Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras

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Abstract Let A, B be square irreducible matrices with entries in $\{0,1\}$. We will show that if the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, then the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent, and hence $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$. As a result, the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if and only if the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic and $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$.

1. Introduction

The interplay between the orbit equivalence of topological dynamical systems and the theory of C^*-algebras has been studied by many authors. Giordano, Putnam, and Skau [7] have proved that two minimal homeomorphisms on a Cantor set are strongly orbit equivalent if and only if the associated C^*-crossed products are isomorphic. Boyle and Tomiyama [3] and Tomiyama [20] have studied relationships between orbit equivalence and C^*-crossed products for topologically free homeomorphisms on compact Hausdorff spaces.

In this paper, we classify one-sided irreducible topological Markov shifts up to continuous orbit equivalence and show that there exists a close connection with the Cuntz–Krieger algebras. The class of one-sided topological Markov shifts is an important class of topological dynamical systems on Cantor sets, though they are not homeomorphisms but local homeomorphisms. The first author [11] introduced the notion of continuous orbit equivalence for one-sided topological Markov shifts (see Definition 2.1) and proved that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) for irreducible matrices A and B with entries in $\{0,1\}$ are continuously orbit equivalent if and only if there exists an isomorphism between the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ preserving their canonical Cartan subalgebras $\mathcal{D}_A$ and $\mathcal{D}_B$. The second author in [15] and [16] studied the associated étale groupoids G_A and their homology groups $H_n(G_A)$ and topological full groups $[[G_A]]$. In fact, the two shifts are continuously orbit equivalent if and only

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if G_A is isomorphic to G_B (see Theorem 2.3). In [12] it was also shown that if $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$ and $\det(\mathrm{id}-A)=\det(\mathrm{id}-B)$, then there exists an isomorphism $\Psi:\mathcal{O}_A\to\mathcal{O}_B$ such that $\Psi(\mathcal{D}_A)=\mathcal{D}_B$, and hence the one-sided topological Markov shifts (X_A,σ_A) and (X_B,σ_B) are continuously orbit equivalent. Since there were no known examples of irreducible matrices A,B such that (X_A,σ_A) and (X_B,σ_B) are continuously orbit equivalent and $\det(\mathrm{id}-A)\neq\det(\mathrm{id}-B)$, the first author [12, Section 6] presented the following conjecture: the determinant $\det(1-A)$ is an invariant for the continuous orbit equivalence class of (X_A,σ_A). In the present article we confirm this conjecture. In other words, we show that (X_A,σ_A) and (X_B,σ_B) are continuously orbit equivalent if and only if $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$ and $\det(\mathrm{id}-A)=\det(\mathrm{id}-B)$ (see Theorem 3.6).

Our proof is closely related to another notion of equivalence for shifts, namely, flow equivalence for two-sided topological Markov shifts. Two-sided topological Markov shifts $(\bar{X}_A,\bar{\sigma}_A)$ and $(\bar{X}_B,\bar{\sigma}_B)$ are said to be *flow equivalent* if there exists an orientation-preserving homeomorphism between their suspension spaces (see [17]). Two characterizations of the flow equivalence are known. One is due to Boyle and Handelman [2] and the other is due to Parry and Sullivan [17], Bowen and Franks [1], and Franks [6] (see Theorems 2.4 and 2.6). By using the former characterization and the groupoid approach, we show that if (X_A,σ_A) and (X_B,σ_B) are continuously orbit equivalent, then $(\bar{X}_A,\bar{\sigma}_A)$ and $(\bar{X}_B,\bar{\sigma}_B)$ are flow equivalent (see Theorem 3.5). This, together with the second characterization, implies that $\det(\mathrm{id}-A)=\det(\mathrm{id}-B)$, and so the conjecture is confirmed. It is known that flow equivalence has a close relationship to stable isomorphisms of Cuntz–Krieger algebras (see [4], [5], [6], [8], [9], [19]). As a corollary of the main result, we also prove that two-sided irreducible topological Markov shifts $(\bar{X}_A,\bar{\sigma}_A)$ and $(\bar{X}_B,\bar{\sigma}_B)$ are flow equivalent if and only if there exists an isomorphism between the stable Cuntz–Krieger algebras $\mathcal{O}_A\otimes\mathbb{K}$ and $\mathcal{O}_B\otimes\mathbb{K}$ preserving their canonical maximal abelian subalgebras (see Corollary 3.8).

2. Preliminaries

We write $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$. The transpose of a matrix A is written A^t. The characteristic function of a set S is denoted by 1_S. We say that a subset of a topological space is *clopen* if it is both closed and open. A topological space is said to be *totally disconnected* if its topology is generated by clopen subsets. By a *Cantor set*, we mean a compact, metrizable, totally disconnected space with no isolated points. It is known that any two such spaces are homeomorphic. A good introduction to symbolic dynamics can be found in the standard textbook [10] by Lind and Marcus.

Let $A=[A(i,j)]_{i,j=1}^N$ be an $N\times N$ matrix with entries in $\{0,1\}$, where $1<N\in\mathbb{N}$. Throughout the paper, we assume that A has no rows or columns identically equal to zero. Define

$$X_A=\big\{(x_n)_{n\in\mathbb{N}}\in\{1,\dots,N\}^{\mathbb{N}}\;\big|\;A(x_n,x_{n+1})=1\ \forall n\in\mathbb{N}\big\}.$$

It is a compact Hausdorff space with natural product topology on $\{1,\ldots,N\}^{\mathbb{N}}$. The shift transformation σ_A on X_A defined by $\sigma_A((x_n)_n) = (x_{n+1})_n$ is a continuous surjective map on X_A. The topological dynamical system (X_A, σ_A) is called the (*right*) *one-sided topological Markov shift* for A. We henceforth assume that A satisfies condition (I) in the sense of [5]. The matrix A satisfies condition (I) if and only if X_A has no isolated points, that is, X_A is a Cantor set.

