Illinois Journal of Mathematics
Volume 52, Number 1, Spring 2008, Pages 17–46
S 0019-2082

MORE MIXED TSIRELSON SPACES THAT ARE NOT ISOMORPHIC TO THEIR MODIFIED VERSIONS

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Abstract. The class of mixed Tsirelson spaces is an important source of examples in the recent development of the structure theory of Banach spaces. The related class of modified mixed Tsirelson spaces has also been well studied. In the present paper, we investigate the problem of comparing isomorphically the mixed Tsirelson space $T[(\mathcal{S}_n, \theta_n)_{n=1}^{\infty}]$ and its modified version $T_M[(\mathcal{S}_n, \theta_n)_{n=1}^{\infty}]$. It is shown that these spaces are not isomorphic for a large class of parameters (θ_n).

1. Introduction

In 1974, Tsirelson [19] settled a fundamental problem in the structure theory of Banach spaces when he gave a surprisingly simple construction of a Banach space that does not contain any isomorphic copy of c_0 or ℓ^p, $1 \leq p < \infty$. Figiel and Johnson [7] provided an analytic description, based on iteration, of the norm of the dual of Tsirelson's original space. Subsequently, other examples of spaces were constructed with norms described iteratively, notable among them were Tzafriri's spaces [20] and Schlumprecht's space [18]. Gowers' and Maurey's solution to the unconditional basic sequence problem [8] is a variation based on the same theme. It has emerged in recent years that far from being isolated examples, Tsirelson's space and its variants form an important class of Banach spaces. Argyros and Deliyanni [2] were the first to provide a general framework for such spaces by defining the class of mixed Tsirelson spaces. Among the earliest variants of Tsirelson's space was its modified version introduced by Johnson [9]. Casazza and Odell [6] showed that Tsirelson's space is isomorphic to its modified version. This isomorphism was

Received June 7, 2006; received in final form January 4, 2007.

Research of the first author was partially supported by AcRF project number R-146-000-086-112.

2000 *Mathematics Subject Classifications.* 46B20, 46B45.

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exploited to study the structure of the space. The modification can be extended directly to the class of mixed Tsirelson spaces, forming the class of modified mixed Tsirelson spaces. It is thus of natural interest to determine if a mixed Tsirelson space is isomorphic to its modified version. This question has been considered by various authors, e.g., [3, 12], who provided answers in what may be considered "extremal" cases. In the present paper, we show that for a large class of parameters, a mixed Tsirelson space and its modified version are not isomorphic.

We shall be concerned exclusively with mixed Tsirelson spaces of the form $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ or $T[(\mathcal{S}_{n_i},\theta_i)_{i=1}^k]$ and their modified versions. We now recall the definitions of these spaces and the various notions involved. Denote by $\mathbb{N}$ the set of natural numbers. For any infinite subset M of $\mathbb{N}$, let $[M]$ and $[M]^{<\infty}$ be the set of all infinite and finite subsets of M, respectively. These are subspaces of the power set of $\mathbb{N}$, which is identified with $2^{\mathbb{N}}$ and endowed with the topology of pointwise convergence. If I and J are nonempty finite subsets of $\mathbb{N}$, we write $I < J$ to mean $\max I < \min J$. We also allow that $\emptyset < I$ and $I < \emptyset$. For a singleton $\{n\}$, $\{n\} < J$ is abbreviated to $n < J$. The general Schreier families $\mathcal{S}_\alpha$, $\alpha < \omega_1$, were introduced by Alspach and Argyros [1]. We shall restrict ourselves to finite parameters. Let $\mathcal{S}_0$ consist of all singleton subsets of $\mathbb{N}$ together with the empty set. Inductively, if $n \in \mathbb{N}$, let $\mathcal{S}_n$ consist of all sets of the form $\bigcup_{i=1}^k G_i$, where $G_i \in \mathcal{S}_{n-1}$, $G_1 < \cdots < G_k$ and $k \leq \min G_1$. The Schreier families are *hereditary*: $G \in \mathcal{S}_n$ whenever $G \subseteq F$ and $F \in \mathcal{S}_n$; *spreading*: for all strictly increasing sequences $(m_i)_{i=1}^k$ and $(n_i)_{i=1}^k$, $(n_i)_{i=1}^k \in \mathcal{S}_n$ if $(m_i)_{i=1}^k \in \mathcal{S}_n$ and $m_i \leq n_i$ for all i; and compact as subspaces of $[\mathbb{N}]^{<\infty}$. A sequence $(E_i)_{i=1}^k$ in $[\mathbb{N}]^{<\infty}$ is said to be $\mathcal{S}_n$*-admissible* if $E_1 < \cdots < E_k$ and $\{\min E_i\}_{i=1}^k \in \mathcal{S}_n$. It is $\mathcal{S}_n$*-allowable* if the E_i's are pairwise disjoint, and $\{\min E_i\}_{i=1}^k \in \mathcal{S}_n$.

Denote by c_{00} the space of all finitely supported real sequences, whose unit vector basis will be denoted by (e_k). For a finite subset E of $\mathbb{N}$ and $x \in c_{00}$, let Ex be the coordinate-wise product of x with the characteristic function of E. The sup norm and the ℓ^1-norm on c_{00} are denoted by $\|\cdot\|_{c_0}$ and $\|\cdot\|_{\ell^1}$, respectively. Given a null sequence $(\theta_n)_{n=1}^\infty$ in $(0,1)$, define sequences of norms $\|\cdot\|_m$ and $|||\cdot|||_m$ on c_{00} as follows. Let $\|x\|_0 = |||x|||_0 = \|x\|_{c_0}$ and

$$\|x\|_{m+1} = \max\left\{ \|x\|_m, \sup_n \theta_n \sup \sum_{i=1}^r \|E_i x\|_m \right\}, \tag{1}$$

where the last sup is taken over all $\mathcal{S}_n$-admissible sequences $(E_i)_{i=1}^r$. The norm $|||x|||_m$ is defined as in (1) except that the last sup is taken over all $\mathcal{S}_n$-allowable sequences $(E_i)_{i=1}^r$. Since these norms are all dominated by the ℓ^1-norm, $\|x\| = \lim_m \|x\|_m$ and $|||x||| = \lim_m |||x|||_m$ exist and are norms on c_{00}. The *mixed Tsirelson space* $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ and the *modified mixed Tsirelson space* $T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ are the completions of c_{00} with respect to the norms

$\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$, respectively. From equation (1), we can deduce that these norms satisfy the implicit equations

$$\|x\| = \max\left\{ \|x\|_{c_0}, \sup_n \theta_n \sup \sum_{i=1}^{r} \|E_i x\| \right\} \tag{2}$$

and

$$|\!|\!|x|\!|\!| = \max\left\{ \|x\|_{c_0}, \sup_n \theta_n \sup \sum_{i=1}^{r} |\!|\!|E_i x|\!|\!| \right\}, \tag{3}$$

where the innermost suprema are taken over all $\mathcal{S}_n$-admissible, respectively, $\mathcal{S}_n$-allowable sequences $(E_i)_{i=1}^r$. The mixed Tsirelson space $T[(\mathcal{S}_{n_i}, \theta_i)_{i=1}^k]$ and modified mixed Tsirelson space $T_M[(\mathcal{S}_{n_i}, \theta_i)_{i=1}^k]$ are defined similarly.

When considering the spaces $T[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$ and $T_M[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$, we may assume without loss of generality that (θ_n) is nonincreasing and that $\theta_{m+n} \geq \theta_m \theta_n$. Such sequences are said to be *regular*. It is known that [17] $\lim \theta_n^{1/n} = \sup \theta_n^{1/n}$ for a regular sequence (θ_n). Let $\theta = \lim_n \theta_n^{1/n}$ and $\varphi_n = \theta_n/\theta^n$. The main result of the paper is the following theorem.

THEOREM 1. *If* $0 < c = \inf \varphi_n \leq \sup \varphi_n = d < 1$, *then* $T[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$ *is not isomorphic to* $T_M[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$.

For standard Banach space terminology and notation, we refer to [15]. Two Banach spaces X and Y are said to be *isomorphic* if they are linearly homeomorphic. A linear homeomorphism from X into Y is called an *embedding*. We say that X *embeds into* Y if such an embedding exists. X and Y are *totally incomparable* if no infinite dimensional subspace of one embeds into the other. A sequence (x_n) in X is said to *dominate* a sequence (y_n) in Y if there is a finite constant K such that $\|\sum a_n y_n\| \leq K \|\sum a_n x_n\|$ for all $(a_n) \in c_{00}$. Two sequences are *equivalent* if they dominate each other.

2. Brief survey of known results

The aim of the present paper is to compare isomorphically the spaces $T[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$ and $T_M[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$ (and also the spaces $T[(\mathcal{S}_{n_i}, \theta_i)_{i=1}^k]$ and $T_M[(\mathcal{S}_{n_i}, \theta_i)_{i=1}^k]$). Let us recall some known results in this direction. Casazza and Odell [6] showed that the Tsirelson space $T[\mathcal{S}_1, \theta]$ is isomorphic to the modified Tsirelson space $T_M[\mathcal{S}_1, \theta]$, with no specific isomorphism constant given in their proof. In [5], Bellenot proved that they are θ^{-1}-isomorphic. Recently, Manoussakis [12] showed that the spaces $T[\mathcal{S}_n, \theta]$ and $T_M[\mathcal{S}_n, \theta]$ are 3-isomorphic for all $n \in \mathbb{N}$ and all $\theta \in (0, 1)$. He also stated without proof in

[11, Section 4] that $T[(\mathcal{S}_{n_i},\theta_i)_{i=1}^k]$ is isomorphic to $T_M[(\mathcal{S}_{n_i},\theta_i)_{i=1}^k]$. A proof of a nominally more general fact will be given below.

Argyros et al. showed that if (θ_n) is regular and $\lim_n \theta_n^{1/n} = 1$, then $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ contains copies of $\ell^\infty(n)$'s uniformly and hereditarily [3, Theorem 1.6]. As a result, they were able to conclude that $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ and $T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ are totally incomparable.

In [13], the authors introduced the condition

$$\lim_m \limsup_n \frac{\theta_{m+n}}{\theta_n} > 0. \tag{†}$$

Condition (†) is weaker than the condition $\lim_n \theta_n^{1/n} = 1$. More precisely, if $\lim_n \theta_n^{1/n} = 1$, then

$$\lim_m \limsup_n \frac{\theta_{m+n}}{\theta_n} = 1.$$

Indeed, if there exist $\delta < 1$, $m \in \mathbb{N}$ and $N \in \mathbb{N}$ such that $\frac{\theta_{n+m}}{\theta_n} < \delta$ for all $n \geq N$, then $\theta_{km+N} < \delta^k \theta_N$ for all $k \in \mathbb{N}$. Thus, $\theta_{km+N}^{\frac{1}{km+N}} < \delta^{\frac{k}{km+N}} \theta_N^{\frac{k}{km+N}}$. Taking $k \to \infty$, we have $\lim_n \theta_n^{1/n} \leq \delta^{1/m} < 1$. It can be shown that the converse is false even for regular sequences.

If (θ_n) satisfies (†), it follows from [14, Proposition 9] that there exists $\varepsilon > 0$ such that for all $V \in [\mathbb{N}]$ and all $k \in \mathbb{N}$, there exists a sequence of pairwise disjoint vectors $(y_j)_{j=1}^k \subseteq \operatorname{span}\{e_k : k \in V\}$ such that $\|\sum_{j=1}^k y_j\| \leq 2 + 1/\varepsilon$ and $\|y_j\| \geq 1$ for all j. In other words, $\ell^\infty(n)$'s uniformly disjointly embeds into the subspace of $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ generated by $(e_k)_{k\in V}$. In particular, the norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are not equivalent on $\operatorname{span}\{e_k : k \in V\}$. This together with the proposition below imply that $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ is not isomorphic to $T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$.

Proposition 2. *If $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ embeds into $T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$, then there exists $V \in [\mathbb{N}]$ such that $|\!|\!|\cdot|\!|\!|$ is equivalent to $\|\cdot\|$ on the subspace* $\operatorname{span}\{e_k : k \in V\}$.

Proof. Let $J : T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty] \to T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ be an embedding. Then (Je_k) is a weakly null sequence. By the Bessaga–Pełczynski selection principle (see, e.g., [15, Proposition 1.a.12]), there is a subsequence (Je_{k_j}) of (Je_k) such that (Je_{k_j}) is equivalent to a seminormalized block sequence (u_j) in $T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$. Let $m_j = \min\operatorname{supp} u_j$. By taking a subsequence if necessary, we may assume that

$$\max\{k_j, m_j\} < \min\{k_{j+1}, m_{j+1}\}.$$

As a result, the sequences (e_{m_j}) and (e_{k_j}) are equivalent in $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$. On the other hand, in $T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$, (e_{m_j}) is dominated by (u_j), which is equivalent to (Je_{k_j}). Hence, there exist finite constants λ and λ' such that for all $(a_i) \in c_{00}$,

$$\begin{aligned}\Big\|\Big|\sum a_j e_{m_j}\Big|\Big\| &\le \lambda \Big\|\sum a_j e_{k_j}\Big\| \\ &\le \lambda' \Big\|\sum a_j e_{m_j}\Big\| \\ &\le \lambda' \Big\|\Big|\sum a_j e_{m_j}\Big|\Big\|.\end{aligned}$$

Thus, $|||\cdot|||$ is equivalent to $\|\cdot\|$ on the subspace $\mathrm{span}\{(e_{m_j})\}$. □

3. Essentially finitely generated spaces

The fact that $T[(\mathcal{S}_{n_i},\theta_i)_{i=1}^k]$ is isomorphic to $T_M[(\mathcal{S}_{n_i},\theta_i)_{i=1}^k]$ was stated by Manoussakis in [11]. We present a nominally more general result here. Let us note that Lopez–Abad and Manoussakis [10] has undertaken a thorough study of mixed Tsirelson spaces generated by finitely many terms.

