On the Cremona contractibility of unions of lines in the plane

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Abstract We discuss the concept of Cremona contractible plane curves, with a historical account on the development of this subject. Then we classify Cremona contractible unions of $d \geq 12$ lines in the plane.

1. Introduction

The *Cremona geometry* of the complex projective space $\mathbb{P}^r$ consists in studying properties of subvarieties of $\mathbb{P}^r$ which are invariant under the action of the *Cremona group* Cr_r, that is, the group of all birational maps $\mathbb{P}^r \dashrightarrow \mathbb{P}^r$. Since $\mathrm{Cr}_1 \cong \mathrm{PGL}(2, \mathbb{C})$, the case $r = 1$ reduces to the (nontrivial, but well-known and widely studied) theory of invariants of finite sets of points of $\mathbb{P}^1$ under the action of the projective linear group. The case $r \geq 3$ has been very little explored, due to the fact that, among other things, very little is known about the structure of Cr_r. Indeed, in this case we do not even know a reasonable set of generators of Cr_r. The intermediate case $r = 2$ is more accessible, and in fact, it has been an object of study over the course of the last 150 years. The reason is that, in this case, we have a good amount of information about Cr_2. The first one is a famous result by Noether and Castelnuovo to the effect that Cr_2 is generated by $\mathrm{PGL}(3, \mathbb{C})$ and the *standard quadratic map*

$$\sigma \colon [x, y, z] \in \mathbb{P}^2 \dashrightarrow [yz, zx, xy] \in \mathbb{P}^2.$$

A classical object of study, from this viewpoint, has been the classification of curves (or, more generally, of linear systems of curves) in $\mathbb{P}^2$ up to the action of Cr_2. If $\mathcal{L}$ is a linear system of curves, its dimension is a *Cremona invariant*, that is, it is the same for all linear systems in the *Cremona orbit* of $\mathcal{L}$, that is, the orbit of $\mathcal{L}$ under the Cr_2-action.

The degree d of the curves in $\mathcal{L}$ instead (called the *degree* of $\mathcal{L}$ and denoted by $\deg(\mathcal{L})$) is not a Cremona invariant: for instance, if one applies to $\mathcal{L}$ a general quadratic transformation (i.e., the composition of σ with a general element of $\mathrm{PGL}(3, \mathbb{C})$), the degree of the transformed linear system is $2d$. However, one can define an important *Cremona invariant* related to the degree, that is, the

Kyoto Journal of Mathematics, Vol. 57, No. 1 (2017), 55–78
DOI 10.1215/21562261-3759513,
Received March 30, 2015. Revised November 30, 2015. Accepted December 3, 2015.
2010 Mathematics Subject Classification: Primary 14H50; Secondary 14E07, 14N20.

Cremona degree of $\mathcal{L}$: this is the minimal degree of a linear system in the Cremona orbit of $\mathcal{L}$. A (not necessarily unique, up to projective transformations) linear system with minimal Cremona degree is called a *Cremona minimal model*.

If $\mathcal{L}$ has dimension 0, that is, it consists of a unique curve C, then the Cremona degree could be 0: this is the case if C can be contracted to a set of points by a Cremona transformation. In this case, one says that C is *Cremona contractible* or simply Cr-*contractible*. If C is Cr-contractible and reducible, it could be contracted to a set of distinct points. However, it is easy to see that any finite set of points in $\mathbb{P}^2$ can be mapped to a single point via a Cremona transformation. Thus, C is Cr-contractible if and only if there is a Cremona transformation which contracts C to a point of the plane. If $\dim(\mathcal{L}) \geq 1$, then the Cremona degree of $\mathcal{L}$ is positive.

The Cremona classification of Cremona minimal models of linear systems is a very classical subject. For example, it is a result which goes back to Noether (though with an incomplete proof) that a pencil of irreducible, rational plane curves is Cremona equivalent to the pencil of lines through a fixed point; that is, pencils of rational plane curves have Cremona degree 1. Similar results for linear systems of positive dimension of curves with positive genus have been classically proved, as we will see in Section 2, which is devoted to a historical account on the subject.

The general problem of classifying Cremona minimal models of *irreducible* plane curves or linear systems (a linear system is said to be *irreducible* if its general curve is as well) has been open for more than one century, with several interesting contributions by various authors, among them it is worth mentioning Marletta [21], [22], who pointed out important properties of adjoint linear systems to such models (see Theorem 2.1; for the definition of adjoint linear systems, see Section 3). This problem, however, has been solved only recently in our paper [4].

The first step in this classification can be considered the characterization of Cr-contractible irreducible plane curves. According to Enriques and Chisini [12, Volume III, Section 21, pp. 191–192], the first result on this subject came in 1900 from Castelnuovo and Enriques [8].

THEOREM 1.1 (CASTELNUOVO AND ENRIQUES)
An irreducible curve C is Cr-*contractible if and only if all adjoint linear systems to C vanish.*

Actually, Castelnuovo and Enriques [8] claimed that the irreducibility assumption on C can be relaxed to C being reduced, but [5, Example 2], namely, a general union of $d \geq 9$ distinct lines with a point of multiplicity $d-3$, shows that this is not true.

Theorem 1.1 is today known as *Coolidge's theorem*, because it appeared also in Coolidge's book [11, p. 398], but the proof therein is not complete (see Section 2). Theorem 1.1 was improved in 1982 by Kumar and Murthy [23].

THEOREM 1.2 (KUMAR–MURTHY)

An irreducible plane curve C is Cr-*contractible if and only if the first two adjoint linear systems to C vanish.*

Using the modern language of *pairs* of a curve on a smooth surface, one considers the pair $(S, \tilde{C})$ where $S \to \mathbb{P}^2$ is a birational morphism which resolves the singularities of C and $\tilde{C}$ is the strict transform of C on S.

Theorem 1.2 implies that the pair $(S, \tilde{C})$ has log-Kodaira dimension $\mathrm{kod}(S, \tilde{C}) = -\infty$ if and only if its second log plurigenus $P_2(S, \tilde{C})$ vanishes (for the definitions, see again Section 3). This can be seen as a log-analogue of Castelnuovo's rationality criterion for regular surfaces. Thus, for an irreducible plane curve C, the following four conditions are equivalent:

(a) C is Cr-contractible,
(b) $\mathrm{kod}(S, \tilde{C}) = -\infty$,
(c) all adjoint linear systems to C vanish,
(d) the first two adjoint linear systems to C vanish.

Condition (d) can be replaced by

(d′) $P_2(S, \tilde{C}) = 0$.

The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are either trivial or easy, and are true even for reducible and reduced plane curves (see Section 3), while (d) $\Rightarrow$ (a) follows from Theorem 1.2.

As for extensions of Kumar and Murthy's theorem to reducible curves, the only known result so far is due to Iitaka [16], [17].

THEOREM 1.3 (IITAKA)

Let C be a reduced plane curve with two irreducible components. Then, C is Cr-*contractible if and only if the first two adjoint linear systems to C vanish.*

By contrast, in [5] we noted that (a), (b), (c), and (d) above are not equivalent for reducible, reduced plane curves. As we said, [5, Example 2] shows that (b) and (c) are not equivalent for reducible curves. Furthermore, an example of Pompilj [27, p. 68] shows that (c) and (d) are not equivalent for curves with three irreducible components (see [5, Example 1]). The same example shows that (a) and (d) are not equivalent for curves with three irreducible components. Note, moreover, that Pompilj's example is the union of three Cr-contractible irreducible curves that turns out to be non-Cr-contractible, and it shows the difficulty of proving the Cr-contractibility of reducible curves by proceeding by induction on the number of irreducible components of the curve, as one may be tempted to do. See the historical account in Section 2 for other difficulties encountered by several mathematicians in tackling this problem.

Concerning reducible curves, the following theorem should also be recalled.

THEOREM 1.4 (KOJIMA–TAKAHASHI [20])

Let (S, D) be a pair where S is a smooth rational surface and D is a reduced curve on S with at most four irreducible components. Then, $\mathrm{kod}(S, D) = -\infty$ *if and only if* $P_6(S, D) = 0$.