We let $(\bar{X}_A, \bar{\sigma}_A)$ denote the two-sided topological Markov shift. Namely,

$$\bar{X}_A = \left\{(x_n)_{n\in\mathbb{Z}} \in \{1,\ldots,N\}^{\mathbb{Z}} \;\middle|\; A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z}\right\}$$

and $\bar{\sigma}_A((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$.

A subset S in X_A (resp., in $\bar{X}_A$) is said to be σ_A-*invariant* (resp., $\bar{\sigma}_A$-*invariant*) if $\sigma_A(S) = S$ (resp., $\bar{\sigma}_A(S) = S$).

2.1. Continuous orbit equivalence

For $x = (x_n)_{n\in\mathbb{N}} \in X_A$, the orbit $\mathrm{orb}_{\sigma_A}(x)$ of x under σ_A is defined by

$$\mathrm{orb}_{\sigma_A}(x) = \bigcup_{k=0}^{\infty}\bigcup_{l=0}^{\infty} \sigma_A^{-k}\big(\sigma_A^{l}(x)\big).$$

DEFINITION 2.1 ([11, SECTION 5])

Let (X_A, σ_A) and (X_B, σ_B) be two one-sided topological Markov shifts. If there exists a homeomorphism $h : X_A \to X_B$ such that $h(\mathrm{orb}_{\sigma_A}(x)) = \mathrm{orb}_{\sigma_B}(h(x))$ for $x \in X_A$, then (X_A, σ_A) and (X_B, σ_B) are said to be *topologically orbit equivalent.* In this case, there exist $k_1, l_1 : X_A \to \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}\big(h(\sigma_A(x))\big) = \sigma_B^{l_1(x)}\big(h(x)\big) \quad \forall x \in X_A.$$

Similarly there exist $k_2, l_2 : X_B \to \mathbb{Z}_+$ such that

$$\sigma_A^{k_2(x)}\big(h^{-1}(\sigma_B(x))\big) = \sigma_A^{l_2(x)}\big(h^{-1}(x)\big) \quad \forall x \in X_B.$$

Furthermore, if we may choose $k_1, l_1 : X_A \to \mathbb{Z}_+$ and $k_2, l_2 : X_B \to \mathbb{Z}_+$ as continuous maps, then the topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *continuously orbit equivalent.*

If two one-sided topological Markov shifts are topologically conjugate, then they are continuously orbit equivalent. For the two matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

the topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, but not topologically conjugate (see [11, Section 5]).

Let $[\sigma_A]$ denote the set of all homeomorphisms τ of X_A such that $\tau(x) \in \mathrm{orb}_{\sigma_A}(x)$ for all $x \in X_A$. It is called the *full group* of (X_A, σ_A). Let Γ_A be the set of all τ in $[\sigma_A]$ such that there exist continuous functions $k, l : X_A \to \mathbb{Z}_+$ satisfying $\sigma_A^{k(x)}(\tau(x)) = \sigma_A^{l(x)}(x)$ for all $x \in X_A$. The set Γ_A is a subgroup of $[\sigma_A]$ and is called the *continuous full group* for (X_A, σ_A). We note that the group Γ_A

has been written as $[\sigma_A]_c$ in the earlier paper [11]. It has been proved in [14] that the isomorphism class of Γ_A as an abstract group is a complete invariant of the continuous orbit equivalence class of (X_A, σ_A) (see [16] for more general results and further studies).

2.2. Étale groupoids

By an *étale groupoid* we mean a second countable locally compact Hausdorff groupoid such that the range map is a local homeomorphism. We refer the reader to [18] for background material on étale groupoids. For an étale groupoid G, we let $G^{(0)}$ denote the unit space, and we let s and r denote the source and range maps, respectively. For $x \in G^{(0)}$, $r(Gx)$ is called the G-*orbit* of x. When every G-orbit is dense in $G^{(0)}$, G is said to be *minimal*. For $x \in G^{(0)}$, we write $G_x = r^{-1}(x) \cap s^{-1}(x)$ and call it the *isotropy group* of x. The isotropy bundle is $G' = \{g \in G \mid r(g) = s(g)\} = \bigcup_{x \in G^{(0)}} G_x$. We say that G is *principal* if $G' = G^{(0)}$. When the interior of G' is $G^{(0)}$, we say that G is *essentially principal*. A subset $U \subset G$ is called a G-*set* if $r|U, s|U$ are injective. For an open G-set U, we let π_U denote the homeomorphism $r \circ (s \mid U)^{-1}$ from $s(U)$ to $r(U)$.

We would like to recall the notion of topological full groups for étale groupoids.

DEFINITION 2.2 ([15, DEFINITION 2.3])

Let G be an essentially principal étale groupoid whose unit space $G^{(0)}$ is compact.

(a) The set of all $\alpha \in \mathrm{Homeo}(G^{(0)})$ such that for every $x \in G^{(0)}$ there exists $g \in G$ satisfying $r(g) = x$ and $s(g) = \alpha(x)$ is called the *full group* of G and is denoted by $[G]$.

(b) The set of all $\alpha \in \mathrm{Homeo}(G^{(0)})$ for which there exists a compact open G-set U satisfying $\alpha = \pi_U$ is called the *topological full group* of G and is denoted by $[[G]]$.

