Entropic solution of the innovation conjecture of T. Kailath

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Abstract On a general filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0,1]), P)$, for a given signal $U_t = B_t + \int_0^t \dot{u}_s \, ds$, where $(B_t, t \in [0,1])$ is a Brownian motion and $\dot{u}$ is adapted and in $L^2(dt \times dP)$, we prove that the filtration of U, denoted $(\mathcal{U}_t, t \in [0,1])$, is equal to the filtration of its innovation process Z, which is defined as $Z_t = U_t - \int_0^t E_P[\dot{u}_s \mid \mathcal{U}_s] \, ds$, $t \in [0,1]$, if and only if

$$H\big(Z(\nu) \mid \mu\big) = \frac{1}{2} E_\nu \Big[\int_0^1 \big| E_P[\dot{u}_s \mid \mathcal{U}_s] \big|^2 \, ds \Big]$$

where $d\nu = \exp(-\int_0^1 E_P[\dot{u}_s \mid \mathcal{U}_s] \, dZ_s - \frac{1}{2} \int_0^1 |E_P[\dot{u}_s \mid \mathcal{U}_s]|^2 \, ds) \, dP$ in the case in which the density has expectation 1; otherwise, we give a localized version of the same strength with a sequence of stopping times of the filtration of U.

1. Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0,1]), P)$ be a probability space satisfying the usual conditions, and denote by (W, H, μ) the classical Wiener space, that is, $W = C_0([0,1], \mathbb{R}^d)$ and H is the corresponding Cameron–Martin space consisting of $\mathbb{R}^d$-valued absolutely continuous functions on $[0,1]$ with square integrable derivatives, which is a Hilbert space under the norm $|h|_H^2 = \int_0^1 |\dot{h}(s)|^2 \, ds$, where $\dot{h}$ denotes the Radon–Nikodym derivative of the absolutely continuous function $t \to h(t)$ with respect to the Lebesgue measure on $[0,1]$. Denote by $(\mathcal{B}_t, t \in [0,1])$ the filtration of the canonical Wiener process, completed with respect to μ-negligible sets. The question that we address in this paper is the following. Assume that $U : \Omega \to W$ is a map of the following form:

$$U(\omega)(t) = U_t(\omega) = B_t(\omega) + \int_0^t \dot{u}_s(\omega) \, ds,$$

where $B = (B_t, t \in [0,1])$ is a Brownian motion on Ω and $(s, \omega) \to \dot{u}_s(\omega)$ is an $\mathbb{R}^d$-valued map belonging to the space $L_a^2(P; H)$, which consists of the elements of $L^2([0,1] \times \Omega, \mathcal{B}([0,1]) \otimes \mathcal{F}, dt \times dP)$ that are $(\mathcal{F}_s, s \in [0,1])$-adapted for almost

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all $s \in [0,1]$. Let us define the innovation process Z associated to U as

$$Z_t = U_t - \int_0^t E_P[\dot{u}_s \,|\, \mathcal{U}_s]\, ds,$$

where $(\mathcal{U}_t, t \in [0,1])$ is the filtration generated by U. It is well known that Z is a P-Brownian motion with respect to $(\mathcal{U}_t, t \in [0,1])$, and Z is naturally adapted to $(\mathcal{U}_t, t \in [0,1])$. This means that the information obtained via Z is included in the information obtained from U. P. Frost [4] and T. Kailath [5] have conjectured that in practical situations the converse of this observation is also true. V. A. Beneš [2] has remarked that this conjecture holds if and only if there is a hidden process which is a strong solution of a certain stochastic differential equation from which one can construct the initial system. This conjecture has also been proved under a restrictive supplementary hypothesis (cf. [1]) where $\dot{u}$ is independent of the Brownian motion B. The main objection to these works lies in the fact that the condition of [2] is unverifiable from the observed data; hence, numerically it is not useful. The second objection is that it uses a hypothesis of independence which is too strong to be encountered in engineering applications. In this paper we give a necessary and sufficient condition in the most general case using the entropic characterization of the almost sure invertibility of adapted perturbations of identity (APIs). Let us explain the idea and the difference from the other works. For simplicity, assume that

