On $\mathrm{SL}(2)$-orbit theorems

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Abstract We extend SL(2)-orbit theorems for the degeneration of mixed Hodge structures to a situation in which we do not assume the polarizability of graded quotients. We also obtain analogous results on Deligne systems.

1. Introduction

1.1.

In this paper, we show that the SL(2)-orbit theorems on the degeneration of Hodge structures (see [10], [3], [9], [7]) hold in a situation in which we do not assume the polarizability of the graded quotients for the weight filtration. We also obtain analogous results on Deligne systems.

1.2.

Recall that a Deligne system of n variables is $(V, W, N_1, \dots, N_n, \alpha)$, where V is a finite-dimensional vector space over a field E of characteristic 0, W is a finite increasing filtration on V (called the weight filtration), $N_1, \dots, N_n : V \to V$ are mutually commuting nilpotent linear operators (called the *monodromy operators*) which respect W, and α is an action of the multiplicative group $\mathbb{G}_m$ on V, satisfying certain conditions (see [11]; see also Section 2.1.2 of this paper for a review).

In this paper, we define a similar notion, namely, a Deligne–Hodge system (DH system for short) of n variables, which is $(V, W, N_1, \dots, N_n, F)$ where $(V, W, N_1, \dots, N_n)$ has the same properties as in the definition of a Deligne system of n variables with $E = \mathbb{R}$, and F is a decreasing filtration on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ (called the *Hodge filtration*) satisfying certain conditions (see Section 2.1.2).

A DH system of zero variables is nothing but a mixed $\mathbb{R}$-Hodge structure.

In general, the notion of a DH system is similar to the notion of an infinitesimal mixed Hodge module (IMHM) of Kashiwara (see [5]; see also Section 2.1.9 of this paper for a review). In fact, if $(V, W, N_1, \dots, N_n, F)$ is an IMHM, then it is a DH system of n variables. In the definition of a DH system, we do not assume the polarizability of the graded quotients for weight filtration, which was

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assumed for IMHM. Another difference is that, in the definition of a DH system, the order of $(N_1, \dots, N_n)$ matters though it does not matter for an IMHM.

1.3.
The SL(2)-orbit theorems are statements on the properties of $\exp(\sum_{j=1}^n iy_jN_j)F$ for an IMHM $(V, W, N_1, \dots, N_n, F)$ of n variables in the situation $y_j/y_{j+1} \to \infty$ $(1 \le j \le n,\ y_{n+1}$ denotes 1). (In specific work, [10] treats the pure case with $n = 1$, [3] treats the pure case in general, [9] treats the mixed case in certain cases, and [7] treats the mixed case in general.) In this paper, we prove the following Theorem 1.4, which shows that the SL(2)-orbit theorems in [10], [3], [9], and [7] are generalizable to DH systems.

THEOREM 1.4
Let $(V, W, N_1, \dots, N_n, F)$ be a DH system of n variables. Then for $N'_j = \sum_{k=1}^j a_{j,k}N_k$ $(1 \le j \le n)$ with $a_{j,k} > 0$ $(1 \le k \le j \le n)$ such that $a_{j,k}/a_{j,k+1} \gg 0$ $(1 \le k < j \le n)$, $(V, W, N'_1, \dots, N'_n, F)$ is an IMHM of n variables.

For example, if (V, W, N_1, N_2, F) is a DH system of two variables, $(V, W, N_1, aN_1 + N_2, F)$ for $a \gg 0$ is an IMHM.

For N'_j as in Theorem 1.4, if $y_j/y_{j+1} \to \infty$ $(1 \le j \le n)$, we have that $\sum_{j=1}^n y_jN_j = \sum_{j=1}^n y'_jN'_j$ with $y'_j/y'_{j+1} \to \infty$ $(1 \le j \le n)$. Hence the property of $\exp(\sum_{j=1}^n iy_jN_j)F$ in the situation $y_j/y_{j+1} \to \infty$ $(1 \le j \le n)$ for a DH system is reduced to the case of IMHM.