Furthermore, if (V, D') *is the almost minimal model of* (S, D) *in the sense of [20, Definition 2.3] and if the support of* D' *is connected, then* $\mathrm{kod}(S, D) = -\infty$ *if and only if* $P_{12}(S, D) = 0$.

However, Kojima and Takahashi do not relate $\mathrm{kod}(S, D) = -\infty$ to the contractibility of D.

In [5] we posed the following problem.

PROBLEM

Is it true that a reduced plane curve C is Cr-contractible if and only if $\mathrm{kod}(S, \tilde{C}) = -\infty$?

In this article we address this problem when C is a reduced union of lines, the first meaningful case, which, we think, presents aspects of general interest. Our main result is the following (see Theorem 4.14 for a more precise statement).

THEOREM 1.5

Let C be the union of $d \geq 12$ distinct lines. Then, all adjoint linear systems to C vanish if and only if C has a point of multiplicity $m \geq d-3$. Moreover, $\mathrm{kod}(S, \tilde{C}) = -\infty$ *if and only if C has a point of multiplicity $m \geq d-2$. Finally, C is* Cr*-contractible if and only if* $\mathrm{kod}(S, \tilde{C}) = -\infty$.

A posteriori, one has that $P_3(S, \tilde{C}) = 0$ implies $\mathrm{kod}(S, \tilde{C}) = -\infty$ for C a union of $d \geq 12$ distinct lines with vanishing adjoints. Moreover, it turns out that, for a union of $d \geq 12$ distinct lines, (d) implies (c). Note that this is not true if $d < 12$: for example, for the dual configurations of the flexes of a smooth cubic plane curve, which has degree 9, 12 triple points, and no other singularity, the first two adjoint linear system vanish, but the third adjoint is trivial and, hence, nonempty.

The case of a reduced union of $d \leq 11$ lines is also interesting but the classification of all cases with vanishing adjoints or with Kodaira dimension $-\infty$ is much more complicated, since it requires the analysis of many dozens of configurations. We performed it for $d \leq 8$ and $d = 11$; the remaining cases are works in progress. So far all cases we have found with Kodaira dimension $-\infty$ are also Cr-contractible. We will not present here this long and tedious classification, but we intend to do it in a forthcoming article.

This article is organized as follows. After the historical Section 2, we fix notation and definitions in Section 3. In Section 4, we classify the union of $d \geq 12$ distinct lines with vanishing adjoints. Among them, we determine those with Kodaira dimension $-\infty$ (the latter set is strictly contained in the former) and we show that these are exactly the Cr-contractible ones.

2. A historical account

The history of Theorem 1.1 is surprisingly intricate, and it is intertwined with that of the Noether–Castelnuovo theorem and with the problem of finding Cremona minimal models of plane curves and linear systems. A short account of some proofs of the Noether–Castelnuovo theorem can be found in [1, pp. 227–228] (see also the historical remarks in [1, Chapter 8]). For more details on the classical literature, see [15, pp. 390–391] and [12, Volume III, Section 20, pp. 175–177].

The study of Cremona transformations γ of $\mathbb{P}^2$ is equivalent to the one of homaloidal nets: such a net is the image of the linear system of lines via γ. The degree of the homaloidal net $\mathcal{L}$ associated to $\gamma \in \mathrm{Cr}_2$ is called the *degree* of γ. Cremona transformations of degree 1 are projective transformations, those of degree 2, the *quadratic transformations*, correspond to homaloidal nets of conics, and so on.

If $\mathcal{L}$ is an irreducible linear system of plane curves of degree d, with base points $P_0, \dots, P_r$ of multiplicity at least $m_0, \dots, m_r$, then we may assume that $m_0 \geq \cdots \geq m_r$. We will use the notation $(d; m_0, \dots, m_r)$ to denote $\mathcal{L}$. We may use exponential notation to denote repeated multiplicities. For instance the homaloidal nets of conics are of the form $(2; 1^3)$, and the related quadratic transformation is said to be *based* at the three simple base points of this net.

The so-called "Noether–Castelnuovo theorem" was apparently first stated in 1869 by Clifford [9]. However, Clifford gave no real proof of it; rather, he presented a plausibility argument based on the analysis of Cremona transformations of degree $d \leq 8$. Immediately after, in 1870, Noether [24] and Rosanes [28] independently came up with a more promising approach. They correctly observed that for a homaloidal net $\mathcal{L} = (d; m_0, \dots, m_r)$ of degree $d > 1$ one has $m_0 + m_1 + m_2 > d$. (This is now called *Noether's inequality.*) Then, they observed that if one performs a quadratic transformation based at P_0, P_1, P_2, the homaloidal net $\mathcal{L}$ is transformed in another of degree $d' = 2d - (m_0 + m_1 + m_2) < d$. By repeating this argument, the degree of $\mathcal{L}$ can be dropped to 1, proving the theorem.

The problem with this argument is the existence of an *irreducible* net of conics through P_0, P_1, P_2. This is certainly the case if P_0, P_1, P_2 are distinct, since then they cannot be collinear by Noether's inequality. The same argument applies also if P_1 is infinitely near to P_0 and P_2 is distinct, but problems may arise if both P_1, P_2 are infinitely near to P_0. The first difficulty appears if P_1, P_2 are infinitely near to P_0 in *different directions*. This was noted by Noether himself, who filled up this gap in [25]. After this, the proof was considered to be correct and the theorem well established. Afterward, a considerable series of papers appeared, by several authors, such as Bertini, Castelnuovo, Guccia, Jung, Martinetti, del Pezzo, de Franchis, Segre (in chronological order), and others. Based on Noether's argument, they pursued the classification of Cremona minimal models of irreducible linear systems of positive dimension of curves of low genus.

It was only in 1901 that Segre [29] pointed out a more subtle gap in Noether's argument when

- P_2 is infinitely near to P_1, which in turn is infinitely near to P_0, and
- P_2 is *satellite* to P_0, that is, P_2 is proximate also to P_0.

In other words, P_2 is infinitely near to P_0 along a *cuspidal* branch. Segre's criticism seemed to be a very serious one, since he presented a series of homaloidal nets of increasing degrees whose degree cannot be lowered by using quadratic transformations.

According to Coolidge [11, p. 447], "it is said that Noether shed tears when he heard of this," but, as Coolidge goes on, "there was no need to do so." Indeed, promptly after Segre's criticism, in the same year 1901, Castelnuovo [7] showed how to decompose a nonlinear Cremona transformations as a composition of *de Jonquières maps* (related to homaloidal nets of the type $(d; d-1, 1^{2(d-1)})$, and the de Jonquières map will be said to be *centered* at the base points of the net), which, in turn, decompose in products of quadratic ones, as shown by Segre in a footnote to [7]. Just one year later, Ferretti [14], a student of Castelnuovo's, also filled up the gap in the aforementioned papers about the classification of linear systems of low genus. It turned out that, even if the proofs were incomplete, all the statements were correct.

Castelnuovo's proof really contains a new idea: it is based on the remark that, if $\mathcal{L}$ is a positive-dimensional linear system of rational plane curves, then all adjoint linear systems to $\mathcal{L}$ vanish. It is this property that ultimately implies that $\mathcal{L}$ has base points of large enough multiplicity so that the degree of $\mathcal{L}$ can be decreased by means of de Jonquières transformations. This idea, according to Castelnuovo himself, came from the joint work [8] with Enriques of the prior year concerning rational and ruled *double planes*, that is, rational and ruled double coverings of $\mathbb{P}^2$. Indeed, Castelnuovo and Enriques stated in [8] that a double plane is rational or ruled if and only if all the adjoint linear systems of index $i \geq 2$ to the branch curve of (the canonical desingularization of) a double plane vanish (see [2], [3] for a more precise statement).

In the last page of [8], Castelnuovo and Enriques stated Theorem 1.1 (with the wrong assumption that C can be reducible): they do not really give a proof; they simply claim that it consists in computations similar to others done in that paper. In addition, they remarked that the same technique could be useful in the classification of linear systems of plane curves with low genus, as Castelnuovo and Ferretti effectively did.

