Decay structure of two hyperbolic relaxation models with regularity loss

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Abstract This article investigates two types of decay structures for linear symmetric hyperbolic systems with nonsymmetric relaxation. Previously, the same authors introduced a new structural condition which is a generalization of the classical Kawashima–Shizuta condition and also analyzed the weak dissipative structure called the regularity-loss type for general systems with nonsymmetric relaxation, which includes the Timoshenko system and the Euler–Maxwell system as two concrete examples. Inspired by the previous work, we further construct in this article two more complex models which satisfy some new decay structure of regularity-loss type. The proof is based on the elementary Fourier energy method as well as the suitable linear combination of different energy inequalities. The results show that the model of type I has a decay structure similar to that of the Timoshenko system with heat conduction via the Cattaneo law, and the model of type II is a direct extension of two models considered previously to the case of higher phase dimensions.

Contents

1. Introduction

In this article, we consider the Cauchy problem on the following linear symmetric hyperbolic system with relaxation (see [5]):

$$u_t + A_m u_x + L_m u = 0, \tag{1.1}$$

with

$$u|_{t=0} = u_0. \tag{1.2}$$

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Here $u = u(t,x) = (u_1,\dots,u_m)^T(t,x) \in \mathbb{R}^m$ over $t > 0$, $x \in \mathbb{R}$, is an unknown function, $u_0 = u_0(x) \in \mathbb{R}^m$ over $x \in \mathbb{R}$ is a given function, and A_m and L_m are $(m \times m)$-real constant matrices. In general we assume A_m is symmetric, and L_m is degenerately dissipative in the sense of $1 \le \dim(\ker L_m) \le m-1$. As pointed out in [33], for a general linear degenerately dissipative system it is interesting to study its decay structure under additional conditions on the coefficient matrices and further investigate the corresponding time-decay property of solutions to the Cauchy problem at the linear level. The purpose of this article is to present two concrete models of A_m and L_m, which do not satisfy the dissipative condition in [33], to derive the decay structures of the corresponding linear systems. We remark that a similar issue has been extensively investigated by Villani [37] for an infinite-dimensional dynamical system, for instance, in the content of kinetic theory.

In what follows let us explain the motivation for dealing with the problem considered here. More generally, one may consider the system in multidimensional space $\mathbb{R}^n$

$$A_m^0 u_t + \sum_{j=1}^n A_m^j u_{x_j} + L_m u = 0, \tag{1.3}$$

where $u = u(t,x) \in \mathbb{R}^m$ over $t \ge 0$, $x \in \mathbb{R}^n$. When the degenerate relaxation matrix L_m is symmetric, Umeda, Kawashima, and Shizuta [36] proved the large-time asymptotic stability of solutions for a class of equations of hyperbolic-parabolic type with applications to both electro-magneto-fluid dynamics and magneto-hydrodynamics. The key idea in [36] and the later generalized work [31] that first introduced the so-called *Kawashima–Shizuta (KS) condition* is to construct the compensating matrix to capture the dissipation of systems over the degenerate kernel space of L_m. The typical feature of the time-decay property of solutions established in those works is that the high-frequency part decays exponentially while the low-frequency part decays polynomially with the same rate as the heat kernel. To precisely state these results, we apply a Fourier transform to (1.3) (or (1.1)). Then we can obtain

$$A_m^0 \hat{u}_t + i|\xi| A_m(\omega)\hat{u} + L_m \hat{u} = 0, \tag{1.4}$$

where $\xi \in \mathbb{R}^n$ denotes the Fourier variable of $x \in \mathbb{R}^n$, $\omega = \xi/|\xi| \in S^{n-1}$, and $A_m(\omega) := \sum_{j=1}^n A_m^j \omega_j$. Moreover, we prepare some notation. Given a real matrix X, we use X^{sy} and X^{asy} to denote the symmetric and skew-symmetric parts of X, respectively, namely, $X^{\mathrm{sy}} = (X + X^T)/2$ and $X^{\mathrm{asy}} = (X - X^T)/2$. Then the decay result in [36] and [31] is stated as in Proposition 1.1.

CONDITION 1.1

We have that A_m^0 is real symmetric and positive definite, A_m^j for each $1 \le j \le n$ is real symmetric, and L_m is real symmetric and nonnegative definite with the nontrivial kernel.

CONDITION 1.2

There is a real compensating matrix $K(\omega) \in C^\infty(S^{n-1})$ with the properties $K(-\omega) = -K(\omega)$, $(K(\omega)A_m^0)^T = -K(\omega)A_m^0$, and

$$[K(\omega)A_m(\omega)]^{\mathrm{sy}} > 0 \quad \text{on } \ker L_m$$

for each $\omega \in S^{n-1}$.

PROPOSITION 1.1 (DECAY PROPERTY OF THE STANDARD TYPE ([36], [31]))

Consider (1.3) with Condition 1.1. For this problem, assume that Condition 1.2 holds. Then the Fourier image $\hat{u}$ of the solution u to (1.3) with initial data $u(0,x) = u_0(x)$ satisfies the pointwise estimate

$$|\hat{u}(t,\xi)| \le Ce^{-c\lambda(\xi)t}|\hat{u}_0(\xi)|, \tag{1.5}$$

where $\lambda(\xi) := |\xi|^2/(1+|\xi|^2)$. Furthermore, let $s \ge 0$ be an integer, and suppose that the initial data u_0 belong to $H^s \cap L^1$. Then the solution u satisfies the decay estimate

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2} \tag{1.6}$$

for $k \le s$. Here C and c are positive constants.

Under Conditions 1.1 and 1.2, we can construct the following energy inequality:

$$\frac{d}{dt}E + cD \le 0,$$

where

$$\begin{aligned} E &= \langle A_m^0\hat{u}, \hat{u}\rangle - \frac{\alpha|\xi|}{1+|\xi|^2}\delta\langle iK(\omega)A_m^0\hat{u}, \hat{u}\rangle, \\ D &= \frac{|\xi|^2}{1+|\xi|^2}|\hat{u}|^2 + |(I-P)\hat{u}|^2, \end{aligned} \tag{1.7}$$

α and δ are suitably small constants, and P denotes the orthogonal projection onto $\ker L_m$.

For the nonlinear system, the global existence of small-amplitude classical solutions was proved by Hanouzet and Natalini [12] in one space dimension and by Yong [38] in several space dimensions, provided that the system is strictly entropy dissipative and satisfies the KS condition. Later on, the large-time behavior of solutions was obtained by Bianchini, Hanouzet, and Natalini [3] and Kawashima and Yong [18] based on the analysis of the Green function of the linearized problem. Those results show that solutions to such nonlinear systems will not develop singularities (e.g., shock waves) in finite time for small smooth initial perturbations (see [5], [20]). Notice that the L^2-stability of a constant equilibrium state in a one-dimensional system of dissipative hyperbolic balance laws endowed with a convex entropy was also studied by Ruggeri and Serre [29]. Moreover, it would be an interesting and important topic to study the relaxation limit of general hyperbolic conservation laws with relaxations (see [4], [17], and references therein).

Recently, it has been found that there exist physical systems which violate the KS condition but still have some kind of time-decay properties. For instance, for the dissipative Timoshenko system (see [14], [15]) and the Euler–Maxwell system (see [8], [35], [34]), the linearized relaxation matrix L_m has a nonzero skew-symmetric part, while it was still proved that solutions decay in time in some different way. Besides those, there are two related works dealing with general partially dissipative hyperbolic systems with zeroth-order source when the KS condition is not satisfied. Beauchard and Zuazua [2] first observed the equivalence of the KS condition with the Kalman rank condition in the context of control theory. They extended the previous analysis to some other situations beyond the KS condition and established the explicit estimate on the solution semigroup in terms of the frequency variable and also the global existence of near-equilibrium classical solutions for some nonlinear balance laws without the KS condition. In the meantime, Mascia and Natalini [25] also made a general study of the same topic for a class of systems without the KS condition. The typical situation considered in [25] is that the nondissipative components are linearly degenerate, which indeed does not hold under the KS condition (see also [16]). Notice that, in both [2] and [25], the rate of convergence of solutions to the equilibrium states for the nonlinear Cauchy problem is still left unknown.

The authors of this article [33] introduced a new structural condition which is a generalization of the KS condition, and they also analyzed the corresponding weak dissipative structure called the *regularity-loss type* for general systems with nonsymmetric relaxation, which includes the Timoshenko system and the Euler–Maxwell system as two concrete examples. Precisely, one has Proposition 1.2.

CONDITION 1.3

We have that A_m^0 is real symmetric and positive definite, and A_m^j for each $1 \le j \le n$ is real symmetric, while L_m is not necessarily real symmetric but is nonnegative definite with nontrivial kernel.

CONDITION 1.4

There is a real matrix S such that $(SA_m^0)^T = SA_m^0$,

$$[SL_m]^{\mathrm{sy}} + [L_m]^{\mathrm{sy}} \ge 0 \quad \text{on } \mathbb{C}^m, \qquad \ker\big([SL_m]^{\mathrm{sy}} + [L_m]^{\mathrm{sy}}\big) = \ker L_m,$$

and moreover, for each $\omega \in S^{n-1}$,

$$i\big[SA_m(\omega)\big]^{\mathrm{asy}} \ge 0 \quad \text{on } \ker[L_m]^{\mathrm{sy}}. \tag{1.8}$$

PROPOSITION 1.2 (DECAY PROPERTY OF THE REGULARITY-LOSS TYPE ([33]))

Consider (1.3) with Condition 1.3. For this problem, assume that Conditions 1.2 and 1.4 hold. Then the Fourier image $\hat{u}$ of the solution u to (1.3) with initial data $u(0,x) = u_0(x)$ satisfies the pointwise estimate

$$\big|\hat{u}(t,\xi)\big| \le Ce^{-c\lambda(\xi)t}\big|\hat{u}_0(\xi)\big|, \tag{1.9}$$

where $\lambda(\xi) := |\xi|^2/(1+|\xi|^2)^2$*. Moreover, let* $s \geq 0$ *be an integer, and suppose that the initial data* u_0 *belong to* $H^s \cap L^1$*. Then the solution* u *satisfies the decay estimate*

$$\big\|\partial_x^k u(t)\big\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}u_0\|_{L^2} \tag{1.10}$$

for $k+\ell \leq s$*. Here* C *and* c *are positive constants.*

Observe that $\lambda(\xi)$ in (1.9) behaves as $|\xi|^2$ as $|\xi| \to 0$ but behaves as $1/|\xi|^2$ as $|\xi| \to \infty$. Thus, estimates (1.9) and (1.10) are weaker than (1.5) and (1.6), respectively. In particular, the decay estimate (1.9) is said to be of the regularity-loss type. Similar decay properties of regularity-loss type have been recently observed for several interesting systems. We refer the reader to [14], [15], and [24] (cf. [1], [28]) for the dissipative Timoshenko system; [8], [35], [34], for the Euler–Maxwell system; [13], [19], for a hyperbolic-elliptic system in radiation gas dynamics; [22], [23], [21], [6], [32], for a dissipative plate equation; and [7], [10] for various kinetic-fluid models.

In fact, one can show that Proposition 1.1 can be regarded as a corollary of Proposition 1.2 after replacing (1.8) in Condition 1.4 by a stronger condition:

$$i\big[SA_m(\omega)\big]^{\mathrm{asy}} \geq 0 \quad \text{on } \mathbb{C}^m,$$

for each $\omega \in S^{n-1}$. The key point for the proof of (1.9) is to derive the matrices S and $K(\omega)$ such that the coercive estimate

$$\delta\big[K(\omega)A_m(\omega)\big]^{\mathrm{sy}} + [SL_m]^{\mathrm{sy}} + [L_m]^{\mathrm{sy}} > 0 \quad \text{on } \mathbb{C}^m \tag{1.11}$$

holds true for suitably small $\delta > 0$. Indeed, under Conditions 1.2, 1.3, and 1.4, estimate (1.11) is satisfied. Then, using (1.11), we get the energy equality

$$\frac{d}{dt}E + cD \leq 0, \tag{1.12}$$

where

$$\begin{aligned} E &= \langle A_m^0\hat{u},\hat{u}\rangle + \frac{\alpha_1}{1+|\xi|^2}\Big(\langle SA_m^0\hat{u},\hat{u}\rangle - \frac{\alpha_2|\xi|}{1+|\xi|^2}\delta\big\langle iK(\omega)A_m^0\hat{u},\hat{u}\big\rangle\Big), \\ D &= \frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{u}|^2 + \frac{1}{1+|\xi|^2}\big|(I-P)\hat{u}\big|^2 + \big|(I-P_1)\hat{u}\big|^2, \end{aligned} \tag{1.13}$$

α_1 and α_2 are suitably small constants, and P and P_1 denote the orthogonal projections onto $\ker L_m$ and $\ker[L_m]^{\mathrm{sy}}$. Interested readers may refer to [33] for more details on this issue and also for the construction of S and $K(\omega)$ for the Timoshenko system and the Euler–Maxwell system. Therefore, Conditions 1.3 and 1.4 are generalizations of the classical KS conditions. We finally remark that it should be interesting to further investigate the nonlinear stability of constant equilibrium states of the systems of regularity-loss type under the structural condition (Conditions 1.2–1.4) postulated in Proposition 1.2.

Inspired by the previous work [33], the goal of this article is to construct much more complex models (1.1) with given A_m and L_m such that they enjoy some new dissipative structure of regularity-loss type. Here we recall the notion of the *uniform dissipativity* of system (1.1) introduced in [33]. Consider the eigenvalue problem for the system (1.1):

$$(\eta A_m^0 + i\xi A_m + L_m)\phi = 0,$$

where $\eta \in \mathbb{C}$ and $\phi \in \mathbb{C}^m$. The corresponding characteristic equation is given by

$$\det(\eta A_m^0 + i\xi A_m + L_m) = 0. \tag{1.14}$$

The solution $\eta = \eta(i\xi)$ of (1.14) is called the *eigenvalue* of system (1.1).

DEFINITION 1.3

System (1.1) is called *uniformly dissipative* of type (p,q) if the eigenvalue $\eta = \eta(i\xi)$ satisfies

$$\Re\eta(i\xi) \le -c|\xi|^{2p}/\bigl(1+|\xi|^2\bigr)^q$$

for all $\xi \in \mathbb{R}^n$, where c is a positive constant and (p,q) is a pair of positive integers.

Note that, as proved in [33, Theorem 4.2], one has $\Re\eta(i\xi) \le -c\lambda(\xi)$ whenever the pointwise estimates in the form of (1.5) or (1.9) hold true. Therefore, we can determine the type (p,q) for a uniformly dissipative system (1.1) in terms of the function $\lambda(\xi)$ obtained from the pointwise estimate on $\hat{u}(t,\xi)$:

$$\bigl|\hat{u}(t,\xi)\bigr| \le Ce^{-c\lambda(\xi)t}\bigl|\hat{u}_0(\xi)\bigr|. \tag{1.15}$$

For example, under the assumptions in Proposition 1.1 or 1.2, the system (1.1) is uniformly dissipative of type $(1,1)$ or $(1,2)$, respectively. Notice that the regularity-loss type corresponds to the situation when p is strictly less than q, that is, $p < q$.

Historically, Shizuta and Kawashima [32] showed that, under Condition 1.1, the strict dissipativity $\Re\eta(i\xi) < 0$ for $\xi \ne 0$ is equivalent to the uniform dissipativity of type $(1,1)$. Moreover, they showed the pointwise estimate (1.5) by using only one compensating skew-symmetric matrix $K(\omega)$ (see (1.7)). The authors [33] formulated a class of systems whose dissipativity is of type $(1,2)$ and obtained Proposition 1.2. Notice that, in this case, we need to use one compensating symmetric matrix S and one compensating skew-symmetric matrix $K(\omega)$ to get the desired pointwise estimate (1.9) (see (1.13)). We note that the dissipative Timoshenko system and the Euler–Maxwell system studied in [14] and [34], respectively, are included in the class of systems with type $(1,2)$, which was formulated in [33]. However, to get the optimal dissipative estimate for these two examples, we need to use one S and two different $K(\omega)$'s (see [26], [34]).

More complicated concrete models have been found. Indeed, Mori and Kawashima [27] considered the Timoshenko–Cattaneo system with heat conduction and showed that its dissipativity is of type $(2,3)$. Moreover, they proved the optimal dissipative estimate by using four different S's and four different $K(\omega)$'s. This means that Proposition 1.2 and the class formulated in [33] is not enough to analyze the dissipativity of general systems (1.3), and we have to study other concrete models.

In this article, we will present a study of two concrete models of system (1.1) related to the above general issue. For Model I, one has (see (2.2) in Theorem 2.1)

$$p = m - 3, \qquad q = m - 2.$$

For Model II, we let m be even, and one has (see (3.2) in Theorem 3.1)

$$p = \frac{1}{2}(3m - 10), \qquad q = 2(m - 3).$$

In both cases we see $p < q$, and hence, the two models that we consider are of regularity-loss type. Compared with energy inequality (1.12), the energy inequalities of Models I and II are much more complicated. More precisely, to control the dissipation term, we must employ a lot of compensating symmetric matrices and skew-symmetric matrices whose numbers depend on the dimension m of the coefficient matrices. Therefore, we cannot apply Proposition 1.2 to Models I and II and need direct calculations (see Sections 2 and 3).

The proof of the estimate in the form of (1.15) is based on the Fourier energy method, and in the meantime, we also give the explicit construction of matrices S and K as used in Proposition 1.2. As seen later on, a series of energy estimates is derived, and their appropriate linear combination leads to a Lyapunov-type inequality of the time-frequency functional equivalent to $|\hat{u}(t,\xi)|^2$, which hence implies (1.15). The most difficult point is that it is unclear to justify whether one choice of (p,q) is optimal (see more discussions in Section 4.1). For that purpose, we also present an alternative approach to find out the value of (p,q) for both Models I and II, and the detailed strategy of the approach is given later.

The rest of the article is organized as follows. In Sections 2 and 3, we study Models I and II, respectively. In each section, for the given model, we first state the main results on the dissipative structure and the decay property of the system (1.1), give the proof by the energy method in the case $m = 6$—which indeed corresponds to some existing physical models—show the proof in the general case $m \geq 6$ still using the energy method, and finally give the explicit construction of matrices S and K. The matrices S and K constructed in Sections 2.3 and 3.3 have a very important role in obtaining a coercive estimate similar to (1.11). Consequently, by employing these matrices, we can derive the desired pointwise estimates through (2.47) and (3.64) to be verified later. In Section 4, we provide another approach to justify the dissipative structure of system (1.1).

Notation

For a nonnegative integer k, we denote by ∂_x^k the totality of all the kth-order derivatives with respect to $x = (x_1, \dots, x_n)$. Let $1 \leq p \leq \infty$. Then $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space over $\mathbb{R}^n$ with norm $\|\cdot\|_{L^p}$. For a nonnegative integer s, $H^s = H^s(\mathbb{R}^n)$ denotes the sth-order Sobolev space over $\mathbb{R}^n$ in the L^2-sense, equipped with the norm $\|\cdot\|_{H^s}$. We note that $L^2 = H^0$. Finally, in this article, we use C and c to denote various positive constants when there is no confusion.

2. Model I

2.1. Main result I

In this section, we consider the Cauchy problem (1.1)–(1.2) with coefficient matrices given by

$$
(2.1)\qquad
A_m = \left(\begin{array}{cccc|cccc}
0 & 1 & 0 & 0 & & & & \\
1 & 0 & 0 & 0 & & & \text{\Large 0} & \\
0 & 0 & 0 & a_4 & 0 & & & \\
0 & 0 & a_4 & 0 & a_5 & 0 & & \\
\hline
 & & 0 & a_5 & 0 & a_6 & & \\
 & & & 0 & a_6 & \ddots & & \\
 & \text{\Large 0} & & & & & & a_m \\
 & & & & & & a_m & 0
\end{array}\right),
\qquad
L_m = \left(\begin{array}{cccc|cccc}
0 & 0 & 0 & 1 & & & & \\
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & \text{\Large 0} & & \\
-1 & 0 & 0 & 0 & & & & \\
\hline
 & & & & 0 & & & \\
 & & & & & \ddots & & \\
 & \text{\Large 0} & & & & & 0 & \\
 & & & & & & & \gamma
\end{array}\right),
$$

where integer $m \geq 6$ is even, $\gamma > 0$, and all elements a_j $(4 \leq j \leq m)$ are nonzero. We note that the system (1.1), (2.1) with $m = 6$ is the Timoshenko system with heat conduction via the Cattaneo law (see [11], [30]). For this problem, we can derive the following decay structure.

THEOREM 2.1

The Fourier image $\hat{u}$ of the solution u to the Cauchy problem (1.1)–(1.2) with (2.1) satisfies the pointwise estimate

$$
(2.2)\qquad \big|\hat{u}(t,\xi)\big| \leq Ce^{-c\lambda(\xi)t}\big|\hat{u}_0(\xi)\big|,
$$

where $\lambda(\xi) := \xi^{2(m-3)}/(1+\xi^2)^{m-2}$. Furthermore, let $s \geq 0$ be an integer, and suppose that the initial data u_0 belong to $H^s \cap L^1$. Then the solution u satisfies the decay estimate

$$
(2.3)\qquad \big\|\partial_x^k u(t)\big\|_{L^2} \leq C(1+t)^{-\frac{1}{2(m-3)}(\frac{1}{2}+k)}\|u_0\|_{L^1} + C(1+t)^{-\frac{\ell}{2}}\|\partial_x^{k+\ell}u_0\|_{L^2}
$$

for $k+\ell \leq s$. Here C and c are positive constants.

We remark that the estimates (2.2) and (2.3) with $m = 6$ are not optimal. Indeed, Mori and Kawashima [27] showed sharper estimates.

The decay estimate (2.3) is derived by the pointwise estimate (2.2) in Fourier space immediately. Thus, readers may refer to [33] (see also [9]), and we omit the proof of (2.3) for brevity. To make the proof more precise, we first consider the special case $m = 6$ in Section 2.2 and then generalize it to the case $m \geq 6$ in Section 2.3. The proof of (2.2) is given in the following two sections.

2.2. Energy method in the case $m = 6$

In this section we first consider the case $m = 6$. In this case, system (1.1) with (2.1) is described as

$$
\begin{aligned}
\partial_t \hat{u}_1 + i\xi \hat{u}_2 + \hat{u}_4 &= 0, \\
\partial_t \hat{u}_2 + i\xi \hat{u}_1 &= 0, \\
\partial_t \hat{u}_3 + i\xi a_4 \hat{u}_4 &= 0, \\
\partial_t \hat{u}_4 + i\xi(a_4 \hat{u}_3 + a_5 \hat{u}_5) - \hat{u}_1 &= 0, \\
\partial_t \hat{u}_5 + i\xi(a_5 \hat{u}_4 + a_6 \hat{u}_6) &= 0, \\
\partial_t \hat{u}_6 + i\xi a_6 \hat{u}_5 + \gamma \hat{u}_6 &= 0.
\end{aligned} \tag{2.4}
$$

For this system we are going to apply the energy method to derive Theorem 2.1 in the case $m = 6$. The proof is organized into the following three steps.

Step 1. We first derive the basic energy equality for system (2.4) in the Fourier space. We multiply all the equations of (2.4) by $\bar{\hat{u}} = (\bar{\hat{u}}_1, \bar{\hat{u}}_2, \bar{\hat{u}}_3, \bar{\hat{u}}_4, \bar{\hat{u}}_5, \bar{\hat{u}}_6)^T$, respectively, and combine the resultant equations. Then we obtain

$$
\sum_{j=1}^{6} \bar{\hat{u}}_j \partial_t \hat{u}_j + 2i\xi \Re(\hat{u}_1 \bar{\hat{u}}_2) + 2i\xi \sum_{j=3}^{5} a_{j+1} \Re(\hat{u}_j \bar{\hat{u}}_{j+1}) + 2i\Im(\hat{u}_4 \bar{\hat{u}}_1) + \gamma |\hat{u}_6|^2 = 0.
$$

Thus, taking the real part for the above equality, we arrive at the basic energy equality

$$
\frac{1}{2} \partial_t |\hat{u}|^2 + \gamma |\hat{u}_6|^2 = 0. \tag{2.5}
$$

Here we use the simple relation $\partial_t (\hat{u}_j^2) = 2\Re(\bar{\hat{u}}_j \partial_t \hat{u}_j)$ for any j. Next we create the dissipation terms.

Step 2. After the dissipation for $\hat{u}_6$ has been established, it remains to obtain the dissipation for the other components $\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5$. The main idea is to make full use of both the hyperbolic terms and the antisymmetric terms in each equation of the system so as to derive corresponding dissipations up to some interactive terms which actually can be controlled after taking an appropriate linear combination of all possible energy identities together with (2.5) from the previous step.