1.5.
We have a canonical functor from the category of DH systems of n variables to the category of Deligne systems of n variables over $\mathbb{R}$, which has the shape $(V, W, N_1, \dots, N_n, F) \mapsto (V, W, N_1, \dots, N_n, \alpha)$ for a canonically defined α (see Section 2.2). We have also a canonical functor from the category of Deligne systems of n variables over $\mathbb{R}$ or over $\mathbb{C}$ to the category of DH systems of n variables which has the shape $(V, W, N_1, \dots, N_n, \alpha) \mapsto (V^{\oplus 2}, W^{\oplus 2}, N_1^{\oplus 2}, \dots, N_n^{\oplus 2}, F)$ for a canonically defined F (Section 2.3). Here in the case of a Deligne system over $\mathbb{C}$, $V^{\oplus 2}$ is regarded as an $\mathbb{R}$-vector space by the restriction of scalars. We study Deligne systems and DH systems by using these two functors and applying the results on one to the other.

From the above theorem on DH systems, we obtain the following theorem on Deligne systems.

THEOREM 1.6
Let $(V, W, N_1, \dots, N_n, \alpha)$ be a Deligne system of n variables over $\mathbb{R}$ or over $\mathbb{C}$. Then for $N'_j = \sum_{k=1}^j a_{j,k}N_k$ with $a_{j,k} > 0$ $(1 \le k \le j \le n)$ such that $a_{j,k}/a_{j,k+1} \gg 0$ $(1 \le k < j \le n)$, the associated DH system $(V^{\oplus 2}, W^{\oplus 2}, (N'_1)^{\oplus 2}, \dots, (N'_n)^{\oplus 2}, F)$ of n variables associated to the Deligne system $(V, W, N'_1, \dots, N'_n, \alpha)$ is an IMHM.

This shows that, roughly speaking, any Deligne system of n variables underlies some IMHM if it is modified in an elementary way.

From Theorem 1.4 (resp., Theorem 1.6) and the SL(2)-orbit theorem in [7, Theorem 0.5], we have the part on DH systems (resp., Deligne systems) in the following theorem.

THEOREM 1.7

(a) *Let $(V, W, N_1, \ldots, N_n, F)$ be a DH system of n variables. If $y_j/y_{j+1} \gg 0$ $(1 \le j \le n$, y_{n+1} denotes $1)$, $(V, W, \exp(\sum_{j=1}^n iy_jN_j)F)$ is a mixed Hodge structure. The splitting of W associated to this mixed Hodge structure (canonical splitting from Section 2.2.1) converges when $y_j/y_{j+1} \to \infty$ $(1 \le j \le n)$.*

(b) *Let E be $\mathbb{R}$ or $\mathbb{C}$, and let $(V, W, N_1, \ldots, N_n, \alpha)$ be a Deligne system of n variables over E. Let W' be the increasing filtration on V defined by α. (For $w \in \mathbb{Z}$, W'_w is defined as the sum of the weight k part for α for all $k \le w$.) If $y_j > 0$ $(1 \le j \le n)$ and $y_j/y_{j+1} \gg 0$ $(1 \le j < n)$, then W' is the relative monodromy filtration (see Section 2.1.1) of $\sum_{j=1}^n y_jN_j$ with respect to W. The splitting τ_0 of W defined by the Deligne system $(V, W, \sum_{j=1}^n y_jN_j, \alpha)$ of one variable (see Section 3.1.3) converges when $y_j > 0$ $(1 \le j \le n)$ and $y_j/y_{j+1} \to \infty$ $(1 \le j < n)$.*

In Theorem 4.2.1 in Section 4.2, we will give more precise descriptions of the convergences in (a) and (b) of this theorem.

The following result is deduced from Theorems 1.4 and 1.6 and from the fact that the category of IMHMs of n variables is an abelian category (see [5]).

PROPOSITION 1.8

The category of Deligne systems of n variables over a field E of characteristic 0 is an abelian category. The category of DH systems of n variables is an abelian category. In these categories, the underlying vector space V of the kernel (resp., cokernel) of a morphism $A \to B$ is the kernel (resp., cokernel) of the map of the underlying vector spaces, and W, N_j, and so on, of the kernel (resp., cokernel) are the ones induced from those of A (resp., B).

1.9.

We expect that results of this paper are useful to generalize the work [8] on classifying spaces of degenerating Hodge structures to a situation where we do not assume the polarizability of the graded quotients for the weight filtration.

We also expect that the study of Deligne systems in this paper is useful in the studies (like [1] and [6]) which treat the degeneration of motives over nonarchimedean local fields. In fact, for a nonarchimedean local field K and for a prime number ℓ which is not the characteristic of the residue field of K, it is expected that the ℓ-adic étale realization of a motive over K with the ℓ-adic monodromy operator produces a Deligne system of one variable over $\mathbb{Q}_\ell$, and degenerations of motives over K yield Deligne systems of many variables over $\mathbb{Q}_\ell$. The results of this paper show that, once we fix a homomorphism $\mathbb{Q}_\ell \to \mathbb{C}$ of fields, the induced Deligne systems over $\mathbb{C}$ have nice real analytic properties. Thus we have real analytic properties of ℓ-adic objects, and such a strange thing should be useful in the study of degeneration.