We shall compute the norm of an element in $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$, respectively, $T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$, with the help of norming trees. This is derived from the implicit description of the norms given in equations (2) and (3) and have been used in [5, 14, 16]. An $((\mathcal{S}_n))$-*admissible tree* (respectively, *allowable tree*) is a finite collection of elements (E_i^m), $0 \le m \le r$, $1 \le i \le k(m)$, in $[\mathbb{N}]^{<\infty}$ with the following properties.

(i) $k(0) = 1$,
(ii) every E_i^{m+1} is a subset of some E_j^m,
(iii) for each j and m, the collection $\{E_i^{m+1} : E_i^{m+1} \subseteq E_j^m\}$ is $\mathcal{S}_n$-admissible ($\mathcal{S}_n$-allowable) for some n.

The set E_1^0 is called the *root* of the tree. The elements E_i^m are called *nodes* of the tree. Given a node E_i^m, $h(E_i^m) = m$ is called the *height* of the node E_i^m. The height of a tree $\mathcal{T}$ is defined by $H(\mathcal{T}) = \max\{h(E) : E \in \mathcal{T}\}$. If $E_i^n \subseteq E_j^m$ and $n > m$, we say that E_i^n is a *descendant* of E_j^m and E_j^m is an *ancestor* of E_i^n. If in the above notation, $n = m+1$, then E_i^n is said to be an *immediate successor* of E_j^m, and E_j^m the *immediate predecessor* of E_i^n. Nodes with no descendants are called *terminal nodes* or *leaves* of the tree. We denote the set of all leaves of a tree $\mathcal{T}$ by $\mathcal{L}(\mathcal{T})$. Nodes that attain maximal height are called *base nodes.*

Assign *tags* to the individual nodes inductively as follows. Let $t(E_1^0) = 1$. If $t(E_i^m)$ has been defined and the collection (E_j^{m+1}) of all immediate successors of E_i^m forms an $\mathcal{S}_k$-admissible ($\mathcal{S}_k$-allowable) collection, then define $t(E_j^{m+1}) = \theta_k t(E_i^m)$ for all immediate successors E_j^{m+1} of E_i^m. If $x \in c_{00}$ and $\mathcal{T}$ is an admissible (allowable) tree, let $\mathcal{T}x = \sum t(E)\|Ex\|_{c_0}$ where the sum is taken over all leaves in $\mathcal{T}$. It follows from the implicit description of the norm in $T[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$ (respectively, $T_M[(\mathcal{S}_n,\theta_n)_{n=1}^\infty]$) that $\|x\| = \max \mathcal{T}x$ (respectively, $|||x||| = \max \mathcal{T}x$), with the maximum taken over the set of all admissible (respectively, allowable) trees. Given a node $E \in \mathcal{T}$ with tag $t(E) = \prod_{i=1}^m \theta_{n_i}$, define $o_{\mathcal{T}}(E) = \sum_{i=1}^m n_i$. When there is no confusion, we write $o(E)$ instead of $o_{\mathcal{T}}(E)$.

To simplify notation, we shall henceforth denote the spaces $T[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$ and $T_M[(\mathcal{S}_n, \theta_n)_{n=1}^\infty]$ by X and X_M, respectively. The norms on these spaces will be denoted by $\|\cdot\|$ and $\|\cdot\|_{X_M}$, respectively.

For a fixed $N \in \mathbb{N}$, an $\mathcal{S}_N$-admissible (-allowable) tree is a tree satisfying conditions (i)–(ii) above and

(iii′) For each j and m, the collection $\{E_i^{m+1} : E_i^{m+1} \subseteq E_j^m\}$ is $\mathcal{S}_N$-admissible (-allowable).

It is well known that an $\mathcal{S}_m$-admissible collection of $\mathcal{S}_n$-admissible sets is $\mathcal{S}_{m+n}$-admissible. The corresponding fact for the “allowable” case comes from [3] (see also [12, Lemma 2.1]).

Lemma 3. *Given an $(\mathcal{S}_n)_{n=1}^\infty$-admissible (-allowable) tree $\mathcal{T}$ of finite height, there exists an $\mathcal{S}_1$-admissible (-allowable) tree $\mathcal{T}'$ with the same root such that $\mathcal{L}(\mathcal{T}) = \mathcal{L}(\mathcal{T}')$, and $o_{\mathcal{T}}(E) = o_{\mathcal{T}'}(E)$ for all $E \in \mathcal{L}(\mathcal{T})$.*

Proof. The proof is by induction on the height $H(\mathcal{T})$ of $\mathcal{T}$. If $H(\mathcal{T}) = 0$, then there is nothing to prove. Assume the statement holds if $H(\mathcal{T}) \le N$ for some N. Let $\mathcal{T}$ be an $(\mathcal{S}_n)_{n=1}^\infty$-admissible (-allowable) tree with $H(\mathcal{T}) = N+1$. Let $\mathcal{E}_1$ be the collection of all nodes of $\mathcal{T}$ at height 1. There exists n_0 such that $\mathcal{E}_1$ is $\mathcal{S}_{n_0}$-admissible (-allowable). It is easy to see that there is an $\mathcal{S}_1$-admissible (-allowable) tree $\mathcal{T}_1$ having the same root as $\mathcal{T}$ and of height n_0 such that $\mathcal{L}(\mathcal{T}_1) = \mathcal{E}_1$ and that every $E \in \mathcal{E}_1$ is a leaf of $\mathcal{T}_1$ at height n_0. If $E \in \mathcal{E}$, then $\mathcal{T}_E = \{F \in \mathcal{T} : F \subseteq E\}$ is an $(\mathcal{S}_n)_{n=1}^\infty$-admissible (-allowable) tree with $H(\mathcal{T}_E) \le N$. By the inductive hypothesis, for each $E \in \mathcal{E}_1$, there exists an $\mathcal{S}_1$-admissible (-allowable) tree $\mathcal{T}'_E$ with root E such that $\mathcal{L}(\mathcal{T}_E) = \mathcal{L}(\mathcal{T}'_E)$ and $o_{\mathcal{T}_E}(F) = o_{\mathcal{T}'_E}(F)$ for all $F \in \mathcal{L}(\mathcal{T}_E)$.

Consider $\mathcal{T}' = \mathcal{T}_1 \cup \bigcup_{E \in \mathcal{E}_1} \mathcal{T}'_E$. Then $\mathcal{T}'$ is an $\mathcal{S}_1$-admissible (-allowable) tree with the same root as $\mathcal{T}$. If $F \in \mathcal{L}(\mathcal{T})$, then $F \subseteq E$ for some $E \in \mathcal{E}_1$ (since the root cannot be a leaf in this case because $H(\mathcal{T}) \ge N+1 \ge 1$). Now $o_{\mathcal{T}}(F) = o_{\mathcal{T}_E}(F) + n_0$ and $F \in \mathcal{L}(\mathcal{T}_E)$. Hence, $F \in \mathcal{L}(\mathcal{T}'_E)$ and $o_{\mathcal{T}'}(F) = o_{\mathcal{T}'_E}(F) + n_0 = o_{\mathcal{T}_E}(F) + n_0 = o_{\mathcal{T}}(F)$. Conversely, if $F \in \mathcal{L}(\mathcal{T}')$, then $F \in \mathcal{L}(\mathcal{T}'_E)$ for some $E \in \mathcal{E}_1$. Thus, $F \in \mathcal{L}(\mathcal{T}_E)$, and hence $F \in \mathcal{L}(\mathcal{T})$. □

Lemma 4. *Let $\mathcal{T}$ be an $(\mathcal{S}_n)_{n=1}^\infty$-admissible (-allowable) tree. If $\mathcal{E}$ is a collection of pairwise disjoint nodes of $\mathcal{T}$ such that $o(E) \le m$ for all $E \in \mathcal{E}$, then $\mathcal{E}$ is $\mathcal{S}_m$-admissible (allowable).*

Proof. The proof is by induction on m. The case $m = 0$ is clear. Now suppose the lemma holds for all $k < m$, $m \ge 1$. If the root of $\mathcal{T}$ belongs to $\mathcal{E}$, then it is the only node in $\mathcal{E}$ and the lemma clearly holds. Otherwise, let $k \in \mathbb{N}$ be such that the nodes $G_1 < \cdots < G_q$ in $\mathcal{T}$ with height 1 is $\mathcal{S}_k$-admissible (-allowable). Since each $E \in \mathcal{E}$ is either equal to or is a descendant of some G_i, $m \geq o(E) \geq o(G_i) = k$. If $m = k$, then $\mathcal{E} \subseteq \{G_1, \ldots, G_q\}$, and thus is $\mathcal{S}_m$-admissible (-allowable). If $k < m$, then for each i, the subtree $\mathcal{T}_i$ with root G_i is an admissible (-allowable) tree such that $o_{\mathcal{T}_i}(E) \leq m - k$ for all $E \in \mathcal{E} \cap \mathcal{T}_i$. By induction, $E \in \mathcal{E} \cap \mathcal{T}_i$ is $\mathcal{S}_{m-k}$-admissible (-allowable). Therefore, $\mathcal{E}$ is an $\mathcal{S}_k$-admissible (-allowable) collection of $\mathcal{S}_{m-k}$-admissible (-allowable) sets, and hence an $\mathcal{S}_m$-admissible (-allowable) set. □

Given $k \in \mathbb{N}$, let $\lceil k \rceil$ denote the least integer greater than or equal to k.

LEMMA 5. *Let $\mathcal{T}$ be an $\mathcal{S}_1$-admissible (-allowable) tree. For any $N \in \mathbb{N}$, there exists an $\mathcal{S}_N$-admissible (-allowable) tree $\mathcal{T}'$ with the same root such that $\mathcal{L}(\mathcal{T}) = \mathcal{L}(\mathcal{T}')$ and $o_{\mathcal{T}'}(E) = N\lceil o_{\mathcal{T}}(E)/N \rceil$ for all $E \in \mathcal{L}(\mathcal{T})$.*

Proof. Note that the statement holds if $H(\mathcal{T}) \leq N$ by Lemma 4. Now suppose that the statement holds if $H(\mathcal{T}) \leq kN$ for some $k \in \mathbb{N}$. Let $\mathcal{T}$ be an $\mathcal{S}_1$-admissible (-allowable) tree with $H(\mathcal{T}) \leq (k+1)N$. Denote by $\mathcal{T}_0$ the tree consisting of all nodes in $\mathcal{T}$ with height $\leq N$. For each $E \in \mathcal{T}$ at height N, $H(\mathcal{T}_E) \leq kN$, where $\mathcal{T}_E$ consists of all nodes F in $\mathcal{T}$ such that $F \subseteq E$. By induction, for each $E \in \mathcal{T}$ at height N, there exists an $\mathcal{S}_N$-admissible (-allowable) tree $\mathcal{T}'_E$ with root E such that $\mathcal{L}(\mathcal{T}_E) = \mathcal{L}(\mathcal{T}'_E)$ and $o_{\mathcal{T}'_E}(F) = N\lceil o_{\mathcal{T}_E}(F)/N \rceil$ for all $F \in \mathcal{L}(\mathcal{T}_E)$. At the same time, there exists an $\mathcal{S}_N$-admissible (-allowable) tree $\mathcal{T}'_0$ with the same root as $\mathcal{T}_0$ such that $\mathcal{L}(\mathcal{T}'_0) = \mathcal{L}(\mathcal{T}_0)$ and $o_{\mathcal{T}'_0}(F) = N\lceil o_{\mathcal{T}_0}(F)/N \rceil$ for all $F \in \mathcal{L}(\mathcal{T}_0)$. Let $\mathcal{T}' = \mathcal{T}'_0 \cup \bigcup \mathcal{T}'_E$, where the second union is taken over all nodes $E \in \mathcal{T}$ at height N. Then $\mathcal{T}'$ is an $\mathcal{S}_N$-admissible (-allowable) tree with the same root as $\mathcal{T}$.