We first construct the dissipation for $\hat{u}_1$. We multiply the first and fourth equations in (2.4) by $-\bar{\hat{u}}_4$ and $-\bar{\hat{u}}_1$, respectively. Then, combining the resultant equations and taking the real part, we have

$$
-\partial_t \Re(\hat{u}_1 \bar{\hat{u}}_4) + |\hat{u}_1|^2 - |\hat{u}_4|^2 - \xi \Re(i\hat{u}_2 \bar{\hat{u}}_4) + a_4 \xi \Re(i\hat{u}_1 \bar{\hat{u}}_3) + a_5 \xi \Re(i\hat{u}_1 \bar{\hat{u}}_5) = 0. \tag{2.6}
$$

We multiply the second and third equations in (2.4) by $-a_4\bar{\hat{u}}_3$ and $-a_4\bar{\hat{u}}_2$, respectively. Then, combining the resultant equations and taking the real part, we have

$$-a_4\partial_t\Re(\hat{u}_2\bar{\hat{u}}_3) - a_4\xi\Re(i\hat{u}_1\bar{\hat{u}}_3) + a_4^2\xi\Re(i\hat{u}_2\bar{\hat{u}}_4) = 0.$$

Therefore, combining the above two equalities, we obtain

$$\begin{aligned} &-\partial_t\Re(\hat{u}_1\bar{\hat{u}}_4 + a_4\hat{u}_2\bar{\hat{u}}_3) + |\hat{u}_1|^2 - |\hat{u}_4|^2 \\ &\quad + (a_4^2-1)\xi\Re(i\hat{u}_2\bar{\hat{u}}_4) + a_5\xi\Re(i\hat{u}_1\bar{\hat{u}}_5) = 0. \end{aligned} \tag{2.7}$$

Furthermore, we multiply the second and fifth equations in (2.4) by $-\bar{\hat{u}}_5$ and $-\bar{\hat{u}}_2$, respectively. Then, combining the resultant equations and taking the real part, we have

$$-\partial_t\Re(\hat{u}_2\bar{\hat{u}}_5) - \xi\Re(i\hat{u}_1\bar{\hat{u}}_5) + a_5\xi\Re(i\hat{u}_2\bar{\hat{u}}_4) + a_6\xi\Re(i\hat{u}_2\bar{\hat{u}}_6) = 0. \tag{2.8}$$

Finally, multiplying (2.7) and (2.8) by a_5^2 and $-a_5(a_4^2-1)$, respectively, and combining the resultant equations, we have

$$\begin{aligned} &\partial_t E_1 + a_5^2\big(|\hat{u}_1|^2 - |\hat{u}_4|^2\big) + a_5(a_4^2 + a_5^2 - 1)\xi\Re(i\hat{u}_1\bar{\hat{u}}_5) \\ &\quad - a_5a_6(a_4^2-1)\xi\Re(i\hat{u}_2\bar{\hat{u}}_6) = 0, \end{aligned} \tag{2.9}$$

where we have defined that $E_1 := -\Re\{a_5^2(\hat{u}_1\bar{\hat{u}}_4 + a_4\hat{u}_2\bar{\hat{u}}_3) - a_5(a_4^2-1)\hat{u}_2\bar{\hat{u}}_5\}$.

Next, we multiply the first and second equations in (2.4) by $-i\xi\bar{\hat{u}}_2$ and $i\xi\bar{\hat{u}}_1$, respectively. Then, combining the resultant equations and taking the real part, we have

$$\xi\partial_t E_2 + \xi^2\big(|\hat{u}_2|^2 - |\hat{u}_1|^2\big) + \xi\Re(i\hat{u}_2\bar{\hat{u}}_4) = 0, \tag{2.10}$$

where $E_2 := -\Re(i\hat{u}_1\bar{\hat{u}}_2)$. Therefore, by the Young inequality, the above equation becomes

$$\xi\partial_t E_2 + \frac{1}{2}\xi^2|\hat{u}_2|^2 \le \xi^2|\hat{u}_1|^2 + \frac{1}{2}|\hat{u}_4|^2. \tag{2.11}$$

We multiply the third and fourth equations in (2.4) by $i\xi a_4\bar{\hat{u}}_4$ and $-i\xi a_4\bar{\hat{u}}_3$, respectively. Then, combining the resultant equations and taking the real part, we have

$$a_4\xi\partial_t\Re(i\hat{u}_3\bar{\hat{u}}_4) + a_4^2\xi^2\big(|\hat{u}_3|^2 - |\hat{u}_4|^2\big) + a_4a_5\xi^2\Re(\hat{u}_3\bar{\hat{u}}_5) + a_4\xi\Re(i\hat{u}_1\bar{\hat{u}}_3) = 0.$$

We multiply the second and third equations in (2.27) by $-a_4\bar{\hat{u}}_3$ and $-a_4\bar{\hat{u}}_2$, respectively. Then, combining the resultant equations and taking the real part, we have

$$-a_4\partial_t\Re(\hat{u}_2\bar{\hat{u}}_3) - a_4\xi\Re(i\hat{u}_1\bar{\hat{u}}_3) + a_4^2\xi\Re(i\hat{u}_2\bar{\hat{u}}_4) = 0.$$

Finally, combining the above two equations, we get

$$\partial_t\{\xi E_3 + F_1\} + a_4^2\xi^2\big(|\hat{u}_3|^2 - |\hat{u}_4|^2\big) + a_4a_5\xi^2\Re(\hat{u}_3\bar{\hat{u}}_5) + a_4^2\xi\Re(i\hat{u}_2\bar{\hat{u}}_4) = 0, \tag{2.12}$$

where $E_3 := a_4\Re(i\hat{u}_3\bar{\hat{u}}_4)$ and $F_1 := -a_4\Re(\hat{u}_2\bar{\hat{u}}_3)$. By using the Young inequality, we can obtain the inequality

$$\partial_t\{\xi E_3 + F_1\} + \frac{1}{2}a_4^2\xi^2|\hat{u}_3|^2 \le a_4^2\xi^2|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^2|\hat{u}_5|^2 + a_4^2|\xi||\hat{u}_2||\hat{u}_4|. \tag{2.13}$$

Multiplying the fourth and fifth equations in (2.27) by $i\xi a_5\bar{\hat{u}}_5$ and $-i\xi a_5\bar{\hat{u}}_4$, respectively, combining the resultant equations, and taking the real part, we have

$$\begin{aligned}(2.14)\qquad &\xi\partial_t E_4 + a_5^2\xi^2\big(|\hat{u}_4|^2 - |\hat{u}_5|^2\big)\\ &\quad - a_4a_5\xi^2\Re(\hat{u}_3\bar{\hat{u}}_5) + a_5a_6\xi^2\Re(\hat{u}_4\bar{\hat{u}}_6) - a_5\xi\Re(i\hat{u}_1\bar{\hat{u}}_5) = 0,\end{aligned}$$

where $E_4 := a_5\Re(i\hat{u}_4\bar{\hat{u}}_5)$. Here, by using the Young inequality, we obtain

$$\begin{aligned}(2.15)\qquad &\xi\partial_t E_4 + \frac{1}{2}a_5^2\xi^2|\hat{u}_4|^2\\ &\quad \le a_5^2\xi^2|\hat{u}_5|^2 + \frac{1}{2}a_6^2\xi^2|\hat{u}_6|^2 + a_4a_5\xi^2\Re(\hat{u}_3\bar{\hat{u}}_5) + a_5\xi\Re(i\hat{u}_1\bar{\hat{u}}_5).\end{aligned}$$

We multiply the fifth equation and the last equation in (2.4) by $i\xi a_6\bar{\hat{u}}_6$ and $-i\xi a_6\bar{\hat{u}}_5$, respectively. Then, combining the resultant equations and taking the real part, we obtain

$$a_6\xi\partial_t\Re(i\hat{u}_5\bar{\hat{u}}_6) + a_6^2\xi^2\big(|\hat{u}_5|^2 - |\hat{u}_6|^2\big) - a_5a_6\xi^2\Re(\hat{u}_4\bar{\hat{u}}_6) + \gamma a_6\xi\Re(i\hat{u}_5\bar{\hat{u}}_6) = 0.$$

By using the Young inequality, this yields

$$(2.16)\qquad a_6\xi\partial_t\Re(i\hat{u}_5\bar{\hat{u}}_6) + \frac{1}{2}a_6^2\xi^2|\hat{u}_5|^2 \le a_6^2\xi^2|\hat{u}_6|^2 + \frac{1}{2}\gamma^2|\hat{u}_6|^2 + a_5a_6\xi^2\Re(\hat{u}_4\bar{\hat{u}}_6).$$

Step 3. In this step, we sum up the energy inequalities derived in the previous step and then get the desired energy estimate. Throughout this step, the β_j's with $j \in \mathbb{N}$ denote the real numbers determined later. We first multiply (2.9) and (2.11) by ξ^2 and β_1, respectively. Then we combine the resultant equations, obtaining

$$\begin{aligned}&\partial_t\{\xi^2E_1 + \beta_1\xi E_2\} + (a_5^2 - \beta_1)\xi^2|\hat{u}_1|^2 + \frac{\beta_1}{2}\xi^2|\hat{u}_2|^2\\ &\quad \le \Big(\frac{\beta_1}{2} + a_5^2\xi^2\Big)|\hat{u}_4|^2 - a_5(a_4^2 + a_5^2 - 1)\xi^3\Re(i\hat{u}_1\bar{\hat{u}}_5) + a_5a_6(a_4^2 - 1)\xi^3\Re(i\hat{u}_2\bar{\hat{u}}_6).\end{aligned}$$

Moreover, combining (2.9), (2.13), and the above inequality, we have

$$\begin{aligned}&\partial_t\big\{(1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1\big\}\\ &\qquad + \big\{a_5^2 + (a_5^2 - \beta_1)\xi^2\big\}|\hat{u}_1|^2 + \frac{\beta_1}{2}\xi^2|\hat{u}_2|^2 + \frac{1}{2}a_4^2\xi^2|\hat{u}_3|^2\\ &\quad \le \Big\{a_5^2 + \frac{\beta_1}{2} + (a_4^2 + a_5^2)\xi^2\Big\}|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^2|\hat{u}_5|^2 + a_4^2|\xi||\hat{u}_2||\hat{u}_4|\\ &\qquad - a_5(a_4^2 + a_5^2 - 1)\xi(1+\xi^2)\Re(i\hat{u}_1\bar{\hat{u}}_5) + a_5a_6(a_4^2 - 1)\xi(1+\xi^2)\Re(i\hat{u}_2\bar{\hat{u}}_6).\end{aligned}$$

For this inequality, letting β_1 be suitably small and employing the Young inequality, we can get

$$\begin{aligned}&\partial_t\big\{(1+\xi^2)E_1 + c\xi E_2 + \xi E_3 + F_1\big\} + c(1+\xi^2)|\hat{u}_1|^2 + \beta_1\xi^2\big(|\hat{u}_2|^2 + |\hat{u}_3|^2\big)\\ (2.17)\qquad &\quad \le C(1+\xi^2)|\hat{u}_4|^2 + C\xi^2|\hat{u}_5|^2\\ &\qquad + |a_4^2 + a_5^2 - 1|C|\xi|^3|\hat{u}_1||\hat{u}_5| + |a_4^2 - 1|C|\xi|(1+\xi^2)|\hat{u}_2||\hat{u}_6|.\end{aligned}$$

Similarly, we multiply (2.15) and (2.17) by $1+\xi^2$ and $\beta_2\xi^2$, respectively. Then we combine the resultant equations, obtaining

$$\begin{aligned}
&\partial_t\big\{\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)+\xi(1+\xi^2)E_4\big\}\\
&\quad+\beta_2 c\xi^2(1+\xi^2)|\hat{u}_1|^2+\beta_2 c\xi^4\big(|\hat{u}_2|^2+|\hat{u}_3|^2\big)+\Big(\frac{1}{2}a_5^2-\beta_2 C\Big)\xi^2(1+\xi^2)|\hat{u}_4|^2\\
&\le \beta_2 C\xi^4|\hat{u}_5|^2+a_5^2\xi^2(1+\xi^2)|\hat{u}_5|^2+\frac{1}{2}a_6^2\xi^2(1+\xi^2)|\hat{u}_6|^2\\
&\quad+a_4a_5\xi^2(1+\xi^2)\Re(\hat{u}_3\bar{\hat{u}}_5)+a_5\xi(1+\xi^2)\Re(i\hat{u}_1\bar{\hat{u}}_5)\\
&\quad+\beta_2|a_4^2+a_5^2-1|C|\xi|^5|\hat{u}_1||\hat{u}_5|+\beta_2|a_4^2-1|C|\xi|^3(1+\xi^2)|\hat{u}_2||\hat{u}_6|.
\end{aligned}$$

Letting β_2 be suitably small and using the Young inequality, we derive that

$$\begin{aligned}
&\partial_t\big\{\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)+\xi(1+\xi^2)E_4\big\}\\
&\quad+c\xi^2(1+\xi^2)\big(|\hat{u}_1|^2+|\hat{u}_4|^2\big)+c\xi^4\big(|\hat{u}_2|^2+|\hat{u}_3|^2\big)\\
&\le C(1+\xi^2)^2|\hat{u}_5|^2+C\xi^2(1+\xi^2)|\hat{u}_6|^2\\
&\quad+|a_4^2+a_5^2-1|C\xi^6|\hat{u}_5|^2+|a_4^2-1|C\xi^2(1+\xi^2)^2|\hat{u}_2||\hat{u}_6|.
\end{aligned}\tag{2.18}$$

If we assume that $a_4^2-1=0$, then estimate (2.18) can be rewritten as

$$\begin{aligned}
&\partial_t\big\{\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)+\xi(1+\xi^2)E_4\big\}\\
&\quad+c\xi^2(1+\xi^2)\big(|\hat{u}_1|^2+|\hat{u}_4|^2\big)+c\xi^4\big(|\hat{u}_2|^2+|\hat{u}_3|^2\big)\\
&\le C(1+\xi^2)^3|\hat{u}_5|^2+C\xi^2(1+\xi^2)|\hat{u}_6|^2.
\end{aligned}\tag{2.19}$$

Then, multiplying (2.16) and the above inequality by $(1+\xi^2)^3$ and $\beta_3\xi^2$, respectively, and combining the resultant equations, we have

$$\begin{aligned}
&\partial_t\big\{\beta_3\xi^2\big(\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)+\xi(1+\xi^2)E_4\big)+\xi(1+\xi^2)^3E_5\big\}\\
&\quad+\beta_3 c\xi^4(1+\xi^2)\big(|\hat{u}_1|^2+|\hat{u}_4|^2\big)+\beta_3 c\xi^6\big(|\hat{u}_2|^2+|\hat{u}_3|^2\big)\\
&\quad+\Big(\frac{1}{2}a_6^2-\beta_3 C\Big)\xi^2(1+\xi^2)^3|\hat{u}_5|^2\\
&\le \beta_3 C\xi^4(1+\xi^2)|\hat{u}_6|^2+\Big(a_6^2\xi^2+\frac{1}{2}\gamma^2\Big)(1+\xi^2)^3|\hat{u}_6|^2\\
&\quad+a_5a_6\xi^2(1+\xi^2)^3\Re(\hat{u}_4\bar{\hat{u}}_6).
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
&\partial_t\big\{\beta_3\xi^2\big(\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)\\
&\qquad+\xi(1+\xi^2)E_4\big)+\xi(1+\xi^2)^3E_5\big\}\\
&\qquad+c\xi^4(1+\xi^2)\big(|\hat{u}_1|^2+|\hat{u}_4|^2\big)+c\xi^6\big(|\hat{u}_2|^2+|\hat{u}_3|^2\big)+c\xi^2(1+\xi^2)^3|\hat{u}_5|^2\\
&\quad\le C(1+\xi^2)^4|\hat{u}_6|^2+C\xi^2(1+\xi^2)^3|\hat{u}_4||\hat{u}_6|.
\end{aligned}$$

Moreover, we multiply (2.13) and (2.15) by $\beta_4\xi^6$ and $\beta_5\xi^6$, respectively, and combine the resultant equations and the above inequality. Then, by letting β_4

and β_5 be suitably small, this yields

$$\begin{aligned}\partial_t E + c\xi^4(1+\xi^2)|\hat{u}_1|^2 + c\xi^6|\hat{u}_2|^2 + c\xi^6(1+\xi^2)|\hat{u}_3|^2 \\ + c\xi^4(1+\xi^2)^2|\hat{u}_4|^2 + c\xi^2(1+\xi^2)^3|\hat{u}_5|^2 \le C(1+\xi^2)^4|\hat{u}_6|^2,\end{aligned} \tag{2.20}$$

where we have defined

$$\begin{aligned}E = \beta_2\beta_3\xi^4(1+\xi^2)E_1 + \beta_1\beta_2\beta_3\xi^5 E_2 + \xi^4(\beta_2\beta_3 + \beta_4\xi^2)(\xi E_3 + F_1) \\ + \xi^3\big(\beta_3(1+\xi^2) + \beta_5\xi^4\big)E_4 + \xi(1+\xi^2)^3 E_5.\end{aligned} \tag{2.21}$$

Finally, by combining the basic energy (2.5) with the above estimate, this yields

$$\begin{aligned}\partial_t\Big\{\frac{1}{2}(1+\xi^2)^4|\hat{u}|^2 + \beta_7 E\Big\} + c\xi^4(1+\xi^2)|\hat{u}_1|^2 \\ + c\xi^6|\hat{u}_2|^2 + c\sum_{j=3}^{6}\xi^{2(6-j)}(1+\xi^2)^{j-2}|\hat{u}_j|^2 \le 0.\end{aligned} \tag{2.22}$$

Thus, integrating the above estimate with respect to t, we obtain the energy estimate

$$\begin{aligned}\big|\hat{u}(t,\xi)\big|^2 + \int_0^t\Big\{\frac{\xi^4}{(1+\xi^2)^3}|\hat{u}_1|^2 + \frac{\xi^6}{(1+\xi^2)^4}|\hat{u}_2|^2 \\ + \sum_{j=3}^{6}\frac{\xi^{2(6-j)}}{(1+\xi^2)^{6-j}}|\hat{u}_j|^2\Big\}\,d\tau \le C\big|\hat{u}(0,\xi)\big|^2,\end{aligned} \tag{2.23}$$

where we use the inequality

$$c|\hat{u}|^2 \le \frac{1}{2}|\hat{u}|^2 + \frac{\beta_7}{(1+\xi^2)^4}E \le C|\hat{u}|^2 \tag{2.24}$$

for suitably small β_7. We note that the energy inequality (2.23) tells us not only the boundedness of the energy part but also the structure property of the dissipation part. More precisely, the estimate (2.22) with (2.24) gives us the pointwise estimate

$$\big|\hat{u}(t,\xi)\big| \le Ce^{-c\lambda(\xi)t}\big|\hat{u}(0,\xi)\big|, \quad \lambda(\xi) = \frac{\xi^6}{(1+\xi^2)^4}. \tag{2.25}$$

If we assume that $a_4^2 + a_5^2 - 1 = 0$, then the estimate (2.18) is rewritten as

$$\begin{aligned}\partial_t\big\{\beta_2\xi^2\big((1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1\big) + \xi(1+\xi^2)E_4\big\} \\ + c\xi^2(1+\xi^2)\big(|\hat{u}_1|^2 + |\hat{u}_4|^2\big) + c\xi^4\big(|\hat{u}_2|^2 + |\hat{u}_3|^2\big) \\ \le C(1+\xi^2)^2|\hat{u}_5|^2 + C(1+\xi^2)^4|\hat{u}_6|^2.\end{aligned} \tag{2.26}$$

Then, multiplying (2.16) and the above inequality by $(1+\xi^2)^2$ and $\beta_3\xi^2$, respectively, and combining the resultant equations, we have

$$\begin{aligned}\partial_t\big\{\beta_3\xi^2\big(\beta_2\xi^2\big((1+\xi^2)E_1 + \beta_1\xi E_2 + \xi E_3 + F_1\big) + \xi(1+\xi^2)E_4\big) + \xi(1+\xi^2)^2E_5\big\} \\ + \beta_3 c\xi^4(1+\xi^2)\big(|\hat{u}_1|^2 + |\hat{u}_4|^2\big) + \beta_3 c\xi^6\big(|\hat{u}_2|^2 + |\hat{u}_3|^2\big)\end{aligned}$$

$$+\Big(\frac{1}{2}a_6^2-\beta_3C\Big)\xi^2(1+\xi^2)^2|\hat{u}_5|^2$$
$$\le \beta_3C(1+\xi^2)^4|\hat{u}_6|^2+\Big(a_6^2\xi^2+\frac{1}{2}\gamma^2\Big)(1+\xi^2)^2|\hat{u}_6|^2+a_5a_6\xi^2(1+\xi^2)^2\Re(\hat{u}_4\bar{\hat{u}}_6).$$

Hence, we arrive at

$$\begin{aligned}
&\partial_t\big\{\beta_3\xi^2\big(\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)\\
&\qquad+\xi(1+\xi^2)E_4\big)+\xi(1+\xi^2)^2E_5\big\}\\
&\qquad+c\xi^4(1+\xi^2)\big(|\hat{u}_1|^2+|\hat{u}_4|^2\big)+c\xi^6\big(|\hat{u}_2|^2+|\hat{u}_3|^2\big)+c\xi^2(1+\xi^2)^2|\hat{u}_5|^2\\
&\quad\le C(1+\xi^2)^4|\hat{u}_6|^2.
\end{aligned}$$

Moreover, we multiply (2.13), (2.15), and (2.16) by $\beta_4\xi^6$, $\beta_5\xi^6$, and $\beta_6\xi^6$, respectively, and combine the resultant equations and the above inequality. Then, by letting β_4 and β_5 be suitably small, this yields

$$\begin{aligned}
&\partial_t\big\{\beta_2\beta_3\xi^4(1+\xi^2)E_1+\beta_1\beta_2\beta_3\xi^5E_2+\xi^4(\beta_2\beta_3+\beta_4\xi^2)(\xi E_3+F_1)\\
&\quad+\xi^3\big(\beta_3(1+\xi^2)+\beta_5\xi^4\big)E_4+\xi\big((1+\xi^2)^2+\beta_6\xi^6\big)E_5\big\}\\
&\quad+c\xi^4(1+\xi^2)|\hat{u}_1|^2+c\xi^6|\hat{u}_2|^2+c\xi^6(1+\xi^2)|\hat{u}_3|^2\\
&\quad+c\xi^4(1+\xi^2)^2|\hat{u}_4|^2+c\xi^2(1+\xi^2)^3|\hat{u}_5|^2\le C(1+\xi^2)^4|\hat{u}_6|^2.
\end{aligned}$$

We note that this estimate is essentially the same as (2.20). Hence, we can obtain the energy estimate (2.23) and the pointwise estimate (2.25). Eventually, we arrive at the estimate for both cases $a_4^2-1=0$ and $a_4^2+a_5^2-1=0$. Moreover, by using a similar argument, we can derive the same estimates in the case $a_4^2-1\ne 0$, $a_4^2+a_5^2-1\ne 0$. Specifically, we multiply (2.16), (2.18), (2.13), and (2.15) by $(1+\xi^2)^3$, $\beta_3\xi^2$, $\beta_4\xi^6$, and $\beta_5\xi^6$, respectively. Then, combining the resultant equalities, we obtain (2.20) with (2.21) and, hence, arrive at (2.23). Thus, we complete the proof of Theorem 2.1 with $m=6$.

2.3. Energy method for Model I

Inspired by the concrete computation in Section 2.2, we consider the more general case $m\ge 6$. Now, our system (1.1) with (2.1) is described as

$$\begin{aligned}
\partial_t\hat{u}_1+i\xi\hat{u}_2+\hat{u}_4&=0,\\
\partial_t\hat{u}_2+i\xi\hat{u}_1&=0,\\
\partial_t\hat{u}_3+i\xi a_4\hat{u}_4&=0,\\
\partial_t\hat{u}_4+i\xi(a_4\hat{u}_3+a_5\hat{u}_5)-\hat{u}_1&=0,\\
\partial_t\hat{u}_j+i\xi(a_j\hat{u}_{j-1}+a_{j+1}\hat{u}_{j+1})&=0,\quad j=5,\dots,m-1,\\
\partial_t\hat{u}_m+i\xi a_m\hat{u}_{m-1}+\gamma\hat{u}_m&=0.
\end{aligned}\tag{2.27}$$

We are going to apply the energy method to this system and derive Theorem 2.1. The proof is organized into the following three steps.

Step 1. We first derive the basic energy equality for the system (1.1) in the Fourier space. Taking the inner product of (1.1) with $\hat{u}$, we have

$$\langle \hat{u}_t, \hat{u}\rangle + i\xi\langle A_m\hat{u}, \hat{u}\rangle + \langle L_m\hat{u}, \hat{u}\rangle = 0.$$

Taking the real part, we get the basic energy equality

$$\frac{1}{2}\frac{\partial}{\partial t}|\hat{u}|^2 + \langle L_m\hat{u}, \hat{u}\rangle = 0,$$

and hence,

$$\frac{1}{2}\partial_t|\hat{u}|^2 + \gamma\hat{u}_m^2 = 0. \tag{2.28}$$

Next we create the dissipation terms in the following two steps.