2. Deligne systems and Deligne–Hodge systems

2.1. Definitions

2.1.1.

We first review the notion of relative monodromy filtration defined by Deligne [4, Proposition 1.6.13].

Let V be an abelian group, let $W = (W_w)_{w\in\mathbb{Z}}$ be a finite increasing filtration on V, and let $N : V \to V$ be a nilpotent homomorphism such that $NW_w \subset W_w$ for all $w \in \mathbb{Z}$.

Then a finite increasing filtration $W' = (W'_w)_{w\in\mathbb{Z}}$ on V is called the *relative monodromy filtration of N with respect to W* if it satisfies the following conditions (a) and (b).

(a) $NW'_w \subset W'_{w-2}$ for any $w \in \mathbb{Z}$.

(b) For any $w \in \mathbb{Z}$ and $m \geq 0$, the map $N^m : \mathrm{gr}^{W'}_{w+m}\mathrm{gr}^W_w \to \mathrm{gr}^{W'}_{w-m}\mathrm{gr}^W_w$ is an isomorphism.

The relative monodromy filtration of N with respect to W need not exist. If it exists, it is unique (see [4, Proposition 1.6.13]).

If V is a vector space over a field E and if the W_w's are E-linear subspaces and N is E-linear, the relative monodromy filtration consists of E-linear subspaces of V if it exists.

2.1.2.

We review the notion of a Deligne system of n variables (see [11]), and define a DH system of n variables.

A Deligne system over a field E of characteristic 0 (resp., DH system) is

$$(V, W, N_1, \dots, N_n, \alpha) \qquad \big(\text{resp., } (V, W, N_1, \dots, N_n, F)\big),$$

where V is a finite-dimensional E-vector (resp., $\mathbb{R}$-vector) space, W is a finite increasing filtration on V by E-linear (resp., $\mathbb{R}$-linear) subspaces, N_j are linear operators $V \to V$, and α is an action of $\mathbb{G}_m$ on V (resp., F is a finite decreasing filtration on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ by $\mathbb{C}$-linear subspaces), satisfying the following conditions (a), (b), (c), (d), and (e) (resp., (a), (b), (c), (d), (f.1), and (f.2)).

(a) The operators $N_1, \dots, N_n$ are nilpotent, mutually commute, and respect W.

(b) There are finite increasing filtrations $W^{(j)}$ $(0 \leq j \leq n)$ such that $W^{(0)} = W$ and such that, for $1 \leq j \leq n$, $W^{(j)}$ is the relative monodromy filtration of N_j with respect to $W^{(j-1)}$.

(c) Let $1 \le j \le n$, let $0 \le k < j-1$, let $w \in \mathbb{Z}$, and let $U = W_w^{(k)}$. Then the restriction $W^{(j)}|_U$ of $W^{(j)}$ to U is the relative monodromy filtration of $N_j|_U$ with respect to $W^{(j-1)}|_U$.

(d) $N_j(W_w^{(k)}) \subset W_w^{(k)}$ for any j, k, w, and $N_j(W_w^{(k)}) \subset W_{w-2}^{(k)}$ if $k \ge j$.

(e) α splits $W^{(n)}$, $W_w^{(j)}$ is stable under the action α of $\mathbb{G}_m$ for any $0 \le j < n$ and $w \in \mathbb{Z}$, and N_j is of weight -2 for α (i.e., $\alpha(a)N_j\alpha(a)^{-1} = a^{-2}N_j$ for any $a \in \mathbb{G}_m$) for any $1 \le j \le n$.

(f.1) $N_j F^p \subset F^{p-1}$ for any $1 \le j \le n$ and $p \in \mathbb{Z}$.

(f.2) $(W^{(n)}, F)$ is a mixed Hodge structure. Furthermore, for $1 \le k < n$, $w \in \mathbb{Z}$ and for $U = W_w^{(k)}$, $(W^{(n)}|_U, F|_U)$ is a mixed Hodge structure.

2.1.3.

We denote the category of Deligne systems of n variables over E by $\mathrm{D}_{n,E}$.

We denote the category of DH systems of n variables by DH_n.

2.1.4.