If $E \in \mathcal{L}(\mathcal{T})$ and $h(E) < N$, then $E \in \mathcal{L}(\mathcal{T}_0) = \mathcal{L}(\mathcal{T}'_0)$ and has no descendants in $\mathcal{T}'$. Hence, $E \in \mathcal{L}(\mathcal{T}')$. Moreover, $o_{\mathcal{T}'}(E) = o_{\mathcal{T}'_0}(E) = N\lceil o_{\mathcal{T}_0}(E)/N \rceil = N\lceil o_{\mathcal{T}}(E)/N \rceil$. If $E \in \mathcal{L}(\mathcal{T})$ and $h(E) \geq N$, then $E \subseteq F$ for some $F \in \mathcal{T}$ at height N. Hence, $E \in \mathcal{L}(\mathcal{T}_F) = \mathcal{L}(\mathcal{T}'_F) \subseteq \mathcal{L}(\mathcal{T}')$ and

$$\begin{aligned} o_{\mathcal{T}'}(E) &= N + o_{\mathcal{T}'_F}(E) = N + N\lceil o_{\mathcal{T}_F}(E)/N \rceil \\ &= N\left\lceil \frac{o_{\mathcal{T}_F}(E) + N}{N} \right\rceil = N\lceil o_{\mathcal{T}}(E)/N \rceil. \end{aligned}$$

Conversely, suppose that $E \in \mathcal{L}(\mathcal{T}')$. Then either $E \in \mathcal{L}(\mathcal{T}'_0) = \mathcal{L}(\mathcal{T}_0)$ with $h(E) < N$ (taken in $\mathcal{T}_0$) or else $E \in \mathcal{L}(\mathcal{T}'_F)$ for some $F \in \mathcal{T}$ at height N. Thus, $E \in \mathcal{L}(\mathcal{T}_F)$. In either case, $E \in \mathcal{L}(\mathcal{T})$. □

Combining Lemmas 3 and 5, we obtain:

PROPOSITION 6. *Let $\mathcal{T}$ be an $(\mathcal{S}_n)_{n=1}^\infty$-admissible (-allowable) tree $\mathcal{T}$ and let $N \in \mathbb{N}$. Then there exists an $\mathcal{S}_N$-admissible (-allowable) tree $\mathcal{T}'$ with the same root such that $\mathcal{L}(\mathcal{T}) = \mathcal{L}(\mathcal{T}')$ and $o_{\mathcal{T}'}(E) = N\lceil o_{\mathcal{T}}(E)/N \rceil$ for all $E \in \mathcal{L}(\mathcal{T})$.*

PROPOSITION 7. *Let $(\theta_n)_{n=1}^\infty$ be a regular sequence. Suppose that there exists $N \in \mathbb{N}$ such that $\theta_N^{1/N} = \theta = \sup \theta_n^{1/n}$, then the spaces X, X_M, Y, and Y_M*

are pairwise isomorphic via the formal identity, where Y and Y_M denote the spaces $T[\mathcal{S}_1,\theta]$ and $T_M[\mathcal{S}_1,\theta]$, respectively.

Proof. It is known that Y and Y_M are isomorphic via the formal identity [5, 6, 12]. We shall show that X_M is isomorphic to Y_M via the formal identity. The proof that X is isomorphic to Y via the formal identity is similar. Let x be a finitely supported vector. There exists an $(\mathcal{S}_n)_{n=1}^{\infty}$-allowable tree $\mathcal{T}$ such that

$$\|x\|_{X_M} = \sum_{E\in\mathcal{L}(\mathcal{T})} t(E)\|Ex\|_{c_0}.$$

By Proposition 6, there exists an $\mathcal{S}_1$-allowable tree $\mathcal{T}'$ with the same root such that $\mathcal{L}(\mathcal{T}) = \mathcal{L}(\mathcal{T}')$ and $o_{\mathcal{T}'}(E) = N\lceil o_{\mathcal{T}}(E)/N\rceil$ for all $E\in\mathcal{L}(\mathcal{T})$. If $E\in\mathcal{L}(\mathcal{T})$ and $t(E) = \theta_{n_1}\cdots\theta_{n_j}$, then

$$t(E) \le \theta^{n_1}\cdots\theta^{n_j} = \theta^{n_1+\cdots+n_j} = \theta^{o_{\mathcal{T}}(E)} < \theta^{-N}\theta^{o_{\mathcal{T}'}(E)}.$$

Therefore,

$$\begin{aligned}\|x\|_{X_M} &= \sum_{E\in\mathcal{L}(\mathcal{T})} t(E)\|Ex\|_{c_0}\\ &< \theta^{-N}\sum_{E\in\mathcal{L}(\mathcal{T}')} \theta^{o_{\mathcal{T}'}(E)}\|Ex\|_{c_0} \le \theta^{-N}\|x\|_{Y_M}.\end{aligned}$$

Conversely, choose an $\mathcal{S}_1$-allowable tree $\mathcal{T}''$ such that

$$\|x\|_{Y_M} = \sum_{E\in\mathcal{L}(\mathcal{T}'')} t(E)\|Ex\|_{c_0}.$$

Since $\mathcal{T}''$ is also $(S_n)_{n=1}^{\infty}$-allowable, there exists an S_N-allowable tree $\mathcal{T}'''$ such that $\mathcal{L}(\mathcal{T}'') = \mathcal{L}(\mathcal{T}''')$ and $o_{\mathcal{T}'''}(E) = N\lceil o_{\mathcal{T}''}(E)/N\rceil$ for all $E\in\mathcal{L}(\mathcal{T}'')$. Hence, $t(E) = \theta^{o_{\mathcal{T}''}(E)} \le \theta^{-N+o_{\mathcal{T}'''}(E)}$. Thus,

$$\|x\|_{Y_M} \le \theta^{-N}\sum_{E\in\mathcal{L}(\mathcal{T}''')} \theta_N^{o_{\mathcal{T}'''}(E)/N}\|Ex\|_{c_0} \le \|x\|_{X_M}.$$

The final inequality holds since $\mathcal{T}'''$ is also $(S_n)_{n=1}^{\infty}$-allowable and the tag of E in $\mathcal{T}'''$ is $\theta_N^{o_{\mathcal{T}'''}(E)/N}$. □

4. Main construction

The main aim of the present paper is to show that the spaces X and X_M are not isomorphic for a large class of regular sequences (θ_n). In view of Proposition 2, it suffices to show that the norms $\|\cdot\|$ and $\|\cdot\|_{X_M}$ are not equivalent on $\operatorname{span}\{e_k : k\in V\}$ for any $V\in[\mathbb{N}]$. Our strategy is to construct, for any $V\in[\mathbb{N}]$, vectors $x\in\operatorname{span}\{e_k : k\in V\}$ where the ratio $\|x\|_{X_M}/\|x\|$ can be made arbitrarily large. The basic units of the construction are the *repeated averages* due to Argyros, Mercourakis, and Tsarpalias [4]. These are then layered together, where each layer consists of repeated averages whose complexities go through a cycle. This variation *within* a layer is the main feature that distinguishes the present construction from related previous constructions that are used in, e.g., [3, 14]. The reason for layered construction of vectors is to dictate that the norming trees that approximately norm the given vector must structurally resemble the vector itself. In the presence of a condition such as (†), one may exploit the large ratio between θ_{m+n} and $\theta_m\theta_n$ to ensure that different layers behave differently. In the absence of such a condition, one must find a way to "lock in" the behavior of the norming tree on the given vector. Our idea is to make the vector cycle through different complexities within each layer so that the norming tree is forced to follow these ups and downs.

If $x, y \in \operatorname{span}\{(e_k)\}$, we define $x < y$, respectively, $x \subseteq y$, to mean $\operatorname{supp} x < \operatorname{supp} y$ and $\operatorname{supp} x \subseteq \operatorname{supp} y$, respectively. We shall also say that $E \subseteq x$ if $E \in [\mathbb{N}]^{<\infty}$ and $E \subseteq \operatorname{supp} x$. An $\mathcal{S}_0$-repeated average is a vector e_k for some $k \in \mathbb{N}$. For any $p \in \mathbb{N}$, an $\mathcal{S}_p$-repeated average is a vector of the form $\frac{1}{k}\sum_{i=1}^{k} x_i$, where $x_1 < \cdots < x_k$ are repeated $\mathcal{S}_{p-1}$-repeated averages and $k = \min\operatorname{supp} x_1$. Observe that any $\mathcal{S}_p$-repeated average x is a convex combination of $\{e_k : k \in \operatorname{supp} x\}$ such that $\|x\|_\infty \le (\min\operatorname{supp} x)^{-1}$ and $\operatorname{supp} x \in \mathcal{S}_p$.

Let $(\theta_n)_{n=1}^{\infty}$ be a given regular decreasing sequence that satisfies the following:

(¬†) $\lim_m \delta_m = 0$, where $\delta_m = \limsup_n \frac{\theta_{m+n}}{\theta_n}$.

(‡) There exists $F : \mathbb{N} \to \mathbb{R}$ with $\lim_{n\to\infty} F(n) = 0$ such that for all $R, t \in \mathbb{N}$ and any arithmetic progression $(s_i)_{i=1}^{R}$ in $\mathbb{N}$,

$$\max_{1\le i\le R} \frac{\theta_{s_i+t}}{\theta_{s_i}} \le F(R)\sum_{i=1}^{R} \frac{\theta_{s_i+t}}{\theta_{s_i}}.$$

Recall from Section 2 that X and X_M are known to be nonisomorphic if condition (†) holds. The condition (‡) is imposed to make the construction work. As we shall see, it is general enough to include many interesting cases.

From here on fix $N \in \mathbb{N}$ and $V \in [\mathbb{N}]$ arbitrarily. Choose sequences $(p_k)_{k=1}^{N}$ and $(L_k)_{k=1}^{N}$ in $\mathbb{N}$, $L_k \ge 2$, that satisfy the following conditions:

(A) $\frac{\theta_{p_{M+1}+n}}{\theta_n} \le \frac{\theta_1}{24N^2}\prod_{i=1}^{M}\theta_{L_i p_i}$ if $0 \le M \le N-2$ and $n \ge p_N$ (the vacuous product $\prod_{i=1}^{0}\theta_{L_i p_i}$ is taken to be 1),

(B) $p_{M+1} > \sum_{i=1}^{M} L_i p_i$ if $0 < M \le N-2$,

(C) $F(L_{M+1}) \le \frac{\theta_1}{144N^2}\prod_{i=1}^{M}\theta_{L_i p_i}$ if $0 < M \le N-2$.

Note that condition (A) may be realized because of (¬†) and condition (C) by way of (‡). Given $k \in \mathbb{N}$ and $1 \le M \le N$, define $r_M(k)$ to be the integer in $\{1, 2, \ldots, L_M\}$ such that $L_M | (k - r_M(k))$. We can construct sequences of vectors $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^N$ with the following properties.

(α) $\mathbf{x}^0$ is a subsequence of $(e_k)_{k\in V}$.

(β) Say $\mathbf{x}^M=(x^M_j)$ and $m_j=\min\operatorname{supp}x^M_j$. Then there is a sequence (I^{M+1}_k) of integer intervals such that $I^{M+1}_k<I^{M+1}_{k+1}$, $\bigcup_{k=1}^\infty I^{M+1}_k=\mathbb{N}$ and each vector $x^{M+1}_k\in\mathbf{x}^{M+1}$ is of the form

$$x^{M+1}_k=\sum_{j\in I^{M+1}_k}a_jx^M_j,$$

where $\theta_{r_{M+1}(k)p_{M+1}}\sum_{j\in I^{M+1}_k}a_je_{m_j}$ is an $\mathcal{S}_{r_{M+1}(k)p_{M+1}}$-repeated average. Moreover, the sequence $(a_j)_{j=1}^\infty$ is decreasing.

Each x^{M+1}_k is made up of components of diverse complexities. In order to estimate its $\|\cdot\|$- and $\|\cdot\|_{X_M}$- norms, we decompose x^{M+1}_k into components of pure forms in the following manner. The coefficients (a_j) are as given in (β).

Notation. Given $1\le r_i\le L_i$, $1\le M\le N-1$, write

$$x^{M+1}_k(r_M)=\sum_{\substack{j\in I^{M+1}_k\\ r_M(j)=r_M}}a_jx^M_j.$$

For $1\le s<M$, define

$$x^{M+1}_k(r_s,\dots,r_M)=\sum_{\substack{j\in I^{M+1}_k\\ r_M(j)\ =r_M}}a_jx^M_j(r_s,\dots,r_{M-1}).$$

If $1\le s\le M$, it is clear that $x^{M+1}_k=\sum x^{M+1}_k(r_s,\dots,r_M)$, where the sum is taken over all possible values of $r_s,\dots,r_M$.

Given a sequence $\mathbf{u}=(u_1,u_2,...)$ of linearly independent vectors, write $[y]_{\mathbf{u}}=(a_k)$ if $y=\sum a_ku_k$. For instance, $\|[x^{M+1}_k]_{\mathbf{x}^M}\|_{\ell^1}=\sum_{j\in I^{M+1}_k}a_j=\theta^{-1}_{r_{M+1}(k)p_{M+1}}$. To compute $\|[x^{M+1}_k]_{\mathbf{x}^{s-1}}\|_{\ell^1}, 1\le s\le M$, calculate the ℓ^1-norms of each of the pure forms $[x^{M+1}_k(r_s,\dots,r_M)]_{\mathbf{x}^{s-1}}$ and sum over all $r_s,\dots,r_M$.

The following simple lemma is useful for our computations. A subset I of $\mathbb{N}$ is said to be L-*skipped* if $|i-j|\ge L$ whenever i and j are distinct elements of I.