$$E_P\Big[\exp\Big(-\int_0^1 E_P[\dot{u}_s \,|\, \mathcal{U}_s]\, dZ_s - \frac{1}{2}\int_0^1 \big|E_P[\dot{u}_s \,|\, \mathcal{U}_s]\big|^2\, ds\Big)\Big] = 1, \tag{1.1}$$

and denote by $\rho(-\delta_Z \hat{u})$ the Girsanov exponential inside the above expectation. Here we use the notation δ_Z to denote the Itô integral of the Lebesgue density of the vector field which is defined as $(t,\omega) \to \hat{u}(t,\omega) = \int_0^t E_P[\dot{u}_s \,|\, \mathcal{U}_s](\omega)\, ds$ with respect to Z. Define a new measure ν by $d\nu = \rho(-\delta_Z \hat{u})\, dP$. Then the observation process U is adapted to the filtration of the innovation process Z up to negligible sets if and only if we have

$$H\big(Z(\nu) \,\big|\, \mu\big) = \frac{1}{2} E_\nu\big[|\hat{u}|_H^2\big] \Big(= \frac{1}{2} E_\nu \int_0^1 \big|E_P[\dot{u}_s \,|\, \mathcal{U}_s]\big|^2\, ds\Big),$$

where $Z(\nu)$ denotes the pushforward of the measure ν under Z, and $H(Z(\nu) \,|\, \mu)$ is the relative entropy of $Z(\nu)$ with respect to the Wiener measure μ, that is,

$$H\big(Z(\nu) \,\big|\, \mu\big) = \int_W \frac{dZ(\nu)}{d\mu} \log \frac{dZ(\nu)}{d\mu}\, d\mu.$$

Clearly, the verification of this condition, namely, the equality of the entropy to the total kinetic energy of $\hat{u}$, requires only the knowledge about the observation process U. However, the calculation of the relative entropy may be time-consuming. In fact, as it follows from Theorem 3, all these results are valid when one works causally with time; in other words, they hold also when one works on the time interval $[0,t]$ for $t \geq 0$, since they are restrictable even to the random time intervals. The final result says that we can also suppress the hypothesis (1.1) by using a sequence of $(\mathcal{U}_t, t \in [0,1])$-stopping times.

2. Characterization of the invertible shifts on the canonical space

We begin with the definition of the notion of almost sure invertibility with respect to a measure. This notion is extremely important since it makes things work. Let us note that in this section all the expectations and conditional expectations are taken with respect to the Wiener measure μ.

DEFINITION 1

Let $T : W \to W$ be a measurable map.

- T is called (μ-)almost surely left invertible if there exists a measurable map $S : W \to W$ such that $S \circ T = I_W$ μ-almost surely.
- Moreover, in this case it is trivial to see that $T \circ S = I_W$ $T\mu$-almost surely, where $T\mu$ denotes the image of the measure μ under the map T.
- If $T\mu$ is equivalent to μ, then we say in short that T is μ-almost surely invertible.
- Otherwise, we may say that T is $(\mu, T\mu)$-invertible in the case when precision is required or just μ-almost surely left invertible, and S is called the μ-left inverse of T.

Let $L^2_a(\mu, H)$ be the μ-square integrable equivalence classes of Cameron–Martin space (denoted by H) valued functions; hence, $t \to (w \to u(w))$ is an absolutely continuous function of $t \in [0, 1]$ with a ds-square integrable Lebesgue density denoted as $\dot{u}_s(w)$. Moreover, we assume that $w \to \dot{u}_s(w)$ is $\mathcal{B}_s$-measurable for ds-almost all $s \in [0, 1]$, which is a Hilbert space. For short we call them adapted vector fields of class L^2. Similarly, $L^0_a(\mu, H)$ denotes the set of adapted vector fields whose Cameron–Martin norm is μ-almost surely finite. Under the topology of convergence in probability, $L^0_a(\mu, H)$ is a nonlocally convex Fréchet space. For the reader's convenience, let us note that $L^0_a(\mu, H)$ is the completion of $L^2_a(\mu, H)$ under the topology of convergence in probability.

Although the following theorem has been proved in [6], for the reader's convenience we give a short and different proof.

