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CLOSED-RANGE COMPOSITION OPERATORS ON $\mathbb{A}^2$

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Dedicated to Alec Matheson

ABSTRACT. For analytic self-maps φ of the unit disk, we develop a necessary and sufficient condition for the composition operator C_φ to be closed-range on the classical Bergman space $\mathbb{A}^2$. This condition is relatively easy to apply. Particular attention is given to the case that φ is an inner function. Included are observations concerning angular derivatives of Blaschke products. In the case that φ is univalent, it is shown that C_φ is closed-range on $\mathbb{A}^2$ only if φ is an automorphism of the disk.

1. Introduction and preliminaries

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ and let $\mathbb{T}$ denote the unit circle $\{z : |z| = 1\}$. Let A denote area measure on $\mathbb{D}$ and let m denote normalized Lebesgue measure on $\mathbb{T}$. The classical Bergman space $\mathbb{A}^2$ is the set of functions f that are analytic in $\mathbb{D}$, such that $\|f\|_{\mathbb{A}^2}^2 := \int_{\mathbb{D}} |f|^2\, dA < \infty$. It is easily seen to be a closed subspace of $L^2(A)$, and as such, forms a Hilbert space with respect to the inner product $\langle f, g\rangle := \int_{\mathbb{D}} f\overline{g}\, dA$. The Bergman space is much less tractable (indeed, much larger) than is the Hardy space $H^2(\mathbb{D})$ and has received considerable attention in recent years, primarily because of its connection to the invariant subspace problem (cf. [3]). If φ is analytic in $\mathbb{D}$ and $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, then we say that φ is an *analytic self-map* of the disk. In this article, we consider the composition operator C_φ on $\mathbb{A}^2$ given by $C_\varphi(f) = f \circ \varphi$, where φ is an analytic self-map of $\mathbb{D}$. This operator is certainly bounded; cf. [8], Section 1.4, problem 5. We investigate when it is closed-range; that is $C_\varphi(\mathbb{A}^2)$ is a closed subspace of $\mathbb{A}^2$. By the Open Mapping Theorem, this occurs precisely when there is a positive constant c such that $\|f \circ \varphi\|_{\mathbb{A}^2} \geq c\|f\|_{\mathbb{A}^2}$ for all f in $\mathbb{A}^2$. Zorboska has given a necessary and sufficient condition for

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C_φ to be closed-range on $H^2(\mathbb{D})$; and she has done likewise in the context of a variety of weighted Bergman spaces (cf. [9]). In the latter setting, her condition is quite difficult to apply. In this paper, we develop an alternative, necessary, and sufficient condition in the context of the Bergman space $\mathbb{A}^2$. We then apply this condition, paying particular attention to the case that φ is an inner function. Zorboska's work (in [9]) shows that every inner function φ induces a closed-range composition operator on the Hardy space $H^2(\mathbb{D})$; also cf. [7]. The situation turns out to be quite different in the context of $\mathbb{A}^2$. First of all, C_φ is a compact operator on $\mathbb{A}^2$ if and only if φ fails to have a so-called angular derivative (to be defined shortly) at each point of $\mathbb{T}$; cf. [8], pages 52 and 195, problems 10 and 15, (respectively), and for related material one may consult [6]. In this case, C_φ cannot be closed-range since it is certainly not finite rank. And, if φ has bounded angular derivative on $\mathbb{T}$, then φ is necessarily a finite Blaschke product, and hence one easily finds that C_φ is closed-range on $\mathbb{A}^2$. The role of the angular derivative is more subtle than these extreme cases indicate (cf. Theorem 2.4). Before we move into the statements and proofs of our results, we first lay some groundwork.

A measure μ called to a *singular measure* on $\mathbb{T}$ if μ is a finite, positive Borel measure on $\mathbb{T}$ and $\mu \perp m$. Define $\tau : [0, 2\pi) \longrightarrow \mathbb{T}$ in the standard way by letting $\tau(t) = e^{it}$. If μ is a singular measure on $\mathbb{T}$, then of course, $\mu \circ \tau$ defines a measure on $[0, 2\pi)$ that is singular with respect to Lebesgue measure on $\mathbb{R}$. We consistently refer to the measure $\mu \circ \tau$ as simply μ, only making reference to $[0, 2\pi)$ to distinguish it from μ on $\mathbb{T}$. The Poisson kernel on $\mathbb{T}$ for evaluation at a point z in $\mathbb{D}$ is given by:

$$\zeta \mapsto P_z(\zeta) := \frac{1 - |z|^2}{|\zeta - z|^2}.$$

If u is continuous and real-valued on $\mathbb{T}$, then

$$\hat{u}(z) := \int_{\mathbb{T}} u(\zeta) P_z(\zeta)\, dm(\zeta)$$

defines a harmonic function on $\mathbb{D}$ that extends continuously to $\overline{\mathbb{D}}$ with boundary values u. The conjugate Poisson kernel on $\mathbb{T}$ for evaluation at a point z in $\mathbb{D}$ is given by:

$$\zeta \mapsto Q_z(\zeta) := \frac{2\Im(\zeta \overline{z})}{|\zeta - z|^2} \left(= \Im\left\{ \frac{z + \zeta}{z - \zeta} \right\} \right).$$

As the name indicates, this kernel provides a harmonic conjugate for $\hat{u}$. Composing with τ, each of these have representations on $[0, 2\pi)$:

$$t \mapsto P_r(t - \theta) := \frac{1 - r^2}{1 - 2r\cos(t - \theta) + r^2}$$

and

$$t \mapsto Q_r(t-\theta) := \frac{2r\sin(t-\theta)}{1-2r\cos(t-\theta)+r^2},$$

respectively for evaluation at $z = re^{i\theta}$.

Let $H^\infty(\mathbb{D})$ denote the set of bounded analytic functions on $\mathbb{D}$. Any function f in $H^\infty(\mathbb{D})$ has nontangential boundary values f^* a.e. (m) on $\mathbb{T}$, and has a unique factorization (unique up to unimodular constants):

$$f = FBS_\mu,$$

where F is a bounded outer function, B is a Blaschke product, and S_μ is a singular inner function, built around a singular measure μ on $\mathbb{T}$. For specifics regarding these factors, the reader may consult [1] or [2]. If F is trivial, that is a unimodular constant, then f is called an inner function. In this case, $|f^*| = 1$ a.e. on $\mathbb{T}$. If f is an inner function, then a theorem of Frostman (cf. [2], page 79) tells us that

$$z \mapsto \frac{f(z)-a}{1-\overline{a}f(z)}$$

defines a Blaschke product on $\mathbb{D}$ for all a in $\mathbb{D}$ except for a set of logarithmic capacity zero. Any such Blaschke product is called a *Frostman shift* of f.