Note, however, that Segre's criticism applied also to the classification in [8], as Castelnuovo admitted in the first page of [7]. Even if Castelnuovo suggested in [7] that the gap could be fixed by arguments similar to those in [7], it seems that nobody did that, until Conforto [10] in 1938 (cf. [13, p. 458]). It turned out only recently that the Castelnuovo–Enriques–Conforto proof of the characterization of rational double planes still had a gap, which was fixed in [3].

To come back to Theorem 1.1, its first correct proof is due to Ferretti [14]. This is essentially exposed by Enriques and Chisini [12, Volume III, Section 21, pp. 187–190]. However, in [12, Volume III, Section 21, p. 190], at the end of the proof of Theorem 1.1 (with the correct statement), Enriques and Chisini insisted on the wrong statement that the irreducibility assumption on C can be weakened to reducedness. A possible explanation for this mistake may reside in the fact that the numerical properties of the multiplicities of the curve C (essentially Noether's inequality) stay the same even if it is reducible. However, the condition that three points P_0, P_1, P_2 of the highest multiplicities are not aligned if $m_0 + m_1 + m_2 > d$, where $d = \deg(C)$, may fail for reducible curves. Moreover, even if one can apply a Cremona transformation decreasing d, one or more components of C could be contracted to points: in that case, if one then applies another Cremona transformation based at those points, such components reappear, causing the argument to become circular. The same considerations suggest that one cannot simply proceed by induction on the number of components of C.

A few years after Ferretti's work, in 1907, Marletta [21] gave a similar proof of Theorem 1.1, by showing the following.

THEOREM 2.1 (MARLETTA)
A curve of Cremona minimal degree $d > 1$, with the point of maximal multiplicity $m_0 > d/3$, has nonvanishing adjoint linear system of index i, with $i = [(d - m_0)/2]$, where $[x]$ denotes the largest integer smaller than or equal to x.

Also Ferretti [14] had given similar interesting results regarding the adjoint linear systems to Cremona minimal models.

Though Theorem 1.1 is today called Coolidge's theorem, its proof in Coolidge's book [11, pp. 396–398] contains the same gap pointed out by Segre for Noether's argument. It is strange how careless Coolidge was in his references: the only one he gives is to a paper of Franciosi of 1918. It is also very strange that Coolidge's wrong proof was repeated *verbatim* by Kumar and Murthy [23], who, however, gave a correct proof of Theorem 1.2 with different methods.

Regarding the minimal degree problem, it seems that Jung in 1889 (see [18]) was the first one who proved the following.

THEOREM 2.2 (G. JUNG)
If an irreducible curve C has degree d and maximal multiplicities $m_0 \geq m_1 \geq m_2$ with $d \geq m_0 + m_1 + m_2$, then C has minimal degree.

The same statement holds mutatis mutandis for irreducible linear systems of plane curves (see [4, Theorem 2.3] for a short proof which uses adjoint linear systems). Theorem 2.2 has been stated by Coolidge [11, p. 403] with the weaker hypothesis $d > m_0 + m_1 + m_2$, but the proof therein works also in the case in which $d = m_0 + m_1 + m_2$.

If an irreducible curve C has Cremona minimal degree d and maximal multiplicities $m_0 \geq m_1 \geq m_2$ with $m_0 + m_1 + m_2 > d$, one sees that the corresponding points P_0, P_1, P_2 are infinitely near, namely, $P_0 \in \mathbb{P}^2$, P_1 is infinitely near to P_0, and either

- P_2 is infinitely near to P_0 (in a different direction with respect to P_1), or

- P_2 is infinitely near to P_1 and P_2 is satellite to P_0.

In both cases, it follows that $m_0 > d/2$.

Marletta [21] gave sufficient conditions on the multiplicities of the singular points of a curve C to ensure that C has Cremona minimal degree with $d < m_0 + m_1 + m_2$.

A key ingredient in the study of (linear systems of) plane curves has been the concept of adjoint linear systems. According to Enriques and Chisini [12, p. 191], they were originally introduced by Brill and Noether in 1873 for the study of linear series on curves (see [26]). Their invariance with respect to Cremona transformations was first used by Kantor in 1883 (but published only in 1891; see [19]) and then by Castelnuovo [6] in 1891.

3. Preliminaries and notation

3.1. Adjoint linear systems

Let C be a reduced plane curve. Let $f\colon S \to \mathbb{P}^2$ be a birational morphism which resolves the singularities of C, and denote by $\tilde{C}$ the strict transform of C on S. For any pair of integers $n \geq 1$ and $m \geq n$, we set

$$\mathrm{ad}_{n,m}(C) = f_*\big(|n\tilde{C} + mK_S|\big)$$

and

$$\mathrm{ad}_m(C) := \mathrm{ad}_{1,m}(C), \quad \text{so that } \mathrm{ad}_{n,m}(C) = \mathrm{ad}_m(nC).$$

We call $\mathrm{ad}_{n,m}(C)$ the *(n,m)-adjoint linear system to C*, and $\mathrm{ad}_m(C)$ is simply the *m-adjoint linear system to C* or the adjoint linear system *of index m* to C.

REMARK 3.1

The reason why we assume $m \geq n$ is that the definition of (n,m)-adjoint systems is independent of the morphism f only in this case. (The reader can easily check this.) Moreover, $\mathrm{ad}_{n,m}(C) = f_*(|n\tilde{C}' + mK_S|)$, with $\tilde{C}' = \tilde{C} + D$, where D is supported on the exceptional divisors of f.

A basic role in Cremona geometry is played by the *n-adjoint sequence* $\{\dim(\mathrm{ad}_{n,m}(C))\}_{m\geq n}$ of C (simply called the *adjoint sequence* if $n = 1$). A crucial remark is that *adjunction extinguishes*, that is, $\mathrm{ad}_{n,m}(C) = \emptyset$ for $m \gg 0$; hence, adjoint sequences stabilize to -1. Therefore, we can consider them as finite sequences, ending with the first -1, after which it stabilizes. From the results in [4] it follows that the adjoint sequence stabilizes to -1 as soon as it reaches the value -1.

Let $\gamma \in \mathrm{Cr}_2$, and let C and C' be reduced curves in $\mathbb{P}^2$. We say that γ *maps C to C'* if there is a commutative diagram

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\gamma"'] & & \mathbb{P}^2 \end{tikzcd} \tag{1}$$

with α, β birational morphisms, and there is a smooth curve $\tilde{C}$ on $\tilde{S}$ such that

$$\alpha_*(\tilde{C}) = C \qquad \text{and} \qquad \beta_*(\tilde{C}) = C'. \tag{2}$$

Note that γ (resp., its inverse) may contract some components of C (resp., of C') to points. In particular, C' could be the zero curve, in which case C is said to be *Cremona contractible* or Cr-*contractible*.

As recalled in Section 2, the following lemma is basically due to Kantor.

LEMMA 3.2 (KANTOR)

If C is a reduced plane curve, then for all integers $n \geq 1$ the n-adjoint sequence of C is a Cremona invariant.

Proof

It follows from diagram (1), from (2), and from

$$\mathrm{ad}_{n,m}(C) = \alpha_*\big(|n\tilde{C} + mK_S|\big), \qquad \mathrm{ad}_{n,m}(C') = \beta_*\big(|n\tilde{C} + mK_S|\big). \qquad \square$$

REMARK 3.3

In [4, Section 4] there is a proof of Kantor's lemma under a useless restrictive hypothesis. Though irrelevant for us, it should be noted that, strictly speaking, the adjoint systems themselves, *are not* Cremona invariant, due to the possible existence for them of (exceptional) fixed components which can be contracted by a Cremona transformation.