Obviously $[G]$ is a subgroup of $\mathrm{Homeo}(G^{(0)})$ and $[[G]]$ is a subgroup of $[G]$.

For $\alpha \in [[G]]$ the compact open G-set U as above uniquely exists, because G is essentially principal. Since G is second countable, it has countably many compact open subsets, and so $[[G]]$ is at most countable. For minimal groupoids on Cantor sets, it is known that the isomorphism class of $[[G]]$ is a complete invariant of G (see [16, Theorem 3.10]).

Let (X_A, σ_A) be a topological Markov shift. The étale groupoid G_A for (X_A, σ_A) is given by

$$G_A = \{(x, n, y) \in X_A \times \mathbb{Z} \times X_A \mid \exists k, l \in \mathbb{Z}_+, n = k - l, \sigma_A^k(x) = \sigma_A^l(y)\}.$$

The topology of G_A is generated by the sets

$$\{(x, k - l, y) \in G_A \mid x \in V, y \in W, \sigma_A^k(x) = \sigma_A^l(y)\},$$

where $V, W \subset X_A$ are open and $k, l \in \mathbb{Z}_+$. Two elements (x, n, y) and (x', n', y') in G_A are composable if and only if $y = x'$, and the multiplication and the inverse are

$$(x,n,y)\cdot(y,n',y')=(x,n+n',y'), \qquad (x,n,y)^{-1}=(y,-n,x).$$

The range and source maps are given by $r(x,n,y)=(x,0,x)$ and $s(x,n,y)=(y,0,y)$, respectively. We identify X_A with the unit space $G_A^{(0)}$ via $x\mapsto(x,0,x)$. The groupoid G_A is essentially principal. The groupoid G_A is minimal if and only if (X_A,σ_A) is irreducible. It is easy to see that the topological full group $[[G_A]]$ is canonically isomorphic to the continuous full group Γ_A.

2.3. Cuntz–Krieger algebras

Let $A=[A(i,j)]_{i,j=1}^N$ be an $N\times N$ matrix with entries in $\{0,1\}$, and let (X_A,σ_A) be the one-sided topological Markov shift. The Cuntz–Krieger algebra $\mathcal{O}_A$, introduced in [5], is the universal C^*-algebra generated by N partial isometries $S_1,\dots,S_N$ subject to the relations

$$\sum_{j=1}^N S_jS_j^*=1 \qquad \text{and} \qquad S_i^*S_i=\sum_{j=1}^N A(i,j)S_jS_j^*.$$

The subalgebra $\mathcal{D}_A$ of $\mathcal{O}_A$ generated by elements $S_{i_1}S_{i_2}\cdots S_{i_k}S_{i_k}^*\cdots S_{i_1}^*$ is naturally isomorphic to $C(X_A)$, and is a Cartan subalgebra in the sense of [18]. It is also well known that the pair $(\mathcal{O}_A,\mathcal{D}_A)$ is isomorphic to the pair $(C_r^*(G_A), C(X_A))$, where $C_r^*(G_A)$ denotes the reduced groupoid C^*-algebra and $C(X_A)$ is regarded as a subalgebra of it. Thus, there exists an isomorphism $\Psi:\mathcal{O}_A\to C_r^*(G)$ such that $\Psi(\mathcal{D}_A)=C(X_A)$.

THEOREM 2.3

Let (X_A,σ_A) and (X_B,σ_B) be two irreducible one-sided topological Markov shifts. The following conditions are equivalent.

(a) *(X_A,σ_A) and (X_B,σ_B) are continuously orbit equivalent.*
(b) *The étale groupoids G_A and G_B are isomorphic.*
(c) *There exists an isomorphism $\Psi:\mathcal{O}_A\to\mathcal{O}_B$ such that $\Psi(\mathcal{D}_A)=\mathcal{D}_B$.*

Proof

The equivalence between (a) and (c) follows from [11, Theorem 1.1]. The equivalence between (b) and (c) follows from [18, Proposition 4.11] (see also [15, Theorem 5.1]). □

2.4. Flow equivalence

In this section, we would like to recall Boyle–Handelman's theorem, which says that the ordered cohomology group is a complete invariant for flow equivalence between irreducible shifts of finite type.

Let $A=[A(i,j)]_{i,j=1}^N$ be an $N\times N$ matrix with entries in $\{0,1\}$, and consider the two-sided topological Markov shift $(\bar X_A,\bar\sigma_A)$. Set

$$\bar H^A=C(\bar X_A,\mathbb{Z})/\{\xi-\xi\circ\bar\sigma_A \mid \xi\in C(\bar X_A,\mathbb{Z})\}.$$

The equivalence class of a function $\xi \in C(\bar{X}_A,\mathbb{Z})$ in $\bar{H}^A$ is written $[\xi]$. We define the positive cone $\bar{H}^A_+$ by

$$\bar{H}^A_+ = \{[\xi] \in \bar{H}^A \mid \xi(x) \geq 0 \ \forall x \in \bar{X}_A\}.$$

The pair $(\bar{H}^A, \bar{H}^A_+)$ is called the *ordered cohomology group* of $(\bar{X}_A, \bar{\sigma}_A)$ (see [2, Section 1.3]). Boyle and Handelman proved the following theorem, which plays a key role in this paper.

THEOREM 2.4 ([2, THEOREM 1.12])
Suppose that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are irreducible two-sided topological Markov shifts. Then the following are equivalent.