For ξ in $\mathbb{T}$ and $0 < a < 1$, let $S_a(\xi)$ denote the interior of the closed convex hull of $\{z : |z| \le a\} \cup \{\xi\}$. We call $S_a(\xi)$ a *Stolz* region based at ξ. Notice that $S_a(\xi)$ forms an angle of $2\arcsin(a)$ at ξ. An analytic self-map φ of $\mathbb{D}$ is said to have an *angular derivative* at some point ξ in $\mathbb{T}$ if there exists α in $\mathbb{T}$ and a complex number β such that

$$\lim_{\substack{z\to\xi \\ z\in S_a(\xi)}} \frac{\varphi(z)-\alpha}{z-\xi} = \beta$$

for any Stolz region $S_a(\xi)$ based at ξ. In this case, β is called the angular derivative of φ at ξ. Two well-known results concerning angular derivatives of analytic self-maps of $\mathbb{D}$ play a central role in our work here, namely, the Julia–Carathéodory theorem and Julia's theorem. Careful statements and proofs of each can be found in [8]. Another important tool in our work is the so called *Nevanlinna Counting Function*, defined for any analytic function φ on $\mathbb{D}$ by:

$$N_\varphi(w) = \begin{cases} \sum_{z\in\varphi^{-1}\{w\}} \log\frac{1}{|z|} & \text{if } w \in \varphi(\mathbb{D}), \\ 0 & \text{if } w \notin \varphi(\mathbb{D}). \end{cases}$$

Our work in this paper is organized as follows. In Section 2, we develop a practicable necessary and sufficient condition for C_φ to be closed-range on $\mathbb{A}^2$ (cf. Theorem 2.4). We then use this result to show that in the case that φ is univalent, C_φ is closed-range on $\mathbb{A}^2$ if and only if φ is a conformal automorphism of $\mathbb{D}$ (cf. Theorem 2.5). In Section 3, we examine the case that φ is an inner function. Most of our analysis occurs in the setting that φ

has analytic continuation across some arc of $\mathbb{T}$. For such φ, we give an analytic condition, which, if satisfied, guarantees that C_φ is closed-range on $\mathbb{A}^2$ (cf. Theorem 3.4). A consequence of this result is that if B is an infinite Blaschke product such that the set of accumulation points (in $\mathbb{T}$) of the zeros of B has an isolated point, then C_B is closed-range on $\mathbb{A}^2$ (cf. Corollary 3.6). We then construct a Blaschke product B^* whose zeros do not accumulate everywhere on $\mathbb{T}$ (and so C_{B^*} is not compact on $\mathbb{A}^2$), such that C_{B^*} fails to be closed-range on $\mathbb{A}^2$ and yet C_{zB^*} is closed-range on $\mathbb{A}^2$ (cf. Example 3.8). After this, we look at the case when φ is a singular inner functions and find that if μ is a singular measure on $\mathbb{T}$ whose support in some arc of $\mathbb{T}$ is nontrivial and is contained in a Cantor set, then C_{S_μ} is closed-range on $\mathbb{A}^2$ (cf. Corollary 3.11). We include in an appendix a result that makes a strong connection between the a.e. existence of the angular derivative of a given Blaschke product and the rate at which its zeros tend to $\mathbb{T}$. This result is relevant since, in light of the Julia–Carathéodory theorem, the a.e. existence of the angular derivative has close ties to the condition provided by Theorem 2.4.

2. A necessary and sufficient condition

Let φ be an analytic self-map of $\mathbb{D}$. In this section, we develop a relatively tractable condition that is necessary and sufficient for the composition operator C_φ on $\mathbb{A}^2$ to be closed-range. We begin with three lemmas whose proofs are very straightforward, and thus are omitted. Recall that any conformal mapping from $\mathbb{D}$ onto $\mathbb{D}$ has the form

$$\psi_a(z) := c \cdot \frac{z-a}{1-\overline{a}z},$$

where c is a unimodular constant (whose role is suppressed in our notation) and $a \in \mathbb{D}$. We call such mappings *conformal automorphisms* of the disk. Notice that $\mathbb{A}_0^2 := \{f \in \mathbb{A}^2 : f(0) = 0\}$ is a closed subspace of $\mathbb{A}^2$ and $\dim(\mathbb{A}^2 \ominus \mathbb{A}_0^2) = 1$.

LEMMA 2.1. *Let φ be an analytic self-map of the disk. Then C_φ is closed-range on $\mathbb{A}^2$ if and only if C_φ is closed-range on $\mathbb{A}_0^2$.*

LEMMA 2.2. *Let φ be an analytic self-map of the disk and let ψ_a be a conformal automorphism of the disk. If one of C_φ, $C_{\varphi\circ\psi_a}$, $C_{\psi_a\circ\varphi}$ is closed-range on $\mathbb{A}^2$, then so are the other two.*

For z and w in $\mathbb{D}$, let $\rho(z,w) = |\frac{z-w}{1-\overline{w}z}|$; which defines the so called *pseudo-hyperbolic metric* on $\mathbb{D}$. For z in $\mathbb{D}$ and $0<r<1$, let $D_r(z) = \{w \in \mathbb{D} : \rho(z,w) < r\}$. Let φ be a nontrivial analytic self-map of the disk. For $\varepsilon > 0$, let $\Omega_\varepsilon(\varphi) = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} \geq \varepsilon\}$ and let $G_\varepsilon(\varphi) = \varphi(\Omega_\varepsilon)$. We often abbreviate our notation of these sets to Ω_ε and G_ε, (respectively), except when there could be confusion as to the associated φ. We say that G_ε satisfies the *reverse Carleson condition* if there is a positive constant η, such that

$$\int_{G_\varepsilon} |f|^2(1-|z|^2)^2\,dA \geq \eta \int_{\mathbb{D}} |f|^2(1-|z|^2)^2\,dA$$

whenever f is analytic in $\mathbb{D}$ and $\int_{\mathbb{D}} |f|^2(1-|z|^2)^2\,dA < \infty$. In [5], Luecking shows that this is equivalent to:

(∗) there are constants c and s, $0 < c, s < 1$, such that $A(G_\varepsilon \cap D_s(z)) \geq c \cdot A(D_s(z))$ for all z in $\mathbb{D}$.