3.2. Pairs

Let (S, D) be a *pair*, namely, D is a reduced curve on a smooth projective surface S. For any nonnegative integer m, the *m-log plurigenus* of (S, D) is

$$P_m(S, D) := h^0\big(S, \mathcal{O}_S\big(m(D + K_S)\big).$$

If $P_m(S, D) = 0$ for all $m \geq 1$, then one says that the *log-Kodaira dimension* of the pair (S, D) is $\mathrm{kod}(S, D) = -\infty$. Otherwise

$$\mathrm{kod}(S, D) = \max\big\{\dim\big(\mathrm{Im}(\phi_{|m(D+K_S)|})\big)\big\},$$

where $\phi_{|m(D+K_S)|}$ is the rational map determined by the linear system $|m(D + K_S)|$ whenever this is not empty.

A pair (S, D) is said to be *contractible* if there exists a birational map $\psi\colon S \dashrightarrow S'$ such that D is contracted by ψ to a union of points, namely, ψ is constant on any irreducible component of D.

REMARK 3.4

Assume S rational. If $\psi : S \to S'$ is a birational morphism which contracts D, then D is contained in the exceptional locus of ψ; therefore, all connected components of D have arithmetic genus 0, in particular, they have normal crossings. Moreover, $\mathrm{kod}(S, D) = -\infty$.

If (S, D) is contractible, then there is a resolution of the indeterminacies of ψ, that is, a commutative diagram

$$\begin{tikzcd} & \tilde{S} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ S \arrow[rr, dashed, "\psi"'] & & S' \end{tikzcd}$$

where α and β are birational morphisms. If $\tilde{D}$ is the strict transform of D via α, then $\tilde{D}$ is contracted to a union of points by the morphism β; hence, $\mathrm{kod}(\tilde{S}, \tilde{D}) = -\infty$.

Let (S, D) and (S', D') be pairs. We will say that (S, D) and (S', D') are *birationally equivalent* if there is a birational map $\phi : S \dashrightarrow S'$ such that ϕ (resp., ϕ^{-1}) does not contract any irreducible component of D (resp., of D') and the image of D via ϕ is D'. (Hence, the image of D' via ϕ^{-1} is D.)

With this definition we immediately have the following result.

LEMMA 3.5

If (S, D) is birationally equivalent to (S', D') and (S', D') is contractible, then (S, D) is contractible too.

REMARK 3.6

If a reduced plane curve C is Cr-contractible, which is condition (a) in the Introduction, then, setting $(S, \tilde{C})$ as at the beginning of the section, one has $\mathrm{kod}(S, \tilde{C}) = -\infty$, which is condition (b) in the Introduction.

Since $\tilde{C}$ is effective, condition (b) is actually equivalent to

$$\mathrm{ad}_{n,m}(C) = \emptyset, \quad \text{for each } m \geq n \geq 1.$$

In particular, (b) implies

$$\mathrm{ad}_m(C) = \emptyset, \quad \text{for each } m \geq 1, \tag{3}$$

which is condition (c) in the Introduction.

Condition (c) trivially implies (d). We see that (d) is equivalent to (d$'$). Indeed, $P_1(S,\tilde{C})=0$ is equivalent to $|\tilde{C}+K_S|=\emptyset$. Then, by adjunction, all irreducible components of $\tilde{C}$ are smooth rational curves. Thus, $2\tilde{C}+2K_S$ intersects all components of $\tilde{C}$ negatively, so that $\tilde{C}$ is in the fixed part of $|2\tilde{C}+2K_S|$ if this is not empty. In conclusion, one has $P_2(S,\tilde{C})=\dim(|2\tilde{C}+2K_S|)+1=0$ if and only if $|\tilde{C}+2K_S|=\emptyset$, that is, $\mathrm{ad}_2(C)=\emptyset$.

3.3. Generalities on unions of lines

In this article we will study curves C which are unions of distinct lines and have vanishing adjoints, that is, such that (3) holds. Then we will see which of them have $\mathrm{kod}(S,\tilde{C})=-\infty$ and which are Cr-contractible. We will now prepare the territory for this.

Let C be a reduced plane curve with $d=\deg(C)$. If C is singular, let $m_0 \geq m_1 \geq \cdots \geq m_r \geq 2$ be the multiplicities of the singular points $P_0,\dots,P_r$ of C, which can be *proper* or *infinitely near*. We set $m_i=1$ if $i>r$. If C is smooth, we assume $m_0=1$, P_0 is a general point of C, and $m_i=1$ for $i>0$. By the *proximity inequality* (see [4] as a general reference for these matters and for notation), we may and will assume that $P_i > P_j$ (i.e., P_i is infinitely near to P_j) implies $i>j$. Therefore, P_0 is proper and either P_1 is also proper or $P_1 >^1 P_0$ (i.e., P_1 is infinitely near of order 1 to P_0).

As announced in the Introduction, we will use the notation $(d;m_0,m_1,\dots,m_r)$ to denote a linear system of plane curves of degree d with assigned base points $P_0,\dots,P_r$ with respective multiplicities at least $m_0,\dots,m_r$. Hence, we may write $C\in(d;m_0,m_1,\dots,m_r)$. We will write $C\cong(d;m_0,m_1,\dots,m_r)$ whenever C has multiplicity exactly m_i at P_i, $i=1,\dots,r$. If $C\cong(d;m_0,m_1,\dots,m_r)$, then $\mathrm{ad}_{n,m}(C)=(nd-3m;nm_0-m,\dots,nm_q-m)$, where q is the maximum such that $nm_q>m$.

LEMMA 3.7

In the above setting, one has $d-m_0\geq 0$ with equality if and only if C consists of d lines in the pencil of center P_0, in which case C is Cr*-contractible.*

Proof

Only the last assertion needs to be justified. Let $d=2\ell+\epsilon$, with $\epsilon\in\{0,1\}$. Let γ by the de Jonquières transformation of degree $\ell+1+\epsilon$ centered at P_0, with multiplicity $\ell+\epsilon$, at d simple basepoints, each one general on a component of C, and, in addition if $\epsilon=1$, at further general simple base point. This γ transformation contracts C to d distinct points of the plane. □

Set

$$d-m_0=2h+\varepsilon, \quad \text{with } \varepsilon\in\{0,1\}. \tag{4}$$

LEMMA 3.8

In the above setting, if $d=m_0+1$, then C consists of $\ell\geq 0$ lines in the pencil of *center P_0 plus an irreducible curve C' of degree $d' = d - \ell$ with P_0 of multiplicity $d' - 1$, and all points of C off P_0 are nonsingular. Then C is* Cr*-contractible.*

Proof
Only the last assertion needs to be justified. Assume $d' > 1$. Then the curve C' is mapped to a line L by a de Jonquières transformation of degree d' centered at P_0, with multiplicity $d' - 1$ and $2d' - 2$ general simple base points of C'. The curve C is then transformed to a curve $\bar{C}$ consisting of ℓ lines $L_1, \ldots, L_\ell$ in a pencil of center $\bar{P}_0$, plus the line L not passing through $\bar{P}_0$. So we are reduced to the case $d' = 1$.

If $\ell = 0$, we finish by contracting L to a point with a quadratic transformation based at three points, two of which lie on L. We proceed similarly if $\ell = 1$. So we may assume $\ell \geq 2$. Then set $\mu = (\ell + 2)/2$ if ℓ is even or $\mu = (\ell + 3)/2$ if ℓ is odd, and note that $\ell \geq \mu \geq 2$. The de Jonquières transformation of degree μ centered at $\bar{P}_0$, with multiplicity $\mu - 1$, plus μ simple base points at $L \cap L_1$, $L \cap L_2$, $\ldots$, $L \cap L_\mu$, plus $\ell - \mu$ simple base points, each general on one of the lines $L_{\mu+1}$, $L_{\mu+2}$, $\ldots$, L_ℓ, and further $2\mu - 2 - \ell$ simple general base points, contracts $\bar{C}$ to $\ell + 1$ distinct points of the plane. □

From now on, we may and will assume $h \geq 1$.

LEMMA 3.9
In the above setting, if (3) holds, one has:

(i) $m_0 > h$ *or, equivalently,* $m_0 > d/3$*;*
(ii) $m_i > h$ *for* $1 \leq i \leq 2$*;*
(iii) $m_0 + m_1 + m_2 \geq d + 1$.