Lemma 8. *If (a_i) is a nonnegative decreasing sequence defined on an interval J in $\mathbb{N}$ and I is an L-skipped set, then*

$$\sum_{i\in I}a_i\le\frac{1}{L}\sum a_i+\sup a_i.$$

Moreover, if there exists r such that $I = \{i \in J : i = r \mod L\}$, then

$$\frac{1}{L}\sum a_i - \sup a_i \le \sum_{i\in I} a_i.$$

PROPOSITION 9. *If $1 \le s \le M < N$ and $k \in \mathbb{N}$, then*

$$\prod_{i=s}^{M}(L_i^{-1} - k^{-1}) \le \frac{\|[x_k^{M+1}(r_s,\dots,r_M)]_{\mathbf{x}^{s-1}}\|_{\ell^1}}{\theta^{-1}_{r_{M+1}(k)p_{M+1}}\prod_{i=s}^{M}\theta^{-1}_{r_ip_i}} \le \prod_{i=s}^{M}(L_i^{-1} + k^{-1}).$$

Proof. The proof is by induction on M. When $M = s$,

$$\begin{aligned}
&\|[x_k^{M+1}(r_s,\dots,r_M)]_{\mathbf{x}^{s-1}}\|_{\ell^1}\\
&\quad= \|[x_k^{M+1}(r_M)]_{\mathbf{x}^{M-1}}\|_{\ell^1}\\
&\quad= \Big\|\Big[\sum_{\substack{j\in I_k^{M+1}\\ r_M(j)=r_M}} a_j x_j^M\Big]_{\mathbf{x}^{M-1}}\Big\|_{\ell^1}\\
&\quad= \sum_{\substack{j\in I_k^{M+1}\\ r_M(j)=r_M}} a_j\|[x_j^M]_{\mathbf{x}^{M-1}}\|_{\ell^1} = \theta^{-1}_{r_Mp_M}\sum_{\substack{j\in I_k^{M+1}\\ r_M(j)=r_M}} a_j.
\end{aligned}$$

Note that $\{j \in I_k^{M+1} : r_M(j) = r_M\}$ is an L_M-skipped subset of the integer interval I_k^{M+1}. It follows from Lemma 8 that

$$\begin{aligned}
\sum_{\substack{j\in I_k^{M+1}\\ r_M(j)=r_M}} a_j &\le \frac{1}{L_M}\sum_{j\in I_k^{M+1}} a_j + \sup a_j\\
&\le (L_M^{-1} + k^{-1})\theta^{-1}_{r_{M+1}(k)p_{M+1}}.
\end{aligned}$$

Therefore, $\|[x_k^{M+1}(r_M)]_{\mathbf{x}^{M-1}}\|_{\ell^1} \le \theta^{-1}_{r_{M+1}(k)p_{M+1}}\theta^{-1}_{r_Mp_M}(L_M^{-1} + k^{-1})$.

Suppose that the proposition holds for $M - 1$. Then

$$\begin{aligned}
&\|[x_k^{M+1}(r_s,\dots,r_M)]_{\mathbf{x}^{s-1}}\|_{\ell^1}\\
&\quad= \Big\|\sum_{\substack{j\in I_k^{M+1}\\ r_M(j)=r_M}} a_j[x_j^M(r_s,\dots,r_{M-1})]_{\mathbf{x}^{s-1}}\Big\|_{\ell^1}\\
&\quad= \sum_{\substack{j\in I_k^{M+1}\\ r_M(j)=r_M}} a_j\|[x_j^M(r_s,\dots,r_{M-1})]_{\mathbf{x}^{s-1}}\|_{\ell^1}\\
&\quad\le \prod_{i=s}^{M-1}\theta^{-1}_{r_ip_i}(L_i^{-1} + k^{-1})\cdot\sum_{\substack{j\in I_k^{M+1}\\ r_M(j)=r_M}}\frac{a_j}{\theta_{r_M(j)p_M}}\\
&\qquad\text{by the inductive hypothesis}
\end{aligned}$$

$$= \prod_{i=s}^{M-1} \theta_{r_i p_i}^{-1}(L_i^{-1}+k^{-1}) \cdot \theta_{r_M p_M}^{-1} \sum_{\substack{j\in I_k^{M+1}\\ r_M(j)=r_M}} a_j$$

$$\leq \prod_{i=s}^{M-1} \theta_{r_i p_i}^{-1}(L_i^{-1}+k^{-1}) \cdot \theta_{r_M p_M}^{-1} \left(\frac{1}{L_M}\sum_{j\in I_k^{M+1}} a_j + \sup a_j\right)$$

by Lemma 8

$$\leq \prod_{i=s}^{M-1} \theta_{r_i p_i}^{-1}(L_i^{-1}+k^{-1})\theta_{r_M p_M}^{-1}\theta_{r_{M+1}(k)p_{M+1}}^{-1}(L_M^{-1}+k^{-1})$$

$$= \theta_{r_{M+1}(k)p_{M+1}}^{-1}\prod_{i=s}^{M}\theta_{r_i p_i}^{-1}(L_i^{-1}+k^{-1}).$$

The other inequality is proved similarly. □

From this point onward, we shall only consider those k's that satisfy

$$k \geq 42N^2 \prod_{i=1}^{N} L_i \theta_{L_i p_i}^{-1}. \tag{4}$$

It follows from the choice of k that for all $1 \leq s \leq M \leq N$,

$$\prod_{i=s}^{M}(L_i^{-1}+k^{-1}) \leq 2\prod_{i=s}^{M} L_i^{-1}. \tag{5}$$

Indeed, since $L_i^{-1}+k^{-1} \leq (1+\frac{1}{42N})L_i^{-1}$ for all i, we have

$$\prod_{i=s}^{M}(L_i^{-1}+k^{-1}) \leq \left(1+\frac{1}{42N}\right)^N \prod_{i=s}^{M} L_i^{-1}$$

$$\leq e^{1/42}\prod_{i=s}^{M} L_i^{-1} < 2\prod_{i=s}^{M} L_i^{-1}.$$

Likewise, for all $1 \leq s \leq M \leq N$,

$$\prod_{i=s}^{M}(L_i^{-1}-k^{-1}) > \frac{1}{2}\prod_{i=s}^{M} L_i^{-1}. \tag{6}$$

COROLLARY 10. *If $1 \leq s \leq M < N$ and k satisfies (4), then*

$$\frac{1}{2} \leq \frac{\|[x_k^{M+1}(r_s,\dots,r_M)]_{\mathbf{x}^{s-1}}\|_{\ell^1}}{\theta_{r_{M+1}(k)p_{M+1}}^{-1}\prod_{i=s}^{M}\theta_{r_i p_i}^{-1}L_i^{-1}} \leq 2.$$

COROLLARY 11. *If k satisfies* (4) *and* $1 \le M \le N$, *then*

$$\|x_k^M\|_{\ell^1} \le 2 \prod_{i=1}^{M} \theta_{L_i p_i}^{-1}.$$

Proof. If $M = 1$, then

$$\|x_k^1\|_{\ell^1} = \|[x_k^1]_{\mathbf{x}^0}\|_{\ell^1} = \theta_{r_1(k)p_1}^{-1} \le \theta_{L_1 p_1}^{-1}.$$

If $M \ge 2$, according to Corollary 10,

$$\begin{aligned}\|x_k^M\|_{\ell^1} &= \sum_{r_1,\dots,r_{M-1}} \|[x_k^M(r_1,\dots,r_{M-1})]_{\mathbf{x}^0}\|_{\ell^1} \\ &\le 2\theta_{r_M(k)p_M}^{-1} \sum_{r_1,\dots,r_{M-1}} \prod_{i=1}^{M-1} \theta_{r_i p_i}^{-1} L_i^{-1} \\ &\le 2\theta_{L_M p_M}^{-1} \prod_{i=1}^{M-1} \theta_{L_i p_i}^{-1} = 2 \prod_{i=1}^{M} \theta_{L_i p_i}^{-1}. \quad \square\end{aligned}$$

We shall employ the same decomposition technique to estimate $\|x_k^N\|_{X_M}$. To simplify notation, let $p(r_M,\dots,r_{M'}) = \sum_{i=M}^{M'} p_i r_i$ if $M \le M'$.

PROPOSITION 12. *If k satisfies* (4), *then*

$$\|x_k^N\|_{X_M} \ge \frac{\theta_1}{2} \sum_{r_1,\dots,r_{N-1}} \theta_{p(r_1,\dots,r_{N-1},r_N(k))} \theta_{r_N(k)p_N}^{-1} \prod_{i=1}^{N-1} \theta_{r_i p_i}^{-1} L_i^{-1}.$$

Proof. We first decompose x_k^N into a sum of pure forms, i.e.,

$$x_k^N = \sum_{r_1,\dots,r_{N-1}} x_k^N(r_1,\dots,r_{N-1}).$$

Now given $r_1,\dots,r_{N-1}$, $\operatorname{supp} x_k^N(r_1,\dots,r_{N-1}) \in \mathcal{S}_{p(r_1,\dots,r_{N-1},r_N(k))}$. Hence,

$$\begin{aligned}\|x_k^N(r_1,\dots,r_{N-1})\|_{X_M} &\ge \theta_{p(r_1,\dots,r_{N-1},r_N(k))} \|x_k^N(r_1,\dots,r_{N-1})\|_{\ell^1} \\ &\ge \frac{\theta_{p(r_1,\dots,r_{N-1},r_N(k))}}{2} \theta_{r_N(k)p_N}^{-1} \prod_{i=1}^{N-1} \theta_{r_i p_i}^{-1} L_i^{-1}\end{aligned}$$

by Corollary 11. Since $k \ge \prod_{i=1}^{N-1} L_i$ by (4), $S \in \mathcal{S}_1$ whenever $S \subseteq \mathbb{N}$ satisfies $k \le \min S$ and $|S| \le \prod_{i=1}^{N-1} L_i$. In particular,

$$\{\operatorname{supp} x_k^N(r_1,\dots,r_{N-1}) : 1 \le r_i \le L_i,\ 1 \le i \le N-1\}$$

is $\mathcal{S}_1$-allowable. Thus,

$$\begin{aligned}\|x_k^N\|_{X_M} &\geq \theta_1 \sum_{r_1,\dots,r_{N-1}} \|x_k^N(r_1,\dots,r_{N-1})\|_{X_M} \\ &\geq \frac{\theta_1}{2} \sum_{r_1,\dots,r_{N-1}} \theta_{p(r_1,\dots,r_{N-1},r_N(k))} \theta^{-1}_{r_N(k)p_N} \prod_{i=1}^{N-1} \theta^{-1}_{r_ip_i} L_i^{-1},\end{aligned}$$

as required. □

The following estimate is easily obtainable from Proposition 12.

COROLLARY 13. *If* $0 \leq M < N-1$ *and* k *satisfies* (4)*, then*

$$\|x_k^N\|_{X_M} \geq \frac{\theta_1}{2} \sum_{r_{M+1},\dots,r_{N-1}} \theta_{p(r_{M+1},\dots,r_{N-1},r_N(k))} \theta^{-1}_{r_N(k)p_N} \prod_{i=M+1}^{N-1} \theta^{-1}_{r_ip_i} L_i^{-1}. \tag{7}$$

Proof. By Proposition 12 and the regularity of (θ_n),

$$\begin{aligned}\|x_k^N\|_{X_M} &\geq \frac{\theta_1}{2} \sum_{r_1,\dots,r_{N-1}} \theta_{p(r_1,\dots,r_N(k))} \theta^{-1}_{r_N(k)p_N} \prod_{i=1}^{N-1} \theta^{-1}_{r_ip_i} L_i^{-1} \\ &\geq \frac{\theta_1}{2} \sum_{r_1,\dots,r_{N-1}} \theta_{p(r_2,\dots,r_N(k))} \theta_{r_1p_1} \theta^{-1}_{r_N(k)p_N} \prod_{i=1}^{N-1} \theta^{-1}_{r_ip_i} L_i^{-1} \\ &= \frac{\theta_1}{2} \sum_{r_2,\dots,r_{N-1}} \theta_{p(r_2,\dots,r_N(k))} \theta^{-1}_{r_N(k)p_N} \prod_{i=2}^{N-1} \theta^{-1}_{r_ip_i} L_i^{-1}.\end{aligned}$$

Repeat the argument M times to obtain the required result. □

The main bulk of the calculations occur in estimating the X-norm of x_k^N. The next lemma is the mechanism behind one of the crucial estimates (Proposition 16). If $x \in c_{00}$ and $p \geq 0$, let $\|x\|_{\mathcal{S}_p} = \sup_{E\in\mathcal{S}_p} \|Ex\|_{\ell^1}$.

LEMMA 14. *Let* $p, q \geq 0$*, and* $P = (m_n) \in [\mathbb{N}]$ *be given. Assume that* $G_1 < G_2 < \cdots$ *is a sequence in* $[P]^{<\infty}$ *such that* $\sum_{m_n\in G_i} a_n e_{m_n}$ *is an* $\mathcal{S}_q$*-repeated average for all* i *and that there exists* $Q = (m_{n_k}) \in [P]$ *so that for each* k*, there is a vector* z_k *satisfying:*

(1) $\operatorname{supp} z_k \subseteq [m_{n_k}, m_{n_k+1})$,
(2) $\|z_k\|_{\ell^1} \leq 1$,
(3) $\|\sum_{k=1}^j z_k\|_{\mathcal{S}_p} \leq 6$ *for all* $j \in \mathbb{N}$.

Set $y_i = \sum_{m_{n_k}\in G_i} a_{n_k} z_k$*. Then*

(i) $\|\sum_{i=1}^j y_i\|_{\mathcal{S}_{p+q}} \leq 6$ *for all* $j \in \mathbb{N}$,
(ii) $\|y_i\|_{\mathcal{S}_{p+q-1}} \leq 6/m$ *if* $q \geq 1$*, where* $m = \min G_i$.