(a) *$(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.*

(b) *The ordered cohomology groups $(\bar{H}^A, \bar{H}^A_+)$ and $(\bar{H}^B, \bar{H}^B_+)$ are isomorphic; that is, there exists an isomorphism $\Phi: \bar{H}^A \to \bar{H}^B$ such that $\Phi(\bar{H}^A_+) = \bar{H}^B_+$.*

We also recall the following from [2] for later use.

PROPOSITION 2.5 ([2, PROPOSITION 3.13(A)])
Let $(\bar{X}_A, \bar{\sigma}_A)$ be a two-sided topological Markov shift, and let $\xi \in C(\bar{X}_A, \mathbb{Z})$. Then $[\xi]$ is in $\bar{H}^A_+$ if and only if

$$\sum_{x \in O} \xi(x) \geq 0$$

holds for any finite $\bar{\sigma}_A$-invariant set $O \subset \bar{X}_A$.

In the same way as above, we introduce (H^A, H^A_+) for the one-sided topological Markov shift (X_A, σ_A) as follows:

$$H^A = C(X_A,\mathbb{Z}) / \{\xi - \xi \circ \sigma_A \mid \xi \in C(X_A,\mathbb{Z})\}$$

and

$$H^A_+ = \{[\xi] \in H^A \mid \xi(x) \geq 0 \ \forall x \in X_A\}.$$

We will show that $(\bar{H}^A, \bar{H}^A_+)$ and (H^A, H^A_+) are actually isomorphic (see Lemma 3.1).

2.5. The Bowen–Franks group

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0,1\}$. The Bowen–Franks group $\mathrm{BF}(A)$ is the abelian group $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N$. Bowen and Franks [1] have proved that the Bowen–Franks group is an invariant of flow equivalence. Parry and Sullivan [17] have proved that the determinant of $\mathrm{id} - A$ is also an invariant of flow equivalence. Evidently, if $\mathrm{BF}(A)$ is an infinite group, then $\det(\mathrm{id} - A)$ is zero. If $\mathrm{BF}(A)$ is a finite group, then $|\det(\mathrm{id} - A)|$ is equal to the cardinality of $\mathrm{BF}(A)$. Therefore it is sufficient to know the Bowen–Franks group and the sign of the determinant in order to find the determinant. The following theorem by Franks shows that these invariants are complete.

THEOREM 2.6 ([6, THEOREM])
Suppose that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are irreducible two-sided topological Markov shifts. Then $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if $\mathrm{BF}(A) \cong \mathrm{BF}(B)$ and $\mathrm{sgn}(\det(\mathrm{id} - A)) = \mathrm{sgn}(\det(\mathrm{id} - B))$.

In what follows, we consider $\mathrm{BF}(A^t) = \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N$. Although $\mathrm{BF}(A^t)$ is isomorphic to $\mathrm{BF}(A)$ as an abelian group, there does not exist a canonical isomorphism between them, and so we must distinguish them carefully.

We denote the equivalence class of $(1, 1, \ldots, 1) \in \mathbb{Z}^N$ in $\mathrm{BF}(A^t)$ by u_A. By [4, Proposition 3.1], $K_0(\mathcal{O}_A)$ is isomorphic to $\mathrm{BF}(A^t)$ and the class of the unit of $\mathcal{O}_A$ maps to u_A under this isomorphism. And $K_1(\mathcal{O}_A)$ is isomorphic to $\mathrm{Ker}(\mathrm{id} - A^t)$ on $\mathbb{Z}^N$. In [15], it has been shown that these groups naturally arise from the homology theory of étale groupoids.

Let G be an étale groupoid whose unit space $G^{(0)}$ is a Cantor set. One can associate the homology groups $H_n(G)$ with G (see [15, Section 3] for the precise definition). The homology group $H_0(G)$ is the quotient of $C(G^{(0)}, \mathbb{Z})$ by the subgroup generated by $1_{r(U)} - 1_{s(U)}$ for compact open G-sets U. We denote the equivalence class of $\xi \in C(G^{(0)}, \mathbb{Z})$ in $H_0(G)$ by $[\xi]$. For the étale groupoid G_A, we have the following.

THEOREM 2.7 ([15, THEOREM 4.14])
Let (X_A, σ_A) be a one-sided topological Markov shift. Then

$$H_n(G_A) \cong \begin{cases} \mathrm{BF}(A^t) = \mathbb{Z}^N/(\mathrm{id} - A^t)\mathbb{Z}^N, & n = 0, \\ \mathrm{Ker}(\mathrm{id} - A^t), & n = 1, \\ 0, & n \geq 2. \end{cases}$$

Moreover, there exists an isomorphism $\Phi : H_0(G_A) \to \mathrm{BF}(A^t)$ such that $\Phi([1_{X_A}]) = u_A$.

In particular, it follows from Theorem 2.3 that the pair $(\mathrm{BF}(A^t), u_A)$ is an invariant for continuous orbit equivalence of one-sided topological Markov shifts (see also [13, Theorem 1.3]). Thus, if (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, then there exists an isomorphism $\Phi : \mathrm{BF}(A^t) \to \mathrm{BF}(B^t)$ such that $\Phi(u_A) = u_B$.

3. Classification up to continuous orbit equivalence

Let (X_A, σ_A) be an irreducible one-sided topological Markov shift. As in the previous section, $(\bar{X}_A, \bar{\sigma}_A)$ denotes the two-sided topological Markov shift corresponding to (X_A, σ_A). Define $\rho : \bar{X}_A \to X_A$ by $\rho((x_n)_{n\in\mathbb{Z}}) = (x_n)_{n\in\mathbb{N}}$. Clearly we have that $\sigma_A \circ \rho = \rho \circ \bar{\sigma}_A$.