THEOREM 1

For any $u \in L^2_a(\mu, H)$, we have the following inequality:

$$H(U\mu \mid \mu) \leq \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 \, ds,$$

where $H(U\mu \mid \mu)$ is the relative entropy of the measure $U\mu$ with respect to μ.

Proof

Let L be the Radon–Nikodym density of $U\mu$ with respect to μ. For any $0 \leq g \in C_b(W)$, using the Girsanov theorem, we have

$$E[g \circ U] = E[gL] \geq E\big[g \circ U L \circ U \rho(-\delta u)\big];$$

hence,

$$L \circ U E\big[\rho(-\delta u) \,\big|\, U\big] \le 1$$

μ-almost surely. Consequently, using the Jensen inequality, we have

$$\begin{aligned} H(U\mu \mid \mu) &= E[L \log L] = E[\log L \circ U] \\ &\le -E\big[\log E\big[\rho(-\delta u) \,\big|\, U\big]\big] \\ &\le -E\big[\log \rho(-\delta u)\big] \\ &= \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 \, ds. \end{aligned}$$

□

THEOREM 2

Assume that $U = I_W + u$ is an API, that is, $u \in L_a^2(\mu, H)$ such that $s \to \dot{u}(s,w)$ is $\mathcal{B}_s$-measurable for almost all s. Then U is almost surely left invertible with a left inverse V if and only if

$$H(U\mu \mid \mu) = \frac{1}{2} E\big[|u|_H^2\big] = \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 \, ds,$$

that is, if and only if the entropy of $U\mu$ is equal to the energy of the drift u.

Proof

Due to Theorem 1, the relative entropy is finite as soon as $u \in L_a^2(\mu, H)$. Let us suppose now that the equality holds, and let us denote by L the Radon–Nikodym derivative of $U\mu$ with respect to μ. Using the Itô representation theorem, we can write

$$L = \exp\Big(-\int_0^1 \dot{v}_s \, dW_s - \frac{1}{2} \int_0^1 |\dot{v}_s|^2 \, ds\Big)$$

$U\mu$-almost surely. Let $V = I_W + v$, as described in [3]. From the Itô formula and Paul Lévy's theorem, it is immediate that V is a $U\mu$-Wiener process; hence,

$$E[L \log L] = \frac{1}{2} E\big[L |v|_H^2\big]. \tag{2.1}$$

Now, for any $f \in C_b(W)$, we have from the Girsanov theorem that

$$E[f \circ U] = E[fL] \ge E\big[f \circ U L \circ U \rho(-\delta u)\big].$$

Consequently,

$$L \circ U E\big[\rho(-\delta u) \,\big|\, U\big] \le 1$$

μ-almost surely. Let us denote $E[\rho(-\delta u) \mid U]$ by $\hat{\rho}$. We then have $\log L \circ U + \log \hat{\rho} \le 0$ μ-almost surely. Taking the expectation with respect to μ and the Jensen inequality gives

$$\begin{aligned} H(U\mu \mid \mu) &= E[L \log L] \le -E[\log \hat{\rho}] \\ &\le -E\big[\log \rho(-\delta u)\big] = \frac{1}{2} E\big[|u|_H^2\big]. \end{aligned}$$

Since log is a strictly concave function, the equality $E[\log \hat{\rho}] = E[\log \rho(-\delta u)]$ implies that $\rho(-\delta u) = \hat{\rho}$ μ-almost surely. Hence we obtain

$$E\big[L \log L + \log \rho(-\delta u)\big] = E\big[\log\big(L \circ U \rho(-\delta u)\big)\big] = 0,$$

and since $L \circ U \rho(-\delta u) \leq 1$ μ-almost surely, we should have that

$$L \circ U \rho(-\delta u) = 1 \tag{2.2}$$

μ-almost surely. Combining the exponential representation of L with (2.2) implies

$$\begin{aligned} 0 &= \Big(\int_0^1 \dot{v}_s \, dW_s\Big) \circ U + \frac{1}{2}|v \circ U|_H^2 + \delta u + \frac{1}{2}|u|_H^2 \\ &= \delta(v \circ U) + \delta u + (v \circ U, u)_H + \frac{1}{2}\big(|u|_H^2 + |v \circ U|_H^2\big) \\ &= \delta(v \circ U + u) + \frac{1}{2}|v \circ U + u|_H^2 \end{aligned} \tag{2.3}$$

μ-almost surely. From (2.1) it follows that $v \circ U \in L_a^2(\mu, H)$; hence taking the expectations on both sides of (2.3) with respect to μ is licit, and this implies that $v \circ U + u = 0$ μ-almost surely, which means that $V = I_W + v$ is the μ-left inverse of U.