Lemma 2.3. *Let φ be an analytic self-map of the disk and let ψ_a be a conformal automorphism of the disk. If there exists $\varepsilon > 0$ such that one of $G_\varepsilon(\varphi)$, $G_\varepsilon(\varphi \circ \psi_a)$, $G_\varepsilon(\psi_a \circ \varphi)$ satisfies condition (∗), then there exists $\varepsilon > 0$ such that the other two also satisfy condition (∗).*

Theorem 2.4. *Let φ be a nontrivial analytic self-map of the disk. Then C_φ is closed-range on $\mathbb{A}^2$ if and only if there exists $\varepsilon > 0$, such that G_ε satisfies condition (∗).*

Proof. By Lemmas 2.2 and 2.3, we may assume that $\varphi(0) = 0$. And, by Lemma 2.1, we may restrict our attention to C_φ on $\mathbb{A}^2_0$. Moreover, by Luecking's work in [5], we need only establish the equivalence between C_φ being closed-range on $\mathbb{A}^2_0$ and the existence of $\varepsilon > 0$ for which G_ε satisfies the reverse Carleson condition. We make use of the fact that $\|f\|^2_{\mathbb{A}^2}$ and $\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^2\,dA$ are boundedly equivalent, independent of f in $\mathbb{A}^2_0$. Let $\mathcal{Z} = \{z \in \mathbb{D} : \varphi'(z) = 0\}$, which is a countable subset of $\mathbb{D}$. First, assume that there exists $\varepsilon > 0$ such that G_ε satisfies the reverse Carleson condition. So, we can find a positive constant η such that $\int_{G_\varepsilon} |f'(z)|^2(1-|z|^2)^2\,dA \geq \eta \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^2\,dA$ for f in $\mathbb{A}^2_0$. In what follows, we use the symbol $\approx$ between two quantities involving f to indicate that these quantities are boundedly equivalent, independent of f in $\mathbb{A}^2_0$. Now, if $f \in \mathbb{A}^2_0$, then

$$\begin{aligned}
\|f\circ\varphi\|^2_{\mathbb{A}^2} &\approx \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^2(1-|z|^2)^2\,dA(z)\\
&= \int_{\mathbb{D}} |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^2\,dA(z)\\
&\geq \int_{\Omega_\varepsilon} |f'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^2\,dA(z)\\
&\geq \varepsilon^2 \int_{\Omega_\varepsilon} |f'(\varphi(z))|^2|\varphi'(z)|^2\big(1-|\varphi(z)|^2\big)^2\,dA(z)\\
&= \varepsilon^2 \cdot \sum_n \int_{R_n\cap\Omega_\varepsilon} |f'(\varphi(z))|^2|\varphi'(z)|^2\big(1-|\varphi(z)|^2\big)^2\,dA(z),
\end{aligned}$$

where $\{R_n\}_n$ is a partition of $\mathbb{D}\setminus\mathcal{Z}$ into at most countably many semi-closed polar rectangles such that φ is univalent on each R_n; cf. [8], Section 10.3, page 186. Let $S_n=\varphi(R_n\cap\Omega_\varepsilon)$ and let ψ_n denote the inverse of $\varphi|_{R_n}$. Then by a standard change of variables involving ψ_n, the last line above becomes

$$\begin{aligned}
\varepsilon^2\cdot\sum_n\int_{G_\varepsilon}&|f'(w)|^2(1-|w|^2)^2\chi_{S_n}(w)\,dA(w)\\
&=\varepsilon^2\int_{G_\varepsilon}|f'(w)|^2(1-|w|^2)^2\Big(\sum_n\chi_{S_n}(w)\Big)\,dA(w)\\
&\ge\varepsilon^2\int_{G_\varepsilon}|f'(w)|^2(1-|w|^2)^2\,dA(w)\\
&\ge\eta\varepsilon^2\int_{\mathbb{D}}|f'(w)|^2(1-|w|^2)^2\,dA(w)\\
&\approx\eta\varepsilon^2\int_{\mathbb{D}}|f(w)|^2\,dA(w).
\end{aligned}$$

And, so we find that C_φ is closed-range on $\mathbb{A}^2_0$.

Conversely, suppose that there does not exist $\varepsilon>0$, such that G_ε satisfies the reverse Carleson condition. Then we can find a sequence $\{f_k\}_{k=1}^\infty$ in $\mathbb{A}^2_0$ such that

$$1=\int_{\mathbb{D}}|f_k'(w)|^2(1-|w|^2)^2\,dA(w)\quad(\approx\|f_k\|^2_{\mathbb{A}^2})$$

for all k, and yet

$$\int_{G_k}|f_k'(w)|^2(1-|w|^2)^2\,dA(w)\longrightarrow0,$$

as $k\to\infty$, where $G_k=\varphi(\Omega_k)$ and $\Omega_k:=\{z\in\mathbb{D}:(1-|\varphi(z)|^2)\le k(1-|z|^2)\}$. Now,

$$\begin{aligned}
\|f_k\circ\varphi\|^2_{\mathbb{A}^2}&\approx\int_{\mathbb{D}}|f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^2\,dA(z)\\
&=\int_{\Omega_k}|f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^2\,dA(z)\\
&\quad+\int_{\mathbb{D}\setminus\Omega_k}|f_k'(\varphi(z))|^2|\varphi'(z)|^2(1-|z|^2)^2\,dA(z).
\end{aligned}$$

And, since φ is an analytic self-map of $\mathbb{D}$, the Nevanlinna counting function N_φ satisfies:

$$N_\varphi(w)=O\Big(\log\Big(\frac{1}{|w|}\Big)\Big)$$

as $|w|\to1^-$; cf. [8], page 180. Using this and another decomposition of the disk into polar rectangles (cf. [8], Section 10.3), one can find positive constants

c_1, c_2, and c_3 (independent of k) such that

$$\begin{aligned}
&\int_{\Omega_k} |f_k'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^2 \, dA(z) \\
&\quad \le c_1 \int_{\Omega_k} |f_k'(\varphi(z))|^2 |\varphi'(z)|^2 \big(1-|\varphi(z)|^2\big) \log\left(\frac{1}{|z|}\right) dA(z) \\
&\quad \le c_2 \int_{G_k} |f_k'(w)|^2 (1-|w|^2) N_\varphi(w) \, dA(w) \\
&\quad \le c_3 \int_{G_k} |f_k'(w)|^2 (1-|w|^2)^2 \, dA(w) \longrightarrow 0,
\end{aligned}$$

as $k \to \infty$. And, there are positive constants c_4 and c_5 (independent of k) such that

$$\begin{aligned}
&\int_{\mathbb{D}\setminus\Omega_k} |f_k'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^2 \, dA(z) \\
&\quad \le \frac{c_4}{k} \int_{\mathbb{D}\setminus\Omega_k} |f_k'(\varphi(z))|^2 |\varphi'(z)|^2 \big(1-|\varphi(z)|^2\big) \log\left(\frac{1}{|z|}\right) dA(z) \\
&\quad \le \frac{c_4}{k} \int_{\mathbb{D}} |f_k'(\varphi(z))|^2 |\varphi'(z)|^2 \big(1-|\varphi(z)|^2\big) \log\left(\frac{1}{|z|}\right) dA(z) \\
&\quad = \frac{c_4}{k} \int_{\mathbb{D}} |f_k'(w)|^2 (1-|w|^2) N_\varphi(w) \, dA(w) \quad \text{(cf., [8], Section 10.3)} \\
&\quad \le \frac{c_5}{k} \int_{\mathbb{D}} |f_k'(w)|^2 (1-|w|^2)^2 \, dA(w) = \frac{c_5}{k} \longrightarrow 0,
\end{aligned}$$

as $k \to \infty$. Evidently, $\|f_k \circ \varphi\|_{\mathbb{A}^2} \longrightarrow 0$, as $k \to \infty$, though $\|f_k\|_{\mathbb{A}^2} = 1$ for all k. It follows that C_φ is not closed-range on $\mathbb{A}_0^2$, and so via the contrapositive, our proof is now complete. □

We conclude this section with an application of Theorem 2.4 in the case that φ is a univalent, analytic self-map of the disk. We show that the only such maps that give rise to closed-range composition operators on the Bergman space $\mathbb{A}^2$ are conformal automorphisms of the disk (cf. Theorem 2.5 below). Coupling Theorem 2.5 with [9], Corollary 4.3, we find that this result carries over to the setting of the classical Hardy space $H^2(\mathbb{D})$ as well as the weighted Bergman spaces

$$\mathbb{A}_\alpha^2 := \left\{ f : f \text{ is analytic in } \mathbb{D} \text{ and } \int_{\mathbb{D}} |f|^2 (1-|z|^2)^\alpha \, dA < \infty \right\};$$

$\alpha > -1$.