LEMMA 3.1

The map $C(X_A,\mathbb{Z})\ni\xi\mapsto\xi\circ\rho\in C(\bar{X}_A,\mathbb{Z})$ gives rise to an isomorphism $\tilde{\rho}$ from H^A to $\bar{H}^A$ satisfying $\tilde{\rho}(H^A_+)=\bar{H}^A_+$.

Proof

For any $\eta\in C(X_A,\mathbb{Z})$, one has that $(\eta-\eta\circ\sigma_A)\circ\rho=\eta\circ\rho-\eta\circ\rho\circ\bar{\sigma}_A$, and so $[\xi]\mapsto[\xi\circ\rho]$ is a well-defined homomorphism $\tilde{\rho}$ from H^A to $\bar{H}^A$.

Let $\zeta\in C(\bar{X}_A,\mathbb{Z})$. Then $\zeta(x)$ depends only on finitely many coordinates of $x\in\bar{X}_A$. Hence, for sufficiently large $n\in\mathbb{N}$, there exists $\xi\in C(X_A,\mathbb{Z})$ such that $\zeta\circ\bar{\sigma}_A^n=\xi\circ\rho$. Thus $\tilde{\rho}$ is surjective.

Clearly $\tilde{\rho}(H^A_+)\subset\bar{H}^A_+$. It follows from the argument above that $\bar{H}^A_+$ is contained in $\tilde{\rho}(H^A_+)$.

It remains for us to show the injectivity. Let $\xi\in C(X_A,\mathbb{Z})$. Suppose that there exists $\zeta\in C(\bar{X}_A,\mathbb{Z})$ such that $\xi\circ\rho=\zeta-\zeta\circ\bar{\sigma}_A$. In the same way as above, for sufficiently large $n\in\mathbb{N}$, there exists $\eta\in C(X_A,\mathbb{Z})$ such that $\zeta\circ\bar{\sigma}_A^n=\eta\circ\rho$. Then

$$\xi\circ\sigma_A^n\circ\rho=\xi\circ\rho\circ\bar{\sigma}_A^n=\zeta\circ\bar{\sigma}_A^n-\zeta\circ\bar{\sigma}_A^{n+1}=(\eta-\eta\circ\sigma_A)\circ\rho.$$

Hence $\xi\circ\sigma_A^n=\eta-\eta\circ\sigma_A$. Thus $[\xi]=[\xi\circ\sigma_A^n]=0$ in H^A. □

LEMMA 3.2

For $\xi\in C(X_A,\mathbb{Z})$, $[\xi]$ is in H^A_+ if and only if $\sum_{x\in O}\xi(x)\geq 0$ holds for every finite σ_A-invariant set $O\subset X_A$.

Proof

Suppose that $[\xi]$ is in H^A_+. By the lemma above, $\tilde{\rho}([\xi])=[\xi\circ\rho]$ is in $\bar{H}^A_+$. Let $O\subset X_A$ be a finite σ_A-invariant set. There exists a finite $\bar{\sigma}_A$-invariant set $\bar{O}\subset\bar{X}_A$ such that $\rho\,|\,\bar{O}$ is a bijection from $\bar{O}$ to O. It follows from Proposition 2.5 that $\sum_{x\in\bar{O}}\xi(\rho(x))\geq 0$. Hence $\sum_{x\in O}\xi(x)\geq 0$.

Suppose that $\sum_{x\in O}\xi(x)\geq 0$ holds for every finite σ_A-invariant set $O\subset X_A$. For any finite $\bar{\sigma}_A$-invariant set $\bar{O}\subset\bar{X}_A$, $O=\rho(\bar{O})\subset X_A$ is a finite σ_A-invariant set and $\rho\,|\,\bar{O}$ is injective. Therefore $\sum_{x\in\bar{O}}\xi(\rho(x))=\sum_{x\in O}\xi(x)\geq 0$. By Proposition 2.5, $[\xi\circ\rho]$ is in $\bar{H}^A_+$. By the lemma above, $[\xi]$ is in H^A_+ as desired. □

Let G be an étale groupoid. We denote by $\mathrm{Hom}(G,\mathbb{Z})$ the set of continuous homomorphisms $\omega:G\to\mathbb{Z}$. We think of $\mathrm{Hom}(G,\mathbb{Z})$ as an abelian group by pointwise addition. For $\xi\in C(G^{(0)},\mathbb{Z})$, we can define $\partial(\xi)\in\mathrm{Hom}(G,\mathbb{Z})$ by $\partial(\xi)(g)=\xi(r(g))-\xi(s(g))$. The cohomology group $H^1(G)=H^1(G,\mathbb{Z})$ is the quotient of $\mathrm{Hom}(G,\mathbb{Z})$ by $\{\partial(\xi)\mid\xi\in C(G^{(0)},\mathbb{Z})\}$. The equivalence class of $\omega:G\to\mathbb{Z}$ is written $[\omega]\in H^1(G)$.

Let $g\in G$ be such that $r(g)=s(g)$, that is, $g\in G'$. Since $\partial(\xi)(g)=0$ for any $\xi\in C(G^{(0)},\mathbb{Z})$, $[\omega]\mapsto\omega(g)$ is a well-defined homomorphism from $H^1(G)$ to $\mathbb{Z}$. We say that g is *attracting* if there exists a compact open G-set U such that $g\in U$, then $r(U) \subset s(U)$ and

$$\lim_{n \to +\infty} (\pi_U)^n(y) = r(g)$$

holds for any $y \in s(U)$.