Proof
(i) The equivalence between $m_0 > h$ and $m_0 > d/3$ is clear. Let us prove that $m_0 > d/3$. The assertion is trivial if $d < 3$, so we assume $d \geq 3$. Suppose $m_0 \leq d/3$. Then $\mathrm{ad}_{[d/3]}(C)$ is the complete linear system of curves of degree $d - 3[\frac{d}{3}] \geq 0$, which is not empty, contradicting (3).

(ii) If $m_2 \leq h$, then $\mathrm{ad}_h(C) = (m_0 - h + \varepsilon; m_0 - h, \overline{m_1 - h})$, where $\overline{m_1 - h} = \max\{m_1 - h, 0\}$ is not empty, contradicting (3).

(iii) This follows from (i) and (ii). □

In the rest of this section, we consider the case in which C is a reduced union of d lines. Since m_i is the number of lines passing through P_i, for $i = 0, \ldots, r$, one has

$$m_0 + m_1 + m_2 \leq d + 3. \tag{5}$$

See Lemma 4.1 below for the description of the case where equality occurs.

If $C \cong (d; m_0, \dots, m_r)$ is a reduced union of lines, then we say that $(d; m_0, \dots, m_r)$ is the *type* of C. Moreover, we say that two reduced unions of lines

$$C = L_1 \cup L_2 \cup \cdots \cup L_d \qquad \text{and} \qquad D = R_1 \cup R_2 \cup \cdots \cup R_d$$

have the same *configuration* if there exists a permutation σ of $\{1, \dots, d\}$ such that

$$L_{i_1} \cap L_{i_2} \cap \cdots \cap L_{i_k} \neq \emptyset \Longleftrightarrow R_{\sigma(i_1)} \cap R_{\sigma(i_2)} \cap \cdots \cap R_{\sigma(i_k)} \neq \emptyset, \quad \text{for each } i_1, \dots, i_k.$$

REMARK 3.10

Clearly, two reduced unions of lines with the same configuration are also of the same type, but, in general, the type does not uniquely determine the configuration. For example, if C is the reduced union of six lines with two triple points, that is, the type of C is $(6; 3^2, 2^9)$, then there are exactly two configurations of this type, according to the following possibilities: either the line passing through the triple points is a component of C or it is not.

We will denote a configuration of a reduced union of lines $C = L_1 \cup \cdots \cup L_d$ as follows:

$$(6) \quad \big(d; \{a_{0,1}, a_{0,2}, \dots, a_{0,m_0}\}, \{a_{1,1}, a_{1,2}, \dots, a_{1,m_1}\}, \dots, \{a_{s,1}, a_{s,2}, \dots, a_{s,m_s}\}\big),$$

where $m_s \geq 3$ and where $P_0 = L_{a_{0,1}} \cap L_{a_{0,2}} \cap \cdots \cap L_{a_{0,m_0}}$ is a point of multiplicity m_0, $P_1 = L_{a_{1,1}} \cap L_{a_{1,2}} \cap \cdots \cap L_{a_{1,m_1}}$ is a point of multiplicity m_1, and so on, for all points of multiplicity greater than or equal to 3. In other words, (6) lists the singular points of C of multiplicity $m \geq 3$ and the lines containing each of them. In particular, we list the singular points according to their multiplicities in nonincreasing order, and among the points of the same multiplicity, we will usually list them in lexicographical order with respect to the given ordering of the lines.

REMARK 3.11

In (6) one does not need to list the nodes, that is, the double points of C. Indeed, $P_{i,j} = L_i \cap L_j$ is a node of C if and only if there exist no $k, h, l \in \{1, \dots, r\}$ such that $a_{k,h} = i$ and $a_{k,l} = j$. The number of nodes of C is

$$\binom{d}{2} - \sum_{i=0}^{s} \frac{m_i(m_i - 1)}{2}.$$

EXAMPLE 3.12

If C is a reduced union of six lines of type $(6; 3^2, 2^9)$ such as in Remark 3.10, then the two possible configurations are $(6, \{1,2,3\}, \{1,4,5\})$ and $(6, \{1,2,3\}, \{4,5,6\})$. The former means that the triple points of C are $P_0 = L_1 \cap L_2 \cap L_3$ and $P_1 = L_1 \cap L_4 \cap L_5$, so that the nodes are $P_{1,6} = L_1 \cap L_6$, $P_{2,4}$, $P_{2,5}$, $P_{2,6}$, $P_{3,4}$, $P_{3,5}$, $P_{3,6}$, $P_{4,6}$, $P_{5,6}$ (see Remark 3.11). The latter configuration means that the triple points of C are $P_0 = L_1 \cap L_2 \cap L_3$ and $P_1 = L_4 \cap L_5 \cap L_6$, so that the nodes are $P_{1,4} = L_1 \cap L_4$, $P_{1,5}$, $P_{1,6}$, $P_{2,4}$, $P_{2,5}$, $P_{2,6}$, $P_{3,4}$, $P_{3,5}$, $P_{3,6}$ (see Remark 3.11).

REMARK 3.13
We will see later in Remark 4.15 that different configurations of the same type may behave quite differently with respect to the adjoint linear systems and Cremona contractibility. Furthermore, two reduced unions of lines with the same configuration are not necessarily projectively equivalent. For example, if the type of C is $(4;4)$, then there is only one possible configuration, but the isomorphism classes of four lines passing through a point depend on one parameter.

4. Configurations of lines with vanishing adjoints

In this section we classify reduced plane curves C which are unions of $d \geq 12$ distinct lines and such that (3) holds, that is, with vanishing adjoint linear systems.

4.1. Basics

We keep the notation introduced above, including (4). The degree d of C is the number of its components, and the singular points of C are all proper. We will assume that (3) holds.

Set $d = 3\delta + \eta$, with $0 \leq \eta \leq 2$. By Lemma 3.9 we have $m_0 > \delta$, and we set $m_0 = \delta + \mu$, with $\mu \geq 1$. Set $\mu = 2\nu + \tau$, with $0 \leq \tau \leq 1$, so that $d - m_0 = 2(\delta - \nu) + (\eta - \tau)$. Thus:

(i) $h = \delta - \nu$ and $\varepsilon = \eta - \tau$, unless either,
(ii) $\eta = 0$, $\tau = 1$, in which case $h = \delta - \nu - 1$ and $\varepsilon = 1$, or
(iii) $\eta = 2$, $\tau = 0$, in which case $h = \delta - \nu + 1$ and $\varepsilon = 0$.

We set

$$m := m_0 + m_1 + m_2.$$

By (5) one has $m \leq d + 3 = 3\delta + \eta + 3$. Since $m_1, m_2 \geq h + 1$ by Lemma 3.9(ii), we have:

- $3\delta + \tau + \varepsilon + 3 \geq m \geq 3\delta + \tau + 2$ in case (i);
- $3\delta + 3 \geq m \geq 3\delta + 1$ in case (ii);
- $3\delta + 5 \geq m \geq 3\delta + 4$ in case (iii).

Thus, the interval in which m lies is $[d+2-\varepsilon, d+3]$ and its length is $\varepsilon+1 \in \{1,2\}$; hence, $d+1 \leq m \leq d+3$. The following table shows the possible values of m_1 and m_2:

(7)

m_1	m_2	ε	m	*Possible cases*
$h+1$	$h+1$	$0,1$	$d+2-\varepsilon$	(i)–(ii)–(iii)
$h+2$	$h+1$	$0,1$	$d+3-\varepsilon$	(i)–(ii)–(iii)
$h+2$	$h+2$	1	$d+3$	(i)–(ii)
$h+3$	$h+1$	1	$d+3$	(i)–(ii)

We will use the notation

$$m_2 = \cdots = m_k, \qquad m_2 - 1 = m_{k+1} = \cdots = m_{k+l} > m_{k+l+1},$$

where $k \geq 2$ and $l \geq 0$. It will be essential for us to consider

$$\mathrm{ad}_h(C) = \big(m_0 - h + \varepsilon; m_0 - h, m_1 - h, (m_2 - h)^{k-1}, (m_2 - h - 1)^l, \ldots\big),$$

which has to be empty, and we note that $1 \leq m_1 - h \leq 3$, whereas $1 \leq m_2 - h \leq 2$.