Step 2. For $\ell = 6, \dots, m-1$, we multiply the fifth equation with $j = \ell - 1$ and $j = \ell$ in (2.27) by $i\xi a_\ell \bar{\hat{u}}_\ell$ and $-i\xi a_\ell \bar{\hat{u}}_{\ell-1}$, respectively. Then, combining the resultant equations and taking the real part, we have

$$\begin{aligned} &a_\ell\xi\partial_t\Re(i\hat{u}_{\ell-1}\bar{\hat{u}}_\ell) + a_\ell^2\xi^2\big(|\hat{u}_{\ell-1}|^2 - |\hat{u}_\ell|^2\big) \\ &\quad - a_\ell a_{\ell-1}\xi^2\Re(\hat{u}_{\ell-2}\bar{\hat{u}}_\ell) + a_\ell a_{\ell+1}\xi^2\Re(\hat{u}_{\ell-1}\bar{\hat{u}}_{\ell+1}) = 0. \end{aligned} \tag{2.29}$$

Here, by using the Young inequality, we obtain

$$\begin{aligned} &\xi\partial_t E_{\ell-1} + \frac{1}{2}a_\ell^2\xi^2|\hat{u}_{\ell-1}|^2 \\ &\quad \le a_\ell^2\xi^2|\hat{u}_\ell|^2 + \frac{1}{2}a_{\ell+1}^2\xi^2|\hat{u}_{\ell+1}|^2 + a_\ell a_{\ell-1}\xi^2\Re(\hat{u}_{\ell-2}\bar{\hat{u}}_\ell) \end{aligned} \tag{2.30}$$

for $\ell = 6, \dots, m-1$, where we have defined $E_{\ell-1} = a_\ell\xi\Re(i\hat{u}_{\ell-1}\bar{\hat{u}}_\ell)$. Then, we multiply the fifth equation with $j = m-1$ and the last equation in (2.27) by $i\xi a_m\bar{\hat{u}}_m$ and $-i\xi a_m\bar{\hat{u}}_{m-1}$, respectively. Then, combining the resultant equations and taking the real part, we obtain

$$\begin{aligned} &a_m\xi\partial_t\Re(i\hat{u}_{m-1}\bar{\hat{u}}_m) + a_m^2\xi^2\big(|\hat{u}_{m-1}|^2 - |\hat{u}_m|^2\big) \\ &\quad - a_m a_{m-1}\xi^2\Re(\hat{u}_{m-2}\bar{\hat{u}}_m) + \gamma a_m\xi\Re(i\hat{u}_{m-1}\bar{\hat{u}}_m) = 0. \end{aligned} \tag{2.31}$$

Using the Young inequality, this yields

$$\begin{aligned} &\xi\partial_t E_{m-1} + \frac{1}{2}a_m^2\xi^2|\hat{u}_{m-1}|^2 \\ &\quad \le a_m^2\xi^2|\hat{u}_m|^2 + \frac{1}{2}\gamma^2|\hat{u}_m|^2 + a_m a_{m-1}\xi^2\Re(\hat{u}_{m-2}\bar{\hat{u}}_m), \end{aligned} \tag{2.32}$$

where we have defined $E_{m-1} = a_m\xi\Re(i\hat{u}_{m-1}\bar{\hat{u}}_m)$.

Step 3. We note that (2.27) with $1 \le j \le 5$ is the same as the five equations in (2.4). Thus, we can adopt the useful estimates derived in Section 2.2. More precisely, we employ (2.13), (2.15), and (2.18) again.

For estimate (2.18), if we assume that $a_4^2 - 1 = 0$, then we can obtain (2.19). Next, by multiplying (2.30) with $\ell = 6$ and (2.19) by $(1+\xi^2)^3$ and $\beta_3\xi^2$, respectively, and combining the resultant equations, we have

$$\begin{aligned}
&\partial_t\big\{\beta_3\xi^2\big(\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)+\xi(1+\xi^2)E_4\big)+\xi(1+\xi^2)^3E_5\big\}\\
&\qquad+\beta_3 c\xi^4(1+\xi^2)\big(|\hat u_1|^2+|\hat u_4|^2\big)+\beta_3 c\xi^6\big(|\hat u_2|^2+|\hat u_3|^2\big)\\
&\qquad+\Big(\frac{1}{2}a_6^2-\beta_3 C\Big)\xi^2(1+\xi^2)^3|\hat u_5|^2\\
&\quad\le \beta_3 C\xi^4(1+\xi^2)|\hat u_6|^2+a_6^2\xi^2(1+\xi^2)^3|\hat u_6|^2\\
&\qquad+\frac{1}{2}a_7^2\xi^2(1+\xi^2)^3|\hat u_7|^2+a_5a_6\xi^2(1+\xi^2)^3\Re(\hat u_4\bar{\hat u}_6).
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
&\partial_t\big\{\beta_3\xi^2\big(\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)\\
&\qquad+\xi(1+\xi^2)E_4\big)+\xi(1+\xi^2)^3E_5\big\}\\
&\qquad+c\xi^4(1+\xi^2)\big(|\hat u_1|^2+|\hat u_4|^2\big)+c\xi^6\big(|\hat u_2|^2+|\hat u_3|^2\big)+c\xi^2(1+\xi^2)^3|\hat u_5|^2\\
&\quad\le C\xi^2(1+\xi^2)^3\big(|\hat u_6|^2+|\hat u_7|^2\big)+C\xi^2(1+\xi^2)^3|\hat u_4||\hat u_6|.
\end{aligned}$$

Moreover, we multiply (2.13) and (2.15) by $\beta_4\xi^6$ and $\beta_5\xi^6$, respectively, and combine the resultant equations and the above inequality. Then, by letting β_4 and β_5 be suitably small, this yields

$$\begin{aligned}
&\partial_t E+c\xi^4(1+\xi^2)|\hat u_1|^2+c\xi^6|\hat u_2|^2+c\xi^6(1+\xi^2)|\hat u_3|^2+c\xi^4(1+\xi^2)^2|\hat u_4|^2\\
&\quad+c\xi^2(1+\xi^2)^3|\hat u_5|^2\le C(1+\xi^2)^4|\hat u_6|^2+C\xi^2(1+\xi^2)^3|\hat u_7|^2,
\end{aligned}\tag{2.33}$$

where E is defined in (2.21).

If we assume that $a_4^2+a_5^2-1=0$, then we employ (2.26). Then, multiplying (2.30) with $\ell=6$ and (2.26) by $(1+\xi^2)^2$ and $\beta_3\xi^2$, respectively, and combining the resultant equations, we have

$$\begin{aligned}
&\partial_t\big\{\beta_3\xi^2\big(\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)+\xi(1+\xi^2)E_4\big)+\xi(1+\xi^2)^2E_5\big\}\\
&\qquad+\beta_3 c\xi^4(1+\xi^2)\big(|\hat u_1|^2+|\hat u_4|^2\big)+\beta_3 c\xi^6\big(|\hat u_2|^2+|\hat u_3|^2\big)\\
&\qquad+\Big(\frac{1}{2}a_6^2-\beta_3 C\Big)\xi^2(1+\xi^2)^2|\hat u_5|^2\\
&\quad\le \beta_3 C(1+\xi^2)^4|\hat u_6|^2+a_6^2\xi^2(1+\xi^2)^2|\hat u_6|^2\\
&\qquad+\frac{1}{2}a_7^2\xi^2(1+\xi^2)^2|\hat u_7|^2+a_5a_6\xi^2(1+\xi^2)^2\Re(\hat u_4\bar{\hat u}_6).
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
&\partial_t\big\{\beta_3\xi^2\big(\beta_2\xi^2\big((1+\xi^2)E_1+\beta_1\xi E_2+\xi E_3+F_1\big)\\
&\qquad+\xi(1+\xi^2)E_4\big)+\xi(1+\xi^2)^2E_5\big\}\\
&\qquad+c\xi^4(1+\xi^2)\big(|\hat u_1|^2+|\hat u_4|^2\big)+c\xi^6\big(|\hat u_2|^2+|\hat u_3|^2\big)+c\xi^2(1+\xi^2)^2|\hat u_5|^2\\
&\quad\le C(1+\xi^2)^4|\hat u_6|^2+C\xi^2(1+\xi^2)^2|\hat u_7|^2.
\end{aligned}$$

Moreover, we multiply (2.13), (2.15), and (2.30) with $\ell = 6$ by $\beta_4\xi^6$, $\beta_5\xi^6$, and $\beta_6\xi^6$, respectively, and combine the resultant equations and the above inequality. Then, by letting β_4, β_5, and β_6 be suitably small, this yields

$$\begin{aligned}\partial_t\big\{&\beta_2\beta_3\xi^4(1+\xi^2)E_1 + \beta_1\beta_2\beta_3\xi^5E_2 + \xi^4(\beta_2\beta_3+\beta_4\xi^2)(\xi E_3 + F_1)\\&+\xi^3\big(\beta_3(1+\xi^2)+\beta_5\xi^4\big)E_4 + \xi\big((1+\xi^2)^2+\beta_6\xi^6\big)E_5\big\}\\&+c\xi^4(1+\xi^2)|\hat{u}_1|^2 + c\xi^6|\hat{u}_2|^2 + c\xi^6(1+\xi^2)|\hat{u}_3|^2\\&+c\xi^4(1+\xi^2)^2|\hat{u}_4|^2 + c\xi^2(1+\xi^2)^3|\hat{u}_5|^2\\&\le C(1+\xi^2)^4|\hat{u}_6|^2 + C\xi^2(1+\xi^2)^3|\hat{u}_7|^2.\end{aligned}$$

Consequently, this estimate is essentially the same as (2.33). Moreover, by using a similar argument, we can derive the same estimate in the cases $a_4^2 - 1 \neq 1$ and $a_4^2 + a_5^2 - 1 \neq 0$ (for details, see Step 3 in Section 2.2).

By using estimate (2.33), we construct the desired estimate. We multiply (2.30) with $\ell = 7$ and (2.33) by $(1+\xi^2)^4$ and $\beta_7\xi^2$, respectively, and combine the resultant equations. Moreover, letting β_7 be suitably small and using the Young inequality, we obtain

$$\begin{aligned}\partial_t\big\{&\beta_7\xi^2E + \xi(1+\xi^2)^4E_6\big\} + c\xi^6(1+\xi^2)|\hat{u}_1|^2 + c\xi^8|\hat{u}_2|^2 + c\xi^8(1+\xi^2)|\hat{u}_3|^2\\&+c\xi^6(1+\xi^2)^2|\hat{u}_4|^2 + c\xi^4(1+\xi^2)^3|\hat{u}_5|^2 + c\xi^2(1+\xi^2)^4|\hat{u}_6|^2\\&\le C(1+\xi^2)^5|\hat{u}_7|^2 + C\xi^2(1+\xi^2)^4|\hat{u}_8|^2.\end{aligned}$$

Eventually, by the induction argument with respect to j in (2.30), we can derive

$$\begin{aligned}&\partial_t\mathcal{E}_{m-2} + c\xi^{2(m-5)}(1+\xi^2)|\hat{u}_1|^2 + c\xi^{2(m-4)}|\hat{u}_2|^2\\&\quad+c\sum_{j=3}^{m-2}\xi^{2(m-j-1)}(1+\xi^2)^{j-2}|\hat{u}_j|^2\\&\le C(1+\xi^2)^{m-3}|\hat{u}_{m-1}|^2 + C\xi^2(1+\xi^2)^{m-4}|\hat{u}_m|^2\end{aligned}\tag{2.34}$$

for $m \ge 7$. Here we define $\mathcal{E}_{m-2}$ as $\mathcal{E}_5 = E$ and

$$\mathcal{E}_{m-2} = \beta_{m-1}\xi^2\mathcal{E}_{m-3} + \xi(1+\xi^2)^{m-4}E_{m-2}, \quad m \ge 8.$$

Now, multiplying (2.32) and (2.34) by $(1+\xi^2)^{m-3}$ and $\beta_m\xi^2$, respectively, and making the appropriate combination, we get

$$\begin{aligned}&\partial_t\mathcal{E}_{m-1} + c\xi^{2(m-4)}(1+\xi^2)|\hat{u}_1|^2 + c\xi^{2(m-3)}|\hat{u}_2|^2\\&\quad+c\sum_{j=3}^{m-1}\xi^{2(m-j)}(1+\xi^2)^{j-2}|\hat{u}_j|^2 \le C(1+\xi^2)^{m-2}|\hat{u}_m|^2.\end{aligned}\tag{2.35}$$

Finally, by combining (2.28) with (2.35), this yields

$$\begin{aligned}\partial_t\Big\{\frac{1}{2}(1+\xi^2)^{m-2}|\hat{u}|^2+\beta_{m+1}\mathcal{E}_{m-1}\Big\}&+c\xi^{2(m-4)}(1+\xi^2)|\hat{u}_1|^2\\ +c\xi^{2(m-3)}|\hat{u}_2|^2&+c\sum_{j=3}^{m}\xi^{2(m-j)}(1+\xi^2)^{j-2}|\hat{u}_j|^2\le 0.\end{aligned}\tag{2.36}$$

Thus, integrating the above estimate with respect to t, we obtain the energy estimate

$$\begin{aligned}\big|\hat{u}(t,\xi)\big|^2&+\int_0^t\Big\{\frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}|\hat{u}_1|^2+\frac{\xi^{2(m-3)}}{(1+\xi^2)^{m-2}}|\hat{u}_2|^2\\ &+\sum_{j=3}^{m}\frac{\xi^{2(m-j)}}{(1+\xi^2)^{m-j}}|\hat{u}_j|^2\Big\}\,d\tau\le C\big|\hat{u}(0,\xi)\big|^2\end{aligned}\tag{2.37}$$

for $m\ge 7$. Here we have used the inequality

$$c|\hat{u}|^2\le\frac{1}{2}|\hat{u}|^2+\frac{\beta_{m+1}}{(1+\xi^2)^{m-2}}\mathcal{E}_{m-1}\le C|\hat{u}|^2$$

for suitably small β_{m+1}. Furthermore, estimate (2.35) with (2.36) gives us the pointwise estimate

$$\big|\hat{u}(t,\xi)\big|\le Ce^{-c\lambda(\xi)t}\big|\hat{u}(0,\xi)\big|,\qquad \lambda(\xi)=\frac{\xi^{2(m-3)}}{(1+\xi^2)^{m-2}}$$

for $m\ge 7$. Therefore, together with the proof in Section 2.2, (2.2) is proved, and we then complete the proof of Theorem 2.1.

2.4. Construction of the matrices K and S

In this section, inspired by the energy method employed in Sections 2.2 and 2.3, we shall derive the matrices K and S. Based on the energy method of Step 2 in Section 2.2, we introduce the $(m\times m)$-matrices

$$S_1=\left(\begin{array}{cccc|c}0&0&0&1&\\0&0&0&0&\\0&0&0&0&0\\1&0&0&0&\\\hline &0&&&0\end{array}\right),\qquad S_2=\left(\begin{array}{cccc|c}0&0&0&0&\\0&0&1&0&\\0&1&0&0&0\\0&0&0&0&\\\hline &0&&&0\end{array}\right),$$

$$S_3=\left(\begin{array}{cccc|cc}0&0&0&0&0&\\0&0&0&0&1&\\0&0&0&0&0&0\\0&0&0&0&0&\\\hline 0&1&0&0&0&\\ &0&&&&0\end{array}\right),$$

and hence,

$$\tilde{S} = -a_5\big\{a_5(S_1 + a_4 S_2) - a_5(a_4^2 - 1)S_3\big\}$$

$$= -a_5 \left(\begin{array}{cccc|ccc} 0 & 0 & 0 & a_5 & 0 & & \\ 0 & 0 & a_4a_5 & 0 & 1-a_4^2 & & \\ 0 & a_4a_5 & 0 & 0 & 0 & & 0 \\ a_5 & 0 & 0 & 0 & 0 & & \\ \hline 0 & 1-a_4^2 & 0 & 0 & 0 & & \\ & 0 & & & & & 0 \end{array}\right).$$

Then, we multiply (1.4) by $\tilde{S}$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

$$\frac{1}{2}\partial_t\langle \tilde{S}\hat{u}, \hat{u}\rangle + \xi\big\langle i[\tilde{S}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\big\rangle + \big\langle [\tilde{S}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\big\rangle = 0, \tag{2.38}$$

where

$$\tilde{S}A_m = -a_5 \left(\begin{array}{cccc|ccc} 0 & 0 & a_4a_5 & 0 & a_5^2 & 0 & \\ 0 & 0 & 0 & a_5 & 0 & a_6(1-a_4^2) & \\ a_4a_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_5 & 0 & 0 & 0 & 0 & \\ \hline 1-a_4^2 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & & & & & 0 \end{array}\right),$$

$$\tilde{S}L_m = a_5^2 \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \\ \hline & 0 & & & 0 \end{array}\right).$$

Equality (2.38) is equivalent to (2.9). We note that the symmetric matrix $S_1 + a_4 S_2$ is the key matrix for the (4×4)-Timoshenko system (see [14], [15]). The symmetric matrix $\tilde{S}$ is one of the key matrices for system (1.4).

We introduce the $(m \times m)$-matrix

$$K_1 = \left(\begin{array}{cccc|c} 0 & -1 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ \hline & 0 & & & 0 \end{array}\right).$$

Then, we multiply (1.4) by $-i\xi K_1$ and take the inner product with $\hat{u}$. Moreover, taking the real part of the resultant equation, we have

$$-\frac{1}{2}\xi\partial_t\langle iK_1\hat{u},\hat{u}\rangle+\xi^2\big\langle[K_1A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle-\xi\big\langle i[K_1L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle=0, \tag{2.39}$$

where

$$K_1A_m=\left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hline & & O & & O \end{array}\right),\qquad K_1L_m=\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hline & & O & & O \end{array}\right).$$

Equality (2.39) is equivalent to (2.10).

We next introduce the $(m\times m)$-matrices

$$K_4=a_4\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & O \\ 0 & 0 & -1 & 0 & \\ \hline & & O & & O \end{array}\right),\qquad S_4=-a_4\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hline & & O & & O \end{array}\right).$$

Then, we multiply (1.4) by $-i\xi K_2$ and S_4, and we take the inner product with $\hat{u}$, respectively. Moreover, taking the real part of the resultant equations and combining these, we have

$$\begin{aligned}&\frac{1}{2}\partial_t\big\langle(S_4-i\xi K_4)\hat{u},\hat{u}\big\rangle+\xi^2\big\langle[K_4A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle+\big\langle[S_4L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle\\&\quad+\xi\big\langle i[S_4A_m-K_4L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle=0,\end{aligned} \tag{2.40}$$

where $S_4L_m=O$ and

$$K_4A_m=\left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & a_4^2 & 0 & a_4a_5 & O \\ 0 & 0 & 0 & -a_4^2 & 0 & \\ \hline 0 & 0 & 0 & 0 & 0 & \\ & O & & & & O \end{array}\right),$$

$$S_4A_m-K_4L_m=\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -a_4^2 & \\ 0 & 0 & 0 & 0 & O \\ 0 & 0 & 0 & 0 & \\ \hline & & O & & O \end{array}\right).$$

Equality (2.40) is equivalent to (2.12).

Similarly, we introduce the $(m \times m)$-matrix

$$K_5 = a_5 \left(\begin{array}{cccc:cc} 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \\ \hdashline 0 & 0 & 0 & -1 & 0 & \\ & & & & & \\ & 0 & & & & 0 \end{array}\right).$$

Then, we multiply (1.4) by $-i\xi K_5$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

$$-\frac{1}{2}\xi\partial_t\langle iK_5\hat{u},\hat{u}\rangle + \xi^2\big\langle [K_5A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle - \xi\big\langle i[K_5L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle = 0, \tag{2.41}$$

where

$$K_5A_m = \left(\begin{array}{cccc:ccc} 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5^2 & 0 & a_5a_6 & \\ \hdashline 0 & 0 & -a_4a_5 & 0 & -a_5^2 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & & & & \\ & 0 & & & & & 0 \end{array}\right),$$

$$K_5L_m = \left(\begin{array}{cccc:cc} 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \\ \hdashline a_5 & 0 & 0 & 0 & 0 & \\ & & & & & \\ & 0 & & & & 0 \end{array}\right).$$

Equality (2.41) is equivalent to (2.14).

Based on the energy method of Step 2 in Section 2.3, we introduce the $(m \times m)$-matrices

$$K_\ell = a_\ell \begin{pmatrix} & & & 0 & 0 & & & \\ & 0 & & \vdots & \vdots & & 0 & \\ & & & 0 & 0 & & & \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ & & & 0 & 0 & & & \\ & 0 & & \vdots & \vdots & & 0 & \\ & & & 0 & 0 & & & \end{pmatrix} \begin{matrix} \\ \\ \\ \ell-1 \\ \ell \\ \\ \\ \\ \end{matrix}$$

$$\qquad\qquad\qquad\qquad\qquad \ell-1 \quad\ \ell$$

for $\ell = 6, \ldots, m-1$. Then, we multiply (1.4) by $-iK_\ell$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

$$-\frac{1}{2}\xi\partial_t\langle iK_\ell\hat{u}, \hat{u}\rangle + \xi^2\big\langle [K_\ell A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\big\rangle = 0 \tag{2.42}$$

for $\ell = 6, \ldots, m-1$, where

$$K_\ell A_m = \left(\begin{array}{ccccccccccc} & & & 0 & 0 & 0 & 0 & & & \\ & \mathbf{0} & & \vdots & \vdots & \vdots & \vdots & & \mathbf{0} & \\ & & & 0 & 0 & 0 & 0 & & & \\ 0 & \cdots & 0 & 0 & a_\ell^2 & 0 & a_\ell a_{\ell+1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -a_{\ell-1}a_\ell & 0 & -a_\ell^2 & 0 & 0 & \cdots & 0 \\ & & & 0 & 0 & 0 & 0 & & & \\ & \mathbf{0} & & \vdots & \vdots & \vdots & \vdots & & \mathbf{0} & \\ & & & 0 & 0 & 0 & 0 & & & \end{array}\right) \begin{array}{c} \\ \\ \\ \ell-1 \\ \ell \\ \\ \\ \\ \end{array}$$
$$\qquad\qquad\qquad\qquad \ell-2 \quad\; \ell-1 \quad\; \ell \quad\; \ell+1$$

Moreover, we have

$$-\frac{1}{2}\xi\partial_t\langle iK_m\hat{u}, \hat{u}\rangle + \xi^2\big\langle [K_m A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\big\rangle - \xi\big\langle i[K_m L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\big\rangle = 0, \tag{2.43}$$

where

$$K_m A_m = \left(\begin{array}{cccccc} & & & 0 & 0 & 0 \\ & \mathbf{0} & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & a_m^2 & 0 \\ 0 & \cdots & 0 & -a_{m-1}a_m & 0 & -a_m^2 \end{array}\right),$$

$$K_m L_m = \left(\begin{array}{cccc} & & & 0 \\ & \mathbf{0} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & a_m\gamma \\ 0 & \cdots & 0 & 0 \end{array}\right).$$

Equalities (2.42) and (2.43) are equivalent to (2.29) and (2.31), respectively.

For the rest of this section, we construct the desired matrices. According to the strategy of Step 3 in Section 2.2, we first combine (2.38) and (2.39). More precisely, multiplying (2.38), (2.40), and (2.39) by $(1+\xi^2)$, $(1+\xi^2)$, and δ_1, respectively, and combining the resultant equations, we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t\big\langle \big\{(1+\xi^2)\mathcal{S} - i\xi\big(\delta_1K_1 + (1+\xi^2)K_4\big)\big\}\hat{u}, \hat{u}\big\rangle \\ &\quad + (1+\xi^2)\big\langle [\mathcal{S}L_m]^{\mathrm{sy}}\hat{u}, \hat{u}\big\rangle + \xi^2\big\langle \big[\big(\delta_1K_1 + (1+\xi^2)K_4\big)A_m\big]^{\mathrm{sy}}\hat{u}, \hat{u}\big\rangle \\ &\quad + \xi(1+\xi^2)\big\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u}, \hat{u}\big\rangle - \xi\big\langle i\big[\big(\delta_1K_1 + (1+\xi^2)K_4\big)L_m\big]^{\mathrm{asy}}\hat{u}, \hat{u}\big\rangle = 0. \end{aligned}$$

Here we define $\mathcal{S} = \tilde{S} + S_4$. We next multiply (2.41) with $\ell = 6$ and the above equation by $(1+\xi^2)^2$ and $\delta_2\xi^2$, respectively, and then combine the resultant equations; we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t\big\langle\big\{\delta_2\xi^2\big((1+\xi^2)\mathcal{S} - i\xi\big(\delta_1K_1 + (1+\xi^2)K_4\big)\big) - \xi(1+\xi^2)^2K_5\big\}\hat{u},\hat{u}\big\rangle\\
&\quad + \delta_2\xi^2(1+\xi^2)\big\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle + \delta_2\xi^3(1+\xi^2)\big\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle\\
&\quad + \xi^2\big\langle\big[\big(\delta_2\xi^2\big(\delta_1K_1 + (1+\xi^2)K_4\big) + (1+\xi^2)^2K_5\big)A_m\big]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle\\
&\quad - \xi\big\langle i\big[\big(\delta_2\xi^2\big(\delta_1K_1 + (1+\xi^2)K_4\big) + (1+\xi^2)^2K_5\big)L_m\big]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle = 0.
\end{aligned}$$

Moreover, multiplying (2.42) and the above equation by $(1+\xi^2)^3$ and $\delta_3\xi^2$, respectively, and combining the resultant equations, we get

$$\begin{aligned}
&\frac{1}{2}\partial_t\big\langle\big\{\delta_3\xi^2\big(\delta_2\xi^2\big((1+\xi^2)\mathcal{S} - i\xi\big(\delta_1K_1 + (1+\xi^2)K_4\big)\big)\\
&\quad - i\xi(1+\xi^2)^2K_5\big) - i\xi(1+\xi^2)^3K_6\big\}\hat{u},\hat{u}\big\rangle\\
&\quad + \delta_2\delta_3\xi^4(1+\xi^2)\big\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle + \delta_2\delta_3\xi^5(1+\xi^2)\big\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle\\
&\quad + \xi^2\big\langle\big[\big(\delta_3\xi^2\big(\delta_2\xi^2\big(\delta_1K_1 + (1+\xi^2)K_4\big) + (1+\xi^2)^2K_5\big)\\
&\quad + (1+\xi^2)^3K_6\big)A_m\big]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle\\
&\quad - \delta_3\xi^3\big\langle i\big[\big(\delta_2\xi^2\big(\delta_1K_1 + (1+\xi^2)K_4\big) + (1+\xi^2)^2K_5\big)L_m\big]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle = 0.
\end{aligned}$$