Proof. We first establish (i). The proof is by induction on q. The case $q=0$ is trivial. Assume the result holds for some q, we shall prove it for $q+1$. If $G_1 < G_2 < \cdots$ is a sequence in $[P]^{<\infty}$ such that $\sum_{m_n \in G_i} a_n e_{m_n}$ is an $\mathcal{S}_{q+1}$-repeated average for all i, then each of these $\mathcal{S}_{q+1}$-repeated averages can be written as $\frac{1}{m_{n(i)}} \sum_{t \in H_i} \sum_{m_n \in F_t} b_n e_{m_n}$, where $m_{n(i)} = \min G_i = |H_i|$, $F_t < F_{t'}$ if $t < t'$ and $\sum_{m_n \in F_t} b_n e_{m_n}$ is an $\mathcal{S}_q$-repeated average for all t. Let $y_i = \sum_{m_{n_k} \in G_i} a_{n_k} z_k$. Then $y_i = \frac{1}{m_{n(i)}} \sum_{t \in H_i} v_t$, where $v_t = \sum_{m_{n_k} \in F_t} b_{n_k} z_k$. Given a set $J \in \mathcal{S}_{p+q+1}$, write $J = \bigcup_{l=1}^{s} J_l$, $J_1 < \cdots < J_s$, $J_l \in \mathcal{S}_{p+q}$, $s \leq \min J$. Note that by induction, $\|J_l(\sum_{t \in H_i} v_t)\|_{\ell^1} \leq 6$ for all l and i. Hence, $\|J_l y_i\|_{\ell^1} \leq \frac{6}{m_{n(i)}}$. Let i_0 be the smallest number such that $J \cap \operatorname{supp} z_k \neq \emptyset$ for some $m_{n_k} \in H_{i_0}$. For any j,

$$\begin{aligned}
\left\| J\left(\sum_{i=1}^{j} y_i\right)\right\|_{\ell^1} &\leq \sum_{i=i_0}^{i_0+2} \|y_i\|_{\ell^1} + \sum_{l} \sum_{i=i_0+3}^{\infty} \|J_l y_i\|_{\ell^1} \\
&\leq 3 + \sum_{l} \sum_{i=i_0+3}^{\infty} \frac{6}{m_{n(i)}} \\
&\leq 3 + \sum_{l} \frac{12}{m_{n(i_0+3)}}, \quad \text{since } m_{n(i+1)} \geq 2m_{n(i)}, \\
&= 3 + \frac{12s}{m_{n(i_0+3)}}.
\end{aligned}$$

But since $\min J < m_{n(i_0+1)}$, $s/m_{n(i_0+3)} < 1/4$. Therefore,

$$\left\| J\left(\sum_{i=1}^{j} y_i\right)\right\|_{\ell^1} < 3 + \frac{12}{4} = 6.$$

To prove (ii), note that an $\mathcal{S}_q$-repeated average $\sum_{m_n \in G_i} a_n e_{m_n}$ may be written as $m^{-1}(u_1 + \cdots + u_m)$, where $u_1 < \cdots < u_m$ are $\mathcal{S}_{q-1}$-repeated averages. If $u_j = \sum_{m_n \in F_j} b_n e_{m_n}$, then $y_i = m^{-1}(w_1 + \cdots + w_m)$, where $w_j = \sum_{m_{n_k} \in F_j} b_{n_k} z_k$. By (i), if $J \in \mathcal{S}_{p+q-1}$, then $\|J(w_1 + \cdots + w_m)\|_{\ell^1} \leq 6$. Hence, $\|J y_i\|_{\ell^1} \leq 6/m$. □

Assume that $0 \leq M < M+s \leq N$ and that $r_1, \ldots, r_{N-1}$ are given. For notational convenience, let $x_k^{M+s}(r_{M+1}, \ldots, r_{M+s-1}) = x_k^{M+s}$ if $s=1$. Taking $m_j = \min \operatorname{supp} x_j^M$, define

$$u_k^{M+s}(r_{M+1}, \ldots, r_{M+s-1}) = \sum b_j e_{m_j}$$

if $x_k^{M+s}(r_{M+1}, \ldots, r_{M+s-1}) = \sum b_j x_j^M$. (The vector is also labeled as u_k^{M+1} if $s=1$.)

Proposition 15. *Let $r_N = r_N(k)$. Then*

$$\|u_k^N(r_{M+1},\dots,r_{N-1})\|_{\mathcal{S}_{p(r_{M+1},\dots,r_N)-1}} \le \frac{6}{k}\prod_{i=M+1}^{N}\theta_{r_ip_i}^{-1}.$$

Proof. We shall apply Lemma 14 repeatedly to show that

$$\prod_{i=M+1}^{M+s}\theta_{r_ip_i}\|u_t^{M+s}(r_{M+1},\dots,r_{M+s-1})\|_{\mathcal{S}_{p(r_{M+1},\dots,r_{M+s})-1}} \le \frac{6}{t} \tag{8}$$

if $r_{M+s}(t) = r_{M+s}$ and

$$\prod_{i=M+1}^{M+s}\theta_{r_ip_i}\|\sum u_t^{M+s}(r_{M+1},\dots,r_{M+s-1})\|_{\mathcal{S}_{p(r_{M+1},\dots,r_{M+s})}} \le 6$$

for any sum over a finite set of t's satisfying $r_{M+s}(t) = r_{M+s}$. Suppose that $s = 1$. Set $p = 0$ and $q = r_{M+1}p_{M+1}$. Let $P = (m_n)$, where $m_n = \min\operatorname{supp} x_n^M$ and $Q = \bigcup_{r_{M+1}(t)=r_{M+1}}\{m_n : x_n^M \subseteq x_t^{M+1}\}$. If $r_{M+1}(t) = r_{M+1}$, let $G_t = \operatorname{supp} u_t^{M+1}$. Also, let $z_j = e_{m_{n_j}}$ if $m_{n_j} \in Q$. Note that if $r_{M+1}(t) = r_{M+1}$, $\theta_{r_{M+1}p_{M+1}}u_t^{M+1}$ is an $\mathcal{S}_q$-repeated average. By Lemma 14,

$$\|\theta_{r_{M+1}p_{M+1}}u_t^{M+1}\|_{\mathcal{S}_{r_{M+1}p_{M+1}-1}} \le \frac{6}{\min G_t} \le \frac{6}{t}$$

if $r_{M+1}(t) = r_{M+1}$ and $\|\sum\theta_{r_{M+1}p_{M+1}}u_t^{M+1}\|_{\mathcal{S}_{r_{M+1}p_{M+1}}} \le 6$ for any sum over a finite set of t's such that $r_{M+1}(t) = r_{M+1}$.

Inductively, suppose that the claim is true for some $s < N - M$. Set $p = p(r_{M+1},\dots,r_{M+s})$ and $q = r_{M+s+1}p_{M+s+1}$. Let $P = (m_n)$, where $m_n = \min\operatorname{supp} x_n^{M+s}$, and

$$Q = \bigcup_{r_{M+s+1}(t)=r_{M+s+1}}\{m_n : x_n^{M+s} \subseteq x_t^{M+s+1}, r_{M+s}(n) = r_{M+s}\}.$$

If $r_{M+s+1}(t) = r_{M+s+1}$, set $G_t = \{m_n : x_n^{M+s} \subseteq x_t^{M+s+1}\}$. Also let $z_j = \prod_{i=M+1}^{M+s}\theta_{r_ip_i}u_{n_j}^{M+s}(r_{M+1},\dots,r_{M+s-1})$ if $m_{n_j} \in Q$. Now

$$\begin{aligned}\|z_j\|_{\ell^1} &= \left\|\prod_{i=M+1}^{M+s}\theta_{r_ip_i}\cdot u_{n_j}^{M+s}(r_{M+1},\dots,r_{M+s-1})\right\|_{\ell^1}\\ &= \prod_{i=M+1}^{M+s}\theta_{r_ip_i}\cdot\|[x_{n_j}^{M+s}(r_{M+1},\dots,r_{M+s-1})]_{\mathbf{x}^M}\|_{\ell^1} \le 1\end{aligned}$$

by Corollary 10. (Note the fact that $L_i \ge 2$.) By the inductive hypothesis, $\|\sum z_j\|_{\mathcal{S}_{p(r_{M+1},\dots,r_{M+s})}} \le 6$ for any sum over a finite set of j's satisfying $r_{M+s}(n_j) = r_{M+s}$. Finally, observe that if $r_{M+s+1}(t) = r_{M+s+1}$ and

$u_t^{M+s+1} = \sum_{m_n \in G_t} c_n u_n^{M+s}$, then $\theta_{r_{M+s+1} p_{M+s+1}} \sum_{m_n \in G_t} c_n e_{m_n}$ is an $\mathcal{S}_q$-repeated average. Thus, it follows from Lemma 14 that

$$\left\| \prod_{i=M+1}^{M+s+1} \theta_{r_i p_i} \sum u_t^{M+s+1}(r_{M+1}, \dots, r_{M+s}) \right\|_{\mathcal{S}_{p(r_{M+1},\dots,r_{M+s+1})}} \leq 6$$

for any sum over a finite set of t's such that $r_{M+s+1}(t) = r_{M+s+1}$ and

$$\left\| \prod_{i=M+1}^{M+s+1} \theta_{r_i p_i} u_t^{M+s+1}(r_{M+1}, \dots, r_{M+s}) \right\|_{\mathcal{S}_{p(r_{M+1},\dots,r_{M+s+1})}} \leq \frac{6}{t}$$

if $r_{M+s+1}(t) = r_{M+s+1}$. This completes the induction. The proposition follows by taking $M + s = N$ and $t = k$ in (8). □

Let $\mathcal{T}$ be an admissible tree and suppose that $0 \leq M \leq N-2$. Say that a collection of nodes $\mathcal{E}$ in $\mathcal{T}$ *is subordinated to* $\mathbf{x}^M$ if they are pairwise disjoint and for each $E \in \mathcal{E}$, there exists j such that $E \subseteq x_j^M$. Note that in this case, for every $E \in \mathcal{E}$, there exist unique $r_{M+1}, \dots, r_{N-1}$ such that $E \subseteq x_k^N(r_{M+1}, \dots, r_{N-1})$. Recall the assumption (4) on k. Note that if $x_j^M \subseteq x_k^N$, then $j \geq k$, and hence j also satisfies (4) in place of k.

Proposition 16. *If $\mathcal{E}$ is a collection of nodes in an admissible tree that is subordinated to $\mathbf{x}^M$ and that $o(E) < p(r_{M+1}, \dots, r_N(k))$ for all $E \in \mathcal{E}$ with $E \subseteq x_k^N(r_{M+1}, \dots, r_{N-1})$, then*

$$\sum_{E \in \mathcal{E}} t(E) \|E x_k^N\| \leq \frac{1}{3N^2}.$$

Proof. Let $\mathcal{E}(r_{M+1}, \dots, r_{N-1})$ be the set of all nodes in $\mathcal{E}$ such that $E \subseteq x_k^N(r_{M+1}, \dots, r_{N-1})$. We have

$$\begin{aligned} &\sum_{E \in \mathcal{E}(r_{M+1},\dots,r_{N-1})} t(E) \|E x_k^N(r_{M+1}, \dots, r_{N-1})\| \\ &\qquad \leq \sum_{j \in G} b_j \|x_j^M\| \leq \sup_{j \in G} \|x_j^M\|_{\ell^1} \sum_{j \in G} b_j, \end{aligned}$$

where $(b_j) = [x_k^N(r_{M+1}, \dots, r_{N-1})]_{\mathbf{x}^M}$ and G consists of all j's such that there exists $E \in \mathcal{E}(r_{M+1}, \dots, r_{N-1})$ with $E \subseteq x_j^M$. Then

$$\{\min \operatorname{supp} x_j^M : j \in G \setminus \{\min G\}\}$$

is a spreading of a subset of $\{\min E : E \in \mathcal{E}\}$. By Lemma 4, $(x_j^M)_{j \in G \setminus \{\min G\}}$ is $\mathcal{S}_{p(r_{M+1},\dots,r_N(k))-1}$-admissible. Thus,

$$\sum_{j \in G \setminus \{\min G\}} b_j \leq \|u_k^N(r_{M+1}, \dots, r_{N-1})\|_{\mathcal{S}_{p(r_{M+1},\dots,r_N(k))-1}}.$$

It follows from Proposition 15 that

$$\sum_{j\in G} b_j \leq \frac{6}{k}\prod_{i=M+1}^{N}\theta_{r_ip_i}^{-1} + \sup_j b_j \leq \frac{7}{k}\prod_{i=M+1}^{N}\theta_{r_ip_i}^{-1}.$$

Hence, using Corollary 11,

$$\begin{aligned}\sum_{E\in\mathcal{E}(r_{M+1},\dots,r_{N-1})} & t(E)\|Ex_k^N(r_{M+1},\dots,r_{N-1})\| \\ &\leq \sup_{j\in G}\|x_j^M\|_{\ell^1}\frac{7}{k}\prod_{i=M+1}^{N}\theta_{r_ip_i}^{-1} \\ &\leq 2\prod_{i=1}^{M}\theta_{L_ip_i}^{-1}\cdot\frac{7}{k}\prod_{i=M+1}^{N}\theta_{r_ip_i}^{-1} \\ &\leq \frac{14}{k}\prod_{i=1}^{N}\theta_{L_ip_i}^{-1}.\end{aligned}$$

Summing over all possible $r_{M+1},\dots,r_{N-1}$, we obtain

$$\sum_{E\in\mathcal{E}} t(E)\|Ex_k^N\| \leq \frac{14}{k}\prod_{i=1}^{N}L_i\theta_{L_ip_i}^{-1} \leq \frac{1}{3N^2}$$

by (4). □

Next, consider a set of nodes $\mathcal{E}'$ in $\mathcal{T}$ that is subordinated to $\mathbf{x}^M$ and that

$$o(E) \geq p\big(r_{M+1}+1, r_{M+2},\dots,r_N(k)\big)$$

for all $E\in\mathcal{E}'$ with $E\subseteq x_k^N(r_{M+1},\dots,r_{N-1})$. In analogy to the above, for given $r_{M+1},\dots,r_{N-1}$, let $\mathcal{E}'(r_{M+1},\dots,r_{N-1})$ be the set of all nodes in $\mathcal{E}'$ such that $E\subseteq x_k^N(r_{M+1},\dots,r_{N-1})$.