The proofs of the following lemmas are elementary and are left to the reader.

LEMMA 4.1

In the above setting, if $m = d + 3$, then:

- *P_0, P_1, P_2 are not collinear and the sides of the triangle with vertices P_0, P_1, P_2 are components of C;*
- *all components of C pass through one of the points P_0, P_1, P_2;*
- *the remaining singular points of C have multiplicity at most 3.*

LEMMA 4.2

In the above setting, if $m = d + 2$, then:

(α) *either P_0, P_1, P_2 are collinear and the line joining them belongs to C, in which case all components of C pass through one of the points P_0, P_1, P_2 and the remaining singular points of C have multiplicity at most 3; or*

(β) *P_0, P_1, P_2 are not collinear and the sides of the triangle with vertices P_0, P_1, P_2 are components of C, in which case all components of C but one pass through one of the points P_0, P_1, P_2, the remaining singular points of C have multiplicity at most 4, and there are at most two of them with multiplicity 4; or*

(γ) *P_0, P_1, P_2 are not collinear and two of the three sides of the triangle with vertices P_0, P_1, P_2 are components of C, in which case all components of C pass through one of the points P_0, P_1, P_2 and the remaining singular points of C have multiplicity at most 3.*

LEMMA 4.3

In the above setting, if $m = d + 1$, then:

(α') *either P_0, P_1, P_2 are collinear and the line joining them belongs to C, in which case all components of C but one pass through one of the points P_0, P_1, P_2 and the remaining singular points of C have multiplicity at most 4; or*

(β') *P_0, P_1, P_2 are not collinear and the sides of the triangle with vertices P_0, P_1, P_2 are components of C, in which case all components of C but two pass through one of the points P_0, P_1, P_2, the remaining singular points of C have multiplicity at most 5, and there is at most one of them with multiplicity 5; or*

(γ') *P_0, P_1, P_2 are not collinear and two of the three sides of the triangle with vertices P_0, P_1, P_2 are components of C, in which case all components of C but one pass through one of the points P_0, P_1, P_2 and the remaining singular points of C have multiplicity at most 4; or*

(δ') *P_0, P_1, P_2 are not collinear and only one side of the triangle with vertices P_0, P_1, P_2 is a component of C, in which case all components of C pass through one of the points P_0, P_1, P_2 and the remaining singular points of C have multiplicity at most* 3.

4.2. The case m maximal

Here we treat the case $m = d + 3$, in which Lemma 4.1 applies.

4.2.1. The subcase $\varepsilon = 0$

We are in either case (i) or case (iii), and in table (7) the second row occurs. Hence, $m_0 \geq m_1 = h + 2$. Thus,

$$\mathrm{ad}_h(C) = (m_0 - h; m_0 - h, 2, 1^{k-1}).$$

LEMMA 4.4

Assume (3) holds, $m = d + 3$, $d \geq 12$, $h \geq 1$, and $\varepsilon = 0$. Then $C \cong (d; d - 2, 3, 2^{2(d-3)})$.

Proof

We claim that $m_0 - h > 2$. Otherwise, $m_0 - h = 2$; hence, $d = 3h + 2$. One has

$$\mathrm{ad}_{h-1}(C) = (5; 3^2, 2^{k-1}, 1^l),$$

which has to be empty. Then either $k \geq 3$ or $l \geq 1$. (Recall that $k \geq 2$.) Taking into account the last item of Lemma 4.1, we see that $h \leq 3$ and, hence, $d \leq 11$, a contradiction.

Then $m_0 - h \geq 3$, and since $\mathrm{ad}_h(C)$ is empty, one has $k \geq 3$. The last item of Lemma 4.1 implies $h \leq 2$. If $h = 2$, then $m_1 = 4$, $m_2 = 3$, and Lemma 4.1 again yields $k \in \{3, 4\}$. The emptiness of $\mathrm{ad}_2(C) = (d - 6; d - 6, 2, 1^{k-1})$ implies $d \leq 10$, a contradiction. If $h = 1$, we obtain the assertion. $\square$

PROPOSITION 4.5

If C is a union of lines and $C \cong (d; d - 2, 3, 2^{2(d-3)})$, then C is Cr*-contractible.*

Proof

The assertion is clear for $d = 3, 4$ by Lemmas 3.7 and 3.8, so we may assume $d \geq 5$, and we proceed by induction on d. Consider the two lines L_1, L_2 through P_1 not passing through P_0, and consider two more lines L_3, L_4 through P_0 and not through P_1. Consider the intersection points $P_{1,3} = L_1 \cap L_3$, $P_{2,4} = L_2 \cap L_4$. Make a quadratic transformation based at $P_0, P_{1,3}, P_{2,4}$. This maps C to a union of lines $C' \cong (d - 2; d - 4, 3, 2^{2(d-4)})$: the lines L_3, L_4 have been contracted to two points of the transforms of the lines L_1, L_2 which do not lie on any other component of C'. The assertion follows by induction. $\square$

4.2.2. The subcase $\varepsilon = 1$

We are now in either case (i) or case (ii), and in table (7) the last two rows occur. Thus, either

$$
\begin{aligned}
&m_0 \geq m_1 = h+3, \\
(8) \quad & \\
&\qquad m_2 = h+1, \quad \text{hence } \mathrm{ad}_h(C) = (m_0 - h + 1; m_0 - h, 3, 1^{k-1}), \quad \text{or}
\end{aligned}
$$

$$
(9) \qquad m_0 \geq m_1 = m_2 = h+2, \quad \text{hence } \mathrm{ad}_h(C) = (m_0 - h + 1; m_0 - h, 2^k, 1^l).
$$

LEMMA 4.6

If (3) holds, $m = d+3$, $d \geq 11$, $h \geq 1$, and $\varepsilon = 1$, then either:

1. $C \cong (d; d-3, 4, 2^{3(d-4)})$, *or*
2. $C \cong (d; d-3, 3^2, 2^{3(d-4)})$ *or* $C \cong (d; d-3, 3^3, 2^{3(d-5)})$.

Proof

Note that in case (8) one has $m_0 - h \geq 3$ and in case (9) one has $m_0 - h \geq 2$. So, to make $\mathrm{ad}_h(C)$ empty we must have:

- $k \geq 5$ in case (8), and
- either $k \geq 3$, or $k = 2$ and $l \geq 1$, in case (9).

In case (8), the last item of Lemma 4.1 yields $h \leq 2$. If $h = 1$, we are in case (a). If $h = 2$, then $m_2 = 3$ implies $k \leq 5$. Then the emptiness of $\mathrm{ad}_h(C) = (d-6; d-7, 3, 1^{k-1})$ requires $d \leq 10$, contrary to the assumption.

In case (9), Lemma 4.1 yields again $h \leq 2$. If $h = 1$, then $m_2 = 3$ implies $k \leq 3$ and one has the two cases in (b). If $h = 2$, then $k = 2$ and $l \leq 4$ and the emptiness of $\mathrm{ad}_h(C) = (d-6; d-7, 2^2, 1^l)$ implies $d \leq 10$, contrary to the assumption. □

PROPOSITION 4.7

If C is either as in Lemma 4.6(a) and $d \geq 11$ or as in Lemma 4.6(b) and $d \geq 12$, then $\mathrm{ad}_{2,3}(C) \neq \emptyset$ and, hence, C is not Cr-contractible.

Proof

In case (a) one has that the fixed part of $\mathrm{ad}_{2,3}(C) = (2d-9; 2d-9, 5, 1^{3(d-4)})$ consists of the $d-3$ components of C through P_0 and the one joining P_0 and P_1 with multiplicity 5, and the movable part consists of $d-10$ general lines through P_0.