THEOREM 2.5. *Let φ be a univalent, analytic self-map of the disk. Then C_φ is closed-range on $\mathbb{A}^2$ if and only if φ is a conformal automorphism of the disk.*

Proof. If φ is a conformal automorphism of the disk, then $C_\varphi(\mathbb{A}^2) = \mathbb{A}^2$, and hence, C_φ is certainly closed-range on $\mathbb{A}^2$. Conversely, suppose that C_φ is closed-range on $\mathbb{A}^2$. Then by Theorem 2.4, there exists $\varepsilon > 0$, such that G_ε satisfies condition $(*)$. Let $K = \mathbb{T} \cap \overline{\Omega}_\varepsilon$. By the Julia–Carathéodory theorem φ has well-defined nontangential boundary values φ^* everywhere on K, and these boundary values are taken in $\mathbb{T}$.

Claim. *φ^* is continuous on K.*

To establish this claim, we proceed via an indirect argument, and suppose to the contrary, that φ^* is not continuous on K. Then we can find ζ_0 in K and a sequence $\{\zeta_n\}_{n=1}^\infty$ in K such that $\zeta_n \longrightarrow \zeta_0$, as $n \to \infty$, yet $\{\varphi^*(\zeta_n)\}_{n=1}^\infty$ converges to some ξ_0 in $\mathbb{T}$, where $\xi_0 \neq \varphi^*(\zeta_0)$. Choose δ, where $0 < \delta < \varepsilon$. By the full statement of the Julia–Carathéodory theorem (cf. [8], page 57), we can find a sequence $\{z_n\}_{n=1}^\infty$ in Ω_δ, such that $|\zeta_n - z_n| \longrightarrow 0$ and $|\varphi^*(\zeta_n) - \varphi(z_n)| \longrightarrow 0$, as $n \to \infty$. Indeed, we may choose z_n to be of the form $r_n\zeta_n$, where $0 < r_n < 1$ and $r_n \longrightarrow 1$ (as $n \to \infty$). Applying Julia's theorem, we find that

$$\varphi^*(\zeta_0) = \lim_{n\to\infty} \varphi(z_n) = \lim_{n\to\infty} \varphi^*(\zeta_n) = \xi_0$$

is a contradiction. Therefore, our claim holds. Now, define $\tilde{\varphi}$ on $\overline{\Omega}_\varepsilon$ by $\tilde{\varphi}(z) = \varphi(z)$ if $z \in \mathbb{D} \cap \overline{\Omega}_\varepsilon$ and $\tilde{\varphi}(\zeta) = \varphi^*(\zeta)$ if $\zeta \in K$. By our claim and Julia's Theorem, $\tilde{\varphi}$ is a continuous extension of φ to all of $\overline{\Omega}_\varepsilon$. And, since we are assuming here that φ is univalent, we may apply [4], Lemma 3.3, to get that φ^* is univalent on K. Therefore, $\tilde{\varphi}$ is a homeomorphism of $\overline{\Omega}_\varepsilon$ onto $\tilde{\varphi}(\overline{\Omega}_\varepsilon)$. Consequently, φ^* is a homeomorphism of K onto $\varphi^*(K)$. Yet, since G_ε satisfies condition $(*)$, each point in $\mathbb{T}$ is an accumulation point of $\varphi(\mathbb{D} \cap \overline{\Omega}_\varepsilon)$. It follows that $\mathbb{T} \cap \tilde{\varphi}(\overline{\Omega}_\varepsilon) = \mathbb{T}$. In other words, $\varphi^*(K)$ must be all of $\mathbb{T}$. Therefore, φ^* is a homeomorphism of K onto $\mathbb{T}$, which forces us to conclude that $K = \mathbb{T}$. Hence, φ has a finite angular derivative at each point of $\mathbb{T}$. In particular, the boundary values of φ are contained in $\mathbb{T}$. It follows that φ maps $\mathbb{D}$ onto $\mathbb{D}$, and thus φ is a conformal automorphism of the disk. □

Remark 2.6. In the proof of Theorem 2.5, we showed that $\tilde{\varphi}$ (as defined there) is continuous on $\overline{\Omega}_\varepsilon$ whether or not φ is univalent. Since $\overline{\Omega}_\varepsilon$ is compact, this extension is uniformly continuous on $\overline{\Omega}_\varepsilon$. This, of course, implies that φ is uniformly continuous on Ω_ε (in general).

3. Regarding inner functions

Now any inner function is an analytic self-map of $\mathbb{D}$. If φ is an inner function with factorization BS_μ, we define σ_φ to be the compact subset of $\mathbb{T}$ consisting of the support of μ along with the set of accumulation points in $\mathbb{T}$ of the zeros of B. If φ is just a singular inner function S_μ, then we often write

σ_μ instead of σ_φ. Notice that $\mathbb{T} \setminus \sigma_\varphi$ is the largest open subset of $\mathbb{T}$ across which φ has analytic continuation. If $\sigma_\varphi \neq \mathbb{T}$ and I is a connected subset of $\mathbb{T} \setminus \sigma_\varphi$, then $\varphi(I)$ is a connected subset of $\mathbb{T}$. In fact, the proofs of Lemmas 3.1 and 3.2 (below) show that the argument of $\varphi(e^{i\theta})$ increases as θ increases, for $e^{i\theta}$ in I. Let $\arg(\varphi(\zeta))$ be a continuous representation of the argument of $\varphi(\zeta)$, as ζ ranges in I; and so the values of $\arg(\cdot)$ might range throughout $\mathbb{R}$. Let $\omega = \sup\{|\arg(\varphi(\zeta)) - \arg(\varphi(\zeta'))| : \zeta, \zeta' \in I\}$. We then say that φ *wraps* I through ω radians. Our analysis is based on separate observations concerning the individual factors B and S_μ. Let B be a Blaschke product with zeros $\{a_n\}_n$, listed according to multiplicity. Let b_n be the factor of B that is built around the zero a_n. So, $b_n(z) = z$ if $a_n = 0$ and $b_n(z) = \frac{|a_n|}{a_n} \cdot \frac{a_n - z}{1 - \overline{a}_n z}$ otherwise. Define h_B on $\mathbb{T}$ by:

$$h_B(\zeta) = \sum_n P_{a_n}(\zeta),$$

where $\zeta \mapsto P_{a_n}(\zeta)$ is the Poisson kernel on $\mathbb{T}$ for evaluation at a_n. A theorem of Frostman (cf. [8], page 183) tells us that B has an angular derivative at some point ξ in $\mathbb{T}$ if and only if $h_B(\xi) < \infty$.