Let (X_A, σ_A) be a one-sided topological Markov shift, and consider the étale groupoid G_A (see Section 2.2 for the definition). We say that $x \in X_A$ is *eventually periodic* if there exist $k, l \in \mathbb{Z}_+$ such that $k \neq l$ and $\sigma_A^k(x) = \sigma_A^l(x)$. This is equivalent to saying that $\{\sigma_A^n(x) \in X_A \mid n \in \mathbb{Z}_+\}$ is a finite set. When x is eventually periodic, we call

$$\min\{k - l \mid k, l \in \mathbb{Z}_+, k > l, \sigma_A^k(x) = \sigma_A^l(x)\}$$

the *period* of x.

LEMMA 3.3

Let $x \in X_A$.

(a) *If x is not eventually periodic, then the isotropy group $(G_A)_x$ is trivial.*

(b) *If x is eventually periodic, then $(G_A)_x = \{(x, np, x) \in G_A \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$, where p is the period of x.*

(c) *When x is eventually periodic and has period p, (x, np, x) is attracting if and only if n is positive.*

Proof

Both (a) and (b) are obvious. We prove (c). Suppose that x is an eventually periodic point whose period is p. Let $(x, np, x) \in (G_A)_x$. Assume that n is positive. Choose $k, l \in \mathbb{Z}_+$ so that $\sigma_A^k(x) = \sigma_A^l(x)$ and $pn = k - l$. Define a clopen neighborhood V and W of x by

$$V = \{(y_n)_n \in X_A \mid y_i = x_i \ \forall i = 1, 2, \dots, k+1\}$$

and

$$W = \{(y_n)_n \in X_A \mid y_i = x_i \ \forall i = 1, 2, \dots, l+1\}.$$

We have that $V \subset W$ and $\sigma_A^k(V) = \sigma_A^l(W)$. Then

$$U = \{(y, np, z) \in G_A \mid y \in V, z \in W, \sigma_A^k(y) = \sigma_A^l(z)\}$$

is a compact open G_A-set such that $(x, np, x) \in U$, $r(U) = V$, $s(U) = W$, and $\pi_U = (\sigma_A^k \mid V)^{-1} \circ (\sigma_A^l \mid W)$. It is easy to see that

$$\lim_{m \to +\infty} (\pi_U)^m(z) = x$$

holds for any $z \in s(U)$. Thus (x, np, x) is attracting.

Suppose that $U \subset G_A$ is a compact open G_A-set containing $(x, 0, x)$. Then $\pi_U(y) = y$ for any y sufficiently close to x, and so $(x, 0, x)$ is not attracting.

Assume that n is negative. Let $U \subset G_A$ be a compact open G_A-set containing (x, np, x). By the argument above, $(x, -np, x)$ is attracting. Hence there exists a clopen neighborhood V of x such that $V \subset s(U)$ and $V \subset \pi_U(V)$. This means that (x, np, x) cannot be an attracting element. □

PROPOSITION 3.4

There exists an isomorphism $\Phi : H^1(G_A) \to H^A$ such that $\Phi([\omega])$ is in H^A_+ if and only if $\omega(g) \geq 0$ for every attracting $g \in G_A$.

Proof

Let $\omega \in \mathrm{Hom}(G_A, \mathbb{Z})$. Define $\xi \in C(X_A, \mathbb{Z})$ by

$$\xi(x) = \omega\big((x, 1, \sigma_A(x))\big).$$

Let us verify that the map $\omega \mapsto \xi$ is surjective. For a given $\xi \in C(X_A, \mathbb{Z})$, we can define $\omega \in \mathrm{Hom}(G_A, \mathbb{Z})$ as follows. Take $(x, n, y) \in G_A$. There exists $k, l \in \mathbb{Z}_+$ such that $k - l = n$ and $\sigma^k_A(x) = \sigma^l_A(y)$. Put

$$\omega\big((x, n, y)\big) = \sum_{i=0}^{k-1} \xi\big(\sigma^i_A(x)\big) - \sum_{j=0}^{l-1} \xi\big(\sigma^j_A(y)\big).$$

Clearly this gives a well-defined continuous homomorphism from G_A to $\mathbb{Z}$. If there exists $\eta \in C(X_A, \mathbb{Z})$ such that $\omega = \partial(\eta)$, then $\xi = \eta - \eta \circ \sigma_A$, that is, $[\xi] = 0$ in H^A. It is also easy to see that the converse holds. Therefore $\Phi : [\omega] \mapsto [\xi]$ is an isomorphism from $H^1(G_A)$ to H^A.

We would like to show that $[\xi]$ is in H^A_+ if and only if $\omega(g) \geq 0$ for every attracting $g \in G_A$. Let $x \in X_A$ be an eventually periodic point whose period is p, and let $g = (x, np, x)$ be an attracting element. By the lemma above, n is positive. There exists $k, l \in \mathbb{Z}_+$ such that $k - l = np$ and $\sigma^k_A(x) = \sigma^l_A(x)$. Then one has

$$\begin{aligned}\omega(g) &= \sum_{i=0}^{k-1} \xi\big(\sigma^i_A(x)\big) - \sum_{j=0}^{l-1} \xi\big(\sigma^j_A(x)\big) \\ &= \sum_{i=l}^{k-1} \xi\big(\sigma^i_A(x)\big) \\ &= n \sum_{i=l}^{l+p-1} \xi\big(\sigma^i_A(x)\big).\end{aligned}$$

Notice that $O = \{\sigma^l_A(x), \sigma^{l+1}_A(x), \ldots, \sigma^{l+p-1}_A(x)\}$ is a finite σ_A-invariant set. By Lemma 3.2, $[\xi]$ belongs to H^A_+ if and only if

$$\sum_{y \in O} \xi(y) \geq 0$$

for any finite σ_A-invariant set $O \subset X_A$, thereby completing the proof. □

Consequently we have the following.