Consequently, by the induction argument with respect to ℓ in (2.42), we have

$$\begin{aligned}
&\frac{1}{2}\partial_t\Big\langle\Big\{\prod_{j=2}^{\ell-3}\delta_j\xi^{2(\ell-4)}(1+\xi^2)\mathcal{S} - i\xi\mathcal{K}_\ell\Big\}\hat{u},\hat{u}\Big\rangle\\
&\quad + \prod_{j=2}^{\ell-3}\delta_j\xi^{2(\ell-4)}(1+\xi^2)\big\langle[\mathcal{S}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle\\
&\quad + \prod_{j=2}^{\ell-3}\delta_j\xi^{2(\ell-4)+1}(1+\xi^2)\big\langle i[\mathcal{S}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle\\
&\quad + \xi^2\big\langle[\mathcal{K}_\ell A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle - \prod_{j=3}^{\ell-3}\delta_j\xi^{2(\ell-5)+1}\big\langle i[\mathcal{K}_5L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle = 0
\end{aligned} \tag{2.44}$$

for $5 \le \ell \le m-1$, where the last term on the left-hand side is replaced by $\xi\langle i[\mathcal{K}_5L_m]^{\mathrm{asy}}\hat{u},\hat{u}\rangle$ for $\ell = 5$. Here we define $\mathcal{K}_\ell$ as $\mathcal{K}_4 = \delta_1K_1 + (1+\xi^2)K_4$ and

$$\mathcal{K}_\ell = \delta_{\ell-3}\xi^2\mathcal{K}_{\ell-1} + (1+\xi^2)^{\ell-3}K_\ell$$

for $\ell \ge 5$. Therefore, we combine (2.43) and (2.44) with $\ell = m-1$. Then we can obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_t\Big\langle\Big\{\prod_{j=2}^{m-4}\delta_j\xi^{2(m-4)}(1+\xi^2)\mathcal{S}-i\xi\mathcal{K}_m\Big\}\hat{u},\hat{u}\Big\rangle+\xi^2\big\langle[\mathcal{K}_mA_m]^{\rm sy}\hat{u},\hat{u}\big\rangle\\
&\quad+\prod_{j=2}^{m-4}\delta_j\xi^{2(m-4)}(1+\xi^2)\big\langle[\mathcal{S}L_m]^{\rm sy}\hat{u},\hat{u}\big\rangle\\
&\quad+\prod_{j=2}^{m-4}\delta_j\xi^{2(m-4)+1}(1+\xi^2)\big\langle i[\mathcal{S}A_m]^{\rm asy}\hat{u},\hat{u}\big\rangle\\
&\quad-\prod_{j=3}^{m-4}\delta_j\xi^{2(m-5)+1}\big\langle i[\mathcal{K}_5L_m]^{\rm asy}\hat{u},\hat{u}\big\rangle\\
&\quad-\xi(1+\xi^2)^{m-3}\big\langle i[K_mL_m]^{\rm asy}\hat{u},\hat{u}\big\rangle=0.
\end{aligned}\tag{2.45}
$$

Finally, multiplying (2.45) by $\delta_{m-3}/(1+\xi^2)^{m-2}$ and combining (2.28) and the resultant equations, we can obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_t\Big\langle\Big[I+\frac{\delta_{m-3}}{(1+\xi^2)^{m-2}}\Big\{\prod_{j=2}^{m-4}\delta_j\xi^{2(m-4)}(1+\xi^2)\mathcal{S}-i\xi\mathcal{K}_m\Big\}\Big]\hat{u},\hat{u}\Big\rangle\\
&\quad+\langle L_m\hat{u},\hat{u}\rangle+\prod_{j=2}^{m-3}\delta_j\frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}\big\langle[\mathcal{S}L_m]^{\rm sy}\hat{u},\hat{u}\big\rangle\\
&\quad+\delta_{m-3}\frac{\xi^2}{(1+\xi^2)^{m-2}}\big\langle[\mathcal{K}_mA_m]^{\rm sy}\hat{u},\hat{u}\big\rangle\\
&\quad+\prod_{j=2}^{m-3}\delta_j\frac{\xi^{2(m-4)+1}}{(1+\xi^2)^{m-3}}\big\langle i[\mathcal{S}A_m]^{\rm asy}\hat{u},\hat{u}\big\rangle\\
&\quad-\prod_{j=3}^{m-3}\delta_j\frac{\xi^{2(m-5)+1}}{(1+\xi^2)^{m-2}}\big\langle i[\mathcal{K}_5L_m]^{\rm asy}\hat{u},\hat{u}\big\rangle\\
&\quad-\delta_{m-3}\frac{\xi}{1+\xi^2}\big\langle i[K_mL_m]^{\rm asy}\hat{u},\hat{u}\big\rangle=0,
\end{aligned}\tag{2.46}
$$

where I denotes an identity matrix. By letting $\delta_1,\dots,\delta_{m-3}$ be suitably small, (2.46) derives energy estimate (2.37). More precisely, noting that

$$
\begin{aligned}
\mathcal{K}_m=&\prod_{j=2}^{m-3}\delta_j\xi^{2(m-4)}\big(\delta_1K_1+(1+\xi^2)K_4\big)+(1+\xi^2)^{m-3}K_m\\
&+\sum_{k=3}^{m-3}\prod_{j=k}^{m-3}\delta_j\xi^{2(m-k-2)}(1+\xi^2)^{k-1}K_{k+2}
\end{aligned}
$$

for $m\geq 6$, we can estimate the dissipation terms as

$$
\begin{aligned}
&\langle L_m\hat{u},\hat{u}\rangle + \prod_{j=2}^{m-3}\delta_j\frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}\big\langle[\mathcal{S}L_m]^{\rm sy}\hat{u},\hat{u}\big\rangle \\
&\qquad + \delta_{m-3}\frac{\xi^2}{(1+\xi^2)^{m-2}}\big\langle[\mathcal{K}_m A_m]^{\rm sy}\hat{u},\hat{u}\big\rangle \\
&\qquad \ge c\Big\{\frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}|\hat{u}_1|^2 + \frac{\xi^{2(m-3)}}{(1+\xi^2)^{m-2}}|\hat{u}_2|^2 + \sum_{j=3}^{m}\frac{\xi^{2(m-j)}}{(1+\xi^2)^{m-j}}|\hat{u}_j|^2\Big\}
\end{aligned}
\tag{2.47}
$$

for suitably small $\delta_1,\dots,\delta_{m-3}$. Consequently, we conclude that our desired symmetric matrix S and skew-symmetric matrix K are described as

$$
S = \frac{\xi^{2(m-4)}}{(1+\xi^2)^{m-3}}\mathcal{S}, \qquad K = \frac{\xi^2}{(1+\xi^2)^{m-2}}\mathcal{K}_m.
$$

3. Model II

3.1. Main result II

In this section, we treat the Cauchy problem (1.1)–(1.2) with

$$
A_m = \left(\begin{array}{cc:cc:cccccc:cc}
0 & 1 & 0 & 0 & & & & & & & & \\
1 & 0 & 0 & 0 & & & & & & & & \\ \hdashline
0 & 0 & 0 & a_4 & 0 & & & & & & & \\
0 & 0 & a_4 & 0 & 0 & 0 & & & & \text{\Large 0} & & \\ \hdashline
 & & 0 & 0 & 0 & a_6 & & & & & & \\
 & & & 0 & a_6 & 0 & & & & & & \\
 & & & & & & \ddots & & & & & \\
 & & & & & & & & 0 & a_{m-2} & & \\
 & & \text{\Large 0} & & & & & & a_{m-2} & 0 & 0 & 0 \\ \hdashline
 & & & & & & & & 0 & 0 & 0 & a_m \\
 & & & & & & & & & 0 & a_m & 0
\end{array}\right),
$$

$$
L_m = \left(\begin{array}{c:cc:cccccc:cc:c}
0 & 0 & 0 & 0 & & & & & & & & \\ \hdashline
0 & \gamma & 1 & 0 & & & & & & & & \\
0 & -1 & 0 & 0 & 0 & & & & & \text{\Large 0} & & \\ \hdashline
0 & 0 & 0 & 0 & a_5 & & & & & & & \\
 & & 0 & -a_5 & 0 & & & & & & & \\
 & & & & & \ddots & & & & & & \\
 & & & & & & & 0 & a_{m-3} & 0 & & \\
 & & & & & & & -a_{m-3} & 0 & 0 & 0 & \\ \hdashline
 & & & & & & & 0 & 0 & 0 & a_{m-1} & 0 \\
 & \text{\Large 0} & & & & & & & 0 & -a_{m-1} & 0 & 0 \\ \hdashline
 & & & & & & & & & 0 & 0 & 0
\end{array}\right),
\tag{3.1}
$$

where integer $m \ge 4$ is even, $\gamma > 0$, and all elements a_j $(4 \le j \le m)$ are nonzero. We note that the system (1.1) with (3.1) for $m = 4$ is the Timoshenko system (see [14], [15]). For this problem, we can derive the following decay structure.

THEOREM 3.1

The Fourier image $\hat{u}$ of the solution u to the Cauchy problem (1.1)–(1.2) with (3.1) satisfies the pointwise estimate

$$\big|\hat{u}(t,\xi)\big| \le Ce^{-c\lambda(\xi)t}\big|\hat{u}_0(\xi)\big|, \tag{3.2}$$

where $\lambda(\xi) := \xi^{3m-10}/(1+\xi^2)^{2(m-3)}$. Furthermore, let $s \ge 0$ be an integer, and suppose that the initial data u_0 belong to $H^s \cap L^1$. Then the solution u satisfies the decay estimate

$$\big\|\partial_x^k u(t)\big\|_{L^2} \le C(1+t)^{-\frac{1}{3m-10}(\frac{1}{2}+k)}\|u_0\|_{L^1} + C(1+t)^{-\frac{\ell}{m-2}}\|\partial_x^{k+\ell}u_0\|_{L^2}$$

for $k+\ell \le s$. Here C and c are positive constants.

3.2. Energy method in the case $m = 6$

Ide, Haramoto, and Kawashima [14] and Ide and Kawashima [15] had already obtained the desired estimates in the case $m = 4$. Thus, we consider the case $m = 6$ in this section, which can shed light on the proof of the general case $m \ge 6$ to be given in Section 3.3. Then we rewrite the system (1.1) with (3.1) as

$$\begin{aligned} \partial_t\hat{u}_1 + i\xi\hat{u}_2 &= 0, \\ \partial_t\hat{u}_2 + i\xi\hat{u}_1 + \gamma\hat{u}_2 + \hat{u}_3 &= 0, \\ \partial_t\hat{u}_3 + i\xi a_4\hat{u}_4 - \hat{u}_2 &= 0, \\ \partial_t\hat{u}_4 + i\xi a_4\hat{u}_3 + a_5\hat{u}_5 &= 0, \\ \partial_t\hat{u}_5 + i\xi a_6\hat{u}_6 - a_5\hat{u}_4 &= 0, \\ \partial_t\hat{u}_6 + i\xi a_6\hat{u}_5 &= 0. \end{aligned} \tag{3.3}$$

Step 1. We first derive the basic energy equality for system (3.3) in the Fourier space. We multiply all the equations of (3.3) by $\bar{\hat{u}} = (\bar{\hat{u}}_1, \bar{\hat{u}}_2, \bar{\hat{u}}_3, \bar{\hat{u}}_4, \bar{\hat{u}}_5, \bar{\hat{u}}_6)^T$, respectively, and combine the resultant equations. Furthermore, taking the real part for the resultant equality, we arrive at the basic energy equality

$$\frac{1}{2}\partial_t|\hat{u}|^2 + \gamma|\hat{u}_2|^2 = 0. \tag{3.4}$$

Next we create the dissipation terms in the following two steps.

Step 2. In this step, we exactly follow the same idea as in Step 2 in Section 2.2 in order to obtain the dissipation of other components besides $\hat{u}_2$.

As the dissipative term $\hat{u}_2$ appears in the second equation of (3.3), we first consider the possibility of obtaining the dissipation of the hyperbolic term $i\xi\hat{u}_1$. For that, we multiply the first and second equations in (3.3) by $i\xi\bar{\hat{u}}_2$ and $-i\xi\bar{\hat{u}}_1$, respectively. Then, combining the resultant equations and taking the real part, we have

$$\xi\partial_t\Re(i\hat{u}_1\bar{\hat{u}}_2) + \xi^2\big(|\hat{u}_1|^2 - |\hat{u}_2|^2\big) + \gamma\xi\Re(i\hat{u}_1\bar{\hat{u}}_2) + \xi\Re(i\hat{u}_1\bar{\hat{u}}_3) = 0. \tag{3.5}$$

Next, we combine the fourth and sixth equations in (3.3), obtaining

$$\partial_t(\xi a_6\hat{u}_4 + ia_5\hat{u}_6) + i\xi^2 a_4 a_6\hat{u}_3 = 0.$$

Then multiplying the first equation in (3.3) and the resultant equation by $\xi a_6\bar{\hat{u}}_4 - ia_5\bar{\hat{u}}_6$ and $\bar{\hat{u}}_1$, combining the resultant equations, and taking the real part, we obtain

$$\begin{aligned}\partial_t\big\{a_6\xi\Re(\hat{u}_1\bar{\hat{u}}_4) - a_5\Re(i\hat{u}_1\bar{\hat{u}}_6)\big\}\\ - a_4a_6\xi^2\Re(i\hat{u}_1\bar{\hat{u}}_3) + a_6\xi^2\Re(i\hat{u}_2\bar{\hat{u}}_4) + a_5\xi\Re(\hat{u}_2\bar{\hat{u}}_6) = 0.\end{aligned} \tag{3.6}$$

To eliminate $\Re(i\hat{u}_1\bar{\hat{u}}_3)$, we multiply (3.5) and (3.6) by $a_4^2a_6^2\xi^2$ and $a_4a_6\xi$, respectively, and add the resultant equations. Then this yields

$$\begin{aligned}a_4a_6\xi\partial_t E_1^{(6)} + a_4^2a_6^2\xi^4\big(|\hat{u}_1|^2 - |\hat{u}_2|^2\big)\\ + a_4a_6^2\xi^3\Re(i\hat{u}_2\bar{\hat{u}}_4) + a_4a_5a_6\xi^2\Re(\hat{u}_2\bar{\hat{u}}_6) + \gamma a_4^2a_6^2\xi^3\Re(i\hat{u}_1\bar{\hat{u}}_2) = 0,\end{aligned} \tag{3.7}$$

where $E_1^{(6)} = a_6\xi\Re(\hat{u}_1\bar{\hat{u}}_4) - a_5\Re(i\hat{u}_1\bar{\hat{u}}_6) + a_4a_6\xi^2\Re(i\hat{u}_1\bar{\hat{u}}_2)$.

We multiply the second and third equations in (3.3) by $\bar{\hat{u}}_3$ and $\bar{\hat{u}}_2$, respectively. Then, combining the resultant equations and taking the real part, we have

$$\partial_t\Re(\hat{u}_2\bar{\hat{u}}_3) + |\hat{u}_3|^2 - |\hat{u}_2|^2 + \xi\Re(i\hat{u}_1\bar{\hat{u}}_3) - a_4\xi\Re(i\hat{u}_2\bar{\hat{u}}_4) + \gamma\Re(\hat{u}_2\bar{\hat{u}}_3) = 0. \tag{3.8}$$

By the Young inequality, (3.8) is estimated as

$$\partial_t E_3 + \frac{1}{2}|\hat{u}_3|^2 \le \xi^2|\hat{u}_1|^2 + (1+\gamma^2)|\hat{u}_2|^2 + a_4\xi\Re(i\hat{u}_2\bar{\hat{u}}_4), \tag{3.9}$$

where $E_3 = \Re(\hat{u}_2\bar{\hat{u}}_3)$.

Furthermore, we multiply the third and fourth equations of (3.3) by $-i\xi a_4\bar{\hat{u}}_4$ and $i\xi a_4\bar{\hat{u}}_3$, respectively. Then, combining the resultant equations and taking the real part, we have

$$-a_4\xi\partial_t\Re(i\hat{u}_3\bar{\hat{u}}_4) + a_4^2\xi^2\big(|\hat{u}_4|^2 - |\hat{u}_3|^2\big) + a_4\xi\Re(i\hat{u}_2\bar{\hat{u}}_4) - a_4a_5\xi\Re(i\hat{u}_3\bar{\hat{u}}_5) = 0. \tag{3.10}$$

By the Young inequality, the above equation is estimated as

$$\xi\partial_t E_4 + \frac{1}{2}a_4^2\xi^2|\hat{u}_4|^2 \le \frac{1}{2}|\hat{u}_2|^2 + a_4^2\xi^2|\hat{u}_3|^2 + a_4a_5\xi\Re(i\hat{u}_3\bar{\hat{u}}_5), \tag{3.11}$$

where $E_4 = -a_4\Re(i\hat{u}_3\bar{\hat{u}}_4)$.

We multiply the fourth and fifth equations in (3.3) by $a_5\bar{\hat{u}}_5$ and $a_5\bar{\hat{u}}_4$, respectively. Then, combining the resultant equations and taking the real part, we have

$$a_5\partial_t\Re(\hat{u}_4\bar{\hat{u}}_5) + a_5^2\big(|\hat{u}_5|^2 - |\hat{u}_4|^2\big) + a_4a_5\xi\Re(i\hat{u}_3\bar{\hat{u}}_5) - a_5a_6\xi\Re(i\hat{u}_4\bar{\hat{u}}_6) = 0.$$

By using the Young inequality, we obtain

$$\partial_t E_5 + \frac{1}{2}a_5^2|\hat{u}_5|^2 \le a_5^2|\hat{u}_4|^2 + \frac{1}{2}a_5^2\xi^2|\hat{u}_3|^2 + a_5a_6\xi\Re(i\hat{u}_4\bar{\hat{u}}_6), \tag{3.12}$$

where $E_5 = a_5\partial_t\Re(\hat{u}_4\bar{\hat{u}}_5)$.

Moreover, we multiply the last equation and the fifth equation in (3.3) by $i\xi a_6\bar{\hat{u}}_5$ and $-i\xi a_6\bar{\hat{u}}_6$, respectively. Then, combining the resultant equations and taking the real part, we have

$$-a_6\xi\partial_t\Re(i\hat{u}_5\bar{\hat{u}}_6)+a_6^2\xi^2\big(|\hat{u}_6|^2-|\hat{u}_5|^2\big)+a_5a_6\xi\Re(i\hat{u}_4\bar{\hat{u}}_6)=0.$$

Using the Young inequality, this yields

$$\xi\partial_t E_6+\frac{1}{2}a_6^2\xi^2|\hat{u}_6|^2\le a_6^2\xi^2|\hat{u}_5|^2+\frac{1}{2}a_5^2|\hat{u}_4|^2, \tag{3.13}$$

where $E_6=-a_6\Re(i\hat{u}_5\bar{\hat{u}}_6)$.

Step 3. In this step, we sum up the energy inequalities and derive the desired energy inequality. For this purpose, we first multiply (3.12) and (3.13) by ξ^2 and β_1, respectively. Then we combine the resultant equations, obtaining

$$\begin{aligned}&\partial_t\{\xi^2E_5+\beta_1\xi E_6\}+\frac{1}{2}\beta_1a_6^2\xi^2|\hat{u}_6|^2+\Big(\frac{1}{2}a_5^2-\beta_1a_6^2\Big)\xi^2|\hat{u}_5|^2\\&\le\Big(\frac{1}{2}\beta_1+\xi^2\Big)a_5^2|\hat{u}_4|^2+\frac{1}{2}a_5^2\xi^4|\hat{u}_3|^2+|a_5||a_6||\xi|^3|\hat{u}_4||\hat{u}_6|.\end{aligned}$$

Letting β_1 be suitably small and using the Young inequality, we get

$$\partial_t\{\xi^2E_5+\beta_1\xi E_6\}+c\xi^2\big(|\hat{u}_5|^2+|\hat{u}_6|^2\big)\le C(1+\xi^2)^2|\hat{u}_4|^2+\frac{1}{2}a_5^2\xi^4|\hat{u}_3|^2.$$

Moreover, combining the above estimate and (3.12), we get

$$\begin{aligned}&\partial_t\big\{(1+\xi^2)E_5+\beta_1\xi E_6\big\}+c(1+\xi^2)|\hat{u}_5|^2+c\xi^2|\hat{u}_6|^2\\&\le C(1+\xi^2)^2|\hat{u}_4|^2+\frac{1}{2}a_5^2\xi^2(1+\xi^2)|\hat{u}_3|^2.\end{aligned} \tag{3.14}$$

Second, we multiply (3.11) and (3.14) by $(1+\xi^2)^2$ and $\beta_2\xi^2$, respectively, and combine the resultant equations. Then we obtain

$$\begin{aligned}&\partial_t\big\{\beta_2\xi^2\big((1+\xi^2)E_5+\beta_1\xi E_6\big)+\xi(1+\xi^2)^2E_4\big\}\\&\quad+\beta_2c\xi^2(1+\xi^2)|\hat{u}_5|^2+\beta_2c\xi^4|\hat{u}_6|^2+\Big(\frac{1}{2}a_4^2-\beta_2C\Big)\xi^2(1+\xi^2)^2|\hat{u}_4|^2\\&\le C(1+\xi^2)^2|\hat{u}_2|^2+C\xi^2(1+\xi^2)^2|\hat{u}_3|^2+C\xi(1+\xi^2)^2\Re(i\hat{u}_3\bar{\hat{u}}_5).\end{aligned}$$

Letting β_2 be suitably small and using the Young inequality, we get

$$\begin{aligned}&\partial_t\big\{\beta_2\xi^2\big((1+\xi^2)E_5+\beta_1\xi E_6\big)+\xi(1+\xi^2)^2E_4\big\}+c\xi^2(1+\xi^2)|\hat{u}_5|^2\\&\quad+c\xi^4|\hat{u}_6|^2+c\xi^2(1+\xi^2)^2|\hat{u}_4|^2\le C(1+\xi^2)^2|\hat{u}_2|^2+C(1+\xi^2)^3|\hat{u}_3|^2.\end{aligned} \tag{3.15}$$

Third, we multiply (3.9) and (3.15) by $(1+\xi^2)^3$ and β_3, respectively, and combine the resultant equations. Then we obtain

$$\begin{aligned}&\partial_t\big\{\beta_3\big(\beta_2\xi^2\big((1+\xi^2)E_5+\beta_1\xi E_6\big)+\xi(1+\xi^2)^2E_4\big)+(1+\xi^2)^3E_3\big\}+\beta_3c\xi^4|\hat{u}_6|^2\\&\quad+\beta_3c\xi^2(1+\xi^2)|\hat{u}_5|^2+\beta_3c\xi^2(1+\xi^2)^2|\hat{u}_4|^2+\Big(\frac{1}{2}-\beta_3C\Big)(1+\xi^2)^3|\hat{u}_3|^2\\&\le C(1+\xi^2)^3|\hat{u}_2|^2+\xi^2(1+\xi^2)^3|\hat{u}_1|^2+a_4\xi(1+\xi^2)^3\Re(i\hat{u}_2\bar{\hat{u}}_4).\end{aligned}$$

Therefore, letting β_3 be suitably small and using the Young inequality, we get

$$\begin{aligned} &\partial_t\{\beta_3(\beta_2\xi^2((1+\xi^2)E_5+\beta_1\xi E_6)+\xi(1+\xi^2)^2E_4)+(1+\xi^2)^3E_3\} \\ &\quad +c\xi^4|\hat{u}_6|^2+c\xi^2(1+\xi^2)|\hat{u}_5|^2+c\xi^2(1+\xi^2)^2|\hat{u}_4|^2+c(1+\xi^2)^3|\hat{u}_3|^2 \\ &\le C(1+\xi^2)^4|\hat{u}_2|^2+\xi^2(1+\xi^2)^3|\hat{u}_1|^2. \end{aligned} \tag{3.16}$$

Fourth, we multiply (3.7) and (3.16) by $(1+\xi^2)^3$ and $\beta_4\xi^2$, respectively, and combine the resultant equalities. Moreover, letting β_4 be suitably small and using the Young inequality, we obtain

$$\begin{aligned} &\partial_t\tilde{E}+c\xi^6|\hat{u}_6|^2+c\xi^4(1+\xi^2)|\hat{u}_5|^2+c\xi^4(1+\xi^2)^2|\hat{u}_4|^2+c\xi^2(1+\xi^2)^3|\hat{u}_3|^2 \\ &\quad +c\xi^4(1+\xi^2)^3|\hat{u}_1|^2\le C(1+\xi^2)^5|\hat{u}_2|^2+a_4a_5a_6\xi^2(1+\xi^2)^3\Re(\hat{u}_2\bar{\hat{u}}_6), \end{aligned} \tag{3.17}$$

where we have defined

$$\begin{aligned} \tilde{E}&=\beta_4\xi^2(\beta_3(\beta_2\xi^2((1+\xi^2)E_5+\beta_1\xi E_6)+\xi(1+\xi^2)^2E_4)+(1+\xi^2)^3E_3) \\ &\quad +a_4a_6\xi(1+\xi^2)^3E_1^{(6)}. \end{aligned}$$

Moreover, to estimate $\Re(\hat{u}_2\bar{\hat{u}}_6)$, we multiply (3.17) by ξ^2 and use the Young inequality again. Then this yields

$$\begin{aligned} &\xi^2\partial_t\tilde{E}+c\xi^8|\hat{u}_6|^2+c\xi^6(1+\xi^2)|\hat{u}_5|^2+c\xi^6(1+\xi^2)^2|\hat{u}_4|^2 \\ &\quad +c\xi^4(1+\xi^2)^3|\hat{u}_3|^2+c\xi^6(1+\xi^2)^3|\hat{u}_1|^2\le C(1+\xi^2)^6|\hat{u}_2|^2. \end{aligned} \tag{3.18}$$

Finally, multiplying the basic energy (3.4) and (3.18) by $(1+\xi^2)^6$ and β_5, respectively, combining the resultant equations, and letting β_5 be suitably small, this yields

$$\begin{aligned} &\partial_t\Big\{\frac{1}{2}(1+\xi^2)^6|\hat{u}|^2+\beta_5\xi^2\tilde{E}\Big\}+c\xi^6(1+\xi^2)^3|\hat{u}_1|^2 \\ &\quad +c(1+\xi^2)^6|\hat{u}_2|^2+c\xi^4(1+\xi^2)^3|\hat{u}_3|^2+c\xi^6(1+\xi^2)^2|\hat{u}_4|^2 \\ &\quad +c\xi^6(1+\xi^2)|\hat{u}_5|^2+c\xi^8|\hat{u}_6|^2\le 0. \end{aligned} \tag{3.19}$$

Thus, integrating the above estimate with respect to t, we obtain the energy estimate

$$\begin{aligned} &|\hat{u}(t,\xi)|^2+\int_0^t\Big\{\frac{\xi^6}{(1+\xi^2)^3}|\hat{u}_1|^2+|\hat{u}_2|^2+\frac{\xi^4}{(1+\xi^2)^3}|\hat{u}_3|^2+\frac{\xi^6}{(1+\xi^2)^4}|\hat{u}_4|^2 \\ &\quad +\frac{\xi^6}{(1+\xi^2)^5}|\hat{u}_5|^2+\frac{\xi^8}{(1+\xi^2)^6}|\hat{u}_6|^2\Big\}\,d\tau\le C|\hat{u}(0,\xi)|^2. \end{aligned} \tag{3.20}$$

Here we have used the inequality

$$c|\hat{u}|^2\le\frac{1}{2}|\hat{u}|^2+\frac{\beta_5\xi^2}{(1+\xi^2)^6}\tilde{E}\le C|\hat{u}|^2 \tag{3.21}$$

for suitably small β_5. We note that the energy inequality (3.20) tells us not only the boundedness of the energy part but also the structure property of the dissipation part. More precisely, estimate (3.19) with (3.21) gives us the pointwise estimate

$$\left|\hat{u}(t,\xi)\right| \le Ce^{-c\lambda(\xi)t}\left|\hat{u}(0,\xi)\right|, \quad \lambda(\xi)=\frac{\xi^8}{(1+\xi^2)^6}.$$

This therefore proves (3.2) in the case $m=6$ for Theorem 3.1.