LEMMA 3.1. *Let B be a Blaschke product. If $\sigma_B \neq \mathbb{T}$ and I is a connected subset of $\mathbb{T} \setminus \sigma_B$, then B wraps I through $\int_I h_B(\zeta)|d\zeta|$ radians; which could be infinite.*

Proof. Let J be a closed subarc of I. By a rotation of $\mathbb{T}$, if necessary, we may assume that $1 \notin J$. Our result follows from the Monotone Convergence theorem if we establish that B wraps J through $\int_J h_B(\zeta)|d\zeta|$ radians. And, since B converges uniformly on J, this reduces to showing that B_N wraps J through $\int_J h_{B_N}(\zeta)|d\zeta|$ radians, where $B_N := \prod_{n=1}^N b_n$ and b_n is the Blaschke factor of B built around the zero a_n. Notice that the derivative of the argument of $b_n(e^{i\theta})$ with respect to θ is:

$$-i \cdot \frac{d(\log(b_n(e^{i\theta})))}{d\theta} = P_{a_n}(e^{i\theta}).$$

Therefore, the argument of $b_n(e^{i\theta})$ increases as θ increases, for $e^{i\theta}$ in J, and $\int_J P_{a_n}(\zeta)|d\zeta| = \text{length}(b_n(J))$ (which is the radian measure of the angle through which b_n wraps J). Since the radians through which B_N wraps J is the sum of the radians through which b_n wraps J, for $1 \leq n \leq N$, we find that B_N wraps J through $\sum_{n=1}^N \int_J P_{a_n}(\zeta)|d\zeta|$ $(= \int_J h_{B_N}(\zeta)|d\zeta|)$ radians. $\square$

Let μ be a singular measure on $\mathbb{T}$. Via $\tau(\theta) := e^{i\theta}$, μ gives rise to a singular measure on $[0, 2\pi)$, which we also call μ. Define g_μ on $[0, 2\pi)$ by

$$g_\mu(t) = \int_0^{2\pi} \frac{1}{1 - \cos(\theta - t)}\, d\mu(\theta).$$

LEMMA 3.2. *Let μ be a singular measure on $\mathbb{T}$. If $\sigma_\mu \neq \mathbb{T}$ and I is a connected subset of $\mathbb{T} \setminus \sigma_\mu$, then S_μ wraps I through $\int_{\tau^{-1}(I)} g_\mu(t)\,dt$ radians; which could be infinite.*

Proof. By the Monotone Convergence theorem, it is sufficient to establish the statement for an arbitrary closed subarc J of I, in place of I. And, under a rotation of $\mathbb{T}$ if necessary (which leaves the value of $\int_{\tau^{-1}(I)} g_\mu(t)\,dt$ unchanged), we may assume that $1 \notin J$. Hence, $\tau^{-1}(J) = [t_1, t_2]$, where $0 < t_1 < t_2 < 2\pi$. Now choose any points a and b such that $t_1 \le a < b \le t_2$. Notice that

$$\begin{aligned}\int_a^b g_\mu(t)\,dt &= \int_0^{2\pi} \left(\int_a^b \frac{1}{1-\cos(\theta - t)}\,dt \right) d\mu(\theta) \\ &= \int_0^{2\pi} \left(\frac{\sin(\theta - b)}{1-\cos(\theta - b)} - \frac{\sin(\theta - a)}{1-\cos(\theta - a)} \right) d\mu(\theta) \\ &= \int_0^{2\pi} \big(Q_1(\theta - b) - Q_1(\theta - a) \big)\,d\mu(\theta),\end{aligned}$$

where $Q_r(\theta) := \frac{2r\sin\theta}{1-2r\cos\theta + r^2}$ is the conjugate Poisson kernel. So, the last integral gives the change in the argument from $S_\mu(\tau(a))$ to $S_\mu(\tau(b))$, and this change is positive since $g_\mu \ge 0$. Since this holds for any such a and b, we conclude that the argument of $S_\mu(e^{i\theta})$ increases as θ increases, for $e^{i\theta}$ in J. Letting $a = t_1$ and $b = t_2$ we find that

$$\int_{\tau^{-1}(J)} g_\mu(t)\,dt = \int_{t_1}^{t_2} g_\mu(t)\,dt$$

gives the full measure in radians through which S_μ wraps J. □

Our next result follows immediately from Lemmas 3.1 and 3.2, and their proofs.

THEOREM 3.3. *Let φ be an inner function with canonical factorization BS_μ. Suppose that $\sigma_\varphi \neq \mathbb{T}$ and let I be a connected subset of $\mathbb{T} \setminus \sigma_\varphi$. Then φ wraps I through $\int_I h_B(\zeta)|d\zeta| + \int_{\tau^{-1}(I)} g_\mu(t)\,dt$ radians; which could be infinite.*

If B is a finite Blaschke product, then for sufficiently small $\varepsilon > 0$, $\Omega_\varepsilon(B)$ contains an annulus of the form $\{z : s < |z| < 1\}$, where $0 < s < 1$. And hence, G_ε contains such an annulus (if $\varepsilon > 0$ is small enough); which is sufficient for condition $(*)$ to hold. So, by Theorem 2.4, finite Blaschke products give rise to closed-range composition operators on $\mathbb{A}^2$. Our next result goes further than this.

THEOREM 3.4. *Let φ be an inner function with canonical factorization BS_μ. If there is a component I of $\mathbb{T} \setminus \sigma_\varphi$, such that $\int_I h_B(\zeta)|d\zeta| + \int_{\tau^{-1}(I)} g_\mu(t)\,dt > 2\pi$, then C_φ is closed-range on $\mathbb{A}^2$.*

Proof. Since $\int_I h_B(\zeta)|d\zeta| + \int_{\tau^{-1}(I)} g_\mu(t)\,dt > 2\pi$, there is a closed subarc J of I, such that $\int_J h_B(\zeta)|d\zeta| + \int_{\tau^{-1}(J)} g_\mu(t)\,dt > 2\pi$. So, by Theorem 3.3, φ wraps J through more than 2π radians. Now J is a compact subset of I and φ has analytic continuation across I. Hence, φ is uniformly continuous on $\{z : \frac{1}{2} \leq |z| < 1 \text{ and } \frac{z}{|z|} \in J\}$ and, in fact, $\{z : \frac{1}{2} \leq |z| < 1 \text{ and } \frac{z}{|z|} \in J\}$ is contained in Ω_ε, provided $\varepsilon > 0$ is sufficiently small. From these observations it follows that G_ε contains an annulus of the form $\{z : s < |z| < 1\}$, where $0 < s < 1$, provided ε is sufficiently small. By Theorem 2.4, this is enough to ensure that C_φ is closed-range on $\mathbb{A}^2$. □

We now spend some time examining the cases: φ is a Blaschke product, and φ is a singular inner function. By a Theorem of Frostman (cf. [2], page 79) and Lemmas 2.2, and 2.3, a study of Blaschke products has much to tell us here about inner functions in general, and so we tackle this case first.