To show the necessity, let us denote by $(L_t, t \in [0,1])$ the martingale

$$L_t = E[L \mid \mathcal{B}_t] = E\Big[\frac{dU\mu}{d\mu} \,\Big|\, \mathcal{B}_t\Big],$$

and let

$$T_n = \inf\Big(t : L_t < \frac{1}{n}\Big).$$

Since $U \circ V = I_W$ $(U\mu)$-almost surely, V can be written as $V = I_W + v$ $(U\mu)$-almost surely, and $v \in L_a^0(U\mu, H)$, that is, $v(t,w) = \int_0^t \dot{v}_s(w)\, ds$, $\dot{v}$ is adapted to the filtration $(\mathcal{B}_t)$ completed with respect to $U\mu$, and $\int_0^1 |\dot{v}_s|^2 \, ds < \infty$ $(U\mu)$-almost surely. Noting that $\{t \leq T_n\} \subset \{L > 0\}$ and that μ and $U\mu$ are equivalent on $\{L > 0\}$, we conclude that

$$\int_0^{T_n} |\dot{v}_s|^2 \, ds < \infty$$

μ-almost surely. Consequently, the inequality

$$E_\mu\big[\rho(-\delta v^n)\big] \leq 1$$

holds true for any $n \geq 1$, where $v^n(t,w) = \int_0^t 1_{[0,T_n]}(s,w) \dot{v}_s(w)\, ds$. By positivity we also have

$$E_\mu\big[\rho(-\delta v^n) 1_{\{L>0\}}\big] \leq 1.$$

Since $\lim_{n\to\infty} T_n = \infty$ $(U\mu)$-almost surely, we also have that $\lim_{n\to\infty} T_n = \infty$ μ-almost surely on the set $\{L > 0\}$, and the Fatou lemma implies that

$$
\begin{aligned}
E_\mu\left[\rho(-\delta v)1_{\{L>0\}}\right] &= E_\mu\left[\lim_n \rho(-\delta v^n)1_{\{L>0\}}\right] \\
&\le \liminf_n E_\mu\left[\rho(-\delta v^n)1_{\{L>0\}}\right] \le 1.
\end{aligned} \tag{2.4}
$$

From the identity $U\circ V = I_W$ $(U\mu)$-almost surely, we have that $v + u\circ V = 0$ $(U\mu)$-almost surely; hence, $v\circ U + u = 0$ μ-almost surely. An algebraic calculation gives immediately that

$$
\rho(-\delta v)\circ U \rho(-\delta u) = 1 \tag{2.5}
$$

μ-almost surely. Now applying the Girsanov theorem to API U and using the relation (2.5), we obtain

$$
\begin{aligned}
E[g\circ U] = E[gL] &= E\left[g\circ U\left(\rho(-\delta v)1_{\{L>0\}}\right)\circ U\rho(-\delta u)\right] \\
&\le E\left[g\rho(-\delta v)1_{\{L>0\}}\right],
\end{aligned}
$$

for any positive $g\in C_b(W)$. (Note that, on the set $\{L>0\}$, $\rho(-\delta v)$ is perfectly well defined with respect to μ.) Therefore,

$$
L \le \rho(-\delta v)1_{\{L>0\}}
$$

μ-almost surely. Now, this last inequality combined with (2.4) gives that

$$
L = \rho(-\delta v)1_{\{L>0\}}
$$

μ-almost surely; hence,

$$
L\circ U\rho(-\delta u) = 1
$$

μ-almost surely. To complete the proof it suffices to remark that

$$
\begin{aligned}
H(U\mu\mid\mu) = E[L\log L] &= E[\log L\circ U] \\
&= E\left[-\log\rho(-\delta u)\right] \\
&= \frac{1}{2}E\left[|u|_H^2\right].
\end{aligned}
$$

□

The following result comes almost for free.