3.3. Energy method for Model II

Inspired by the concrete calculation in Section 3.2, we consider the more general situation $m\ge 6$. Then we rewrite our system (1.4) with (3.1) as

$$\begin{aligned}
&\partial_t\hat{u}_1 + i\xi\hat{u}_2 = 0,\\
&\partial_t\hat{u}_2 + i\xi\hat{u}_1 + \gamma\hat{u}_2 + \hat{u}_3 = 0,\\
&\partial_t\hat{u}_3 + i\xi a_4\hat{u}_4 - \hat{u}_2 = 0,\\
&\partial_t\hat{u}_j + i\xi a_j\hat{u}_{j-1} + a_{j+1}\hat{u}_{j+1} = 0, \quad j=4,6,\dots,m-2 \text{ (for even)},\\
&\partial_t\hat{u}_j + i\xi a_{j+1}\hat{u}_{j+1} - a_j\hat{u}_{j-1} = 0, \quad j=5,7,\dots,m-1 \text{ (for odd)},\\
&\partial_t\hat{u}_m + i\xi a_m\hat{u}_{m-1} = 0.
\end{aligned} \tag{3.22}$$

Step 1. We first derive the basic energy equality for system (1.4) in the Fourier space. Taking the inner product of (1.4) with $\hat{u}$, we have

$$\langle\hat{u}_t,\hat{u}\rangle + i\xi\langle A_m\hat{u},\hat{u}\rangle + \langle L_m\hat{u},\hat{u}\rangle = 0.$$

Taking the real part, we get the basic energy equality

$$\frac{1}{2}\partial_t|\hat{u}|^2 + \langle L_m\hat{u},\hat{u}\rangle = 0,$$

and hence,

$$\frac{1}{2}\partial_t|\hat{u}|^2 + \gamma|\hat{u}_2|^2 = 0. \tag{3.23}$$

Next we create the dissipation terms in the following two steps.

Step 2. We note that we had already derived some useful equations in Section 3.2. Indeed, (3.5), (3.9), (3.11), and (3.12) are valid for our general problem. Therefore, we adopt these equations in this section.

To eliminate $\Re(i\hat{u}_1\bar{\hat{u}}_3)$ in (3.5), we first prepare a useful equation. We combine the fourth equations with $j=4,\dots,2\ell$ in (3.22) inductively. Then we obtain

$$\partial_t\mathcal{U}_{2\ell} + i\xi(-i\xi)^{\ell-2}\prod_{j=2}^{\ell}a_{2j}\hat{u}_3 + \prod_{j=2}^{\ell}a_{2j+1}\hat{u}_{2\ell+1} = 0, \tag{3.24}$$

for $4\le 2\ell\le m-2$, where we have defined $\mathcal{U}_4=\hat{u}_4$ and

$$\mathcal{U}_{2\ell} = -i\xi a_{2\ell}\mathcal{U}_{2\ell-2} + \prod_{j=2}^{\ell-1}a_{2j+1}\hat{u}_{2\ell}.$$

Moreover, by combining the last equation in (3.22) and (3.24), this yields

$$i^{m/2}\partial_t\mathcal{U}_m - i\xi^{m/2-1}\prod_{j=2}^{m/2} a_{2j}\hat{u}_3 = 0. \tag{3.25}$$

Multiplying (3.25) by $-\bar{\hat{u}}_1$ and the first equation in (3.22) by $-\overline{i^{m/2}\mathcal{U}_m}$, combining the resultant equations, and taking the real part, we obtain

$$-\partial_t\Re(i^{m/2}\mathcal{U}_m\bar{\hat{u}}_1) - \prod_{j=2}^{m/2} a_{2j}\xi^{m/2-1}\Re(i\hat{u}_1\bar{\hat{u}}_3) + \xi\Re(i^{m/2+1}\mathcal{U}_m\bar{\hat{u}}_2) = 0. \tag{3.26}$$

To eliminate $\Re(i\hat{u}_1\bar{\hat{u}}_3)$, we multiply (3.5) by $\prod_{j=2}^{m/2} a_{2j}\xi^{m/2-2}$ and combine the resultant equation and (3.26). Then we obtain

$$\begin{aligned}&\partial_t E_1^{(m)} + \prod_{j=2}^{m/2} a_{2j}\xi^{m/2}\big(|\hat{u}_1|^2 - |\hat{u}_2|^2\big)\\ &\quad + \gamma\prod_{j=2}^{m/2} a_{2j}\xi^{m/2-1}\Re(i\hat{u}_1\bar{\hat{u}}_2) + \xi\Re(i^{m/2+1}\mathcal{U}_m\bar{\hat{u}}_2) = 0,\end{aligned} \tag{3.27}$$

where we have defined

$$E_1^{(m)} = \prod_{j=2}^{m/2} a_{2j}\xi^{m/2-1}\Re(i\hat{u}_1\bar{\hat{u}}_2) - \Re(i^{m/2}\mathcal{U}_m\bar{\hat{u}}_1).$$

For $\ell = 4, 6, \dots, m-2$, we multiply the fourth equation and fifth equation with $j=\ell$ and $j=\ell+1$ in (3.22) by $a_{\ell+1}\bar{\hat{u}}_{\ell+1}$ and $a_{\ell+1}\bar{\hat{u}}_\ell$, respectively. Then, combining the resultant equations and taking the real part, we have

$$\begin{aligned}&a_{\ell+1}\partial_t\Re(\hat{u}_\ell\bar{\hat{u}}_{\ell+1}) + a_{\ell+1}^2\big(|\hat{u}_{\ell+1}|^2 - |\hat{u}_\ell|^2\big)\\ &\quad + a_\ell a_{\ell+1}\xi\Re(i\hat{u}_{\ell-1}\bar{\hat{u}}_{\ell+1}) - a_{\ell+1}a_{\ell+2}\xi\Re(i\hat{u}_\ell\bar{\hat{u}}_{\ell+2}) = 0.\end{aligned} \tag{3.28}$$

By using the Young inequality, we obtain

$$\begin{aligned}&\partial_t E_{\ell+1} + \frac{1}{2}a_{\ell+1}^2|\hat{u}_{\ell+1}|^2\\ &\quad \le a_{\ell+1}^2|\hat{u}_\ell|^2 + \frac{1}{2}a_{\ell+1}^2\xi^2|\hat{u}_{\ell-1}|^2 + a_{\ell+1}a_{\ell+2}\xi\Re(i\hat{u}_\ell\bar{\hat{u}}_{\ell+2}),\end{aligned} \tag{3.29}$$

where $E_{\ell+1} = a_{\ell+1}\Re(\hat{u}_\ell\bar{\hat{u}}_{\ell+1})$.

For $\ell = 4, \dots, m-4$, we multiply the fourth and fifth equations with $j = \ell+2$ and $j=\ell+1$ in (3.22) by $i\xi a_{\ell+2}\bar{\hat{u}}_{\ell+1}$ and $-i\xi a_{\ell+2}\bar{\hat{u}}_{\ell+2}$, respectively. Then, combining the resultant equations and taking the real part, we have

$$\begin{aligned}&-a_{\ell+2}\xi\partial_t\Re(i\hat{u}_{\ell+1}\bar{\hat{u}}_{\ell+2}) + a_{\ell+2}^2\xi^2\big(|\hat{u}_{\ell+2}|^2 - |\hat{u}_{\ell+1}|^2\big)\\ &\quad + a_{\ell+1}a_{\ell+2}\xi\Re(i\hat{u}_\ell\bar{\hat{u}}_{\ell+2}) - a_{\ell+2}a_{\ell+3}\xi\Re(i\hat{u}_{\ell+1}\bar{\hat{u}}_{\ell+3}) = 0.\end{aligned} \tag{3.30}$$

Here, by using the Young inequality, we obtain

$$\begin{aligned} &\xi\partial_t E_{\ell+2} + \frac{1}{2}a_{\ell+2}^2\xi^2|\hat{u}_{\ell+2}|^2 \\ &\quad \le a_{\ell+2}^2\xi^2|\hat{u}_{\ell+1}|^2 + \frac{1}{2}a_{\ell+1}^2|\hat{u}_\ell|^2 + a_{\ell+2}a_{\ell+3}\xi\Re(i\hat{u}_{\ell+1}\bar{\hat{u}}_{\ell+3}), \end{aligned} \tag{3.31}$$

where $E_{\ell+2} = -a_{\ell+2}\Re(i\hat{u}_{\ell+1}\bar{\hat{u}}_{\ell+2})$.

Moreover, we multiply the last equation and the fifth equation with $j = m-1$ in (3.22) by $i\xi a_m\bar{\hat{u}}_{m-1}$ and $-i\xi a_m\bar{\hat{u}}_m$, respectively. Then, combining the resultant equations and taking the real part, we have

$$\begin{aligned} &-a_m\xi\partial_t\Re(i\hat{u}_{m-1}\bar{\hat{u}}_m) + a_m^2\xi^2\big(|\hat{u}_m|^2 - |\hat{u}_{m-1}|^2\big) \\ &\quad + a_{m-1}a_m\xi\Re(i\hat{u}_{m-2}\bar{\hat{u}}_m) = 0. \end{aligned} \tag{3.32}$$

By using the Young inequality, this yields

$$\xi\partial_t E_m + \frac{1}{2}a_m^2\xi^2|\hat{u}_m|^2 \le a_m^2\xi^2|\hat{u}_{m-1}|^2 + \frac{1}{2}a_{m-1}^2|\hat{u}_{m-2}|^2, \tag{3.33}$$

where $E_m = -a_m\Re(i\hat{u}_{m-1}\bar{\hat{u}}_m)$.

Step 3. In this step, we sum up the energy inequalities constructed in the previous step and then make the desired energy inequality. The strategy is essentially the same as in Section 3.2.

For this purpose, we first multiply (3.29) with $\ell = m-2$ and (3.33) by ξ^2 and β_1, respectively. Then we combine the resultant equations, obtaining

$$\begin{aligned} &\partial_t\{\xi^2 E_{m-1} + \beta_1\xi E_m\} + \frac{1}{2}\beta_1 a_m^2\xi^2|\hat{u}_m|^2 + \Big(\frac{1}{2}a_{m-1}^2 - \beta_1 a_m^2\Big)\xi^2|\hat{u}_{m-1}|^2 \\ &\quad \le \Big(\frac{1}{2}\beta_1 + \xi^2\Big)a_{m-1}^2|\hat{u}_{m-2}|^2 + \frac{1}{2}a_{m-1}^2\xi^4|\hat{u}_{m-3}|^2 + |a_{m-1}||a_m||\xi|^3|\hat{u}_{m-2}||\hat{u}_m|. \end{aligned}$$

Letting β_1 be suitably small and using the Young inequality, we get

$$\begin{aligned} &\partial_t\{\xi^2 E_{m-1} + \beta_1\xi E_m\} + c\xi^2\big(|\hat{u}_m|^2 + |\hat{u}_{m-1}|^2\big) \\ &\quad \le C(1+\xi^2)^2|\hat{u}_{m-2}|^2 + \frac{1}{2}a_{m-1}^2\xi^4|\hat{u}_{m-3}|^2. \end{aligned}$$

Moreover, combining the above estimate and (3.29) with $\ell = m-2$, we get

$$\begin{aligned} &\partial_t\big\{(1+\xi^2)E_{m-1} + \beta_1\xi E_m\big\} + c\xi^2|\hat{u}_m|^2 + c(1+\xi^2)|\hat{u}_{m-1}|^2 \\ &\quad \le C(1+\xi^2)^2|\hat{u}_{m-2}|^2 + \frac{1}{2}a_{m-1}^2\xi^2(1+\xi^2)|\hat{u}_{m-3}|^2. \end{aligned} \tag{3.34}$$

Second, we multiply (3.34) and (3.31) with $\ell = m-4$ by $\beta_2\xi^2$ and $(1+\xi^2)^2$, respectively, and combine the resultant equations. Then we obtain

$$\begin{aligned} &\partial_t\big\{\beta_2\xi^2\big((1+\xi^2)E_{m-1} + \beta_1\xi E_m\big) + \xi(1+\xi^2)^2E_{m-2}\big\} \\ &\qquad + \beta_2 c\xi^4|\hat{u}_m|^2 + \beta_2 c\xi^2(1+\xi^2)|\hat{u}_{m-1}|^2 + \Big(\frac{1}{2}a_{m-2}^2 - \beta_2 C\Big)\xi^2(1+\xi^2)^2|\hat{u}_{m-2}|^2 \\ &\quad \le C\xi^2(1+\xi^2)^2|\hat{u}_{m-3}|^2 + \frac{1}{2}a_{m-3}^2(1+\xi^2)^2|\hat{u}_{m-4}|^2 \\ &\qquad + C|\xi|(1+\xi^2)^2|\hat{u}_{m-3}||\hat{u}_{m-1}|. \end{aligned}$$

Letting β_2 be suitably small and using the Young inequality, we get

$$\begin{aligned}&\partial_t\big\{\beta_2\xi^2\big((1+\xi^2)E_{m-1}+\beta_1\xi E_m\big)+\xi(1+\xi^2)^2E_{m-2}\big\}\\&\qquad+c\xi^4|\hat{u}_m|^2+c\xi^2(1+\xi^2)|\hat{u}_{m-1}|^2+c\xi^2(1+\xi^2)^2|\hat{u}_{m-2}|^2\\&\quad\leq C(1+\xi^2)^3|\hat{u}_{m-3}|^2+\frac{1}{2}a_{m-3}^2(1+\xi^2)^2|\hat{u}_{m-4}|^2.\end{aligned}\tag{3.35}$$

Third, we multiply (3.35) and (3.29) with $\ell=m-4$ by β_3 and $(1+\xi^2)^3$, respectively, and combine the resultant equations. Then we obtain

$$\begin{aligned}&\partial_t\big\{\beta_3\big(\beta_2\xi^2\big((1+\xi^2)E_{m-1}+\beta_1\xi E_m\big)+\xi(1+\xi^2)^2E_{m-2}\big)+(1+\xi^2)^3E_{m-3}\big\}\\&\quad+\beta_3c\xi^4|\hat{u}_m|^2+\beta_3c\xi^2(1+\xi^2)|\hat{u}_{m-1}|^2+\beta_3c\xi^2(1+\xi^2)^2|\hat{u}_{m-2}|^2\\&\quad+\Big(\frac{1}{2}a_{m-3}^2-\beta_3C\Big)(1+\xi^2)^3|\hat{u}_{m-3}|^2\\&\leq C(1+\xi^2)^3|\hat{u}_{m-4}|^2+\frac{1}{2}a_{m-3}^2\xi^2(1+\xi^2)^3|\hat{u}_{m-5}|^2\\&\quad+C|\xi|(1+\xi^2)^3|\hat{u}_{m-4}||\hat{u}_{m-2}|.\end{aligned}$$

Therefore, letting β_3 be suitably small and using the Young inequality, we get

$$\begin{aligned}&\partial_t\big\{\beta_3\big(\beta_2\xi^2\big((1+\xi^2)E_{m-1}+\beta_1\xi E_m\big)+\xi(1+\xi^2)^2E_{m-2}\big)\\&\qquad+(1+\xi^2)^3E_{m-3}\big\}+c\xi^4|\hat{u}_m|^2+c\xi^2(1+\xi^2)|\hat{u}_{m-1}|^2\\&\qquad+c\xi^2(1+\xi^2)^2|\hat{u}_{m-2}|^2+c(1+\xi^2)^3|\hat{u}_{m-3}|^2\\&\quad\leq C(1+\xi^2)^4|\hat{u}_{m-4}|^2+\frac{1}{2}a_{m-3}^2\xi^2(1+\xi^2)^3|\hat{u}_{m-5}|^2.\end{aligned}\tag{3.36}$$

Inspired by the derivation of (3.34), (3.35), and (3.36), we can conclude that the inequality

$$\begin{aligned}&\partial_t\mathcal{E}_{m-5}+c\sum_{\ell=5}^{m}\xi^{2([\ell/2]-2)}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2\\&\quad\leq C(1+\xi^2)^{m-4}|\hat{u}_4|^2+\frac{1}{2}a_5^2\xi^2(1+\xi^2)^{m-5}|\hat{u}_3|^2\end{aligned}\tag{3.37}$$

is derived by the induction argument. Here $[\cdot]$ denotes the greatest integer function, $\mathcal{E}_1=\beta_1\xi E_m+(1+\xi^2)E_{m-1}$, and

$$\begin{aligned}\mathcal{E}_\ell&=\beta_\ell\xi^2\mathcal{E}_{\ell-1}+\xi(1+\xi^2)^\ell E_{m-\ell},\\\mathcal{E}_{\ell+1}&=\beta_{\ell+1}\mathcal{E}_\ell+(1+\xi^2)^{\ell+1}E_{m-(\ell+1)},\end{aligned}\tag{3.38}$$

for even integers ℓ with $\ell\geq 2$.

Furthermore, we multiply (3.37) and (3.11) by $\beta_{m-4}\xi^2$ and $(1+\xi^2)^{m-4}$, respectively, and combine the resultant equations. Then we obtain

$$
\begin{aligned}
&\partial_t \mathcal{E}_{m-4} + \beta_{m-4} c \sum_{\ell=5}^{m} \xi^{2([\ell/2]-1)} (1+\xi^2)^{m-\ell} |\hat{u}_\ell|^2 \\
&\quad + \Big(\frac{1}{2} a_4^2 - \beta_{m-4} C\Big) \xi^2 (1+\xi^2)^{m-4} |\hat{u}_4|^2 \\
&\le C\xi^2 (1+\xi^2)^{m-4} |\hat{u}_3|^2 + \frac{1}{2} (1+\xi^2)^{m-4} |\hat{u}_2|^2 + C|\xi| (1+\xi^2)^{m-4} |\hat{u}_3||\hat{u}_5|,
\end{aligned}
$$

where $\mathcal{E}_{m-4}$ is defined by (3.38) with $\ell = m-4$. Thus, letting β_{m-4} be suitably small and using the Young inequality, we obtain

$$
\begin{aligned}
&\partial_t \mathcal{E}_{m-4} + c \sum_{\ell=4}^{m} \xi^{2([\ell/2]-1)} (1+\xi^2)^{m-\ell} |\hat{u}_\ell|^2 \\
&\quad \le C(1+\xi^2)^{m-3} |\hat{u}_3|^2 + \frac{1}{2} (1+\xi^2)^{m-4} |\hat{u}_2|^2.
\end{aligned} \tag{3.39}
$$

Similarly, we multiply (3.39) and (3.9) by β_{m-3} and $(1+\xi^2)^{m-3}$, combine the resultant equalities, and take β_{m-3} suitably small. Then we have

$$
\begin{aligned}
&\partial_t \mathcal{E}_{m-3} + c \sum_{\ell=3}^{m} \xi^{2([\ell/2]-1)} (1+\xi^2)^{m-\ell} |\hat{u}_\ell|^2 \\
&\quad \le C(1+\xi^2)^{m-2} |\hat{u}_2|^2 + \xi^2 (1+\xi^2)^{m-3} |\hat{u}_1|^2,
\end{aligned} \tag{3.40}
$$

where $\mathcal{E}_{m-3}$ is defined by (3.38) with $\ell = m-3$.

To estimate $|\hat{u}_1|^2$ in (3.40), we next employ (3.27). Namely, we multiply (3.27) and (3.40) by $(1+\xi^2)^{m-3}$ and $\beta_{m-2}\alpha_m \xi^{m/2-2}$, respectively. Then we combine the resultant equations, obtaining

$$
\begin{aligned}
&\partial_t \big\{ \beta_{m-2}\alpha_m \xi^{m/2-2} \mathcal{E}_{m-3} + (1+\xi^2)^{m-3} E_1^{(m)} \big\} \\
&\quad + \beta_{m-2}\alpha_m c \xi^{m/2-2} \sum_{\ell=3}^{m} \xi^{2([\ell/2]-1)} (1+\xi^2)^{m-\ell} |\hat{u}_\ell|^2 \\
&\quad + \alpha_m (1-\beta_{m-2}) \xi^{m/2} (1+\xi^2)^{m-3} |\hat{u}_1|^2 \\
&\le C\xi^{m/2-2} (1+\xi^2)^{m-2} |\hat{u}_2|^2 + \gamma \alpha_m \xi^{m/2-1} (1+\xi^2)^{m-3} \Re(i\hat{u}_1 \bar{\hat{u}}_2) \\
&\quad + \xi (1+\xi^2)^{m-3} \Re(i^{m/2+1} \mathcal{U}_m \bar{\hat{u}}_2),
\end{aligned}
$$

where we have defined $\alpha_m = \prod_{j=2}^{m/2} a_{2j}$. Here, taking β_{m-2} suitably small and using the Young inequality, we get

$$
\begin{aligned}
&\partial_t \big\{ \beta_{m-2}\alpha_m \xi^{m/2-2} \mathcal{E}_{m-3} + (1+\xi^2)^{m-3} E_1^{(m)} \big\} \\
&\quad + c\xi^{m/2-2} \sum_{\ell=3}^{m} \xi^{2([\ell/2]-1)} (1+\xi^2)^{m-\ell} |\hat{u}_\ell|^2 + c\xi^{m/2} (1+\xi^2)^{m-3} |\hat{u}_1|^2 \\
&\le C\xi^{m/2-2} (1+\xi^2)^{m-2} |\hat{u}_2|^2 + \xi (1+\xi^2)^{m-3} \Re(i^{m/2+1} \mathcal{U}_m \bar{\hat{u}}_2).
\end{aligned} \tag{3.41}
$$

For the last term on the right-hand side of (3.41), we note that

$$\mathcal{U}_m = \Big(\prod_{j=0}^{m/2-3} a_{m-2j}\Big)(-i\xi)^{m/2-2}\hat{u}_4 + \Big(\prod_{j=2}^{m/2-1} a_{2j+1}\Big)\hat{u}_m$$
$$+ \sum_{k=3}^{m/2-1}\Big(\prod_{j=2}^{k-1} a_{2j+1}\Big)\Big(\prod_{j=0}^{m/2-1-k} a_{m-2j}\Big)(-i\xi)^{m/2-k}\hat{u}_{2k},$$

for $m \geq 6$, where the last term on the right-hand side is neglected in the case $m=6$. Then, substituting the above equality into (3.41), we obtain

$$\begin{aligned}&\partial_t\big\{\beta_{m-2}\alpha_m\xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)}\big\}\\ &\quad + c\xi^{m/2-2}\sum_{\ell=3}^{m}\xi^{2([\ell/2]-1)}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 + c\xi^{m/2}(1+\xi^2)^{m-3}|\hat{u}_1|^2\\ &\leq C\xi^{m/2-2}(1+\xi^2)^{m-2}|\hat{u}_2|^2 + C\sum_{k=2}^{m/2}|\xi|^{m/2+1-k}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_{2k}|.\end{aligned} \tag{3.42}$$

In order to control the term of $|\hat{u}_m|$ on the right-hand side of (3.42) we introduce the inequality

$$|\xi|^{3m/2-5}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_m| \leq \varepsilon\xi^{3m-10}|\hat{u}_m|^2 + C_\varepsilon(1+\xi^2)^{2(m-3)}|\hat{u}_2|^2.$$

Inspired by the above inequality, we multiply (3.42) by $\xi^{3m/2-6}$ and employ this inequality. Then we obtain

$$\begin{aligned}&\xi^{3m/2-6}\partial_t\big\{\beta_{m-2}\alpha_m\xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)}\big\} + (c-\varepsilon)\xi^{3m-10}|\hat{u}_m|^2\\ &\quad + c\xi^{2m-10}\sum_{\ell=3}^{m-1}\xi^{2[\ell/2]}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 + c\xi^{2m-6}(1+\xi^2)^{m-3}|\hat{u}_1|^2\\ &\leq \big\{C\xi^{2m-8} + C_\varepsilon(1+\xi^2)^{m-3}\big\}(1+\xi^2)^{m-3}|\hat{u}_2|^2\\ &\quad + C\sum_{k=2}^{m/2-1}|\xi|^{2m-5-k}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_{2k}|.\end{aligned}$$

Therefore, letting ε be suitably small, we have

$$\begin{aligned}&\xi^{3m/2-6}\partial_t\big\{\beta_{m-2}\alpha_m\xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)}\big\}\\ &\quad + c\xi^{2m-10}\sum_{\ell=3}^{m}\xi^{2[\ell/2]}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 + c\xi^{2m-6}(1+\xi^2)^{m-3}|\hat{u}_1|^2\\ &\leq C(1+\xi^2)^{2(m-3)}|\hat{u}_2|^2 + C\sum_{k=2}^{m/2-1}|\xi|^{2m-5-k}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_{2k}|.\end{aligned} \tag{3.43}$$

Moreover, applying the inequality

$$\begin{aligned}&|\xi|^{2m-5-k}(1+\xi^2)^{m-3}|\hat{u}_2||\hat{u}_{2k}|\\ &\quad\leq \varepsilon\xi^{2m-10+2k}(1+\xi^2)^{m-2k}|\hat{u}_{2k}|^2 + C_\varepsilon\xi^{2m-4k}(1+\xi^2)^{m-6+2k}|\hat{u}_2|^2\end{aligned}$$

to (3.43), we can get

$$\partial_t\mathcal{E}_{m-2} + c\xi^{2m-10}\sum_{\ell=3}^{m}\xi^{2[\ell/2]}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 + c\xi^{2m-6}(1+\xi^2)^{m-3}|\hat{u}_1|^2 \le C(1+\xi^2)^{2(m-3)}|\hat{u}_2|^2, \tag{3.44}$$

where we have defined $\mathcal{E}_{m-2} = \xi^{3m/2-6}(\beta_{m-2}\alpha_m\xi^{m/2-2}\mathcal{E}_{m-3} + (1+\xi^2)^{m-3}E_1^{(m)})$.