In cases (b), one has $\mathrm{ad}_{2,3}(C) = (2d-9; 2d-9, 3^2, 1^{3(d-4)})$ and $\mathrm{ad}_{2,3}(C) = (2d-9; 2d-9, 3^3, 1^{3(d-5)})$, respectively. The latter is nonempty: its fixed part consists of the $d-3$ components of C through P_0 and the ones joining P_0 with P_1, P_2, P_3 with multiplicity 3, and the movable part consists of $d-12$ general lines through P_0. The former is also nonempty: its fixed part consists of the $d-3$ components of C through P_0, the ones joining P_0 with P_1, P_2 with multiplicity 3, and the line joining P_0 with the intersection of the two distinct lines in C containing P_1, P_2, and the movable part consists of $d-11$ general lines through P_0. □

4.3. The case m minimal

Now we treat the different extremal case in which $m = d + 2 - \varepsilon$ and, hence, $m_1 = m_2 = h + 1$ (first line of (7)).

4.3.1. The subcase $\varepsilon = 0$

We are in either case (i) or case (iii). We have $m = d + 2$ and

$$\mathrm{ad}_h(C) = (m_0 - h; m_0 - h, 1^k).$$

LEMMA 4.8

Assume (3) holds, $m = d+2$, $d \geq 12$, $h \geq 1$, and $\varepsilon = 0$. Then $C \cong (d; d-2, 2^{2d-3})$.

Proof

Assume first $m_0 - h = 1$. Hence, $d = 3h + 1$. Recall that $k \geq 2$. If $k > 2$, then Lemma 4.2 implies $h \leq 3$ and, hence, $d \leq 10$, a contradiction. If $k = 2$, then $\mathrm{ad}_{h-1}(C) = (4; 2^3, 1^l)$ has to be empty, so we must have $l \geq 6$. Lemma 4.2 implies $h \leq 4$, which can happen only in case (β) of Lemma 4.2, in which case $l \leq 2$, a contradiction. If $h \leq 3$, then $d \leq 10$ and we have a contradiction again. Hence, $m_0 - h \geq 2$ and $\mathrm{ad}_h(C) = \emptyset$ yield $k > m_0 - h \geq 2$. We discuss separately the various cases in Lemma 4.2.

Case (α). We have $h \leq 2$. If $h = 2$, then $k \leq 6$. Since $\mathrm{ad}_2(C) = (d - 6; d - 6, 1^k)$ is empty, we have $6 \geq k \geq d - 5$, a contradiction. If $h = 1$, we have the assertion.

Case (β). We have $h \leq 3$. If $h = 3$, then $\mathrm{ad}_3(C) = (d - 9; d - 9, 1^k)$. On the other hand, one has $k \leq 4$, which leads to $d \leq 11$, a contradiction. If $h = 2$, then $k \leq 6$ and $d \leq 11$. So we are left with the case $h = 1$, leading to the assertion.

Case (γ). We have $h \leq 2$. If $h = 2$, one has $k \leq 6$. The emptiness of $\mathrm{ad}_2(C)$ implies $d \leq 11$, a contradiction. If $h = 1$, we have the assertion. □

PROPOSITION 4.9

If C is a union of lines and $C \cong (d; d - 2, 2^{2d-3})$, then C is Cr*-contractible.*

Proof

If $d \leq 3$, the assertion is trivial. We then argue by induction on d. Let L_1, L_2 be the two lines not passing through P_0, and let L_3, L_4 be two lines through P_0. Consider the intersection points $P_{1,3} = L_1 \cap L_3$, $P_{2,4} = L_2 \cap L_4$. Make a quadratic transformation based at $P_0, P_{1,3}, P_{2,4}$. This maps C to a union of lines $C' \cong (d - 2; d - 4, 2^{2d-7})$: the lines L_3, L_4 have been contracted to two points of the transforms of the lines L_1, L_2 which do not lie on any other component of C'. The assertion follows by induction. □

4.3.2. The subcase $\varepsilon = 1$

We are in either case (i) or case (ii). We have $m = d + 1$ and

$$\mathrm{ad}_h(C) = (m_0 - h + 1; m_0 - h, 1^k).$$

LEMMA 4.10

Assume (3) holds, $m = d+1$, $d \geq 12$, $h \geq 1$, and $\varepsilon = 1$. Then $C \cong (d; d-3, 2^{3(d-2)})$.

Proof

Since $m_0 - h + 1 \geq 2$ and $\mathrm{ad}_h(C)$ is empty, one has $k \geq 5$. Lemma 4.3 applies.

Case (α'). One has $h \leq 3$. If $h = 3$, then $k \leq 5$. Hence, $k = 5$ and $\mathrm{ad}_2(C) = (2; 1^6)$, but the six points do lie on a (reducible) conic, a contradiction. If $h = 2$, then $\mathrm{ad}_h(C) = (d-6; d-7, 1^k)$ and to make $\mathrm{ad}_h(C)$ empty we need $k > 2(d-6)$. On the other hand, one sees that $k \leq 8$ and, hence, $d \leq 9$, a contradiction. In case $h = 1$ we obtain the assertion.

Case (β'). One has $h \leq 4$. If $h = 4$, then $k \leq 3$, a contradiction. If $h = 3$, then $k \leq 6$ and, hence, $k \in \{5, 6\}$. Since $\mathrm{ad}_3(C) = (d-9; d-10, 1^k)$, we have $k > 2(d-9)$ and, thus, $d \leq 11$, a contradiction. If $h = 2$, then $k \leq 10$ and, hence, $d \leq 10$, a contradiction again. Therefore, $h = 1$ and we obtain the assertion.

Case (γ'). One has $h \leq 3$. If $h = 3$, then $k \leq 4$, a contradiction. If $h = 2$, then $k \leq 10$, which forces $d \leq 10$, a contradiction. Hence, $h = 1$ and we obtain the assertion.

Case (δ'). One has $h \leq 2$. If $h = 2$, then $k \leq 6$, which leads to a contradiction as above. Hence, $h = 1$ and we obtain the assertion. □

PROPOSITION 4.11

If C is a union of lines and $C \cong (d; d-3, 2^{3(d-2)})$ with $d \geq 9$, then $\mathrm{ad}_{2,3}(C) \neq \emptyset$ and, hence, C is not Cr-contractible.

Proof

Let P_1, P_2, P_3 be the vertices of the triangle formed by the three lines of C not passing through P_0. One has $\mathrm{ad}_{2,3}(C) = (2d-9; 2d-9, 1^{3(d-2)}) \neq \emptyset$: its fixed part consists of the $d-3$ components of C through P_0 plus the three lines joining P_0 with P_i, $i = 1, 2, 3$, and the movable part consists of $d-9$ general lines through P_0. □

4.4. The intermediate case for m

Now we treat the intermediate case in which the length of the interval $[d+2-\varepsilon, d+3]$ in which m lies is 2, which forces $\varepsilon = 1$, and $m = d+2$ is the intermediate value. Hence, we are in the case described in Lemma 4.2. The values of m_1, m_2 are given by the second row of table (7). The relevant adjoint is

$$\mathrm{ad}_h(C) = (m_0 - h + 1; m_0 - h, 2, 1^{k-1}).$$

LEMMA 4.12

Assume (3) holds, $m = d+2$, $d \geq 11$, $h \geq 1$, and $\varepsilon = 1$. Then $C \cong (d; d-3, 3, 2^{3(d-3)})$.

Proof
As in the proof of Lemma 4.10, we have $k \geq 5$. Again we make a case-by-case discussion according to the possibilities listed in Lemma 4.2.

Case (α). One has $h \leq 2$. If $h=2$, then $k \leq 7$. Since $\mathrm{ad}_2(C) = (d-6; d-7, 2, 1^{k-1}) = \emptyset$, it follows that $d \leq 10$, a contradiction. If $h=1$, we have the assertion.

Case (β). One has $h \leq 3$. If $h=3$, then $k \leq 4$, a contradiction. If $h=2$, then $k \leq 7$, which forces $d \leq 10$ as above, a contradiction. Hence, $h=1$ and we have the assertion.