Proposition 3.5. *Let B be an infinite Blaschke product and let $\{a_n\}_{n=1}^\infty$ be the zeros of B, listed according to multiplicity. If there is a component I of $\mathbb{T} \setminus \sigma_B$ and a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ such that*

$$\inf_k \frac{1 - |a_{n_k}|}{dist(a_{n_k}, I)} > 0,$$

then C_B is closed-range on $\mathbb{A}^2$.

Proof. By elementary methods, there is a positive constant c that depends only on the length of I such that $\int_I P_{a_{n_k}}(\zeta)|d\zeta| \geq c \cdot \frac{1-|a_{n_k}|}{dist(a_{n_k}, I)}$. Therefore, by our hypothesis, $\int_I h_B(\zeta)|d\zeta| = \infty$. So, the result follows from Theorem 3.4. □

The next result is an immediate consequence of Proposition 3.5.

Corollary 3.6. *Let B be an infinite Blaschke product. If σ_B has an isolated point, then C_B is closed-range on $\mathbb{A}^2$.*

Example 3.7. We now construct an example of a Blaschke product B such that σ_B omits an arc of $\mathbb{T}$, yet C_B is not closed-range on $\mathbb{A}^2$. One can accomplish this as a Frostman shift of an appropriately chosen singular inner function. We instead take a more direct approach making use of an example found in [8], page 185. In this reference, the author describes a Blaschke product, call it B_0, whose zeros $\{a_n\}_{n=1}^\infty$ are of multiplicity one and that fails to have an angular derivative at any point in $\mathbb{T}$. Let $\Gamma = \{\zeta \in \mathbb{T} : \Re(\zeta) \leq 0\}$. Now, choose θ_0, $0 < \theta_0 < \frac{\pi}{2}$, and let β be the Blaschke product that is a factor of B, and whose (simple) zeros are precisely those of B_0 in $\{re^{i\theta} : 0 \leq r < 1 \text{and} - \theta_0 \leq \theta \leq \theta_0\}$. Then β converges uniformly on Γ and so $\int_\Gamma h_\beta(\zeta)|d\zeta| < \infty$. Thus, by the definition of h_β, for any $\delta > 0$, we can find r_0,

$0 < r_0 < 1$, such that

$$\sum_{a_n \in W} \int_\Gamma P_{a_n}(\zeta)|d\zeta| < \delta,$$

where $W := \{re^{i\theta} : r_0 \le r < 1 \text{ and } -\theta_0 \le \theta \le \theta_0\}$. Armed with this, we can find r_1, $0 < r_1 < 1$, such that

$$\sum_{a_n \in W_1} \int_\Gamma P_{a_n}(\zeta)|d\zeta| < \frac{\pi}{4},$$

where $W_1 := \{re^{i\theta} : r_1 \le r < 1 \text{ and } -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\}$. And, we proceed for $k = 2, 3, 4, \ldots$ to find r_k, $r_{k-1} < r_k < 1$, such that

$$\sum_{a_n \in W_k \setminus W_{k-1}} \int_\Gamma P_{a_n}(\zeta)|d\zeta| < \frac{\pi}{2^{k+1}},$$

where $W_k := \{re^{i\theta} : r_k \le r < 1 \text{ and } -\frac{(2^k-1)\pi}{2^{k+1}} \le \theta \le \frac{(2^k-1)\pi}{2^{k+1}}\}$. Letting $V = \bigcup_{k=1}^\infty W_k$, we find that

$$\sum_{a_n \in V} \int_\Gamma P_{a_n}(\zeta)|d\zeta| < \frac{\pi}{2}.$$

We let B_1 be the Blaschke product that is a factor of B_0 such that the zeros of B_1 are precisely those of B_0 in V. By the argument supporting the example in [8], page 185, B_1 fails to have an angular derivative at every point in $\mathbb{T} \setminus \Gamma$. Moreover,

$$\int_\Gamma h_{B_1}(\zeta)|d\zeta| < \frac{\pi}{2}$$

and so B_1 wraps Γ through an angle of less than $\frac{\pi}{2}$ radians. The Blaschke product B_1 might itself suffice as an example of what we are looking for, but if h_{B_1} is bounded on Γ, then our job in showing this turns out to be unnecessarily complicated. We therefore construct an additional factor B_2 that will add very little to the wrapping of Γ but will give us a product $B := B_1 B_2$ with the property that h_B is not bounded on Γ near either i or $-i$. This is accomplished using the following elementary fact concerning Poisson Kernels. Let $\gamma = \{w \in \mathbb{D} : \Im(w) \le 0 \text{ and } P_w(1) = 1\}$. Then γ is a Jordan arc that approaches $\mathbb{T}$ tangentially at 1 and

$$\int_{\mathbb{T}^+} P_w(\zeta)|d\zeta| \longrightarrow 0$$

as w converges to 1 in γ, where $\mathbb{T}^+ := \{\zeta \in \mathbb{T} : \Im(\zeta) \ge 0\}$. So, under a rotation by i, we can find a Blaschke sequence $\{c_n\}_{n=1}^\infty$ in $\{z \in \mathbb{D} : \Re(z) > 0\}$ that converges tangentially to i such that (for all n) $P_{c_n}(i) = 1$ and

$$\int_\Gamma P_{c_n}(\zeta)|d\zeta| < \frac{\pi}{2^{n+2}}.$$

Let $d_n = \overline{c}_n$ (for $n = 1, 2, 3, \ldots$) and let B_2 be the Blaschke product whose (simple) zeros are the points $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$. Then h_{B_2} is unbounded on Γ near i and $-i$, yet

$$\int_{\Gamma} h_{B_2}(\zeta)|d\zeta| < \frac{\pi}{2}.$$

Now, let $B = B_1 B_2$. Then

$$\int_{\Gamma} h_B(\zeta)|d\zeta| < \pi$$

and so B wraps Γ through less than π radians. Now, since B_1 is a factor of B, B itself has no angular derivative at any point in $\{\zeta \in \mathbb{T} : \Re(\zeta) \geq 0\}$. And, though the angular derivative of B exists and is bounded on any compact subset of $\{\zeta \in \mathbb{T} : \Re(\zeta) < 0\}$ $(= \Gamma \setminus \{i, -i\})$, in modulus it tends to infinity as ζ in $\Gamma \setminus \{i, -i\}$ tends to i or $-i$; cf. [8], Frostman's theorem, page 183. So, by the Julia–Carathéodory theorem (cf. [8], page 57), for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\Omega_\varepsilon(B) \cap \{re^{i\theta} : 0 \leq r < 1 \text{ and } -\frac{\pi}{2} - \delta \leq \theta \leq \frac{\pi}{2} + \delta\}$ is a compact subset of $\mathbb{D}$. Yet B converges uniformly on $\overline{\mathbb{D}} \setminus \{re^{i\theta} : 0 \leq r < 1 \text{ and } -\frac{\pi}{2} - \delta \leq \theta \leq \frac{\pi}{2} + \delta\}$ and its boundary values there wrap through less than π radians. We can therefore find ζ_0 in $\mathbb{T}$ and $\eta > 0$, such that $G_\varepsilon(B) \cap \{z \in \mathbb{D} : |z - \zeta_0| < \eta\} = \emptyset$. It now follows from Theorem 2.4 that C_B is not closed-range on $\mathbb{A}^2$.