Proposition 17. $\sum_{E\in\mathcal{E}'} t(E)\|Ex_k^N\| \leq \frac{1}{3N^2}\|x_k^N\|_{X_M}$.

Proof. We have

$$\begin{aligned}\sum_{E\in\mathcal{E}'(r_{M+1},\dots,r_{N-1})} & t(E)\|Ex_k^N(r_{M+1},\dots,r_{N-1})\| \\ &\leq \theta_{p(r_{M+1}+1,\dots,r_N(k))}\sum_{j\in G'} a_j\|x_j^M\| \\ &\leq \theta_{p(r_{M+1}+1,\dots,r_N(k))}\sup_{j\in G'}\|x_j^M\|_{\ell^1}\sum_{j\in G'} a_j,\end{aligned}$$

where $(a_j) = [x_k^N(r_{M+1},\dots,r_{N-1})]_{\mathbf{x}^M}$ and G' consists of all j's such that there exists $E \in \mathcal{E}'(r_{M+1},\dots,r_{N-1})$ with $E \subseteq x_j^M$. But

$$\sum_{j\in G'} a_j \le \|[x_k^N(r_{M+1},\dots,r_{N-1})]_{\mathbf{x}^M}\|_{\ell^1}$$
$$\le 2\theta^{-1}_{r_N(k)p_N}\prod_{i=M+1}^{N-1}\theta^{-1}_{r_ip_i}L_i^{-1} \quad \text{by Corollary 10.}$$

Applying Corollary 11 to the above, we have

$$\sum_{E\in\mathcal{E}'(r_{M+1},\dots,r_{N-1})} t(E)\|Ex_k^N(r_{M+1},\dots,r_{N-1})\| \tag{9}$$
$$\le 4\theta_{p(r_{M+1}+1,\dots,r_N(k))}\theta^{-1}_{r_N(k)p_N}\prod_{i=1}^{M}\theta^{-1}_{L_ip_i}\prod_{i=M+1}^{N-1}\theta^{-1}_{r_ip_i}L_i^{-1}.$$

Recall the lower estimate for $\|x_k^N\|_{X_M}$ given by (7) in Corollary 13. For fixed $r_{M+1},\dots,r_{N-1}$, the ratio of (9) with the $(r_{M+1},\dots,r_{N-1})$-indexed term in (7) is

$$\le \frac{8}{\theta_1}\frac{\theta_{p(r_{M+1}+1,\dots,r_N(k))}}{\theta_{p(r_{M+1},\dots,r_N(k))}}\prod_{i=1}^{M}\theta^{-1}_{L_ip_i}$$
$$= \frac{8}{\theta_1}\frac{\theta_{p_{M+1}+p(r_{M+1},\dots,r_N(k))}}{\theta_{p(r_{M+1},\dots,,r_N(k))}}\prod_{i=1}^{M}\theta^{-1}_{L_ip_i}$$
$$\le \frac{8}{\theta_1}\frac{\theta_1}{24N^2}\prod_{i=1}^{M}\theta_{L_ip_i}\prod_{i=1}^{M}\theta^{-1}_{L_ip_i} \quad \text{by condition (A)}$$
$$= \frac{1}{3N^2}.$$

Hence,

$$\sum_{E\in\mathcal{E}'} t(E)\|Ex_k^N\| \le \frac{1}{3N^2}\|x_k^N\|_{X_M}. \qquad \square$$

In the next two results, let $(d_j) = [x_k^N]_{\mathbf{x}^{M+1}}$. Recall the convention that $x_k^N(r_{M+2},\dots,r_{N-1}) = x_k^N$ if $M = N-2$.

LEMMA 18. *Suppose that $0 \le M \le N-2$. Given $r_{M+2},\dots,r_{N-1}$, write*

$$K = \{j : x_j^{M+1} \subseteq x_k^N(r_{M+2},\dots,r_{N-1})\}.$$

If J is an L_{M+1}-skipped set, then

$$\sum_{j\in J\cap K} d_j \le (L^{-1}_{M+1} + k^{-1})\sum_{j\in K} d_j \le \frac{3}{2}L^{-1}_{M+1}\sum_{j\in K} d_j. \tag{10}$$

Proof. The second inequality follows from the choice of k since $k \geq 2L_{M+1}$ by (4). Recall the notation from (β) expressing

$$x_i^{M+2} = \sum_{j \in I_i^{M+2}} a_j x_j^{M+1}.$$

For each i such that $x_i^{M+2} \subseteq x_k^N$, let $J_i = J \cap I_i^{M+2}$. Then J_i is an L_{M+1}-skipped subset of the integer interval I_i^{M+2}. By Lemma 8,

$$\begin{aligned}\sum_{j \in J_i} d_j &\leq L_{M+1}^{-1} \sum_{j \in I_i^{M+2}} d_j + \sup_{j \in I_i^{M+2}} d_j \\ &= L_{M+1}^{-1} \sum_{j \in I_i^{M+2}} d_j + d_{\min I_i^{M+2}}.\end{aligned}$$

Now $J \cap K = \bigcup_{i \in K'} J_i$, where $K' = \{i : x_i^{M+2} \subseteq x_k^N(r_{M+2}, \ldots, r_{N-1})\}$. Thus,

$$\begin{aligned}\sum_{j \in J \cap K} d_j &\leq L_{M+1}^{-1} \sum_{i \in K'} \sum_{j \in I_i^{M+2}} d_j + \sum_{i \in K'} d_{\min I_i^{M+2}} \\ &= L_{M+1}^{-1} \sum_{j \in K} d_j + \sum_{i \in K'} d_{\min I_i^{M+2}}.\end{aligned}$$

If $[x_k^N]_{\mathbf{x}^{M+2}} = (b_i)$, then for all $j \in I_i^{M+2}$, we can express $d_j = b_i a_j$, where $\theta_{r_{M+2}(i) p_{M+2}} \sum_{j \in I_i^{M+2}} a_j e_{m_j}$ is an $\mathcal{S}_{r_{M+2}(i) p_{M+2}}$-repeated average, with $m_j = \min \operatorname{supp} x_j^{M+1}$. In particular,

$$\begin{aligned}\theta_{r_{M+2}(i) p_{M+2}} a_{j_0} &\leq i^{-1} \leq k^{-1} \\ &= k^{-1} \theta_{r_{M+2}(i) p_{M+2}} \sum_{j \in I_i^{M+2}} a_j\end{aligned}$$

for all $j_0 \in I_i^{M+2}$. Thus,

$$\begin{aligned}d_{\min I_i^{M+2}} = b_i a_{\min I_i^{M+2}} &\leq b_i k^{-1} \sum_{j \in I_i^{M+2}} a_j \\ &\leq k^{-1} \sum_{j \in I_i^{M+2}} b_i a_j = k^{-1} \sum_{j \in I_i^{M+2}} d_j.\end{aligned}$$

Therefore,

$$\sum_{i \in K'} d_{\min I_i^{M+2}} \leq k^{-1} \sum_{i \in K'} \sum_{j \in I_i^{M+2}} d_j = k^{-1} \sum_{j \in K} d_j.$$

Hence,

$$\sum_{j\in J\cap K} d_j = (L_{M+1}^{-1}+k^{-1})\sum_{j\in K} d_j. \qquad \square$$

We say that an admissible tree $\mathcal{T}$ *is subordinated to* $\mathbf{x}^M$ if its set of base nodes is subordinated to $\mathbf{x}^M$ and any leaf that is not at the base is a singleton. Given an admissible tree that is subordinated to $\mathbf{x}^M$, let $\mathcal{E}''$ be the collection of all base nodes E in $\mathcal{T}$ such that $p(r_{M+1},\dots,r_N(k)) \le o(E) < p(r_{M+1}+1,\dots,r_N(k))$ if $E \subseteq x_k^N(r_{M+1},\dots,r_{N-1})$. It follows from condition (B) that for $E\in\mathcal{E}''$, $o(E)$ uniquely determines $r_{M+1},\dots,r_{N-1}$ such that $E\subseteq x_k^N(r_{M+1},\dots,r_{N-1})$. Let $\mathcal{D}$ denote the set of all D's that are immediate predecessors of some $E\in\mathcal{E}''$. We say that D *effectively intersects* x_j^{M+1} for some j if there exists $E\in\mathcal{E}''$ such that $E\subseteq \operatorname{supp} x_j^{M+1}\cap D$. Let $\tilde{\mathcal{D}}$ be the subcollection of all $D\in\mathcal{D}$ such that D effectively intersects at least two x_j^{M+1}'s. For each $D\in\tilde{\mathcal{D}}$, let $J(D)=\{j : D \text{ effectively intersects } x_j^{M+1}\}$, then $J(D)$ is an L_{M+1}-skipped set. Indeed, if $D\in\mathcal{D}$ and E_1,E_2 are successors of D in $\mathcal{E}''$ such that $E_i\subseteq \operatorname{supp} x_{j_i}^{M+1}\cap D$, $i=1,2$, and $j_1<j_2$, then $o(E_1)=o(E_2)$, and hence $r_{M+1}(j_1)=r_{M+1}(j_2)$. Thus, $j_2-j_1\ge L_{M+1}$. Let $J=\bigcup_{D\in\tilde{\mathcal{D}}} J(D)$. If the elements of $\tilde{\mathcal{D}}$ are arranged in order, then the union of $J(D)$ taken over every other $D\in\tilde{\mathcal{D}}$ is an L_{M+1}-skipped set. Hence, J is the union of at most two L_{M+1}-skipped sets.

Proposition 19.

$$\sum_{D\in\tilde{\mathcal{D}}}\sum_{\substack{E\in\mathcal{E}''\\ E\subseteq D}} t(E)\|Ex_k^N\| \le \frac{1}{3N^2}\|x_k^N\|_{X_M}.$$

Proof. Let (d_j) be as in Lemma 18 and $g(j)=p(r_{M+1},\dots,r_N(k))$ if $x_j^{M+1}\subseteq x_k^N(r_{M+1},\dots,r_{N-1})$. Then

$$\sum_{D\in\tilde{\mathcal{D}}}\sum_{\substack{E\in\mathcal{E}''\\ E\subseteq D}} t(E)\|Ex_k^N\| \le \sum_{D\in\tilde{\mathcal{D}}}\sum_{j\in J(D)} \theta_{g(j)} d_j \|x_j^{M+1}\|_{\ell^1}.$$

But $x_j^{M+1}=\sum_{\ell\in I_j^{M+1}} a_\ell x_\ell^M$ with $\sum a_\ell = \theta^{-1}_{r_{M+1}(j)p_{M+1}}$. Hence,

$$(11)\quad \sum_{D\in\tilde{\mathcal{D}}}\sum_{\substack{E\in\mathcal{E}''\\ E\subseteq D}} t(E)\|Ex_k^N\| \le \sup_\ell \|x_\ell^M\|_{\ell^1}\sum_{D\in\tilde{\mathcal{D}}}\sum_{j\in J(D)} \theta_{g(j)} d_j \theta^{-1}_{r_{M+1}(j)p_{M+1}}$$

$$\le 2\sup_\ell \|x_\ell^M\|_{\ell^1}\sum_{j\in J}\theta_{g(j)} d_j \theta^{-1}_{r_{M+1}(j)p_{M+1}},$$

since each j belongs to at most two $J(D)$. Fix $r_{M+2},\dots,r_{N-1}$ and let K be as in Lemma 18. Since J is the union of at most two L_{M+1}-skipped sets,

$$\begin{aligned}\sum_{j\in J\cap K}\theta_{g(j)}d_j\theta^{-1}_{r_{M+1}(j)p_{M+1}} &\le \sup_{j\in K}\frac{\theta_{g(j)}}{\theta_{r_{M+1}(j)p_{M+1}}}\sum_{j\in J\cap K}d_j \\ &\le \frac{3}{L_{M+1}}\sup_{j\in K}\frac{\theta_{g(j)}}{\theta_{r_{M+1}(j)p_{M+1}}}\sum_{j\in K}d_j \quad \text{by (10).}\end{aligned}$$