Finally, multiplying the basic energy (3.4) and (3.44) by $(1+\xi^2)^{2(m-3)}$ and β_{m-1}, respectively, combining the resultant equations, and letting β_{m-1} be suitably small, we obtain

$$\partial_t\Big\{\frac{1}{2}(1+\xi^2)^{2(m-3)}|\hat{u}|^2 + \beta_{m-1}\mathcal{E}_{m-2}\Big\} + c\xi^{2m-6}(1+\xi^2)^{m-3}|\hat{u}_1|^2 + c(1+\xi^2)^{2(m-3)}|\hat{u}_2|^2 + c\xi^{2m-10}\sum_{\ell=3}^{m}\xi^{2[\ell/2]}(1+\xi^2)^{m-\ell}|\hat{u}_\ell|^2 \le 0. \tag{3.45}$$

Thus, integrating the above estimate with respect to t, we obtain the energy estimate

$$\big|\hat{u}(t,\xi)\big|^2 + \int_0^t\Big\{\frac{\xi^{2m-6}}{(1+\xi^2)^{m-3}}|\hat{u}_1|^2 + |\hat{u}_2|^2 + \frac{\xi^{2m-10}}{(1+\xi^2)^{m-3}}\sum_{\ell=3}^{m}\frac{\xi^{2[\ell/2]}}{(1+\xi^2)^{\ell-3}}|\hat{u}_\ell|^2\Big\}\,d\tau \le C\big|\hat{u}(0,\xi)\big|^2, \tag{3.46}$$

where we use the inequality

$$c|\hat{u}|^2 \le \frac{1}{2}|\hat{u}|^2 + \frac{\beta_{m-1}}{(1+\xi^2)^{2(m-3)}}\mathcal{E}_{m-2} \le C|\hat{u}|^2 \tag{3.47}$$

for suitably small β_{m-1}. Furthermore, estimate (3.45) with (3.47) gives us the pointwise estimate

$$\big|\hat{u}(t,\xi)\big| \le Ce^{-c\lambda(\xi)t}\big|\hat{u}(0,\xi)\big|, \quad \lambda(\xi) = \frac{\xi^{3m-10}}{(1+\xi^2)^{2(m-3)}}.$$

This therefore proves (3.2) and completes the proof of Theorem 3.1.

3.4. Construction of the matrices K and S

In this section, inspired by the energy method stated in Sections 3.2 and 3.3, we derive the desired matrices K and S. Based on the energy method of Step 2 in Section 3.2, we first introduce the following $(m\times m)$-matrices:

$$K_1 = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & \\ -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ \hline & & 0 & & 0 \end{array}\right), \qquad K_4 = a_4\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & \\ \hline & & 0 & & 0 \end{array}\right).$$

Then, we multiply (1.4) by $-i\xi K_1$ and take the inner product with $\hat{u}$. Moreover, taking the real part of the resultant equation, we have

$$-\frac{1}{2}\xi\partial_t\langle iK_1\hat{u},\hat{u}\rangle+\xi^2\big\langle [K_1A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle-\xi\big\langle i[K_1L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle=0, \tag{3.48}$$

where

$$K_1A_m=\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ \hline & & 0 & & 0 \end{array}\right),\qquad K_1L_m=\left(\begin{array}{cccc|c} 0 & \gamma & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ \hline & & 0 & & 0 \end{array}\right).$$

Equality (3.48) is equivalent to (3.5). Similarly, by using the matrix K_4, we obtain

$$-\frac{1}{2}\xi\partial_t\langle iK_4\hat{u},\hat{u}\rangle+\xi^2\big\langle [K_4A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle-\xi\big\langle i[K_4L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle=0, \tag{3.49}$$

where

$$K_4A_m=a_4^2\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \\ \hline & & 0 & & 0 \end{array}\right),$$

$$K_4L_m=-a_4\left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 1 & 0 & 0 & 0 & \\ \hline & & 0 & & & 0 \end{array}\right).$$

Equality (3.49) is equivalent to (3.10).

We next introduce

$$S_3=\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \\ \hline & & 0 & & 0 \end{array}\right),\qquad \tilde{S}_\ell=\begin{pmatrix} & 0 & & 1 & & \\ & & & 0 & 0 & \\ & & & \vdots & & \\ 1 & 0 & \cdots & 0 & \cdots & 0 \\ & 0 & & \vdots & 0 & \\ & & & 0 & & \end{pmatrix}\ell \quad (\text{column } \ell)$$

for $2\le \ell\le m-1$. Then, by using the same argument, we can show that the equality

$$\frac{1}{2}\partial_t\langle S_3\hat{u},\hat{u}\rangle+\xi\big\langle i[S_3A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle+\big\langle [S_3L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle=0, \tag{3.50}$$

which satisfies

$$S_3A_m = \left(\begin{array}{cccc:c} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & a_4 & \\ 1 & 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \\ \hdashline & \mathbf{0} & & & \mathbf{0} \end{array}\right), \qquad S_3L_m = \left(\begin{array}{cccc:c} 0 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & \\ 0 & \gamma & 1 & 0 & \mathbf{0} \\ 0 & 0 & 0 & 0 & \\ \hdashline & \mathbf{0} & & & \mathbf{0} \end{array}\right),$$

is equivalent to (3.8). Similarly, we derive that

$$\frac{1}{2}\partial_t\langle \tilde{S}_{2j}\hat{u},\hat{u}\rangle + \xi\big\langle i[\tilde{S}_{2j}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle + \big\langle [\tilde{S}_{2j}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle = 0, \tag{3.51}$$

which satisfies

$$\tilde{S}_{2j}A_m = \begin{pmatrix} & & & & a_{2j} & & \\ & & \mathbf{0} & & 0 & \mathbf{0} & \\ & & & & \vdots & & \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ & & \mathbf{0} & & \vdots & \mathbf{0} & \\ & & & & 0 & & \end{pmatrix} 2j$$
$$\qquad\qquad\qquad 2j-1$$

and

$$\tilde{S}_{2j}L_m = \begin{pmatrix} 0 & \cdots & 0 & a_{2j+1} & 0 & \cdots & 0 \\ & & & 0 & & & \\ & \mathbf{0} & & \vdots & & \mathbf{0} & \\ & & & 0 & & & \end{pmatrix},$$
$$\qquad\qquad 2j+1$$

is equivalent to

$$\partial_t\Re(\hat{u}_1\bar{\hat{u}}_{2j}) - a_{2j}\xi\Re(i\hat{u}_1\bar{\hat{u}}_{2j-1}) + a_{2j+1}\Re(\hat{u}_1\bar{\hat{u}}_{2j+1}) + \xi\Re(i\hat{u}_2\bar{\hat{u}}_{2j}) = 0,$$

for $2 \le j \le (m-2)/2$. Therefore, to construct (3.26), we sum up (3.51) with respect to j with $2 \le j \le (m-2)/2$ and find that

$$\frac{1}{2}\partial_t\langle \tilde{\mathcal{S}}_{m-2}\hat{u},\hat{u}\rangle + \xi\big\langle i[\tilde{\mathcal{S}}_{m-2}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle + \big\langle [\tilde{\mathcal{S}}_{m-2}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle = 0 \tag{3.52}$$

is equivalent to (3.26). Here we define $\tilde{\mathcal{S}}_{2\ell}$ as $\tilde{\mathcal{S}}_4 = \tilde{S}_4$ and

$$\tilde{\mathcal{S}}_{2\ell} = a_{2\ell}\xi\tilde{\mathcal{S}}_{2\ell-2} + \prod_{j=2}^{\ell-1} a_{2j+1}\tilde{S}_{2\ell}$$

for $\ell \ge 3$. Consequently, multiplying (3.48) by $\prod_{j=2}^{m/2} a_{2j}\xi^{m/2-2}$ and combining the resultant equality and (3.52), we obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_t\Big\langle\Big(\tilde{\mathcal{S}}_{m-2} - i\prod_{j=2}^{m/2} a_{2j}\xi^{m/2-1}K_1\Big)\hat{u},\hat{u}\Big\rangle \\
&+\Big\langle\Big[\tilde{\mathcal{S}}_{m-2}L_m + \prod_{j=2}^{m/2} a_{2j}\xi^{m/2}K_1A_m\Big]^{\mathrm{sy}}\hat{u},\hat{u}\Big\rangle \\
&+\xi\big\langle i[\tilde{\mathcal{S}}_{m-2}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle - \prod_{j=2}^{m/2} a_{2j}\xi^{m/2-1}\big\langle i[K_1L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle = 0.
\end{aligned}
\tag{3.53}
$$

This equality is the same as (3.27).

Based on the energy method of Step 3 in Section 3.3, we next introduce the following $(m\times m)$-matrices:

$$
S_{\ell+1} = a_{\ell+1}\begin{pmatrix}
 & \mathbf{0} & & 0 & 0 & & \mathbf{0} & \\
 & & & \vdots & \vdots & & & \\
 & & & 0 & 0 & & & \\
0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 0 & & & \\
 & \mathbf{0} & & \vdots & \vdots & & \mathbf{0} & \\
 & & & 0 & 0 & & &
\end{pmatrix}
\begin{matrix} \\ \\ \\ \ell \\ \ell+1 \\ \\ \\ \\ \end{matrix}
$$
$$\qquad\qquad\qquad \ell \quad \ell+1$$

for $\ell = 4,6,\dots,m-2$. Then, we multiply (1.4) by $S_{\ell+1}$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

$$
\frac{1}{2}\partial_t\langle S_{\ell+1}\hat{u},\hat{u}\rangle + \xi\big\langle i[S_{\ell+1}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle + \big\langle [S_{\ell+1}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle = 0 \tag{3.54}
$$

for $\ell = 4,6,\dots,m-2$, where

$$
S_{\ell+1}A_m = a_\ell\begin{pmatrix}
 & \mathbf{0} & & 0 & 0 & 0 & 0 & & \mathbf{0} & \\
 & & & \vdots & \vdots & \vdots & \vdots & & & \\
 & & & 0 & 0 & 0 & 0 & & & \\
0 & \cdots & 0 & 0 & 0 & 0 & a_{\ell+2} & 0 & \cdots & 0 \\
0 & \cdots & 0 & a_\ell & 0 & 0 & 0 & 0 & \cdots & 0 \\
 & & & 0 & 0 & 0 & 0 & & & \\
 & \mathbf{0} & & \vdots & \vdots & \vdots & \vdots & & \mathbf{0} & \\
 & & & 0 & 0 & 0 & 0 & & &
\end{pmatrix}
\begin{matrix} \\ \\ \\ \ell \\ \ell+1 \\ \\ \\ \\ \end{matrix}
$$
$$\qquad\qquad\qquad \ell-1 \quad \ell \quad \ell+1 \quad \ell+2$$

and

$$
S_{\ell+1}L_m = a_{\ell+1}^2 \begin{pmatrix} & \mathbf{0} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \mathbf{0} & \\ 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ & \mathbf{0} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \mathbf{0} & \end{pmatrix} \begin{matrix} \\ \ell \\ \ell+1 \\ \\ \end{matrix} .
$$
$$
\hphantom{S_{\ell+1}L_m = a_{\ell+1}^2} \qquad \qquad \ell \qquad \ell+1
$$

We note that (3.54) is equivalent to (3.28).

Additionally, we introduce the following $(m \times m)$-matrices:

$$
K_{\ell+2} = a_{\ell+2} \begin{pmatrix} & \mathbf{0} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \mathbf{0} & \\ 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ & \mathbf{0} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \mathbf{0} & \end{pmatrix} \begin{matrix} \\ \ell+1 \\ \ell+2 \\ \\ \end{matrix}
$$
$$
\hphantom{K_{\ell+2} = a_{\ell+2}} \qquad \qquad \ell+1 \qquad \ell+2
$$

for $\ell = 4, 6, \dots, m-2$. Then, we multiply (1.4) by $-iK_{\ell+2}$ and take the inner product with $\hat{u}$. Furthermore, taking the real part of the resultant equation, we obtain

$$
-\frac{1}{2}\xi\partial_t\langle iK_{\ell+2}\hat{u}, \hat{u}\rangle + \xi^2\big\langle [K_{\ell+2}A_m]^{\mathrm{sy}}\hat{u}, \hat{u}\big\rangle - \xi\big\langle i[K_{\ell+2}L_m]^{\mathrm{asy}}\hat{u}, \hat{u}\big\rangle = 0, \tag{3.55}
$$

for $\ell = 4, 6, \dots, m-4$, where

$$
K_{\ell+2}A_m = a_{\ell+2}^2 \begin{pmatrix} & \mathbf{0} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \mathbf{0} & \\ 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ & \mathbf{0} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \mathbf{0} & \end{pmatrix} \begin{matrix} \\ \ell+1 \\ \ell+2 \\ \\ \end{matrix}
$$
$$
\hphantom{K_{\ell+2}A_m = a_{\ell+2}^2} \qquad \qquad \ell+1 \qquad \ell+2
$$

and

$$K_{\ell+2}L_m = a_{\ell+2}\begin{pmatrix} & & & 0 & 0 & 0 & 0 & & & \\ & 0 & & \vdots & \vdots & \vdots & \vdots & & 0 & \\ & & & 0 & 0 & 0 & 0 & & & \\ 0 & \cdots & 0 & 0 & 0 & 0 & -a_{\ell+3} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -a_{\ell+1} & 0 & 0 & 0 & 0 & \cdots & 0 \\ & & & 0 & 0 & 0 & 0 & & & \\ & 0 & & \vdots & \vdots & \vdots & \vdots & & 0 & \\ & & & 0 & 0 & 0 & 0 & & & \end{pmatrix} \begin{matrix} \\ \\ \\ \ell+1 \\ \ell+2 \\ \\ \\ \\ \end{matrix}.$$
$$\qquad\qquad\qquad\qquad \ell-1 \quad \ell \quad \ell+1 \quad \ell+3$$

Moreover, we have

$$-\frac{1}{2}\xi\partial_t\langle iK_m\hat{u},\hat{u}\rangle + \xi^2\big\langle [K_mA_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle - \xi\big\langle i[K_mL_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle = 0, \tag{3.56}$$

where

$$K_mA_m = a_m^2\begin{pmatrix} & & & 0 & 0 \\ & 0 & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

$$K_mL_m = a_{m-1}a_m\begin{pmatrix} & & & 0 & 0 & 0 \\ & 0 & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then, (3.55) and (3.56) are equivalent to (3.30) and (3.32), respectively.

For the rest of this section, we construct the desired matrices. According to the strategy of Step 3 in Section 3.2, we first combine (3.54) and (3.56). More precisely, multiplying (3.54) with $\ell = m-2$ and (3.56) by $(1+\xi^2)$ and δ_1, respectively, and combine the resultant equations, and we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t\big\langle\big\{(1+\xi^2)S_{m-1} - \delta_1 i\xi K_m\big\}\hat{u},\hat{u}\big\rangle \\ &\quad + (1+\xi^2)\big\langle[S_{m-1}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle + \delta_1\xi^2\big\langle[K_mA_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle \\ &\quad + \xi(1+\xi^2)\big\langle i[S_{m-1}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle - \delta_1\xi\big\langle i[K_1L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle = 0. \end{aligned}$$

We next multiply (3.55) with $\ell = m-4$ and the above equation by $(1+\xi^2)^2$ and $\delta_2\xi^2$, respectively, and combining the resultant equations, we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t\big\langle\big\{\delta_2\xi^2\big((1+\xi^2)S_{m-1} - \delta_1 i\xi K_m\big) - i\xi(1+\xi^2)^2K_{m-2}\big\}\hat{u},\hat{u}\big\rangle \\ &\quad + \delta_2\xi^2(1+\xi^2)\big\langle[S_{m-1}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle \end{aligned}$$

$$+\xi^2\big\langle\big[\big(\delta_1\delta_2\xi^2 K_m+(1+\xi^2)^2K_{m-2}\big)A_m\big]^{\rm sy}\hat u,\hat u\big\rangle$$
$$+\delta_2\xi^3(1+\xi^2)\big\langle i[S_{m-1}A_m]^{\rm asy}\hat u,\hat u\big\rangle$$
$$-\xi\big\langle i\big[\big(\delta_1\delta_2\xi^2 K_m+(1+\xi^2)^2K_{m-2}\big)L_m\big]^{\rm asy}\hat u,\hat u\big\rangle=0.$$

Furthermore, multiplying (3.54) with $\ell=m-4$ and the above equation by $(1+\xi^2)^3$ and δ_3, respectively, and combining the resultant equations, we get

$$\begin{aligned}&\frac12\partial_t\big\langle\big\{\delta_3\big(\delta_2\xi^2\big((1+\xi^2)S_{m-1}-\delta_1 i\xi K_m\big)\\&\qquad-i\xi(1+\xi^2)^2K_{m-2}\big)+(1+\xi^2)^3S_{m-3}\big\}\hat u,\hat u\big\rangle\\&\qquad+(1+\xi^2)\big\langle\big[\big(\delta_2\delta_3\xi^2S_{m-1}+(1+\xi^2)^2S_{m-3}\big)L_m\big]^{\rm sy}\hat u,\hat u\big\rangle\\&\qquad+\delta_3\xi^2\big\langle\big[\big(\delta_1\delta_2\xi^2K_m+(1+\xi^2)^2K_{m-2}\big)A_m\big]^{\rm sy}\hat u,\hat u\big\rangle\\&\qquad+\xi(1+\xi^2)\big\langle i\big[\big(\delta_2\delta_3\xi^2S_{m-1}+(1+\xi^2)^2S_{m-3}\big)A_m\big]^{\rm asy}\hat u,\hat u\big\rangle\\&\qquad-\delta_3\xi\big\langle i\big[\big(\delta_1\delta_2\xi^2K_m+(1+\xi^2)^2K_{m-2}\big)L_m\big]^{\rm asy}\hat u,\hat u\big\rangle=0.\end{aligned}\tag{3.57}$$

Now, we introduce the new matrices $\mathcal{K}_\ell$ and $\mathcal{S}_\ell$ as $\mathcal{K}_0=K_m$,

$$\mathcal{K}_\ell=\delta_{\ell-1}\delta_\ell\xi^2\mathcal{K}_{\ell-2}+(1+\xi^2)^\ell K_{m-\ell}$$

for $\ell\ge 2$, $\mathcal{S}_1=S_{m-1}$, and

$$\mathcal{S}_\ell=\delta_{\ell-1}\delta_\ell\xi^2\mathcal{S}_{\ell-2}+(1+\xi^2)^{\ell-1}S_{m-\ell}$$

for $\ell\ge 3$. Then (3.57) is rewritten as

$$\begin{aligned}&\frac12\partial_t\big\langle\big\{(1+\xi^2)\mathcal{S}_3-\delta_3 i\xi\mathcal{K}_2\big\}\hat u,\hat u\big\rangle+(1+\xi^2)\big\langle[\mathcal{S}_3L_m]^{\rm sy}\hat u,\hat u\big\rangle+\delta_3\xi^2\big\langle[\mathcal{K}_2A_m]^{\rm sy}\hat u,\hat u\big\rangle\\&\quad+\xi(1+\xi^2)\big\langle i[\mathcal{S}_3A_m]^{\rm asy}\hat u,\hat u\big\rangle-\delta_3\xi\big\langle i[\mathcal{K}_2L_m]^{\rm asy}\hat u,\hat u\big\rangle=0.\end{aligned}$$

Consequently, by the induction argument with respect to ℓ in (3.54) and (3.55), we arrive at

$$\begin{aligned}&\frac12\partial_t\big\langle\big\{(1+\xi^2)\mathcal{S}_{m-5}-\delta_{m-5}i\xi\mathcal{K}_{m-6}\big\}\hat u,\hat u\big\rangle+(1+\xi^2)\big\langle[\mathcal{S}_{m-5}L_m]^{\rm sy}\hat u,\hat u\big\rangle\\&\quad+\delta_{m-5}\xi^2\big\langle[\mathcal{K}_{m-6}A_m]^{\rm sy}\hat u,\hat u\big\rangle+\xi(1+\xi^2)\big\langle i[\mathcal{S}_{m-5}A_m]^{\rm asy}\hat u,\hat u\big\rangle\\&\quad-\delta_{m-5}\xi\big\langle i[\mathcal{K}_{m-6}L_m]^{\rm asy}\hat u,\hat u\big\rangle=0.\end{aligned}\tag{3.58}$$

Applying the Young inequality to (3.58), we can obtain (3.37).

Moreover, we multiply (3.49) and (3.58) by $(1+\xi^2)^{m-4}$ and $\delta_{m-4}\xi^2$, respectively, and combine the resultant equations. Then this yields

$$\begin{aligned}&\frac12\partial_t\big\langle\big\{\delta_{m-4}\xi^2(1+\xi^2)\mathcal{S}_{m-5}-i\xi\mathcal{K}_{m-4}\big\}\hat u,\hat u\big\rangle\\&\quad+\delta_{m-4}\xi^2(1+\xi^2)\big\langle[\mathcal{S}_{m-5}L_m]^{\rm sy}\hat u,\hat u\big\rangle+\xi^2\big\langle[\mathcal{K}_{m-4}A_m]^{\rm sy}\hat u,\hat u\big\rangle\\&\quad+\delta_{m-4}\xi^3(1+\xi^2)\big\langle i[\mathcal{S}_{m-5}A_m]^{\rm asy}\hat u,\hat u\big\rangle-\xi\big\langle i[\mathcal{K}_{m-4}L_m]^{\rm asy}\hat u,\hat u\big\rangle=0.\end{aligned}$$

Similarly, we multiply (3.50) and the above equation by $(1+\xi^2)^{m-3}$ and δ_{m-3}, respectively, and combine the resultant equations. Then we get

$$\begin{aligned}&\frac{1}{2}\partial_t\big\langle\big\{(1+\xi^2)\mathcal{S}_{m-3}-\delta_{m-3}i\xi\mathcal{K}_{m-4}\big\}\hat{u},\hat{u}\big\rangle+(1+\xi^2)\big\langle[\mathcal{S}_{m-3}L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle\\&\quad+\delta_{m-3}\xi^2\big\langle[\mathcal{K}_{m-4}A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle+\xi(1+\xi^2)\big\langle i[\mathcal{S}_{m-3}A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle\\&\quad-\delta_{m-3}\xi\big\langle i[\mathcal{K}_{m-4}L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle=0.\end{aligned}\tag{3.59}$$

By the Young inequality applied to (3.59), we can derive (3.40).

We next employ (3.53) as constructed before. Multiplying (3.53) and (3.59) by $(1+\xi^2)^{m-3}$ and $\delta_{m-2}\alpha_m\xi^{m/2-2}$, respectively, and combining the resultant equations, we get

$$\begin{aligned}&\frac{1}{2}\partial_t\big\langle\big\{(1+\xi^2)\mathcal{S}'-\alpha_m i\xi^{m/2-1}\mathcal{K}'\big\}\hat{u},\hat{u}\big\rangle+(1+\xi^2)\big\langle[\mathcal{S}'L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle\\&\quad+\alpha_m\xi^{m/2}\big\langle[\mathcal{K}'A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle+\xi(1+\xi^2)\big\langle i[\mathcal{S}'A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle\\&\quad-\alpha_m\xi^{m/2-1}\big\langle i[\mathcal{K}'L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle=0,\end{aligned}\tag{3.60}$$

where we have defined

$$\mathcal{S}'=\delta_{m-2}\alpha_m\xi^{m/2-2}\mathcal{S}_{m-3}+(1+\xi^2)^{m-4}\tilde{\mathcal{S}}_{m-2},$$
$$\mathcal{K}'=\delta_{m-2}\delta_{m-3}\mathcal{K}_{m-4}+(1+\xi^2)^{m-3}K_1$$

and had already defined $\alpha_m=\prod_{j=2}^{m/2}a_{2j}$. By (3.60), we can get (3.44).