THEOREM 3

Assume that $U = I_W + u$ is an API which is μ-almost surely left invertible, and let τ be any stopping time such that, for u^τ defined as $u^\tau(t,w) = u(t\wedge\tau(w),w)$,

$$
E\left[\rho(-\delta u^\tau)\right] = 1.
$$

Then $U^\tau = I_W + u^\tau$ is μ-almost surely invertible; in other words, there exists some API, say, V' such that $V'\circ U^\tau = U^\tau\circ V' = I_W$ μ-almost surely.

Proof

Since $E[\rho(-\delta u^\tau)] = 1$, $U^\tau\mu$ is equivalent to the Wiener measure μ; hence its Radon–Nikodym density can be written as

$$
\frac{dU^\tau\mu}{d\mu} = \rho(-\delta\xi).
$$

From the Girsanov theorem it follows that

$$\rho(-\delta\xi)\circ U^\tau E\big[\rho(-\delta u^\tau)\,\big|\,U^\tau\big]=1 \tag{2.6}$$

μ-almost surely. Let z be the innovation process of U^τ, which is defined as $z_t=U_t^\tau-\int_0^t E[\dot u_s^\tau\,|\,\mathcal{U}_s^\tau]\,ds$, where $(\mathcal{U}_s^\tau, s\in[0,1])$ denotes the filtration corresponding to U^τ. Applying the Girsanov theorem again, this time using the Brownian motion z (cf. [6] for the details), we find that

$$E\big[\rho(-\delta u^\tau)\,\big|\,U^\tau\big]=\exp\Big(-\int_0^1 E[\dot u_s^\tau\,|\,\mathcal{U}_s^\tau]\,dz_s-\frac{1}{2}\int_0^1\big|E[\dot u_s^\tau\,|\,\mathcal{U}_s^\tau]\big|^2\,ds\Big).$$

This relation combined with (2.6) gives the relation

$$\dot\xi_t\circ U^\tau+E\big[\dot u 1_{[0,\tau]}(t)\,\big|\,\mathcal{U}_t^\tau\big]=0$$

$(dt\times d\mu)$-almost surely. Besides, for any $A\in L^\infty(\mu)$, we have that

$$\begin{aligned}E\big[AE\big[\dot u_t 1_{[0,\tau]}(t)\,\big|\,\mathcal{U}_t^\tau\big]\big]&=E\big[E[A\,|\,\mathcal{U}_t^\tau]\dot u_t 1_{[0,\tau]}(t)\big]\\&=E\big[E[A\,|\,\mathcal{U}_t]\dot u_t 1_{[0,\tau]}(t)\big]\\&=E\big[AE[\dot u_t^\tau\,|\,\mathcal{U}_t]\big]\\&=E[A\dot u_t^\tau],\end{aligned}$$

where the last equality follows from the left invertibility of U. Hence, we obtain

$$\dot\xi_t\circ U^\tau+\dot u 1_{[0,\tau]}(t)=\dot\xi_t\circ U^\tau+\dot u_t^\tau=0$$

$(dt\times d\mu)$-almost surely, which is equivalent to the μ-almost sure invertibility of U^τ. □

3. The case of a general probability space

The following result is essential for the proof of the conjecture. We use the notations explained in the introduction, and we differentiate carefully between the Wiener measure μ and the probability P as well as the respective expectations and conditional expectations to avoid any ambiguity. In particular, we denote by $L_a^2(P,H)$ the space of adapted, P-square integrable vector fields; this is exactly the same space as $L_a^2(\mu,H)$, where the Wiener space is replaced by a general probability space Ω, μ is replaced by a probability P defined on $(\Omega,\mathcal{F})$, and the canonical filtration $(\mathcal{B}_t, t\in[0,1])$ of the Wiener space is replaced by a general filtration $(\mathcal{F}_t, t\in[0,1])$ of $(\Omega,\mathcal{F})$. Similarly, $L_a^0(P,H)$ denotes the version of $L_a^0(\mu,H)$ on this general probability space; we remark that it is the completion of $L_a^2(P,H)$ with respect to the convergence in probability P.