For example, a Deligne system of zero variables over E is nothing but a finite-dimensional E-vector space endowed with an action of $\mathbb{G}_m$.

A DH system of zero variables is just a mixed $\mathbb{R}$-Hodge structure. In this paper, we call a mixed $\mathbb{R}$-Hodge structure just a mixed Hodge structure.

2.1.5.

A Deligne system of one variable over E is nothing but (V, W, N, α) where V is a finite-dimensional E-vector space, W is a finite increasing filtration on V, N is a nilpotent linear map $V \to V$ such that $N(W_w) \subset W_w$ for any $w \in \mathbb{Z}$, and α is an action of $\mathbb{G}_m$ on V such that W_w is stable under the action α of $\mathbb{G}_m$ for any $w \in \mathbb{Z}$, N is of weight -2 for α, and if we define W'_w to be the sum of the weight k part of α for all $k \le w$, then W' is the relative monodromy filtration of N with respect to W.

2.1.6.

Both the categories $\mathrm{D}_{n,E}$ and DH_n have direct sum, tensor products, symmetric powers, exterior powers, duals, and Tate twists defined in the evident manners.

The following is easy to see.

LEMMA 2.1.7

Let E be a field of characteristic 0, and let E' be a subfield of E. Let $H = (V, W, N_1, \ldots, N_n, \alpha)$ be as in the hypothesis of the definition of a Deligne system of n variables over E. (We do not assume (a)–(e).)

(a) *Assume that $H = H' \otimes_{E'} E$ for some $H' = (V', W', N'_1, \ldots, N'_n, \alpha')$ over E'. Then H is in $\mathrm{D}_{n,E}$ if and only if H' is in $\mathrm{D}_{n,E'}$.*

(b) *Assume that E is a finite extension of E'. Let H' be H but V in H' is regarded as an E'-vector space by the restriction of scalars. Then H is in $\mathrm{D}_{n,E}$ if and only if H' is in $\mathrm{D}_{n,E'}$.*

The following is also easy to see.

LEMMA 2.1.8

Let $(V,W,N_1,\dots,N_n,\alpha)$ (resp., $(V,W,N_1,\dots,N_n,F)$) be an object of $\mathrm{D}_{n,E}$ (resp., DH_n). Then for any $a_{j,k} \in E$ (resp., $\mathbb{R}$) $(1 \le k \le j \le n)$ such that $a_{j,j} \ne 0$ $(1 \le j \le n)$, if we put $N'_j = \sum_{k=1}^{j} a_{j,k} N_k$ for $1 \le j \le n$, then $(V,W,N'_1,\dots,N'_n,\alpha)$ (resp., $(V,W,N'_1,\dots,N'_n,F)$) belongs to $\mathrm{D}_{n,E}$ (resp., DH_n).

2.1.9.

The notion of a DH system of n variables is similar to the notion of an IMHM of Kashiwara. We review the notion of an IMHM. (In fact, we consider in this paper only IMHMs which have $\mathbb{R}$-structure, and we call such an IMHM just IMHM in this paper.)

An IMHM of n variables is $(V,W,N_1,\dots,N_n,F)$ as in the hypothesis of the definition of a DH system of n variables, satisfying the conditions (a), (f.1), (g), and (h).

(a) The same as (a) in Section 2.1.2.

(f.1) The same as (f.1) in Section 2.1.2.

(g) For each $w \in \mathbb{Z}$, there is a nondegenerate $\mathbb{R}$-bilinear form $\langle \cdot,\cdot \rangle_w : \mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbb{R}$ which is symmetric if w is even and antisymmetric if w is odd such that $\langle N_j u, v \rangle_w + \langle u, N_j v \rangle_w = 0$ for any j and any $u, v \in \mathrm{gr}^W_w$ and such that if $y_j \gg 0$ $(1 \le j \le n)$, then $(\mathrm{gr}^W_w, \langle \cdot,\cdot \rangle_w, \exp(\sum_{j=1}^{n} iy_jN_j)F(\mathrm{gr}^W_w))$ is a polarized Hodge structure of weight w. Here $F(\mathrm{gr}^W_w)$ denotes the filtration on $\mathrm{gr}^W_{w,\mathbb{C}}$ induced by F.

(h) For $1 \le j \le n$, the relative monodromy filtration of N_j with respect to W exists.

By the arguments in [11, Section 3, Example 2], we have the following.

PROPOSITION 2.1.10

An IMHM of n variables is a DH system of n variables.