THEOREM 3.5

Let (X_A, σ_A) and (X_B, σ_B) be two irreducible one-sided topological Markov shifts. If (X_A, σ_A) is continuously orbit equivalent to (X_B, σ_B), then there exists an isomorphism $\Phi : H^A \to H^B$ such that $\Phi(H_+^A) = H_+^B$. In particular, $(\bar{X}_A, \bar{\sigma}_A)$ is flow equivalent to $(\bar{X}_B, \bar{\sigma}_B)$.

Proof

Consider the étale groupoids G_A and G_B. By Theorem 2.3, G_A and G_B are isomorphic. Let $\varphi : G_A \to G_B$ be an isomorphism. For $g \in G_A$, g is attracting in G_A if and only if $\varphi(g)$ is attracting in G_B. It follows from Proposition 3.4 above that (H^A, H_+^A) is isomorphic to (H^B, H_+^B). Then, Lemma 3.1 implies that $(\bar{H}^A, \bar{H}_+^A)$ is isomorphic to $(\bar{H}^B, \bar{H}_+^B)$. By Theorem 2.4, $(\bar{X}_A, \bar{\sigma}_A)$ is flow equivalent to $(\bar{X}_B, \bar{\sigma}_B)$. □

THEOREM 3.6

Let (X_A, σ_A) and (X_B, σ_B) be two irreducible one-sided topological Markov shifts. The following conditions are equivalent.

(a) *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
(b) *The étale groupoids G_A and G_B are isomorphic.*
(c) *There exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathcal{D}_A) = \mathcal{D}_B$.*
(d) *$\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$ and* $\operatorname{sgn}(\det(\operatorname{id} - A)) = \operatorname{sgn}(\det(\operatorname{id} - B))$.
(e) *There exists an isomorphism $\Phi : \mathrm{BF}(A^t) \to \mathrm{BF}(B^t)$ such that $\Phi(u_A) = u_B$ and* $\operatorname{sgn}(\det(\operatorname{id} - A)) = \operatorname{sgn}(\det(\operatorname{id} - B))$.

Proof

The equivalence between (a), (b), and (c) is already known (see Theorem 2.3). As mentioned in Section 2.5, $(K_0(\mathcal{O}_A), [1])$ is isomorphic to $(\mathrm{BF}(A^t), u_A)$, and so (d) $\Rightarrow$ (e) holds. The implication (e) $\Rightarrow$ (a) follows from [12, Theorem 1.1].

Suppose that (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent. It follows from the theorem above that the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. Therefore, by [17], we have that $\det(\operatorname{id} - A) = \det(\operatorname{id} - B)$ (see Theorem 2.6). Since (a) $\Rightarrow$ (c) is already known, $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$. Thus we have obtained (d). This completes the proof. □

As mentioned in Section 2.5, $\det(\operatorname{id} - A) = 0$ when $\mathrm{BF}(A^t)$ is infinite, and $|\det(\operatorname{id} - A)|$ equals the cardinality of $\mathrm{BF}(A^t)$ when $\mathrm{BF}(A^t)$ is finite. Hence, our invariant of the continuous orbit equivalence consists of a finitely generated abelian group F, an element $u \in F$, and $s \in \{-1, 0, 1\}$ such that F is an infinite group if and only if $s = 0$. Conversely, for any such triplet (F, u, s), there exists an irreducible one-sided topological Markov shift whose invariant is equal to (F, u, s). This is probably known to experts, but the authors are not aware of a specific reference and thus include a proof for completeness.

LEMMA 3.7

Let F be a finitely generated abelian group, and let $u \in F$. Let $s = 0$ when F is infinite, and let s be either -1 or 1 when F is finite. There exists an irreducible one-sided topological Markov shift (X_A, σ_A) such that $(F, u) \cong (\mathrm{BF}(A^t), u_A)$ and the sign of $\det(\mathrm{id} - A)$ equals $s \in \{-1, 0, 1\}$.

Proof

Suppose that we are given (F, u, s). It suffices to find a square irreducible matrix A with entries in $\mathbb{Z}_+$ satisfying the desired properties (see [10, Section 2.3] or [5, Remark 2.16]). Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\mathbb{Z}_+$ such that $A(1,1) = 2$, $A(i,i) \geq 2$, and $A(i,j) = 1$ for all i, j with $i \neq j$. Let $d_i = A(i,i) - 2$, and let $r = |\{i \mid d_i = 0\}| - 1$. Then it is straightforward to see that

$$\mathrm{BF}(A^t) \cong \mathbb{Z}^r \oplus \bigoplus_{d_i \geq 2} \mathbb{Z}/d_i\mathbb{Z} \qquad \text{and} \qquad \det(\mathrm{id} - A) = (-1)^N \prod_{i=2}^{N} d_i.$$

Therefore we can construct such A so that $\mathrm{BF}(A^t) \cong F$ and the sign of $\det(\mathrm{id} - A)$ equals s. In what follows, we identify $\mathrm{BF}(A^t)$ with F. Note that $u_A \in \mathrm{BF}(A^t)$ is zero. Choose $(c_1, c_2, \ldots, c_N) \in \mathbb{Z}^N$ whose equivalence class in $\mathrm{BF}(A^t)$ equals u. Since u_A is zero, we may assume that $c_i \in \mathbb{Z}_+$ for all i. We now construct a new matrix B as follows. Set

$$\Sigma = \bigl\{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \bigm| 1 \leq i \leq N, 0 \leq j \leq c_i\bigr\}.$$