THEOREM 4

Let $U=B+u=B+\int_0^\cdot \dot u_s\,ds$ be an API mapping Ω to W with $u\in L^2(P,H)$. Then

$$H\big(U(P)\,|\,\mu\big)=\frac{1}{2}E_P\big[|u|_H^2\big]$$

if and only if there exists some $v: W \to H$ (of the form $v = \int_0^\cdot \dot{v}_s\, ds$) with $\dot{v}$ adapted ds-almost surely to the filtration $(\mathcal{B}_t(W))$ such that

$$U(\omega) = B(\omega) - v \circ U(\omega),$$

which implies in particular that $B = Z$, where Z is the innovation process associated to U. In other words, U is a solution of the following stochastic differential equation:

$$dU_t = -\dot{v}_t \circ U\, dt + dB_t.$$

Proof
Note first that U is not necessarily a strong solution. Let us now prove the necessity. Since U is an API, $U(P)$ is absolutely continuous with respect to the Wiener measure μ. Let l be the corresponding Radon–Nikodym derivative. We can represent it as a Girsanov exponential $U(P)$-almost surely; that is, we have

$$\begin{aligned} l = \frac{dU(P)}{d\mu} &= \rho(-\delta v) \\ &= \exp\Big(-\int_0^1 \dot{v}_s\, dW_s - \frac{1}{2}\int_0^1 |\dot{v}_s|^2\, ds\Big), \end{aligned}$$

$U(P)$-almost surely, where (W_t) is the canonical Wiener process. For any positive $f \in C_b(W)$, it follows from the Girsanov theorem that

$$E_P[f \circ U] = E_\mu[fl] \geq E_P\big[f \circ U l \circ U \rho(-\delta_B u)\big],$$

where

$$\rho(-\delta_B u) = \exp\Big(-\int_0^1 \dot{u}_s\, dB_s - \frac{1}{2}\int_0^1 |\dot{u}_s|^2\, ds\Big).$$

This inequality, which is valid for any positive, measurable f, implies that

$$l \circ U E_P\big[\rho(-\delta_B u) \bigm| U\big] \leq 1$$

P-almost surely. Therefore,

$$\begin{aligned} H\big(U(P) \bigm| \mu\big) &= E_\mu[l \log l] = E_P[\log l \circ U] \\ &\leq -E_P\big[\log E_P\big[\rho(-\delta_B u) \bigm| U\big]\big] \leq \frac{1}{2} E_P\big[|u|_H^2\big]. \end{aligned}$$

The equality hypothesis $H(U(P) \mid \mu) = \frac{1}{2}E_P[|u|_H^2]$ and the strict convexity of the function $x \to -\log x$ imply that

$$l \circ U \rho(-\delta_B u) = 1$$

P-almost surely. Therefore,

$$\begin{aligned} 1 &= \rho(-\delta v) \circ U \rho(-\delta B) \\ &= \exp -\Big[(\delta v) \circ U + \frac{1}{2}|v \circ U|_H^2 + \delta_B u + \frac{1}{2}|u|_H^2\Big] \\ &= \exp -\Big[\delta_B(v \circ U) + (v \circ U, u)_H + \frac{1}{2}|v \circ U|_H^2 + \delta_B u + \frac{1}{2}|u|_H^2\Big], \end{aligned}$$

which implies that

$$\delta_B(u+v\circ U)+\frac{1}{2}|v\circ U+u|_H^2=0$$

P-almost surely. Since $E_P[|v\circ U|_H^2]=E_\mu[l|v|_H^2]=2E_\mu[l\log l]$, it follows that $v\circ U+u=0$ P-almost surely. Note that we can write

$$U=B+u=Z+\hat{u},\qquad \hat{u}=\int_0^{\cdot} E_P[\dot{u}_s\,|\,\mathcal{U}_s]\,ds,$$

since $u=-v\circ U$, $\dot{u}$ is adapted to the filtration of U, and therefore $B=Z$.