EXAMPLE 3.8. We now use Example 3.7 to find a Blaschke product B^* such that C_{B^*} is not closed-range on $\mathbb{A}^2$ and yet C_{zB^*} is. Let B be the Blaschke product constructed in Example 3.7. Then C_B is not closed-range on $\mathbb{A}^2$ and yet, by Theorem 3.4, $C_{z^n B}$ is closed-range on $\mathbb{A}^2$ provided n is sufficiently large. So, there is a nonnegative integer k such that $C_{z^k B}$ is not closed-range on $\mathbb{A}^2$ and yet $C_{z^{k+1} B}$ is closed-range on $\mathbb{A}^2$. We let $B^* = z^k B$.

Let us now turn to the case that φ is a singular inner function.

DISCUSSION 3.9. Let μ be a singular measure on $\mathbb{T}$ and assume that there is a nontrivial arc I contained in $\mathbb{T}$ such that $\mu(I) > 0$ and $m(I \cap \sigma_\mu) = 0$. Let $\{I_n\}_n$ be the components of $I \setminus \sigma_\mu$. If there are only finitely many $I_{n's}$, then $\mu|_I$ is a finite sum of point masses and thus we can find an N so that $\int_{\tau^{-1}(I_N)} g_\mu(t)\, dt = \infty$. And so, by Theorem 3.4, C_{S_μ} is closed-range on $\mathbb{A}^2$. Therefore, the interesting case is that there are infinitely many $I_{n's}$. Under a rotation of $\mathbb{T}$ (if necessary), we may assume that $1 \in \sigma_\mu$, and hence $\tau^{-1}(I_n) = (a_n, b_n)$, where $0 < a_n < b_n < 2\pi$, for all values of n. Now, suppose that there exists $M > 0$, such that $\int_{a_n}^{b_n} g_\mu(t)\, dt \leq M$ for all n. Let $\{c_n\}_{n=1}^{\infty}$ be an l^1-sequence of positive real numbers and define ν on $[0, 2\pi)$ by

$$\nu = \sum_{n=1}^{\infty} c_n \lambda|_{(a_n, b_n)},$$

where λ denotes Lebesgue measure on $\mathbb{R}$. Then we can find a positive constant c, such that

$$\int_0^{2\pi}\left(\int \frac{1}{(\theta-t)^2}\,d\nu(t)\right)d\mu(\theta) \le c\int_0^{2\pi}\left(\int \frac{1}{1-\cos(\theta-t)}\,d\nu(t)\right)d\mu(\theta)$$
$$= c\int_0^{2\pi} g_\mu(t)\,d\nu(t) < \infty.$$

THEOREM 3.10. *Assuming the context and terminology in Discussion 3.9, if there exists ν as described there such that $\int \frac{1}{(\theta-t)^2}\,d\nu(t)$ diverges for each θ in $\tau^{-1}(I\cap\sigma_\mu)$, then C_{S_μ} is closed-range on $\mathbb{A}^2$.*

Proof. Since $\mu(I)>0$ and $\int \frac{1}{(\theta-t)^2}\,d\nu(t)$ diverges for each θ in $\tau^{-1}(I\cap\sigma_\mu)$, we conclude that $\int_0^{2\pi}(\int \frac{1}{(\theta-t)^2}\,d\nu(t))\,d\mu(\theta)$ diverges. So, by Discussion 3.9, there cannot be a bound on $\int_{a_n}^{b_n} g_\mu(t)\,dt$ independent of n. So, there exists N such that $\int_{a_N}^{b_N} g_\mu(t)\,d(t) > 2\pi$. The result now follows from Theorem 3.4. □

Let $\mathcal{C}$ be the Cantor set in $[0,1]$. We say that a compact subset F of $\mathbb{T}$ is a Cantor set in $\mathbb{T}$ if it is of the form $\psi_a(\tau(\mathcal{C}))$, where ψ_a is a conformal automorphism of the disk.

COROLLARY 3.11. *Let μ be a singular measure on $\mathbb{T}$. If there is a nontrivial arc I in $\mathbb{T}$ such that $\mu(I)>0$ and $I\cap\sigma_\mu$ is contained in a Cantor set F in $\mathbb{T}$, then C_{S_μ} is closed-range on $\mathbb{A}^2$.*

Proof. Since composition with a conformal automorphism of the disk does not change the status of C_{S_μ} on $\mathbb{A}^2$ with regard to being closed-range, we may assume that $F=\tau(\mathcal{C})$. Let $\{I_n\}_{n=1}^\infty$ be the components of $I\setminus F$, let $c_n = \text{length}(\tau^{-1}(I_n))$ and define ν on $[0,2\pi)$ by $\nu=\sum_{n=1}^\infty c_n\lambda|_{\tau^{-1}(I_n)}$. Now, the Cantor set has the property that every point θ in it is within a distance of $\frac{1}{3^n}$ from a component of $[0,1]\setminus\mathcal{C}$ of length at least $\frac{1}{3^n}$ (for $n=1,2,3,\ldots$). From this, it follows that $\int \frac{1}{(\theta-t)^2}\,d\nu(t)$ diverges for each θ in $\mathcal{C}$. The conclusion now follows from Theorem 3.10. □

The proof of Corollary 3.11 cannot be extended much beyond sets of the Cantor type. For example, let $E=\{1\}\cup\{\frac{2k\pi i+1}{2k\pi i-1}: k\in\mathbb{Z}\}$ and let $\mu=\delta_{\{1\}}$—the point mass at 1. Then support$(\mu)\subseteq E$ and $\int_{\tau^{-1}(I)} g_\mu(t)\,dt = 2\pi$ for any component I of $\mathbb{T}\setminus E$, and so the proof of Corollary 3.11 does not carry over with E in place of F. Yet we know (by Corollary 3.11) that in this example, C_{S_μ} is closed-range on $\mathbb{A}^2$, since the support of μ is a single point. Now, by Corollary 3.11, if B is a Frostman shift of a singular inner function whose singular measure has support in a Cantor set of the unit circle, then C_B is closed-range on $\mathbb{A}^2$. Yet, we still do not know if this carries over for any Blaschke product B such that σ_B is contained in a Cantor set of the unit circle.