Define $B = [B((i,j),(k,l))]_{(i,j),(k,l) \in \Sigma}$ by

$$B\bigl((i,j),(k,l)\bigr) = \begin{cases} A(i,k), & j = c_i, l = 0, \\ 1, & i = k, j+1 = l, \\ 0, & \text{otherwise.} \end{cases}$$

The group $\mathrm{BF}(A^t)$ is the abelian group with generators $e_1, \ldots, e_N$ and relations

$$e_i = \sum_{j=1}^{N} A(i,j) e_j,$$

and u equals $\sum c_i e_i$. The group $\mathrm{BF}(B^t)$ is the abelian group with generators $\{f_{i,j} \mid (i,j) \in \Sigma\}$ and relations

$$f_{i,j} = f_{i,j'} \qquad \text{and} \qquad f_{i,c_i} = \sum_{k=1}^{N} A(i,k) f_{k,0},$$

and u_B equals $\sum f_{i,j}$. Hence $(\mathrm{BF}(A^t), u)$ is isomorphic to $(\mathrm{BF}(B^t), u_B)$. It is also easy to see that $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$. The proof is completed. □

For $i = 1, 2$, let G_i be a minimal essentially principal étale groupoid whose unit space is a Cantor set. It has been shown that the following conditions are mutually equivalent (see [16, Theorem 3.10]). For a group Γ, we let $D(\Gamma)$ denote the commutator subgroup.

- G_1 and G_2 are isomorphic as étale groupoids.
- $[[G_1]]$ and $[[G_2]]$ are isomorphic as discrete groups.
- $D([[G_1]])$ and $D([[G_2]])$ are isomorphic as discrete groups.

The étale groupoid G_A arising from (X_A, σ_A) is minimal, essentially principal, and purely infinite (see [16, Lemma 6.1]). Hence $D([[G_A]])$ is simple by [16, Theorem 4.16]. Moreover, $D([[G_A]])$ is finitely generated (see [16, Corollary 6.25]), $[[G_A]]$ is of type F_∞ (see [16, Theorem 6.21]), and $[[G_A]]/D([[G_A]])$ is isomorphic to $(H_0(G_A) \otimes \mathbb{Z}_2) \oplus H_1(G_A)$ (see [16, Corollary 6.24]). Theorem 3.6 tells us that the isomorphism class of $[[G_A]]$ (and $D([[G_A]])$) is determined by $(H_0(G_A), [1_{X_A}], \det(\mathrm{id} - A))$ (see also Theorem 2.7). By Lemma 3.7, for each triplet (F, u, s) there exists (X_A, σ_A) whose invariant agrees with it. In particular, the simple finitely generated groups $D([[G_A]])$ are parameterized by such triplets (F, u, s).

We conclude this article by giving a corollary. We denote by $\mathbb{K}$ the C^*-algebra of all compact operators on $\ell^2(\mathbb{Z})$. Let $\mathcal{C} \cong c_0(\mathbb{Z})$ be the maximal abelian subalgebra of $\mathbb{K}$ consisting of diagonal operators.

COROLLARY 3.8

Let $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ be two irreducible two-sided topological Markov shifts. The following conditions are equivalent.

(a) *$(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.*

(b) *There exists an isomorphism $\Psi : \mathcal{O}_A \otimes \mathbb{K} \to \mathcal{O}_B \otimes \mathbb{K}$ such that $\Psi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$.*

Proof

From [5, Theorem 4.1], we know that (a) $\Rightarrow$ (b). Let us assume (b). In what follows, we identify the Bowen–Franks group with the K_0-group of the Cuntz–Krieger algebra. We have the isomorphism $K_0(\Psi) : \mathrm{BF}(A^t) \to \mathrm{BF}(B^t)$. By Lemma 3.7, there exists an irreducible one-sided topological Markov shift (X_C, σ_C) such that $(\mathrm{BF}(B^t), K_0(\Psi)(u_A)) \cong (\mathrm{BF}(C^t), u_C)$ and $\det(\mathrm{id} - B) = \det(\mathrm{id} - C)$. It follows from Theorem 2.6 that $(\bar{X}_B, \bar{\sigma}_B)$ is flow equivalent to $(\bar{X}_C, \bar{\sigma}_C)$. Moreover, by Huang's theorem (see [8, Theorem 2.15]) and its proof, there exists an isomorphism $\Phi : \mathcal{O}_B \otimes \mathbb{K} \to \mathcal{O}_C \otimes \mathbb{K}$ such that $\Phi(\mathcal{D}_B \otimes \mathcal{C}) = \mathcal{D}_C \otimes \mathcal{C}$ and $K_0(\Phi)(K_0(\Psi)(u_A)) = u_C$. Then $\Phi \circ \Psi$ is an isomorphism from $\mathcal{O}_A \otimes \mathbb{K}$ to $\mathcal{O}_C \otimes \mathbb{K}$ such that $(\Phi \circ \Psi)(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_C \otimes \mathcal{C}$ and $K_0(\Phi \circ \Psi)(u_A) = u_C$. In the same way as the proof of [12, Theorem 4.1], we can conclude that $(\mathcal{O}_A, \mathcal{D}_A)$ is isomorphic to $(\mathcal{O}_C, \mathcal{D}_C)$. By virtue of Theorem 3.6, we get that $\det(\mathrm{id} - A) = \det(\mathrm{id} - C)$. Therefore $\det(\mathrm{id} - A) = \det(\mathrm{id} - B)$. Hence, by Theorem 2.6, $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. □

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