Sufficiency. If $U=B-v\circ U$, then $Z=B$ and $v\circ U+u=0$. Let l again denote the Radon–Nikodym derivative of $U(P)$ with respect to μ. As before we can write $l=\rho(-\delta\xi)$ $U(P)$-almost surely, for some $\xi:W\to H$ such that $\xi=\int_0^{\cdot}\dot{\xi}_s\,ds$, $\int_0^1|\dot{\xi}_s|^2\,ds<\infty$ $U(P)$-almost surely, and $\dot{\xi}_s$ is $\mathcal{B}_s(W)$-measurable ds-almost surely. Using the Girsanov theorem as above, we find that

$$l\circ U E_P\big[\rho(-\delta_B u)\,\big|\,U\big]\le 1$$

but the hypothesis implies that $\rho(-\delta_B u)$ is U-measurable. It then follows that

$$\delta_B(u+\xi\circ U)+\frac{1}{2}|\xi\circ U+u|_H^2\le 0$$

P-almost surely. Since $E_P[|\xi\circ U|_H^2]=2H(U(P)\,|\,\mu)<\infty$, it follows that $\xi\circ U=v\circ U$ P-almost surely. Consequently

$$H\big(U(P)\,|\,\mu\big)=E_\mu[l\log l]=E_P[\log l\circ U]=-E_P\big[\log\rho(-\delta_B u)\big]=\frac{1}{2}E_P\big[|u|_H^2\big],$$

and this completes the proof. □

Theorem 4 says that $U=B+u$ with $u\in L_a^0(P,H)$ is the weak solution of the stochastic differential equation

$$dU_t=dB_t-\dot{v}_t\circ U\,dt$$

if and only if we have equality between the entropy $H(U(P)\,|\,\mu)$ and the total kinetic energy of u with respect to the probability P. A natural question is: When is this solution strong? The following theorem gives the answer.

THEOREM 5

Assume that U is a weak solution of the stochastic differential equation

$$dU_t=dB_t-\dot{v}_t\circ U\,dt,$$

with the hypothesis that $v\circ U\in L_a^0(P,H)$. Define the sequence of stopping times $(t_n,n\ge 1)$ as

$$t_n=\inf\Big(t:\int_0^t|\dot{v}_s|^2\,ds>n\Big).$$

Let

$$\dot{u}_n(t,\omega)=-\dot{v}_t\circ U(\omega)1_{[0,t_n\circ U]}(t),$$

and let $U^n = B + u_n$ *where* $u_n(t,\omega) = \int_0^t \dot{u}_n(s,\omega)\,ds$. *Define a new probability* Q_n *by* $dQ_n = \rho(-\delta_B(u_n))\,dP$. *Then*

$$H\big(B(Q_n)\mid \mu\big) = \frac{1}{2} E_{Q_n}\big[|u_n|_H^2\big]$$

for any $n \geq 1$ *if and only if* U *is a strong solution.*

Proof

Necessity. Since, under Q_n, U_n is a Brownian motion, since the hypothesis combined with Theorem 4 implies that $v_n \circ U$ is measurable with respect to the filtration of B up to Q_n-negligible sets, and since Q_n is equivalent to P, it follows that $v_n \circ U$ is adapted to the same filtration completed with P-negligible sets. Since $\lim_{n\to\infty} v_n \circ U = v \circ U$, U is also adapted to the P-completion of the filtration of B; hence U is a strong solution of the above stochastic differential equation.

Sufficiency. If U is a strong solution, then it is of the form $U = \hat{U}(B) = B - v \circ \hat{U}(B)$ and $\hat{U} : W \to W$ has a μ-almost surely left inverse $V = I_W + v$. Since Q_n is equivalent to P, we also have that $U = \hat{U}(B)$ Q_n-almost surely. Moreover, $B = U_n + v_n \circ U$ and $v_n \circ U$ is adapted to the filtration of B up to Q_n-negligible sets for any $n \geq 1$. Due to Theorem 4 this is equivalent to the equality

$$H\big(B(Q_n)\mid \mu\big) = \frac{1}{2} E_{Q_n}\big[|u_n|_H^2\big],$$

for any $n \geq 1$. □

4. Proof of the innovation conjecture

We are now at a position to give the proof of the conjecture. We shall do it in two steps by using the notation explained in the introduction. The first step is with a supplementary hypothesis to explain clearly the idea; the second one is in full generality.