Case (γ). One has $h \leq 2$. If $h=2$, then $k \leq 7$, which forces $d \leq 10$, a contradiction. Hence, $h=1$ and we have the assertion. □

PROPOSITION 4.13
If C is a union of lines and $C \cong (d; d-3, 3, 2^{3(d-3)})$ with $d \geq 10$, then $\mathrm{ad}_{2,3}(C) \neq \emptyset$ and, hence, C is not Cr*-contractible.*

Proof
There are two configurations of C. Either C contains the line passing through P_0 and P_1 or it does not. In the former case, let P_2, P_3 be the intersection points of the line not passing through P_0 and P_1 with the two lines through P_1 not passing through P_0. In both cases, one has $\mathrm{ad}_{2,3}(C) = (2d-9; 2d-9, 3, 1^{3(d-3)}) \neq \emptyset$. Indeed, in the former case, its fixed part consists of the $d-3$ components of C through P_0, the one joining P_0 with P_1 with multiplicity 3, plus the two lines joining P_0 with P_2 and P_3, and the movable part consists of $d-10$ general lines through P_0. In the latter case, the fixed part consists of the $d-3$ components of C through P_0 plus the line joining P_0 with P_1 with multiplicity 3, and the movable part consists of $d-9$ general lines through P_0. □

We collect the previous results in the following theorem.

THEOREM 4.14
Let C be a reduced union of $d \geq 12$ lines. If condition (3) holds, then C has one of the following types:

$$(d;d), \qquad (d;d-1,2^{d-1}), \qquad (d;d-2,3,2^{2(d-3)}), \qquad (d;d-2,2^{2d-3}), \tag{10}$$

$$(d;d-3,4,2^{3(d-4)}), \qquad (d;d-3,3^3,2^{3(d-5)}), \qquad (d;d-3,3^2,2^{3(d-4)}), \tag{11}$$

$$(d;d-3,3,2^{3(d-3)}), \qquad (d;d-3,2^{3(d-2)}). \tag{12}$$

The types in (10) are Cr*-contractible, while the types in (11) and (12) are not* Cr*-contractible. If $S \to \mathbb{P}^2$ is a birational morphism which resolves the singularities of C and we denote by $\tilde{C}$ the strict transform of C on S, for the types in (10) one has $\mathrm{kod}(S,\tilde{C}) = -\infty$, while for the types in (11) and (12) one has $P_3(S,\tilde{C}) > 0$; thus, $\mathrm{kod}(S,\tilde{C}) \geq 0$. In particular, C is* Cr*-contractible if and only if $\mathrm{kod}(S,\tilde{C}) = -\infty$.*

Proof
Types $(d;d)$ and $(d;d-1,2^{d-1})$ are Cr-contractible by Lemmas 3.7 and 3.8. Types $(d;d-2,3,2^{2(d-3)})$ and $(d;d-2,2^{2d-3})$ are Cr-contractible by Propositions 4.5 and 4.9. The fact that $P_3(S,\tilde{C})>0$ for the types in (11) and (12) follows since $\mathrm{ad}_{2,3}(C)\neq\emptyset$ for them by Propositions 4.7, 4.11, and 4.13. □

REMARK 4.15
It is easy to check that each of the types in (10), (11), and (12) except $(d;d-3,3,2^{3(d-3)})$ has exactly one configuration, whereas the latter has exactly two configurations, namely, (see the proof of Proposition 4.13)

$$\big(d;\{4,5,\dots,d\},\{1,2,3\}\big) \qquad \text{and} \qquad \big(d;\{4,5,\dots,d\},\{2,3,4\}\big).$$

By Proposition 4.13, neither is Cr-contractible if $d\geq 10$.

It is interesting to note that instead, for $d=9$, the two configurations above behave quite differently with respect to Cr-contractibility. Indeed, the latter one is not Cr-contractible, whereas we will see in a moment that the former is instead Cr-contractible. Both configurations have vanishing adjoint linear systems, but the former one also has $\mathrm{ad}_{n,m}(C)=\emptyset$ for every $m\geq n\geq 1$, whereas the latter one has, as we saw, $\mathrm{ad}_{2,3}(C)\neq\emptyset$.

LEMMA 4.16
Let C be a reduced plane curve. Suppose that there is a Cremona transformation γ such that $\gamma(C)=B\cup Z$, where B is either a line or a conic and Z is either $\emptyset$ or a union of points. Then C is Cr-*contractible.*

Proof
Suppose that B is a line. Choose two general points $Q_1,Q_2\in B$ and a general point $Q_3\in\mathbb{P}^2$. Let ω be the Cremona quadratic transformation centered at Q_1,Q_2,Q_3. Then, $\omega\circ\gamma$ contracts C to points.

Suppose that B is a conic. If B is irreducible, choose five general points $Q_1,Q_2,\dots,Q_5\in B$ and a general point $Q_6\in\mathbb{P}^2$. If B is a reducible conic, say, B is the union of two lines R_1 and R_2, choose two general points $Q_1,Q_2\in R_1$, three general points $Q_3,Q_4,Q_5\in R_2$, and a general point $Q_6\in\mathbb{P}^2$. In both cases, the Cremona map ω defined by the homaloidal net $|4L-2Q_1-2Q_2-2Q_3-Q_4-Q_5-Q_6|$ is such that $\omega\circ\gamma$ contracts C to points. □

PROPOSITION 4.17
Let $C\cong(9;6,3,2^{18})$ be the configuration

$$\big(9;\{1,2,3,4,5,6\},\{1,7,8\}\big).$$

Then C is Cr-*contractible.*

Proof
Let P_0 be the point of multiplicity 6, and let P_1 be the triple point. Denote, as usual, $P_{i,j}=L_i\cap L_j$, for $i\neq j$.

Take the de Jonquières map γ_1 defined by the homaloidal net $|4L-3P_0-P_1-P_{4,7}-P_{5,8}-P_{6,9}-P_{7,9}-P_{8,9}|$. Note that γ_1 contracts L_1, L_4, L_5, L_6 to points and maps the other five lines to a pentagon. Setting $\bar{L}_i=\gamma_1(L_i)$, $i=2,3,7,8,9$, and $\bar{P}_{i,j}=\bar{L}_i\cap\bar{L}_j$, for $i\neq j$, one sees that $\gamma_1(L_4)=\bar{P}_{8,9}$, $\gamma_1(L_5)=\bar{P}_{7,9}$, $\gamma_1(L_6)=\bar{P}_{7,8}$, and that $\gamma_1(L_1)$ is a point lying on $\bar{L}_9$, different from the vertices of the pentagon.

Now the quadratic map γ_2 centered at $\bar{P}_{2,8}$, $\bar{P}_{3,7}$, $\bar{P}_{3,9}$ contracts $\bar{L}_3$ to a point and maps the other four lines to a quadrilateral. Setting $\tilde{L}_i=\gamma_2(\bar{L}_i)$, for $i=2,7,8,9$, and $\tilde{P}_{i,j}=\tilde{L}_i\cap\tilde{L}_j$, $i\neq j$, one sees that $\gamma_2(\bar{L}_3)=\tilde{P}_{2,8}$.

Then take the quadratic map γ_3 centered at $\tilde{P}_{2,7}$, $\tilde{P}_{2,9}$ and a general point $Q_8\in\tilde{L}_8$. One sees that γ_3 contracts $\tilde{L}_2$ and maps the other three lines to a triangle. Setting $\hat{L}_i=\gamma_3(\tilde{L}_i)$, $i=7,8,9$, and $\hat{P}_{i,j}=\hat{L}_i\cap\hat{L}_j$, for $i\neq j$, one sees that $\gamma_3(\tilde{L}_2)$ is a point lying on $\hat{L}_8$.

Finally, consider the quadratic map γ_4 centered at $\hat{P}_{7,8}$, at the point infinitely near to $\hat{P}_{7,8}$ in the direction of the line $\hat{L}_7$, and at a general point $Q_9\in\hat{L}_9$. One sees that γ_4 contracts the line $\hat{L}_7$ to a point and maps the other two lines to a reducible conic. The choice of the fundamental points of the Cremona maps γ_i, $i=1,2,3,4$, implies that $\gamma_4\circ\gamma_3\circ\gamma_2\circ\gamma_1$ maps C to a conic; hence, C is Cr-contractible by Lemma 4.16. $\square$

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