Finally, multiplying (3.60) by $\delta_{m-1}\xi^{3m/2-6}/(1+\xi^2)^{2(m-3)}$ and combining (3.23) and the resultant equation, we can obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\Big\langle\Big[I+\frac{\delta_{m-1}}{(1+\xi^2)^{2(m-3)}}\big\{\xi^{3m/2-6}(1+\xi^2)\mathcal{S}'-\alpha_m i\xi^{2m-7}\mathcal{K}'\big\}\Big]\hat{u},\hat{u}\Big\rangle\\&\quad+\langle L_m\hat{u},\hat{u}\rangle+\delta_{m-1}\frac{\xi^{3m/2-6}}{(1+\xi^2)^{2m-7}}\big\langle[\mathcal{S}'L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle\\&\quad+\alpha_m\delta_{m-1}\frac{\xi^{2(m-3)}}{(1+\xi^2)^{2(m-3)}}\big\langle[\mathcal{K}'A_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle\\&\quad-\alpha_m\delta_{m-1}\frac{\xi^{2m-7}}{(1+\xi^2)^{2(m-3)}}\big\langle i[\mathcal{K}'L_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle\\&\quad+\delta_{m-1}\frac{\xi^{3m/2-5}}{(1+\xi^2)^{2m-7}}\big\langle i[\mathcal{S}'A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle=0,\end{aligned}\tag{3.61}$$

where I denotes an identity matrix. By letting $\delta_1,\dots,\delta_{m-1}$ be suitably small, (3.61) derives the energy estimate (3.46). To be more precise, we introduce

$$\mathcal{K}_{m-4}=(1+\xi^2)^{m-4}K_4+\sum_{k=3}^{m/2}\prod_{j=2}^{k-1}\delta_{m-2j}\delta_{m-2j-1}\xi^{2(k-2)}(1+\xi^2)^{m-2k}K_{2k}$$

for $m\geq 6$, and hence,

$$\begin{aligned}\mathcal{K}' &= (1+\xi^2)^{m-3}K_1 + \delta_{m-2}\delta_{m-3}(1+\xi^2)^{m-4}K_4 \\ &\quad + \delta_{m-2}\delta_{m-3}\sum_{k=3}^{m/2}\prod_{j=2}^{k-1}\delta_{m-2j}\delta_{m-2j-1}\xi^{2(k-2)}(1+\xi^2)^{m-2k}K_{2k}.\end{aligned} \tag{3.62}$$

Moreover, we find that

$$\mathcal{S}_{m-3} = (1+\xi^2)^{m-4}S_3 + \sum_{k=3}^{m/2}\prod_{j=2}^{k-1}\delta_{m-2j}\delta_{m-2j+1}\xi^{2(k-2)}(1+\xi^2)^{m-2k}S_{2k-1}$$

for $m \geq 6$, $\tilde{\mathcal{S}}_4 = \tilde{S}_4$, $\tilde{\mathcal{S}}_6 = a_5\tilde{S}_6 + a_6\xi\tilde{S}_4$, and

$$\begin{aligned}\tilde{\mathcal{S}}_{m-2} &= \prod_{j=2}^{m/2-2} a_{2j+1}\tilde{S}_{m-2} + \prod_{j=1}^{m/2-3} a_{m-2j}\xi^{m/2-3}\tilde{S}_4 \\ &\quad + \sum_{k=2}^{m/2-3}\Big(\prod_{j=2}^{m/2-k-1} a_{2j+1}\Big)\Big(\prod_{j=1}^{k-1} a_{m-2j}\Big)\xi^{k-1}\tilde{S}_{m-2k}\end{aligned}$$

for $m \geq 10$, and also

$$\begin{aligned}\mathcal{S}' &= \delta_{m-2}\alpha_m\xi^{m/2-2}(1+\xi^2)^{m-4}S_3 \\ &\quad + \alpha_m\sum_{k=3}^{m/2}\prod_{j=1}^{k-1}\delta_{m-2j}\delta_{m-2j+1}\xi^{m/2+2(k-3)}(1+\xi^2)^{m-2k}S_{2k-1} \\ &\quad + \prod_{j=2}^{m/2-2} a_{2j+1}(1+\xi^2)^{m-4}\tilde{\mathcal{S}}_{m-2} + \prod_{j=1}^{m/2-3} a_{m-2j}\xi^{m/2-3}(1+\xi^2)^{m-4}\tilde{S}_4 \\ &\quad + \sum_{k=2}^{m/2-3}\Big(\prod_{j=2}^{m/2-k-1} a_{2j+1}\Big)\Big(\prod_{j=1}^{k-1} a_{m-2j}\Big)\xi^{k-1}(1+\xi^2)^{m-4}\tilde{\mathcal{S}}_{m-2k}.\end{aligned} \tag{3.63}$$

Therefore, by using (3.62) and (3.63), we can estimate the dissipation terms as

$$\begin{aligned}&\langle L_m\hat{u},\hat{u}\rangle + \delta_{m-1}\frac{\xi^{3(m-4)/2}}{(1+\xi^2)^{2m-7}}\big\langle[\mathcal{S}'L_m]^{\mathrm{sy}}\hat{u},\hat{u}\big\rangle \\ &\quad + \delta_{m-1}\frac{\xi^{2(m-3)}}{(1+\xi^2)^{2(m-3)}}\big\langle[\mathcal{K}'A_m]^{\mathrm{asy}}\hat{u},\hat{u}\big\rangle \\ &\geq c\Big\{\frac{\xi^{2(m-3)}}{(1+\xi^2)^{m-3}}|\hat{u}_1|^2 + |\hat{u}_2|^2 + \sum_{j=2}^{m/2}\frac{\xi^{2(m+j-6)}}{(1+\xi^2)^{m+2j-7}}|\hat{u}_{2j-1}|^2 \\ &\quad + \sum_{j=2}^{m/2}\frac{\xi^{2(m+j-5)}}{(1+\xi^2)^{m+2j-6}}|\hat{u}_{2j}|^2\Big\},\end{aligned} \tag{3.64}$$

for suitably small $\delta_1, \ldots, \delta_{m-1}$. We note that this estimate is the same as the dissipation part of (3.46). Consequently, we conclude that our desired symmetric matrix S and skew-symmetric matrix K are described as

$$S = \frac{\xi^{3(m-4)/2}}{(1+\xi^2)^{2m-7}}\mathcal{S}', \qquad K = \frac{\xi^{2(m-3)}}{(1+\xi^2)^{2(m-3)}}\mathcal{K}'.$$

4. Alternative approach

4.1. General strategy

In this section, by using the Fourier energy method, we provide an alternative way to justify the dissipative structure of the linear symmetric hyperbolic system with relaxation (1.1). The key point of the approach is to derive from the above system a new system of m equations or inequalities

$$(I_1), (I_2), \dots, (I_j), \dots, (I_m),$$

in the Fourier space, such that their appropriate linear combination can capture the dissipation rate of all the degenerate components only over the frequency domain far from $|\xi| = 0$ and $|\xi| = \infty$. Precisely, for any $0 < \epsilon < M < \infty$, by considering

$$\sum_{j=1}^{m} c_j I_j \tag{4.1}$$

for an appropriate choice of constants $c_j > 0$ $(1 \le j \le m)$ which may depend on ϵ and M, we expect to obtain that, for $\epsilon \le |\xi| \le M$,

$$\partial_t \{ |\hat{u}|^2 + \Re E_1^{\mathrm{int}}(\hat{u}) \} + c_{\epsilon,M} |\hat{u}|^2 \le 0, \tag{4.2}$$

where $c_{\epsilon,M} > 0$ depending on ϵ and M is a constant and $E_1^{\mathrm{int}}(\hat{u})$ is an interactive functional such that $|\hat{u}|^2 + \Re E_1^{\mathrm{int}}(\hat{u}) \sim |\hat{u}|^2$ over $\epsilon \le |\xi| \le M$. To deal with the dissipation rate around $|\xi| = 0$ or $|\xi| = \infty$, instead of (4.1), we reconsider the frequency-weighted linear combination in the form of

$$\sum_{j=1}^{m} c_j \frac{|\xi|^{\alpha_j}}{(1+|\xi|)^{\alpha_j+\beta_j}} I_j. \tag{4.3}$$

Here $\alpha_j \ge 0$ and $\beta_j \ge 0$ $(1 \le j \le m)$ are constants to be chosen such that similar computations to those used for deriving (4.2) can be applied so as to obtain a Lyapunov inequality taking the form

$$\partial_t \{ |\hat{u}|^2 + \Re E^{\mathrm{int}}(\hat{u}) \} + c \sum_{j=1}^{m} \lambda_j(\xi) |\hat{u}_j|^2 \le 0, \tag{4.4}$$

for all $t \ge 0$ and all $\xi \in \mathbb{R}$, where $c > 0$ is a constant, $\lambda_j(\xi)$ $(j = 1, 2, \dots, m)$ are nonnegative rational functions of $|\xi|$, and $E^{\mathrm{int}}(\hat{u})$ is an interactive functional such that $|\hat{u}|^2 + \Re E^{\mathrm{int}}(\hat{u}) \sim |\hat{u}|^2$ for all $\xi \in \mathbb{R}$. If (4.4) was proved, then by defining

$$\lambda_{\min}(\xi) = \min_{1 \le j \le m} \lambda_j(\xi), \quad \xi \in \mathbb{R},$$

it follows that

$$|\hat{u}(t,\xi)|^2 \le C e^{-c\lambda_{\min}(\xi)t} |\hat{u}(0,\xi)|^2,$$

for all $t \geq 0$ and all $\xi \in \mathbb{R}$, which thus implies the dissipative structure of the considered system (1.1). Observe that $\lambda_j(\xi)$ $(1 \leq j \leq m)$ and hence $\lambda_{\min}(\xi)$ may depend on $\alpha_j \geq 0$ and $\beta_j \geq 0$ $(1 \leq j \leq m)$. In general, α_j and β_j are required to satisfy a series of inequalities such that (4.3) indeed can be applied to deduce (4.4) by using the Cauchy–Schwarz inequalities. Therefore, we always expect to choose constants α_j and β_j such that $\lambda_{\min}(\xi)$ is optimal in the sense that $\lambda_{\min}(\xi)$ may tend to zero when $|\xi| \to 0$ or $|\xi| \to \infty$ at the slowest rate. Finally, we remark that due to (4.2), which holds over $\epsilon \leq |\xi| \leq M$, considering (4.3) is equivalent to considering $\sum_{j=1}^m c_j|\xi|^{\alpha_j} I_j$ over $|\xi| \leq \epsilon$ with $0 < \epsilon \leq 1$ and $\sum_{j=1}^m c_j|\xi|^{-\beta_j} I_j$ over $|\xi| \geq M$ with $M \geq 1$. In this way, it is more convenient to derive those inequalities satisfied by $\lambda_j(\xi)$ $(1 \leq j \leq m)$.

Finally, we remark that, although the current section provides an alternative approach for the justification of decay structures obtained in the previous sections for two types of models, it is still far from being understood how this approach can be extended to the general hyperbolic systems by using similar computations.

4.2. Revisiting Model I

By using the same strategy as in Sections 2.2 and 2.3, one can obtain m identities (I_j) with $j = 1, 2, \ldots, m$ as follows:

$$
\begin{aligned}
(I_1):&\quad \partial_t\langle i\xi\hat{u}_2, \hat{u}_1\rangle + |\xi|^2|\hat{u}_2|^2 = -\langle i\xi\hat{u}_2, \hat{u}_4\rangle + |\xi|^2|\hat{u}_1|^2,\\
(I_2):&\quad \partial_t\langle -\hat{u}_1, \hat{u}_4\rangle + |\hat{u}_1|^2 = |\hat{u}_4|^2 + \langle i\xi\hat{u}_2, \hat{u}_4\rangle + \langle \hat{u}_1, i\xi a_4\hat{u}_3 + i\xi a_5\hat{u}_5\rangle,\\
(I_3):&\quad \partial_t\big\{\langle i\xi a_4\hat{u}_3, \hat{u}_4\rangle - \langle a_4\hat{u}_3, \hat{u}_2\rangle\big\} + a_4^2|\xi|^2|\hat{u}_3|^2\\
&\qquad = a_4^2|\xi|^2|\hat{u}_4|^2 + \langle i\xi a_4\hat{u}_3, -i\xi a_5\hat{u}_5\rangle + a_4^2\langle i\xi\hat{u}_4, \hat{u}_2\rangle,\\
(I_4):&\quad \partial_t\langle i\xi a_5\hat{u}_4, \hat{u}_5\rangle + a_5^2|\xi|^2|\hat{u}_4|^2\\
&\qquad = \langle i\xi a_5\hat{u}_4, -i\xi a_6\hat{u}_6\rangle + a_5^2|\xi|^2|\hat{u}_5|^2 + a_5a_4|\xi|^2\langle \hat{u}_3, \hat{u}_5\rangle + \langle i\xi a_5\hat{u}_1, \hat{u}_5\rangle,\\
(I_{j-1}):&\quad \partial_t\langle i\xi a_j\hat{u}_{j-1}, \hat{u}_j\rangle + a_j^2|\xi|^2|\hat{u}_{j-1}|^2\\
&\qquad = \langle i\xi a_j\hat{u}_{j-1}, -i\xi a_{j+1}\hat{u}_{j+1}\rangle + a_j^2|\xi|^2|\hat{u}_j|^2 + a_ja_{j-1}|\xi|^2\langle \hat{u}_{j-2}, \hat{u}_j\rangle,\\
&\qquad\quad j = 6, 7, \ldots, m-1,\\
(I_{m-1}):&\quad \partial_t\langle i\xi a_m\hat{u}_{m-1}, \hat{u}_m\rangle + a_m^2|\xi|^2|\hat{u}_{m-1}|^2\\
&\qquad = \langle i\xi a_m\hat{u}_{m-1}, -\gamma\hat{u}_m\rangle + a_m^2|\xi|^2|\hat{u}_m|^2 + a_{m-1}a_m|\xi|^2\langle \hat{u}_{m-2}, \hat{u}_m\rangle,\\
(I_m):&\quad \frac{1}{2}\partial_t|\hat{u}|^2 + \gamma|\hat{u}_m|^2 = 0.
\end{aligned}
$$

We note that the equations $(I_1), (I_2), (I_3), (I_4), (I_{j-1}), (I_{m-1}), (I_m)$ are parallel to (2.10), (2.6), (2.12), (2.14), (2.29), (2.29), (2.28), respectively. Hence, we omit the proof for the derivation of these equations.

Step 1. We claim that, for any $0 < \epsilon < M < \infty$, there is $c_{\epsilon,M} > 0$ such that, for all $\epsilon \leq |\xi| \leq M$,

$$\partial_t\{|\hat{u}|^2 + \Re E_1^{\rm int}(\hat{u})\} + c_{\epsilon,M}|\hat{u}|^2 \le 0, \tag{4.5}$$

where $E_1^{\rm int}(\hat{u})$ is an interactive functional chosen such that

$$|\hat{u}|^2 + \Re E_1^{\rm int}(\hat{u}) \sim |\hat{u}|^2. \tag{4.6}$$

Proof of claim

The key observation is that all the right-hand terms of identities (I_j) $(1 \le j \le m)$ can be absorbed by the left-hand dissipative terms after taking an appropriate linear combination of all identities. In fact, let us define

$$\begin{aligned} E_1^{\rm int}(\hat{u}) &= c_1\langle i\xi\hat{u}_2, \hat{u}_1\rangle + c_2\langle -\hat{u}_1, \hat{u}_4\rangle \\ &\quad + c_3\{\langle i\xi a_4\hat{u}_3, \hat{u}_4\rangle - \langle a_4\hat{u}_3, \hat{u}_2\rangle\} + \sum_{j=4}^{m-1} c_j\langle i\xi a_j\hat{u}_{j-1}, \hat{u}_j\rangle. \end{aligned}$$

By taking the real part of each identity (I_j), taking the sum $\sum_{j=1}^m c_j I_j$ with an appropriate choice of constants c_j $(1 \le j \le m)$, and applying the Cauchy–Schwarz inequality to the right-hand product terms, one can obtain (4.5), where constants c_j $(1 \le j \le m)$ depending on ϵ and M are chosen such that

$$0 < c_1 \ll c_2 \ll \cdots \ll c_{m-2} \ll c_{m-1} \ll 1 = c_m.$$

The detailed representation of the proof is omitted for brevity. Note that (4.6) holds true due to $|E^{\rm int}(\hat{u})| \le C_M c_{m-1}|\hat{u}|^2$ for some constant C_M depending on M and also due to the smallness of c_{m-1}. □

Step 2. Let $|\xi| \ge M$ for $M \ge 1$. We consider the weighted linear combination of identities (I_j) $(1 \le j \le m)$ in the form of

$$I_m + \sum_{j=1}^{m-1} c_j|\xi|^{-\beta_j} I_j,$$

where c_j $(1 \le j \le m-1)$ are chosen in terms of Step 2 and $\beta_j \ge 0$ are chosen such that all the right-hand product terms can be absorbed after using the Cauchy–Schwarz inequality. In fact, multiplying (I_j) by $|\xi|^{-\beta_j}$, one has

$$\begin{aligned}
(I_{\beta_1}):\quad & \partial_t\langle i\xi|\xi|^{-\beta_1}\hat{u}_2, \hat{u}_1\rangle + |\xi|^{2-\beta_1}|\hat{u}_2|^2 = -\langle i\xi|\xi|^{-\beta_1}\hat{u}_2, \hat{u}_4\rangle + |\xi|^{2-\beta_1}|\hat{u}_1|^2,\\
(I_{\beta_2}):\quad & \partial_t\langle -|\xi|^{-\beta_2}\hat{u}_1, \hat{u}_4\rangle + |\xi|^{-\beta_2}|\hat{u}_1|^2\\
&\quad = |\xi|^{-\beta_2}|\hat{u}_4|^2 + \langle i\xi|\xi|^{-\beta_2}\hat{u}_2, \hat{u}_4\rangle\\
&\qquad + \langle \hat{u}_1, i\xi|\xi|^{-\beta_2}a_4\hat{u}_3 + i\xi|\xi|^{-\beta_2}a_5\hat{u}_5\rangle,\\
(I_{\beta_3}):\quad & \partial_t\{\langle i\xi|\xi|^{-\beta_3}a_4\hat{u}_3, \hat{u}_4\rangle - \langle a_4|\xi|^{-\beta_3}\hat{u}_3, \hat{u}_2\rangle\} + a_4^2|\xi|^{2-\beta_3}|\hat{u}_3|^2\\
&\quad = a_4^2|\xi|^{2-\beta_3}|\hat{u}_4|^2 + \langle i\xi|\xi|^{-\beta_3}a_4\hat{u}_3, -i\xi a_5\hat{u}_5\rangle + a_4^2\langle i\xi|\xi|^{-\beta_3}\hat{u}_4, \hat{u}_3\rangle,\\
(I_{\beta_4}):\quad & \partial_t\langle i\xi|\xi|^{-\beta_4}a_5\hat{u}_4, \hat{u}_5\rangle + a_5^2|\xi|^{2-\beta_4}|\hat{u}_4|^2\\
&\quad = \langle i\xi|\xi|^{-\beta_4}a_5\hat{u}_4, -i\xi a_6\hat{u}_6\rangle
\end{aligned}$$

$$+ a_5^2|\xi|^{2-\beta_4}|\hat{u}_5|^2 + a_5a_4|\xi|^{2-\beta_4}\langle\hat{u}_3,\hat{u}_5\rangle + \big\langle i\xi|\xi|^{-\beta_4}a_5\hat{u}_1,\hat{u}_5\big\rangle,$$

$$(I_{\beta_{j-1}}):\quad \partial_t\big\langle i\xi|\xi|^{-\beta_{j-1}}a_j\hat{u}_{j-1},\hat{u}_j\big\rangle + a_j^2|\xi|^{2-\beta_{j-1}}|\hat{u}_{j-1}|^2$$
$$= \big\langle i\xi|\xi|^{-\beta_{j-1}}a_j\hat{u}_{j-1},-i\xi a_{j+1}\hat{u}_{j+1}\big\rangle + a_j^2|\xi|^{2-\beta_{j-1}}|\hat{u}_j|^2$$
$$+ a_ja_{j-1}|\xi|^{2-\beta_{j-1}}\langle\hat{u}_{j-2},\hat{u}_j\rangle,\quad j=6,7,\dots,m-1,$$

$$(I_{\beta_{m-1}}):\quad \partial_t\big\langle i\xi|\xi|^{-\beta_{m-1}}a_m\hat{u}_{m-1},\hat{u}_m\big\rangle + a_m^2|\xi|^{2-\beta_{m-1}}|\hat{u}_{m-1}|^2$$
$$= \big\langle i\xi|\xi|^{-\beta_{m-1}}a_m\hat{u}_{m-1},-\gamma\hat{u}_m\big\rangle + a_m^2|\xi|^{2-\beta_{m-1}}|\hat{u}_m|^2$$
$$+ a_{m-1}a_m|\xi|^{2-\beta_{m-1}}\langle\hat{u}_{m-2},\hat{u}_m\rangle.$$

We then require β_j $(1\le j\le m-1)$ to satisfy the following relations. From (I_{β_1}),

$$\beta_1-1\ge 0,\qquad \beta_1-2\ge 0,$$
$$2(\beta_1-1)\ge(\beta_1-2)+(\beta_4-2),\qquad \beta_1-2\ge\beta_2,$$

where, since $|\xi|\ge M$, $\beta_1-1\ge 0$ is such that $\xi|\xi|^{-\beta_1}$ in the left first product term of (I_{β_1}) is bounded, $\beta_1-2\ge 0$ is such that $|\xi|^{2-\beta_1}$ in the left second product term of (I_{β_1}) is bounded, $2(\beta_1-1)\ge(\beta_1-2)+(\beta_4-2)$ is such that the product term $\langle i\xi|\xi|^{-\beta_1}\hat{u}_2,\hat{u}_4\rangle$ on the right first term of (I_{β_1}) can be bounded by the linear combination of the dissipative terms $|\xi|^{2-\beta_1}|\hat{u}_2|^2$ in (I_{β_1}) and $|\xi|^{2-\beta_4}|\hat{u}_4|^2$ in (I_{β_4}), and $\beta_1-2\ge\beta_2$ is such that the term $|\xi|^{2-\beta_1}|\hat{u}_1|^2$ on the right second term of (I_{β_1}) can be bounded by the dissipative term $|\xi|^{-\beta_2}|\hat{u}_1|^2$ of (I_{β_2}). In the same way, from (I_{β_j}) for $j=2,3,\dots,m-1$, respectively, we require

$$\beta_2\ge 0,$$
$$\beta_2\ge\beta_4-2,\qquad 2(\beta_2-1)\ge(\beta_1-2)+(\beta_4-2),\qquad \beta_2\ge\beta_3,\qquad \beta_2\ge\beta_5,$$
$$\beta_3-1\ge 0,\qquad \beta_3\ge 0,\qquad \beta_3-2\ge 0,$$
$$\beta_3-2\ge\beta_4-2,\qquad \beta_3\ge\beta_5,\qquad 2(\beta_3-1)\ge(\beta_3-2)+(\beta_4-2),$$
$$\beta_4\ge 1,\qquad \beta_4\ge 2,\qquad \beta_4\ge\beta_6,\qquad \beta_4\ge\beta_5,$$
$$2(\beta_4-2)\ge q(\beta_3-2)+(\beta_5-2),\qquad 2(\beta_4-1)\ge\beta_2+(\beta_5-2),$$

for $j=6,\dots,m-1$,

$$\beta_{j-1}\ge 1,\qquad \beta_{j-1}\ge 2,$$
$$\beta_{j-1}\ge\beta_{j+1},\qquad \beta_{j-1}\ge\beta_j,\qquad 2(\beta_{j-1}-2)\ge(\beta_{j-2}-2)+(\beta_j-2),$$

and

$$\beta_{m-1}\ge 1,\qquad \beta_{m-1}\ge 2,$$
$$2(\beta_{m-1}-1)\ge\beta_{m-1}-2,\qquad \beta_{m-1}-2\ge 0,\qquad 2(\beta_{m-1}-2)\ge\beta_{m-2}-2.$$

Let us choose

$$\beta_1=4,\qquad \beta_2=\beta_3=\cdots=\beta_{m-1}=2,$$

which satisfy all the above inequalities of β_j $(1\le j\le m-1)$.