We have the relation

$$U = B + u = Z + \hat{u},$$

and we shall denote by $(\mathcal{Z}_t, t \in [0,1])$ the filtration generated by the innovation process Z. We use also the notation

$$\rho(-\delta_Z \hat{u}) = \exp\Big(-\int_0^1 E_P[\dot{u}_s \mid \mathcal{U}_s]\,dZ_s - \frac{1}{2}\int_0^1 \big|E_P[\dot{u}_s \mid \mathcal{U}_s]\big|^2\,ds\Big).$$

First we give a proof with a supplementary hypothesis which will be suppressed in the final proof.

PROPOSITION 1

Assume that

$$E_P\big[\rho(-\delta_Z \hat{u})\big] = 1.$$

Denote by ν *the probability defined by* $d\nu = \rho(\delta_Z \hat{u})\,dP$. *Then* $\mathcal{U}_t = \mathcal{Z}_t$ *for any* $t \geq 0$ *up to negligible sets and* $\hat{u} = v \circ Z$, *with* $v \in L^0(\mu, H)$ *and with* $\dot{v}_s$ *being* $\mathcal{B}_s(W)$*-measurable* ds*-almost surely, if and only if*

$$H\big(Z(\nu)\,|\,\mu\big) = \frac{1}{2}E_\nu\big[|\hat{u}|_H^2\big].$$

Proof
By Paul Lévy's theorem, U is a Brownian motion under the measure ν and $Z = U - \hat{u}$. Then Theorem 4 says that (replacing B by U and P by ν), $\hat{u}$ is a functional of Z and that $s \to E_P[\dot{u}_s \,|\, \mathcal{U}_s]$ is adapted to the filtration $(\mathcal{Z}_s, s \in [0,1])$ ds-almost surely. Hence, U is Z-measurable. Moreover, the same theorem implies the existence of some $v \in L^0(\mu, H)$ which is defined as

$$\frac{dZ(\nu)}{d\mu} = \rho(\delta v)$$

such that $\hat{u} = v \circ Z$. □

Now we are ready to give the full proof.

THEOREM 6
Let $T_n = \inf(t : \int_0^t |E_P[\dot{u}_s \,|\, \mathcal{U}_s]|^2\,ds > n)$. *Define*

$$\hat{u}_n(t,\omega) = \hat{u}(t \wedge T_n, \omega),$$
$$U_n = Z + \hat{u}_n.$$

Then $\mathcal{Z}_t = \mathcal{U}_t$ *for any* $t \geq 0$ *up to negligible sets, and hence* $\hat{u}$ *should be of the form* $\tilde{u} \circ Z$ *with some* $\tilde{u} \in L_a^0(\mu, H)$ *if and only if we have*

$$H\big(Z(\nu_n)\,|\,\mu\big) = \frac{1}{2}E_{\nu_n}\big[|\hat{u}_n|_H^2\big] \tag{4.1}$$

for any $n \geq 1$, *where* $d\nu_n = \rho(-\delta_Z \hat{u}_n)\,dP$ *and*

$$\rho(-\delta_Z \hat{u}_n) = \exp\Big(-\int_0^{T_n} E_P[\dot{u}_s \,|\, \mathcal{U}_s]\,dZ_s - \frac{1}{2}\int_0^{T_n} \big|E_P[\dot{u}_s \,|\, \mathcal{U}_s]\big|^2\,ds\Big).$$

Proof
Sufficiency. Under the measure ν_n, U_n is a Brownian motion and $Z = U_n - \hat{u}_n$. It follows from Theorem 4 that $\hat{u}_n$ is $((\mathcal{Z}_t), \nu_n)$-adapted if and only if the relation (4.1) holds true. Since ν_n is equivalent to P, U_n is also $((\mathcal{Z}_t), P)$-adapted for any $n \geq 1$; since $U_n \to U$ in $L^0(P, W)$ (i.e., P-equivalence classes of W-valued measurable functions under the topology of convergence in probability P), U is also $((\mathcal{Z}_t), P)$-adapted.

Necessity. Assume that U is $((\mathcal{Z}_t), P)$-adapted; then it is also $((\mathcal{Z}_t), \nu_n)$-adapted since $\nu_n \sim P$ for any $n \geq 1$. Hence U_n is also $((\mathcal{Z}_t), \nu_n)$-adapted for any $n \geq 1$ and this is equivalent to (4.1) for any $n \geq 1$. □

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