However,

$$\begin{aligned}\sup_{j\in K}\frac{\theta_{g(j)}}{\theta_{r_{M+1}(j)p_{M+1}}} &\le \sup_{1\le j\le L_{M+1}}\frac{\theta_{r_{M+1}(j)p_{M+1}+p(r_{M+2},\dots,r_N(k))}}{\theta_{r_{M+1}(j)p_{M+1}}} \\ &\le F(L_{M+1})\sum_{r_{M+1}}\frac{\theta_{r_{M+1}p_{M+1}+p(r_{M+2},\dots,r_N(k))}}{\theta_{r_{M+1}p_{M+1}}}\end{aligned}$$

by condition (‡). Therefore,

$$\sum_{j\in J\cap K}\theta_{g(j)}d_j\theta^{-1}_{r_{M+1}(j)p_{M+1}} \le \frac{3F(L_{M+1})}{L_{M+1}}\sum_{r_{M+1}}\frac{\theta_{p(r_{M+1},\dots,r_N(k))}}{\theta_{r_{M+1}p_{M+1}}}\sum_{j\in K}d_j.$$

Note that

$$\sum_{j\in K}d_j = \|[x^N_k(r_{M+2},\dots,r_{N-1})]_{\mathbf{x}^{M+1}}\|_{\ell^1} \le 2\theta^{-1}_{r_N(k)p_N}\prod_{i=M+2}^{N-1}\theta^{-1}_{r_ip_i}L_i^{-1}$$

by Corollary 10. Summing over all $r_{M+2},\dots,r_{N-1}$, we have

$$\begin{aligned}(12)\qquad &\sum_{j\in J}\theta_{g(j)}d_j\theta^{-1}_{r_{M+1}(j)p_{M+1}} \\ &\le \frac{6F(L_{M+1})}{L_{M+1}}\sum_{r_{M+1},\dots,r_{N-1}}\frac{\theta_{p(r_{M+1},\dots,r_{N-1},r_N(k))}}{\theta_{r_{M+1}p_{M+1}}}\cdot\theta^{-1}_{r_N(k)p_N} \\ &\quad\times\prod_{i=M+2}^{N-1}\theta^{-1}_{r_ip_i}L_i^{-1} \\ &= 6F(L_{M+1})\sum_{r_{M+1},\dots,r_{N-1}}\theta_{p(r_{M+1},\dots,r_{N-1},r_N(k))}\theta^{-1}_{r_N(k)p_N} \\ &\quad\times\prod_{i=M+1}^{N-1}\theta^{-1}_{r_ip_i}L_i^{-1}.\end{aligned}$$

Comparing (11) and (12) with (7) in Corollary 13, we see that

$$\sum_{D\in\tilde{\mathcal{D}}}\sum_{\substack{E\in\mathcal{E}''\\ E\subseteq D}} t(E)\|Ex\| \le 24\theta_1^{-1}F(L_{M+1})\|x_k^N\|_{X_M}\sup_\ell \|x_\ell^M\|_{\ell^1}$$

$$\le 48F(L_{M+1})\theta_1^{-1}\prod_{i=1}^{M}\theta_{L_ip_i}^{-1}\|x_k^N\|_{X_M}$$

by Corollary 11,

$$\le \frac{1}{3N^2}\|x_k^N\|_{X_M}$$

by condition (C). □

Definition 20. *Given $N, p \in \mathbb{N}$ define*

$$\Theta_p = \Theta_p(N) = \max\left\{\prod_{i=1}^{N}\theta_{\ell_i} : \ell_i \in \mathbb{N}, \sum_{i=1}^{N}\ell_i = p\right\}.$$

For any $N \in \mathbb{N}$ and $V \in [\mathbb{N}]$, choose integer sequences $(p_k)_{k=1}^N$ and $(L_k)_{k=1}^N$, and sequences of vectors $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^N$ as above.

Theorem 21. *There exists a finitely supported vector $x \in \operatorname{span}\{e_k : k \in V\}$ such that*

$$\|x\| \le \left(\frac{2}{N} + 4\theta_1^{-1}\sup_{r_1,\ldots,r_{N-1}}\frac{\Theta_{p(r_1,\ldots,r_N(k))}}{\theta_{p(r_1,\ldots,r_N(k))}}\right)\|x\|_{X_M}. \tag{13}$$

Proof. Consider an admissible tree $\mathcal{T}$ that is subordinated to $\mathbf{x}^M$, $0 \le M \le N-2$. Let $\mathcal{E}$ and $\mathcal{E}'$ be the set of all base nodes such that $o(E) < p(r_{M+1},\ldots,r_N(k))$, respectively, $o(E) \ge p(r_{M+1}+1,\ldots,r_K(k))$ if $E \subseteq x_k^M(r_{M+1},\ldots,r_{N-1})$. Also, define $\mathcal{E}'', \mathcal{D}$ and $\tilde{\mathcal{D}}$ as in the discussion preceding Proposition 19. Finally, let $\mathcal{E}'''$ be the set of all leaves of $\mathcal{T}$ not at the base. By Proposition 19,

$$\sum_{E\in\mathcal{E}''} t(E)\|Ex_k^N\| = \sum_{D\in\mathcal{D}}\sum_{\substack{E\in\mathcal{E}''\\ E\subseteq D}} t(E)\|Ex_k^N\|$$

$$\le \frac{1}{3N^2}\|x_k^N\|_{X_M} + \sum_{D\in\mathcal{D}\setminus\tilde{\mathcal{D}}}\sum_{\substack{E\in\mathcal{E}''\\ E\subseteq D}} t(E)\|Ex_k^N\|.$$

If $D \in \mathcal{D}\setminus\tilde{\mathcal{D}}$, D effectively intersects at most one x_j^{M+1}. Set $D' = D \cap \operatorname{supp} x_j^{M+1}$ ($D' = \emptyset$ if no such j exists). Then

$$\sum_{\substack{E\in\mathcal{E}''\\ E\subseteq D}} t(E)\|Ex_k^N\| = \sum_{\substack{E\in\mathcal{E}''\\ E\subseteq D'}} t(E)\|Ex_k^N\| \le t(D)\|D'x_k^N\|.$$

Now let $\mathcal{T}'$ be a tree obtained from $\mathcal{T}$ by taking all $D \in \mathcal{D}\backslash\tilde{\mathcal{D}}$, all $E \in \mathcal{E}'''$ and all their ancestors, with each $D \in \mathcal{D}\backslash\tilde{\mathcal{D}}$ modified into D' as described above. Then $\mathcal{T}'$ is an admissible tree that is subordinated to $\mathbf{x}^{M+1}$ and $H(\mathcal{T}') < H(\mathcal{T})$. (Note that every node in $\mathcal{E}'''$ is a singleton.) By Propositions 16 and 17 and the above,

$$\begin{aligned}\sum_{E\in\mathcal{L}(\mathcal{T})} t(E)\|Ex_k^N\| &\le \frac{1}{3N^2} + \frac{1}{3N^2}\|x_k^N\|_{X_M} + \Big(\sum_{E\in\mathcal{E}''} + \sum_{E\in\mathcal{E}'''}\Big)t(E)\|Ex_k^N\| \\ &\le \frac{1}{N^2}\|x_k^N\|_{X_M} + \sum_{D\in\mathcal{D}\backslash\tilde{\mathcal{D}}} t(D)\|D'x_k^N\| + \sum_{E\in\mathcal{E}'''} t(E)\|Ex_k^N\| \\ &= \frac{1}{N^2}\|x_k^N\|_{X_M} + \sum_{E\in\mathcal{L}(\mathcal{T}')} t(E)\|Ex_k^N\|.\end{aligned}$$

Now let $\mathcal{T}$ be an admissible tree all of whose leaves are singletons. Let $\mathcal{T}_1$ be the subtree of $\mathcal{T}$ consisting of leaves E in $\mathcal{T}$ with $h(E) < N$ and their ancestors. Then $\mathcal{T}_1$ is subordinated to $\mathbf{x}^0$ and $H(\mathcal{T}_1) \le N-1$. By the above argument, there is an admissible tree $\mathcal{T}_1'$ subordinated to $\mathbf{x}^1$ with $H(\mathcal{T}_1') \le N-2$ so that

$$\sum_{E\in\mathcal{L}(\mathcal{T}_1)} t(E)\|Ex_k^N\| \le \sum_{E\in\mathcal{L}(\mathcal{T}_1')} t(E)\|Ex_k^N\| + \frac{1}{N^2}\|x_k^N\|_{X_M}.$$

Repeating the argument, we reach an admissible tree $\mathcal{T}_1^{(N-1)}$ subordinated to $\mathbf{x}^{N-1}$ with $H(\mathcal{T}_1^{(N-1)}) = 0$ such that

$$\sum_{E\in\mathcal{L}(\mathcal{T}_1)} t(E)\|Ex_k^N\| \le \sum_{E\in\mathcal{L}(\mathcal{T}_1^{(N-1)})} t(E)\|Ex_k^N\| + \frac{N-1}{N^2}\|x_k^N\|_{X_M}.$$

Since $H(\mathcal{T}_1^{(N-1)}) = 0$ and $\mathcal{T}_1^{(N-1)}$ is subordinated to $\mathbf{x}^{N-1}$, $\mathcal{T}_1^{(N-1)}$ consists of a single node E such that $E \subseteq x_{j_0}^{N-1}$ for some j_0. Recall that $x_k^N = \sum_{j\in I_k^N} a_j x_j^{N-1}$, where $0 \le \theta_{r_N(k)p_N} a_j \le k^{-1}$ for all $j \in I_k^N$. Hence,

$$\begin{aligned}\sum_{E\in\mathcal{L}(\mathcal{T}_1^{(N-1)})} t(E)\|Ex_k^N\| &\le a_{j_0}\|x_{j_0}^{N-1}\|_{\ell^1} \\ &\le 2\theta^{-1}_{r_N(k)p_N} k^{-1}\prod_{i=1}^{N-1}\theta^{-1}_{L_ip_i} \quad \text{by Corollary 11} \\ &\le \frac{1}{N^2} \quad \text{by (4)}.\end{aligned}$$

Therefore,

$$\sum_{E\in\mathcal{L}(\mathcal{T}_1)} t(E)\|Ex_k^N\| \le \frac{1}{N}\|x_k^N\|_{X_M}. \tag{14}$$

Let $\mathcal{T}_2$ be the subtree of $\mathcal{T}$ consisting of leaves E in $\mathcal{T}$ with $h(E) \geq N$ and their ancestors. Since every leaf in $\mathcal{T}_2$ is a singleton, the set of all leaves is subordinated to $\mathbf{x}^0$. Let $\mathcal{G}$ be the collection of all leaves E of $\mathcal{T}_2$ such that $o(E) < p(r_1, \ldots, r_N(k))$ if $E \subseteq x_k^N(r_1, \ldots, r_N(k))$. Then

$$\sum_{E \in \mathcal{G}} t(E)\|Ex_k^N\| \leq \frac{1}{3N^2} \quad \text{by Proposition 16.}$$

Hence,

$$\sum_{E \in \mathcal{L}(\mathcal{T}_2)} t(E)\|Ex_k^N\| \leq \frac{1}{3N^2} + \sum_{E \in \mathcal{G}'} t(E)\|Ex_k^N\|,$$

where $\mathcal{G}'$ consists of all leaves of $\mathcal{T}_2$ that are not in $\mathcal{G}$. If $E \in \mathcal{G}'$ and $E \subseteq x_k^N(r_1, \ldots, r_{N-1})$, then $o(E) \geq p(r_1, \ldots, r_N(k))$ and $h(E) \geq N$. Thus, $t(E) = \prod_{i=1}^{j} \theta_{\ell_i}$ with $j \geq N$ and $\sum_{i=1}^{j} \ell_i \geq p(r_1, \ldots, r_N(k))$. Choose $(\ell_i')_{i=1}^N$ so that $1 \leq \ell_i' \leq \ell_i$ for $1 \leq i < N$, $1 \leq \ell_N' \leq \ell = \sum_{i=N}^{j} \ell_i$ and $\sum_{i=1}^{N} \ell_i' = p(r_1, \ldots, r_N(k))$. Since (θ_n) is regular, $t(E) = \prod_{i=1}^{j} \theta_{\ell_i} \leq \prod_{i=1}^{N-1} \theta_{\ell_i} \cdot \theta_\ell \leq \prod_{i=1}^{N} \theta_{\ell_i'} \leq \Theta_{p(r_1, \ldots, r_N(k))}$. Therefore, using the estimates from Corollary 10 and Proposition 12, we have

$$\begin{aligned} &\sum_{E \in \mathcal{L}(\mathcal{T}_2)} t(E)\|Ex_k^N\| \\ &\leq \frac{1}{3N^2} + \sum_{r_1, \ldots, r_{N-1}} \sum_{\substack{E \in \mathcal{G}' \\ E \subseteq x_k^N(r_1, \ldots, r_{N-1})}} t(E)\|Ex_k^N\| \\ &\leq \frac{1}{3N^2} + \sum_{r_1, \ldots, r_{N-1}} \Theta_{p(r_1, \ldots, r_N(k))} \|x_k^N(r_1, \ldots, r_{N-1})\|_{\ell^1} \\ &\leq \left(\frac{1}{3N^2} + 4\theta_1^{-1} \sup_{r_1, \ldots, r_{N-1}} \frac{\Theta_{p(r_1, \ldots, r_N(k))}}{\theta_{p(r_1, \ldots, r_N(k))}} \right) \|x_k^N\|_{X_M}. \end{aligned} \tag{15}$$