We now define

$$E_\infty^{\rm int}(\hat u) = c_1\big\langle i\xi|\xi|^{-4}\hat u_2, \hat u_1\big\rangle + c_2\big\langle -|\xi|^{-2}\hat u_1, \hat u_4\big\rangle$$
$$+ c_3\big\{\big\langle i\xi|\xi|^{-2}a_4\hat u_3, \hat u_4\big\rangle - \big\langle a_4|\xi|^{-2}\hat u_3, \hat u_2\big\rangle\big\}$$
$$+\sum_{j=4}^{m-1} c_j\big\langle i\xi|\xi|^{-2}a_j\hat u_{j-1}, \hat u_j\big\rangle.$$

Then, as in Step 1, one can show that, for any $M\geq 1$, there is $c_M>0$ such that, for all $|\xi|\geq M$,

$$\partial_t\big\{|\hat u|^2 + \Re E_\infty^{\rm int}(\hat u)\big\} + c_M\Big\{|\xi|^{-2}\big(|\hat u_1|^2+|\hat u_2|^2\big) + \sum_{j=3}^m |\hat u_j|^2\Big\}\leq 0.$$

Step 3. Let $|\xi|\leq \epsilon$ for $0<\epsilon\leq 1$. As in Step 2, we consider the weighted linear combination of identities (I_j) $(1\leq j\leq m)$ in the form of

$$I_m + \sum_{j=1}^{m-1} c_j|\xi|^{\alpha_j} I_j,$$

where c_j $(1\leq j\leq m-1)$ are chosen in terms of Step 1 and $\alpha_j\geq 0$ are chosen such that all the right-hand product terms can be absorbed after using the Cauchy–Schwarz inequality. In fact, as in Step 2, multiplying (I_j) by $|\xi|^{\alpha_j}$, one has

$$(I_{\alpha_1}):\quad \partial_t\big\langle i\xi|\xi|^{\alpha_1}\hat u_2, \hat u_1\big\rangle + |\xi|^{2+\alpha_1}|\hat u_2|^2 = -\big\langle i\xi|\xi|^{\alpha_1}\hat u_2, \hat u_4\big\rangle + |\xi|^{2+\alpha_1}|\hat u_1|^2,$$

$$(I_{\alpha_2}):\quad \partial_t\big\langle -|\xi|^{\alpha_2}\hat u_1, \hat u_4\big\rangle + |\xi|^{\alpha_2}|\hat u_1|^2$$
$$= |\xi|^{\alpha_2}|\hat u_4|^2 + \big\langle i\xi|\xi|^{\alpha_2}\hat u_2, \hat u_4\big\rangle + \big\langle \hat u_1, i\xi|\xi|^{\alpha_2}a_4\hat u_3 + i\xi|\xi|^{\alpha_2}a_5\hat u_5\big\rangle,$$

$$(I_{\alpha_3}):\quad \partial_t\big\{\big\langle i\xi|\xi|^{\alpha_3}a_4\hat u_3, \hat u_4\big\rangle - \big\langle a_4|\xi|^{\alpha_3}\hat u_3, \hat u_2\big\rangle\big\} + a_4^2|\xi|^{2+\alpha_3}|\hat u_3|^2$$
$$= a_4^2|\xi|^{2+\alpha_3}|\hat u_4|^2 + \big\langle i\xi|\xi|^{\alpha_3}a_4\hat u_3, -i\xi a_5\hat u_5\big\rangle + a_4^2\big\langle i\xi|\xi|^{\alpha_3}\hat u_4, \hat u_3\big\rangle,$$

$$(I_{\alpha_4}):\quad \partial_t\big\langle i\xi|\xi|^{\alpha_4}a_5\hat u_4, \hat u_5\big\rangle + a_5^2|\xi|^{2+\alpha_4}|\hat u_4|^2$$
$$= \big\langle i\xi|\xi|^{\alpha_4}a_5\hat u_4, -i\xi a_6\hat u_6\big\rangle + a_5^2|\xi|^{2+\alpha_4}|\hat u_5|^2$$
$$+ a_5a_4|\xi|^{2+\alpha_4}\langle \hat u_3, \hat u_5\rangle + \big\langle i\xi|\xi|^{\alpha_4}a_5\hat u_1, \hat u_5\big\rangle,$$

$$(I_{\alpha_{j-1}}):\quad \partial_t\big\langle i\xi|\xi|^{\alpha_{j-1}}a_j\hat u_{j-1}, \hat u_j\big\rangle + a_j^2|\xi|^{2+\alpha_{j-1}}|\hat u_{j-1}|^2$$
$$= \big\langle i\xi|\xi|^{\alpha_{j-1}}a_j\hat u_{j-1}, -i\xi a_{j+1}\hat u_{j+1}\big\rangle + a_j^2|\xi|^{2+\alpha_{j-1}}|\hat u_j|^2$$
$$+ a_ja_{j-1}|\xi|^{2+\alpha_{j-1}}\langle \hat u_{j-2}, \hat u_j\rangle,\quad j=6,7,\dots,m-1,$$

$$(I_{\alpha_{m-1}}):\quad \partial_t\big\langle i\xi|\xi|^{\alpha_{m-1}}a_m\hat u_{m-1}, \hat u_m\big\rangle + a_m^2|\xi|^{2+\alpha_{m-1}}|\hat u_{m-1}|^2$$
$$= \big\langle i\xi|\xi|^{\alpha_{m-1}}a_m\hat u_{m-1}, -\gamma\hat u_m\big\rangle + a_m^2|\xi|^{2+\alpha_{m-1}}|\hat u_m|^2$$
$$+ a_{m-1}a_m|\xi|^{2+\alpha_{m-1}}\langle \hat u_{m-2}, \hat u_m\rangle.$$

As in the case of the large frequency domain, for $|\xi| \le \epsilon$ with $\epsilon > 0$, in order for all the right product terms to be bounded, from equations (I_{α_j}) $(j = 1, 2, \ldots, m-1)$ above, respectively, we have to require

$$\alpha_1 + 1 \ge 0, \qquad 2(\alpha_1 + 1) \ge (\alpha_1 + 2) + (\alpha_4 + 2), \qquad \alpha_1 + 2 \ge \alpha_2,$$

$$\alpha_2 \ge \alpha_4 + 2, \qquad 2(\alpha_2 + 1) \ge (\alpha_1 + 2) + (\alpha_4 + 2), \qquad \alpha_2 \ge \alpha_3, \qquad \alpha_2 \ge \alpha_5,$$

$$\alpha_3 \ge \alpha_4, \qquad \alpha_3 \ge \alpha_5, \qquad 2(\alpha_3 + 1) \ge (\alpha_4 + 2) + (\alpha_1 + 2),$$

$$\alpha_4 \ge \alpha_6, \qquad \alpha_4 \ge \alpha_5,$$

$$2(\alpha_4 + 2) \ge (\alpha_3 + 2) + (\alpha_5 + 2), \qquad 2(\alpha_4 + 1) \ge \alpha_2 + (\alpha_5 + 2),$$

for $j = 6, \ldots, m-1$,

$$\alpha_{j-1} \ge \alpha_{j+1}, \qquad \alpha_{j-1} \ge \alpha_j, \qquad 2(\alpha_{j-1} + 2) \ge (\alpha_{j-2} + 2) + (\alpha_j + 2),$$

and

$$\alpha_{m-1} \ge 0, \qquad \alpha_{m-1} + 2 \ge 0, \qquad 2(\alpha_{m-1} + 2) \ge \alpha_{m-2} + 2.$$

To consider the best choice of $\{\alpha_j\}_{j=1}^{m-1}$, one can see that

$$\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_j \ge \alpha_{j+1} \ge \cdots \ge \alpha_{m-2} \ge \alpha_{m-1} \ge 0 := \alpha_m,$$

with

$$\begin{aligned} &\alpha_1 - \alpha_4 \ge 2, \\ &\alpha_2 - \alpha_4 \ge 2, \\ &\alpha_3 - \alpha_4 \ge 2, \\ &\alpha_{j-1} - \alpha_j \le \alpha_j - \alpha_{j+1}, \quad 4 \le j \le m-1. \end{aligned}$$

Therefore, the possible best choice satisfies

$$\begin{aligned} &\alpha_1 - \alpha_4 = 2, \\ &\alpha_2 - \alpha_4 = 2, \\ &\alpha_3 - \alpha_4 = 2, \\ &2 = \alpha_3 - \alpha_4 \le \alpha_4 - \alpha_5 \le \cdots \le \alpha_{m-1} - \alpha_m = \alpha_{m-1} = 2, \end{aligned}$$

which implies

$$\begin{aligned} &\alpha_1 = \alpha_2 = \alpha_3 = 2(m-4), \\ &\alpha_j = 2(m-j-1), \quad 4 \le j \le m-1. \end{aligned}$$

We now define

$$\begin{aligned} E_0^{\rm int}(\hat{u}) = {} & c_1 \langle i\xi|\xi|^{2(m-4)} \hat{u}_2, \hat{u}_1 \rangle + c_2 \langle -|\xi|^{2(m-4)} \hat{u}_1, \hat{u}_4 \rangle \\ & + c_3 \{ \langle i\xi|\xi|^{2(m-4)} a_4 \hat{u}_3, \hat{u}_4 \rangle - \langle a_4 |\xi|^{2(m-4)} \hat{u}_3, \hat{u}_2 \rangle \} \\ & + \sum_{j=4}^{m-1} c_j \langle i\xi|\xi|^{2(m-j-1)} a_j \hat{u}_{j-1}, \hat{u}_j \rangle. \end{aligned}$$

Then, as in Step 1, one can show that, for any $0<\epsilon\leq 1$, there is $c_\epsilon>0$ such that, for all $|\xi|\leq\epsilon$,

$$\partial_t\big\{|\hat u|^2+\Re E_0^{\rm int}(\hat u)\big\}+c_\epsilon\bigg\{|\xi|^{2m-8}|\hat u_1|^2+|\xi|^{2m-6}|\hat u_2|^2+\sum_{j=3}^m|\xi|^{2(m-j)}|\hat u_j|^2\bigg\}\leq 0,$$

which further implies that, for $|\xi|\leq\epsilon$,

$$\partial_t\big\{|\hat u|^2+\Re E_0^{\rm int}(\hat u)\big\}+c_\epsilon|\xi|^{2m-6}|\hat u|^2\leq 0.$$

Step 4. For $\xi\in\mathbb{R}$ let us define

$$\begin{aligned}E^{\rm int}(\hat u)&=c_1\frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-4}}\langle i\xi\hat u_2,\hat u_1\rangle+c_2\frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-6}}\langle-\hat u_1,\hat u_4\rangle\\&\quad+c_3\frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-6}}\big\{\langle i\xi a_4\hat u_3,\hat u_4\rangle-\langle a_4\hat u_3,\hat u_2\rangle\big\}\\&\quad+\sum_{j=4}^{m-1}c_j\frac{|\xi|^{2(m-j-1)}}{(1+|\xi|)^{2(m-j)}}\langle i\xi a_j\hat u_{j-1},\hat u_j\rangle.\end{aligned}$$

As in Steps 2 and 3, we consider the weighted linear combination of identities (I_j) $(1\leq j\leq m)$ in the form of

$$\begin{aligned}&I_m+c_1\frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-4}}I_1+c_2\frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-6}}I_2\\&\quad+c_3\frac{|\xi|^{2(m-4)}}{(1+|\xi|)^{2m-6}}I_3+\sum_{j=4}^{m-1}c_j\frac{|\xi|^{2(m-j-1)}}{(1+|\xi|)^{2(m-j)}}I_j,\end{aligned}$$

where c_j $(1\leq j\leq m-1)$ are chosen in terms of Step 1. Thanks to computations in Steps 1, 2, and 3, in the same way, one can deduce that, for $\xi\in\mathbb{R}$,

$$\begin{aligned}&\partial_t\big\{|\hat u|^2+\Re E^{\rm int}(\hat u)\big\}+c\bigg\{\frac{|\xi|^{2m-8}}{(1+|\xi|)^{2m-6}}|\hat u_1|^2\\&\quad+\frac{|\xi|^{2m-6}}{(1+|\xi|)^{2m-4}}|\hat u_2|^2+\sum_{j=3}^m\frac{|\xi|^{2(m-j)}}{(1+|\xi|)^{2(m-j)}}|\hat u_j|^2\bigg\}\leq 0,\end{aligned}$$

which further gives

$$\partial_t\big\{|\hat u|^2+\Re E^{\rm int}(\hat u)\big\}+c\frac{|\xi|^{2m-6}}{(1+|\xi|)^{2m-4}}|\hat u|^2\leq 0.$$

By noticing $|\hat u|^2+\Re E^{\rm int}(\hat u)\sim|\hat u|^2$, it follows that

$$\big|\hat u(t,\xi)\big|\leq Ce^{-c\eta(\xi)t}\big|\hat u(0,\xi)\big|,\quad \eta(\xi)=\frac{|\xi|^{2m-6}}{(1+|\xi|)^{2m-4}},$$

for all $t\geq 0$ and all $\xi\in\mathbb{R}$. Notice that the result here is consistent with (2.2) proved in Section 2.3.

4.3. Revisiting Model II

In this section, we revisit Model II (1.1) with coefficient matrices A_m and L_m defined in (3.1). For simplicity of representation, we rewrite A_m with $m = 2n$ as

$$A_{2n} = \begin{pmatrix} 0 & a_{12} & & & & & \\ a_{21} & 0 & & & & & \\ & & 0 & a_{34} & & & \\ & & a_{43} & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & a_{2n-1,2n} \\ & & & & & a_{2n,2n-1} & 0 \end{pmatrix},$$

with $a_{2j-1,2j} = a_{2j,2j-1} = a_j$ for $1 \le j \le n$, and also choose L_m with $m = 2n$ as

$$L_{2n} = \begin{pmatrix} 0 & & & & & & & & \\ & 1 & 1 & & & & & & \\ & -1 & 0 & & & & & & \\ & & & 0 & 1 & & & & \\ & & & -1 & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & 1 & \\ & & & & & & -1 & 0 & \\ & & & & & & & & 0 \end{pmatrix}.$$

With notation as above, system (1.1) can read

$$\partial_t \hat{u}_{2j-1} + i\xi a_j \hat{u}_{2j} - \hat{u}_{2j-2} = 0,$$

$$\partial_t \hat{u}_{2j} + i\xi a_j \hat{u}_{2j-1} + \hat{u}_{2j+1} + \delta_{2,2j}\hat{u}_2 = 0, \quad j = 1, 2, \ldots, n,$$

with the convention that $\hat{u}_{2n+1} \equiv 0$ and $\hat{u}_0 \equiv 0$. As for Model I, we can obtain the estimates

$$\frac{1}{2}\partial_t |\hat{u}|^2 + |\hat{u}_2|^2 = 0, \tag{4.7}$$

$$\begin{aligned} &\partial_t \Re\Big\langle i\xi a_1 \hat{u}_1, \sum_{j=1}^{n} (-i\xi)^{1-j} \Big(\prod_{k=2}^{j} a_k\Big)^{-1} \hat{u}_{2j}\Big\rangle + c a_1^2 \xi^2 |\hat{u}_1|^2 \\ &\quad \lesssim \big(1 + |\xi|\big)^2 |\hat{u}_2|^2 + \Re\Big\langle \xi^2 a_1^2 \hat{u}_2, \sum_{j=2}^{n} (-i\xi)^{1-j} \Big(\prod_{k=2}^{j} a_k\Big)^{-1} \hat{u}_{2j}\Big\rangle, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\partial_t \Re\langle \hat{u}_{2j-1}, u_{2j-2}\rangle + c|\hat{u}_{2j-1}|^2 \\ &\quad \lesssim |\hat{u}_{2j-2}|^2 + \xi^2 |\hat{u}_{2j-3}|^2 + \Re\langle -i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-2}\rangle, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\partial_t \Re\langle i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-1}\rangle + c a_j^2 \xi^2 |\hat{u}_{2j}|^2 \\ &\quad \lesssim |\hat{u}_{2j-2}|^2 + a_j^2 \xi^2 |\hat{u}_{2j-1}|^2 + \Re\langle -i\xi a_j \hat{u}_{2j+1}, \hat{u}_{2j-1}\rangle, \end{aligned} \tag{4.10}$$

for $j = 2, 3, \ldots, n$. Indeed, by using (3.23), (3.27), (3.29), and (3.31) derived in Section 3.3, we can get (4.7), (4.8), (4.9), and (4.10) immediately.

Let us denote (4.7), (4.8), (4.9), and (4.10) by (I_1), (I_2), (I_{2j-1}), and (I_{2j}), respectively, where $j = 2, 3, \ldots, n$. Consider the linear combination of all $2n$ equations

$$\sum_{j=1}^{n} (c_{2j-1} I_{2j-1} + c_{2j} I_{2j}),$$

where $c_1 = 1$ and $c_k > 0$ $(k = 2, 3, \ldots, 2n)$ are constants to be properly chosen. It is straightforward to verify that, for any $0 < \epsilon < M < \infty$, one can choose constants c_k $(1 \le k \le 2n)$ depending on ϵ and M, with

$$0 < c_{2n} \ll c_{2n-1} \ll \cdots \ll c_{2j} \ll c_{2j-1} \ll \cdots \ll c_3 \ll c_2 \ll 1 = c_1,$$

such that there is $c_{\epsilon,M} > 0$ such that, for all $\epsilon \le |\xi| \le M$,

$$\partial_t \{ |\hat{u}|^2 + \Re E_1^{\rm int}(\hat{u}) \} + c_{\epsilon,M} |\hat{u}|^2 \le 0,$$

where $E_1^{\rm int}(\hat{u})$ is an interactive functional given by

$$\begin{aligned} E_1^{\rm int}(\hat{u}) = c_2 \Big\langle i\xi a_1 \hat{u}_1, \sum_{j=1}^{n} (-i\xi)^{1-j} \Big(\prod_{k=2}^{j} a_k \Big)^{-1} \hat{u}_{2j} \Big\rangle \\ + \sum_{j=2}^{n} \big\{ c_{2j-1} \langle \hat{u}_{2j-1}, u_{2j-2} \rangle + c_{2j} \langle i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-1} \rangle \big\}, \end{aligned}$$

satisfying

$$|\hat{u}|^2 + \Re E_1^{\rm int}(\hat{u}) \sim |\hat{u}|^2, \quad \text{for } \epsilon \le |\xi| \le M.$$

Furthermore, we can consider the frequency-weighted linear combination in the form of

$$\sum_{j=1}^{n} \Big\{ c_{2j-1} \frac{|\xi|^{\alpha_{2j-1}}}{(1+|\xi|)^{\alpha_{2j-1}+\beta_{2j-1}}} I_{2j-1} + c_{2j} \frac{|\xi|^{\alpha_{2j}}}{(1+|\xi|)^{\alpha_{2j}+\beta_{2j}}} I_{2j} \Big\}, \tag{4.11}$$

where $\alpha_1 = \beta_1 = 0$. As for Model I, we use the same strategy to determine the choice of constants

$$\alpha_2, \alpha_3, \ldots, \alpha_{2n}, \qquad \beta_2, \beta_3, \ldots, \beta_{2n}.$$

In fact, by considering the low-frequency domain $|\xi| \le \epsilon$ with $\epsilon \le 1$, $\alpha_2, \alpha_3, \ldots, \alpha_{2n}$ are required to satisfy inequalities

$$\begin{aligned} 2 - j + \alpha_2 \ge 0, &\quad j = 2, 3, \ldots, n, \\ \alpha_2 \ge 0, & \\ 2 + \alpha_2 \ge 0, & \\ \alpha_3 \ge 0, &\qquad 2 + \alpha_3 \ge 2 + \alpha_2, \\ \alpha_4 \ge 0, &\qquad 2 + \alpha_4 \ge \alpha_3, \end{aligned}$$

$$\alpha_{2j} \geq 2 + \alpha_{2j-2}, \qquad 2 + \alpha_{2j} \geq \alpha_{2j-1},$$
$$\alpha_{2j-1} \geq 2 + \alpha_{2j-2}, \qquad 2 + \alpha_{2j-1} \geq \alpha_{2j-3}, \quad j = 3, 4, \dots, n,$$

and

$$\begin{aligned} 2(3 - j + \alpha_2) &\geq \alpha_{2j} + 2, \quad j = 2, \dots, n, \\ 1 + \alpha_3 &\geq \frac{2 + \alpha_4}{2}, \\ \alpha_{2j} &\geq \frac{1}{2}(\alpha_{2j+1} + \alpha_{2j-1}) - 1, \\ \alpha_{2j+1} &\geq \frac{1}{2}(\alpha_{2j+2} + \alpha_{2j}) - 1, \quad j = 2, \dots, n-1. \end{aligned}$$

One can take the best choice

$$\begin{aligned} \alpha_2 &= 4(n-2), \\ \alpha_{2j-1} = \alpha_{2j} &= 4(n-2) + 2(j-2), \quad j = 2, 3, \dots, n. \end{aligned}$$

Similarly, by considering the high-frequency domain $|\xi| \geq M$ with $M \geq 1$, constants $\beta_2, \beta_3, \dots, \beta_{2n}$ are required to satisfy inequalities

$$\begin{aligned} &\beta_2 - 2 \geq 0, \\ &\beta_3 \geq 0, \qquad \beta_3 - 2 \geq \beta_2 - 2, \\ &\beta_4 \geq 0, \qquad \beta_4 - 2 \geq \beta_3, \\ &\beta_{2j} \geq \beta_{2j-2}, \qquad \beta_{2j} - 2 \geq \beta_{2j-1}, \\ &\beta_{2j-1} \geq \beta_{2j-2} - 2, \qquad \beta_{2j-1} - 2 \geq \beta_{2j-3}, \quad j = 3, \dots, n, \end{aligned}$$

and

$$\begin{aligned} &2(\beta_3 - 1) \geq \beta_4 - 2, \\ &\beta_2 + j - 2 \geq 0, \qquad 2(\beta_2 + j - 3) \geq \beta_{2j} - 2, \quad j = 2, \dots, n, \\ &2(\beta_{2j} - 1) \geq \beta_{2j+1} + \beta_{2j-1}, \\ &2(\beta_{2j+1} - 1) \geq (\beta_{2j+2} - 2) + (\beta_{2j} - 2), \quad j = 2, \dots, n-1. \end{aligned}$$

One can take the best choice

$$\beta_{2j} = \beta_{2j+1} = 2j, \quad j = 1, 2, \dots, n.$$

Now, by (4.11), let us define the interactive functional

$$\begin{aligned} E^{\text{int}}(\hat{u}) = c_2 \frac{|\xi|^{\alpha_2}}{(1+|\xi|)^{\alpha_2+\beta_2}} \Big\langle i\xi a_1 \hat{u}_1, \sum_{j=1}^{n} (-i\xi)^{1-j} \Big(\prod_{k=2}^{j} a_k\Big)^{-1} \hat{u}_{2j} \Big\rangle \\ + \sum_{j=2}^{n} \Big\{ c_{2j-1} \frac{|\xi|^{\alpha_{2j-1}}}{(1+|\xi|)^{\alpha_{2j-1}+\beta_{2j-1}}} \langle \hat{u}_{2j-1}, \hat{u}_{2j-2} \rangle \\ + c_{2j} \frac{|\xi|^{\alpha_{2j}}}{(1+|\xi|)^{\alpha_{2j}+\beta_{2j}}} \langle i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-1} \rangle \Big\}, \end{aligned}$$

that is,

$$E^{\rm int}(\hat{u}) = c_2 \frac{|\xi|^{4n-8}}{(1+|\xi|)^{4n-6}} \Big\langle i\xi a_1 \hat{u}_1, \sum_{j=1}^{n} (-i\xi)^{1-j} \Big(\prod_{k=2}^{j} a_k\Big)^{-1} \hat{u}_{2j} \Big\rangle$$
$$+ \sum_{j=2}^{n} \Big\{ c_{2j-1} \frac{|\xi|^{4n+2j-12}}{(1+|\xi|)^{4n+4j-14}} \langle \hat{u}_{2j-1}, \hat{u}_{2j-2} \rangle$$
$$+ c_{2j} \frac{|\xi|^{4n+2j-12}}{(1+|\xi|)^{4n+4j-12}} \langle i\xi a_j \hat{u}_{2j}, \hat{u}_{2j-1} \rangle \Big\},$$

and also define the energy dissipation rate

$$D(\hat{u}) = |\hat{u}_2|^2 + \frac{|\xi|^{2+\alpha_2}}{(1+|\xi|)^{\alpha_2+\beta_2}} |\hat{u}_1|^2$$
$$+ \sum_{j=2}^{n} \Big\{ \frac{|\xi|^{\alpha_{2j-1}}}{(1+|\xi|)^{\alpha_{2j-1}+\beta_{2j-1}}} |\hat{u}_{2j-1}|^2 + \frac{|\xi|^{2+\alpha_{2j}}}{(1+|\xi|)^{\alpha_{2j}+\beta_{2j}}} |\hat{u}_{2j}|^2 \Big\},$$

that is,

$$D(\hat{u}) = |\hat{u}_2|^2 + \frac{|\xi|^{4n-6}}{(1+|\xi|)^{4n-6}} |\hat{u}_1|^2$$
$$+ \sum_{j=2}^{n} \Big\{ \frac{|\xi|^{4n+2j-12}}{(1+|\xi|)^{4n+4j-14}} |\hat{u}_{2j-1}|^2 + \frac{|\xi|^{4n+2j-10}}{(1+|\xi|)^{4n+4j-12}} |\hat{u}_{2j}|^2 \Big\}.$$

Then it follows that

$$\partial_t \big\{ |\hat{u}|^2 + \Re E^{\rm int}(\hat{u}) \big\} + cD(\hat{u}) \le 0,$$

for all $t \ge 0$ and all $\xi \in \mathbb{R}$. Noticing

$$|\hat{u}|^2 + \Re E^{\rm int}(\hat{u}) \sim |\hat{u}|^2$$

and

$$D(\hat{u}) \gtrsim \frac{|\xi|^{6n-10}}{(1+|\xi|)^{8n-12}} |\hat{u}|^2,$$

one can see that Model II (1.1), where coefficient matrices A_m and L_m are defined in (3.1) with $m = 2n$, enjoys the dissipative structure

$$\big|\hat{u}(t,\xi)\big|^2 \le Ce^{-c\eta(\xi)t} \big|\hat{u}(0,\xi)\big|,$$

with

$$\eta(\xi) = \frac{|\xi|^{6n-10}}{(1+|\xi|)^{8n-12}} = \frac{|\xi|^{3m-10}}{(1+|\xi|)^{4m-12}}.$$

Hence, the derived result here is consistent with (3.2) proved in Theorem 3.1.

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