QUESTION 3.12. Let F be a Cantor set contained in $\mathbb{T}$. Does there exist a Blaschke product B with $\sigma_B \subseteq F$ such that C_B is not closed-range on $\mathbb{A}^2$?

The case that φ is an outer function can be handled in much the same way as the singular inner case, though outer functions are less likely to give rise to closed-range composition operators since, if nontrivial, they do not have unimodular boundary values a.e. on $\mathbb{T}$.

We close this paper with a result (cf. the Appendix) that exposes a link between the a.e. existence of the angular derivative of a given Blaschke product B and the rate at which its zeros tend to $\mathbb{T}$. We see this as fitting into the theme of this paper since the Julia–Carathéodory theorem provides strong connection between the notion of angular derivative and the sets Ω_ε and G_ε of Theorem 2.4.

Appendix

Let B be a Blaschke product with zeros $\{a_n\}_n$, listed according to multiplicity. By a theorem of Frostman (cf. [8], page 183), B has an angular derivative at some ξ in $\mathbb{T}$ if and only if

$$h_B(\xi) := \sum_n P_{a_n}(\xi) < \infty.$$

In this section, we make use of this theorem of Frostman to link the convergence of $\sum_{n=1}^{\infty} \sqrt{1-|a_n|^2}$ with B having an angular derivative a.e. on $\mathbb{T}$, which, by Theorem 2.4 and the Julia–Carathéodory theorem, increases the likelihood that C_B is closed-range on $\mathbb{A}^2$. We first need a lemma.

LEMMA 3.13. *Suppose* $0 < a < 1$ *and* $0 < \theta < \frac{\pi}{2}$. *Then*

$$\frac{1}{2\pi}\int_\theta^\pi P_a(t)\,dt \le \frac{1}{\pi}\left(\frac{1-a}{1+a}\right)\cdot\frac{1+\cos\theta}{\sin\theta}.$$

In particular,

$$\frac{1}{2\pi}\int_{\sqrt{1-a}}^\pi P_a(t)\,dt = O(\sqrt{1-a}),$$

as $a \to 1^-$.

Proof. Let ψ be the Möbius transformation from $\mathbb{D}$ onto $\mathbb{C}^+ := \{z : \Im(z) > 0\}$ given by

$$\psi(z) = i\left(\frac{1+a}{1-a}\right)\cdot\frac{1-z}{1+z}.$$

Notice that ψ maps a to i, -1 to ∞ and 1 to 0. And, for $0 < \theta < \pi$,

$$\psi(e^{i\theta}) = \Re(\psi(e^{i\theta})) = \left(\frac{1+a}{1-a}\right)\cdot\frac{\sin\theta}{1+\cos\theta} =: x_{a,\theta}.$$

By conformal invariance of harmonic measure,

$$\frac{1}{2\pi}\int_{\theta}^{\pi} P_a(t)\,dt = \frac{1}{\pi}\int_{x_{a,\theta}}^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan(x_{a,\theta})\right).$$

And, by elementary methods we find that $\frac{\pi}{2} - \arctan(x) \le \frac{1}{x}$, whenever $0 < x < \infty$ (and this inequality is only meaningful for large x). Hence,

$$\frac{1}{2\pi}\int_{\theta}^{\pi} P_a(t)\,dt \le \frac{1}{\pi x_{a,\theta}}. \qquad \square$$

THEOREM 3.14. *Let $\{r_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $0 < r_n \le 1$ for all n and $\sum_{n=1}^{\infty} r_n < \infty$.*

(i) *If $\sum_{n=1}^{\infty}\sqrt{r_n}$ diverges, then there is a Blaschke sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that $(1-|a_n|^2) = r_n$ for all n and $P_{a_n}(\zeta)$ does not converge to 0 for any ζ in $\mathbb{T}$. And therefore, the Blaschke product B with zeros $\{a_n\}_{n=1}^{\infty}$ fails to have an angular derivative at any point in $\mathbb{T}$.*

(ii) *If $\sum_{n=1}^{\infty}\sqrt{r_n}$ converges and $\{a_n\}_{n=1}^{\infty}$ is any sequence in $\mathbb{D}$ such that $(1-|a_n|^2) = r_n$ for all n, then the Blaschke product B with zeros $\{a_n\}_{n=1}^{\infty}$ has an angular derivative a.e. on $\mathbb{T}$.*

Proof. We first establish (i), following closely the proof of the example in [8], on page 185. Construct a sequence $\{I_n\}_{n=1}^{\infty}$ of contiguous arcs in $\mathbb{T}$ such that length$(I_n) = \sqrt{r_n}$, for $n = 1, 2, 3, \ldots$. Let ζ_n be the midpoint of I_n and let $a_n = (1-r_n)\zeta_n$. If $\zeta \in I_n$, which, for any ζ in $\mathbb{T}$, occurs for infinitely many values of n, then

$$|\zeta - a_n|^2 < \left(\frac{\sqrt{r_n}}{2} + r_n\right)^2 \le r_n,$$

if n is sufficiently large. And, so for infinitely many values of n,

$$P_{a_n}(\zeta) := \frac{1-|a_n|^2}{|\zeta - a_n|^2} > \frac{1-|a_n|^2}{r_n} = 1.$$

Evidently, $P_{a_n}(\zeta)$ does not converge to zero (as $n \to \infty$) for any ζ in $\mathbb{T}$.

And, now we address (ii). If n is sufficiently large, then $a_n \ne 0$. For such n, let $\alpha_n = \frac{a_n}{|a_n|}$ and let $E_n = \{\zeta \in \mathbb{T} : |\arg(\zeta - \alpha_n)| < \sqrt{1-|a_n|^2}\}$. Since (by our hypothesis) $\sum_n m(E_n) < \infty$, for any $\varepsilon > 0$ there exists N such that $\sum_{n \ge N} m(E_n) < \varepsilon$. Let $F_N = \bigcup_{n \ge N} E_n$; and so $m(F_N) < \varepsilon$. Now, by Lemma 3.13, there is a positive constant c such that

$$\sum_{n \ge N}\int_{\mathbb{T}\setminus F_N} P_{a_n}(\zeta)\,dm(\zeta) \le \sum_{n \ge N}\int_{\mathbb{T}\setminus E_n} P_{a_n}(\zeta)\,dm(\zeta) \le \sum_{n \ge N} c\sqrt{1-|a_n|^2} < \infty.$$

Hence, $h_B(\zeta) := \sum_n P_{a_n}(\zeta)$ converges a.e. on $\mathbb{T} \setminus F_N$. And, since $m(F_N) < \varepsilon$ and $\varepsilon > 0$ is arbitrary, we conclude that $h_B(\zeta)$ converges a.e. on $\mathbb{T}$. Therefore, by a theorem of Frostman (cf. [8], page 183), the angular derivative of B exists a.e. on $\mathbb{T}$. $\square$

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