Combining (14) and (15) and maximizing over all admissible trees gives

$$\begin{aligned} \|x_k^N\| &= \max_{\mathcal{T}} \mathcal{T} x_k^N \\ &\leq \left(\frac{2}{N} + 4\theta_1^{-1} \sup_{r_1, \ldots, r_{N-1}} \frac{\Theta_{p(r_1, \ldots, r_N(k))}}{\theta_{p(r_1, \ldots, r_N(k))}} \right) \|x_k^N\|_{X_M}. \quad \square \end{aligned}$$

Remark. Note that the term $\|x_k^N\|_{X_M}$ enters the arguments leading up to the proof of Theorem 21 only via the lower estimate established in Proposition 12. Therefore, if we define

$$\Phi_k^N = \frac{\theta_1}{2} \sum_{r_1, \ldots, r_{N-1}} \theta_{p(r_1, \ldots, r_{N-1}, r_N(k))} \theta_{r_N(k) p_N}^{-1} \prod_{i=1}^{N-1} \theta_{r_i p_i}^{-1} L_i^{-1},$$

then we actually obtain the inequality

$$\|x_k^N\| \le \left(\frac{2}{N} + 4\theta_1^{-1} \sup_{r_1,\dots,r_{N-1}} \frac{\Theta_{p(r_1,\dots,r_N(k))}}{\theta_{p(r_1,\dots,r_N(k))}}\right)\Phi_k^N.$$

5. Proof of main theorem and examples

In this section, we give a proof for Theorem 1. Recall that we define $\theta = \lim \theta_n^{1/n} = \sup \theta_n^{1/n}$ for a regular sequence (θ_n) and let $\varphi_n = \theta_n/\theta^n$. It was mentioned in the discussion at the beginning of Section 2 that X and X_M are not isomorphic if $\theta = 1$. If $\theta < 1$ and $\varphi_N = 1$ for some N, then X and X_M are isomorphic by Proposition 7. We shall presently show that X and X_M are not isomorphic under some mild conditions on (φ_n). For the remainder of the section, assume that $\theta < 1$.

PROPOSITION 22. *If* $\inf \varphi_n = c > 0$. *Then* (θ_n) *satisfies* $(\neg\dagger)$ *and* $(\ddagger)$.

Proof. Indeed,

$$\frac{\theta_{m+n}}{\theta_n} = \frac{\varphi_{m+n}}{\varphi_n}\theta^m \le \frac{1}{c}\theta^m \quad \text{for all } m, n \in \mathbb{N}.$$

Thus, $(\neg\dagger)$ holds. Also,

$$\sum_{i=1}^{R} \frac{\theta_{s_i+t}}{\theta_{s_i}} = \sum_{i=1}^{R} \frac{\varphi_{s_i+t}}{\varphi_{s_i}}\theta^t \ge cR\theta^t.$$

On the other hand,

$$\max_{1\le i\le R} \frac{\theta_{s_i+t}}{\theta_{s_i}} = \max_{1\le i\le R} \frac{\varphi_{s_i+t}}{\varphi_{s_i}}\theta^t \le \frac{\theta^t}{c}.$$

Thus, $(\ddagger)$ holds with $F(R) = \frac{1}{c^2 R}$. □

Proof of Theorem 1. Let $\varepsilon > 0$ and $V \in [\mathbb{N}]$ be given. Choose $N \in \mathbb{N}$ such that $\frac{2}{N} + 4\theta_1^{-1}\frac{d^N}{c} < \varepsilon$. Obtain from Theorem 21 a vector $x \in \operatorname{span}\{e_k : k \in V\}$ that satisfies (13). Let $p \in \mathbb{N}$, if $(\ell_i)_{i=1}^N$ is a sequence of positive integers such that $\sum_{i=1}^N \ell_i = p$, then

$$\prod_{i=1}^{N} \theta_{\ell_i} = \theta^p \prod_{i=1}^{N} \varphi_{\ell_i} \le \theta^p d^N$$

and

$$\theta_p = \varphi_p \theta^p \ge c\theta^p.$$

Thus,

$$\sup_p \frac{\Theta_p}{\theta_p} \le \frac{d^N}{c}.$$

It follows from (13) that

$$\|x\| \leq \left(\frac{2}{N} + 4\theta_1^{-1}\frac{d^N}{c}\right)\|x\|_{X_M} < \varepsilon\|x\|_{X_M}.$$

Hence, according to Proposition 2, X and X_M are not isomorphic. □

In the next two examples, we show that neither $\inf \varphi_n > 0$ nor $\sup \varphi_n < 1$ is a necessary condition for X and X_M to be nonisomorphic.

Example 23. If $\theta < 1$ and $\varphi_n = \frac{1}{n+1}$, then X and X_M are not isomorphic.

Proof. It suffices to show that (θ_n) satisfies $(\neg\dagger)$, $(\ddagger)$ and $\lim_N \sup_p \frac{\Theta_p(N)}{\theta_p} = 0$. Note that

$$\frac{\theta_{m+n}}{\theta_n} = \frac{n+1}{m+n+1}\theta^m.$$

Hence,

$$\delta_m = \limsup_n \frac{\theta_{m+n}}{\theta_n} = \theta^m \to 0$$

as $m \to \infty$. Thus, $(\neg\dagger)$ holds.

To see that (θ_n) satisfies $(\ddagger)$, let $s_1 < s_2 < \cdots < s_R$ be an arithmetic progression in $\mathbb{N}$. Note that $s \mapsto \frac{s+1}{s+t+1}$ is a concave increasing function for $s \geq 0$. Let $g(s)$ be the linear function interpolating $(s_1, \frac{s_1+1}{s_1+t+1})$ and $(s_R, \frac{s_R+1}{s_R+t+1})$. Then

$$\begin{aligned}\sum_{i=1}^{R} \frac{\theta_{s_i+t}}{\theta_{s_i}} &= \theta^t \sum_{i=1}^{R} \frac{s_i+1}{s_i+t+1} \geq \theta^t \sum_{i=1}^{R} g(s_i)\\ &= \theta^t \frac{R}{2}[g(s_1) + g(s_R)]\\ &\qquad \text{since } (g(s_i))_{i=1}^R \text{ is an arithmetic progression}\\ &\geq \theta^t \frac{R}{2} \max\{g(s_1), g(s_R)\} = \frac{R}{2} \max_{1\leq i\leq R} \frac{\theta_{s_i+t}}{\theta_{s_i}}.\end{aligned}$$

Hence, $(\ddagger)$ holds with $F(R) = \frac{2}{R}$.

Finally, if If $(\ell_i)_{i=1}^N$ is a sequence of positive integers such that $\sum_{i=1}^N \ell_i = p$, then at least one ℓ_i is $\geq \frac{p}{N}$. Without loss of generality, assume that $\ell_1 \geq \frac{p}{N}$. Then

$$\frac{1}{\ell_1+1} \leq \frac{N}{p+1}.$$

Hence,

$$\prod_{i=1}^{N} \theta_{\ell_i} = \theta^p \prod_{i=1}^{N} \frac{1}{\ell_i+1} \leq \theta^p \left(\frac{N}{p+1}\right)\left(\frac{1}{2}\right)^{N-1} = \frac{N}{2^{N-1}}\theta_p.$$

Thus,

$$\sup_p \frac{\Theta_p(N)}{\theta_p} \le \frac{N}{2^{N-1}}.$$

It follows from Proposition 2 and Theorem 21 that X and X_M are not isomorphic. □

EXAMPLE 24. There exists a regular sequence (θ_n) with $0 < \theta < 1$ and $\lim_n \varphi_n = 1$ such that X and X_M are not isomorphic.

Proof. Let $0 < \theta_1 < \theta < 1$ be given. Choose sequences (q_n) and (K_n) in $\mathbb{N}$ such that

$$\theta^{q_{M+N+1}} \le \frac{1}{24N^2}\theta_1^{2+s(M,N)}$$

and

$$\frac{1}{K_{M+N+1}} \le \frac{1}{144N^2}\theta_1^{3+s(M,N)}$$

if $0 \le M \le N$, where $s(M,N) = \sum_{i=1}^{M} K_{N+i}q_{N+i}$ if $0 < M \le N$ and $s(0,N) = 0$. Then choose a sequence (φ_n) such that $\varphi_1 = \frac{\theta_1}{\theta}$, (φ_n) increases to 1, $\varphi_{n+1} \le \frac{\varphi_n}{\theta}$ and $\lim_N \varphi_{s(N,N)}^N = 0$.

Define $\theta_n = \varphi_n\theta^n$. Then (θ_n) is a regular sequence such that $\lim \theta_n^{1/n} = \lim \varphi_n^{1/n}\theta = \theta$. Since $\inf \varphi_n = \varphi_1 > 0$, $(\neg\dagger)$ and $(\ddagger)$ hold with $F(R) = \frac{1}{\varphi_1^2 R}$ according to Proposition 22.

Given $N \in \mathbb{N}$, we claim that the sequences $(p_k)_{k=1}^N = (q_{N+k})_{k=1}^N$ and $(L_k)_{k=1}^N = (K_{N+k})_{k=1}^N$ satisfy conditions (A), (B), and (C). Indeed,

$$\frac{\theta_{p_{M+1}+n}}{\theta_n} = \theta^{p_{M+1}}\frac{\varphi_{p_{M+1}+n}}{\varphi_n} \le \frac{\theta^{p_{M+1}}}{\varphi_1} \le \frac{\theta^{q_{N+M+1}}}{\theta_1} \le \frac{1}{24N^2}\theta_1^{s(M,N)}.$$

By regularity, $\theta_n \ge \theta_1^n$. Hence,

$$\prod_{i=1}^{M} \theta_{L_ip_i} \ge \theta_1^{\sum_{i=1}^M L_ip_i} = \theta_1^{s(M,N)}$$

if $M > 0$. Thus,

$$\frac{\theta_{p_{M+1}+n}}{\theta_n} \le \frac{1}{24N^2}\prod_{i=1}^{M}\theta_{L_ip_i}.$$

Therefore, condition (A) is satisfied if $M > 0$. If $M = 0$, then $s(M,N) = 0$ and the vacuous product $\prod_{i=1}^{M}\theta_{L_ip_i} = 1$ and the result is clear.

To see that condition (B) is satisfied, we note that by the choice of (q_n), $q_{M+N+1} \ge 2 + s(M,N)$, which is equivalent to saying that $p_{M+1} \ge 2 + \sum_{i=1}^{M} L_ip_i$ if $M > 0$.

If $M > 0$,

$$F(L_{M+1}) = \frac{1}{\varphi_1^2 L_{M+1}} = \frac{1}{\varphi_1^2 K_{M+N+1}} \le \frac{1}{\theta_1^2 \cdot 144N^2}\theta_1^{3+s(M,N)}$$
$$\le \frac{\theta_1}{144N^2}\theta_1^{\sum_{i=1}^M L_i p_i} \le \frac{\theta_1}{144N^2}\prod_{i=1}^M \theta_{L_i p_i}.$$

Therefore, condition (C) is also satisfied. Finally, we consider the ratio

$$\frac{\Theta_{p(r_1,\ldots,r_N(k))}}{\theta_{p(r_1,\ldots,r_N(k))}}.$$

If $(\ell_i)_{i=1}^N$ is a sequence in $\mathbb{N}$ such that $\sum_{i=1}^N \ell_i = p(r_1,\ldots,r_N(k))$, then

$$\prod_{i=1}^N \theta_{\ell_i} = \theta^{p(r_1,\ldots,r_N(k))}\prod_{i=1}^N \varphi_{\ell_i}$$
$$\le \theta^{p(r_1,\ldots,r_N(k))}\varphi^N_{p(r_1,\ldots,r_N(k))}$$

since (φ_n) is increasing and $0 < \varphi_n < 1$. Now

$$p(r_1,\ldots,r_N(k)) = r_1p_1 + \cdots + r_{N-1}p_{N-1} + r_N(k)p_N$$
$$\le L_1p_1 + \cdots + L_Np_N = \sum_{i=1}^N K_{N+i}q_{N+i} = s(N,N).$$

Thus,

$$\prod_{i=1}^N \theta_{\ell_i} \le \theta^{p(r_1,\ldots,r_N(k))}\varphi^N_{s(N,N)}$$
$$= \frac{\varphi^N_{s(N,N)}}{\varphi_{p(r_1,\ldots,r_N(k))}}\theta_{p(r_1,\ldots,r_N(k))}$$
$$\le \varphi^N_{s(N,N)}\varphi_1^{-1}\theta_{p(r_1,\ldots,r_N(k))}.$$

Hence,

$$\sup_{r_1,\ldots,r_{N-1}} \frac{\Theta_{p(r_1,\ldots,r_N(k))}}{\theta_{p(r_1,\ldots,r_N(k))}} \le \varphi^N_{s(N,N)}\varphi_1^{-1}.$$

Since (φ_N) is chosen such that $\lim_N \varphi^N_{s(N,N)} = 0$, we see that

$$\lim_N \sup_{r_1,\ldots,r_{N-1}} \frac{\Theta_{p(r_1,\ldots,r_N(k))}}{\theta_{p(r_1,\ldots,r_N(k))}} = 0.$$

Arguing as in the proof of Theorem 1, we may conclude that X and X_M are not isomorphic. □

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