 185–199. MR 2033150
- [LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92, Springer Verlag, Berlin–Heidelberg–New York, 1977. MR 0500056
- [N] C. P. Niculescu, *Absolute continuity in Banach space theory*, Rev. Roumaine Math. Pures Appl. **24** (1979), 413–422. MR 0542855
- [O1] E. Oja, *Lifting bounded approximation properties from Banach spaces to their dual spaces*, J. Math. Anal. Appl. **323** (2006), 666–679. MR 2262236
- [O2] E. Oja, *The impact of the Radon–Nikodým property on the weak bounded approximation property*, Rev. R. Acad. Cienc. Serie A. Mat. **100** (2006), 325–331. MR 2267414
- [OPe] E. Oja and A. Pelander, *The approximation property in terms of the approximability of weak*-weak continuous operators*, J. Math. Anal. Appl. **286** (2003), 713–723. MR 2008860
- [Pe] A. Pelander, *The approximation property in terms of density in operator topologies*, Acta Comment. Univ. Tartu. Math. **7** (2003), 37–44. MR 2039105
- [P] A. Pietsch, *Operator ideals*, North-Holland, Amsterdam–New York–Oxford, 1980. MR 0582655
- [R1] O. I. Reinov, *Operators of RN type in Banach spaces*, Sibirsk. Mat. Zh. **19** (1978), 857–865 (in Russian). MR 0503624
- [R2] O. I. Reinov, *How bad can a Banach space with the approximation property be?*, Mat. Zametki **33** (1983), 833–846 (in Russian); English translation in Math. Notes **33** (1983), 427–434. MR 0709222
- [R3] O. I. Reinov, *A survey of some results in connection with Grothendieck approximation property*, Math. Nachr. **119** (1984), 257–264. MR 0774195
- [SW] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, Graduate Texts in Mathematics, vol. 3, Springer Verlag, New York, 1999. MR 1741419

- [T1] H.-O. Tylli, *The essential norm of an operator is not self-dual*, Israel J. Math. **91** (1995), 93–110. [MR 1348307](#)
- [T2] H.-O. Tylli, *Duality of the weak essential norm*, Proc. Amer. Math. Soc. **129** (2001), 1437–1443. [MR 1814170](#)
- [T3] H.-O. Tylli, *Duality of the distance to closed operator ideals*, Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), 197–212. [MR 1960055](#)
- [W] G. Willis, *The compact approximation property does not imply the approximation property*, Studia Math. **103** (1992), 99–108. [MR 1184105](#)

ALEKSEI LISSITSIN, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, TARTU UNIVERSITY, J. LIIVI 2, EE-50409 TARTU, ESTONIA

E-mail address: a30364@ut.ee

KRISTEL MIKKOR, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, TARTU UNIVERSITY, J. LIIVI 2, EE-50409 TARTU, ESTONIA

E-mail address: kristel.mikkor@ut.ee

EVE OJA, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, TARTU UNIVERSITY, J. LIIVI 2, EE-50409 TARTU, ESTONIA

E-mail address: eve.oja@ut.ee