2.2. A functor $\mathrm{DH}_n \to \mathrm{D}_{n,\mathbb{R}}$

We define a functor $\mathrm{DH}_n \to \mathrm{D}_{n,\mathbb{R}}$.

2.2.1.

We review that, for a mixed Hodge structure (V,W,F), we have a canonical splitting of W. (This canonical splitting is called the SL(2)-*splitting* in [2].)

There is a unique pair (s',δ) of a splitting $s' : \mathrm{gr}^W = \bigoplus_{w\in\mathbb{Z}} \mathrm{gr}^W_w \overset{\cong}{\to} V$ of W and a linear map $\delta : \mathrm{gr}^W \to \mathrm{gr}^W$ such that the Hodge (p,q)-component $\delta_{p,q}$ of δ

for $F(\mathrm{gr}^W)$ is zero unless $p<0$ and $q<0$ and such that $F=s'(\exp(i\delta)F(\mathrm{gr}^W))$ (see [3, Proposition 2.20]).

The canonical splitting s of W is a modification of this s'. It is defined by $s=s'\circ\exp(\zeta)$ where $\zeta:\mathrm{gr}^W\to\mathrm{gr}^W$ is the linear map which is determined by δ as a Lie polynomial of $\delta_{p,q}$ as in [3, Lemma 6.60].

Any morphisms of mixed Hodge structures commute with the canonical splittings.

LEMMA 2.2.2

Let $(V,W,N_1,\dots,N_n,F)$ be a DH system of n variables, and let α be the canonical splitting of $W^{(n)}$ associated to the mixed Hodge structure $(W^{(n)},F)$. Then $(V,W,N_1,\dots,N_n,\alpha)$ is a Deligne system of n variables.

Proof

It is sufficient to prove the following (a) and (b).

(a) For any $0\le j<n$ and $w\in\mathbb{Z}$, $W_w^{(j)}$ is stable under the action α of $\mathbb{G}_m$.
(b) For any $1\le j\le n$, N_j is of weight -2 for α.

We prove (a). Let $U=W_w^{(j)}$. The inclusion map $U\to V$ is a morphism of mixed Hodge structures $(W^{(n)}|_U,F|_U)\to(W^{(n)},F)$. Hence the canonical splitting of $W^{(n)}|_U$ associated to the mixed Hodge structure $(W^{(n)}|_U,F|_U)$ and the canonical splitting of $W^{(n)}$ associated to the mixed Hodge structure $(W^{(n)},F)$ (i.e., α) are compatible. This proves (a).

We prove (b). By $N_jF^p\subset F^{p-1}$ for any p, N_j is a morphism of mixed Hodge structures $(W^{(n)},F)\to(W^{(n)}(-1),F(-1))$ where (-1) is the Tate twist. Hence via N_j, the canonical splitting of $W^{(n)}$ associated to the mixed Hodge structure $(W^{(n)},F)$ is compatible with the canonical splitting of $W^{(n)}(-1)$ associated to the mixed Hodge structure $(W^{(n)}(-1),F(-1))$. This proves (b). □

2.2.3.

Thus we obtained the functor

$$\mathrm{DH}_n\to\mathrm{D}_{n,\mathbb{R}};\qquad (V,W,N_1,\dots,N_n,F)\mapsto(V,W,N_1,\dots,N_n,\alpha).$$

2.3. A functor $\mathrm{D}_{n,E}\to\mathrm{DH}_n$ for $E=\mathbb{R}$ or $\mathbb{C}$

For $E=\mathbb{R}$ or $\mathbb{C}$, we define a functor $\mathrm{D}_{n,E}\to\mathrm{DH}_n$. We consider the case $E=\mathbb{R}$ in Sections 2.3.1–2.3.3 and the case $E=\mathbb{C}$ in Section 2.3.4.

2.3.1.

Let $(V,W,N_1,\dots,N_n,\alpha)$ be a Deligne system of n variables over $\mathbb{R}$. We define a decreasing filtration F on $V_{\mathbb{C}}^{\oplus 2}$ as follows. For $w\in\mathbb{Z}$, let V_w be the weight w part of V with respect to the action α of $\mathbb{G}_m$. We define F as a direct sum of the following decreasing filtrations on $V_{w,\mathbb{C}}^{\oplus 2}$. If w is an even integer $2r$, then define the filtration F on $V_{w,\mathbb{C}}^{\oplus 2}$ by $F^r=V_{w,\mathbb{C}}^{\oplus 2}$ and $F^{r+1}=0$. If w is an odd integer $2r+1$, then define the filtration F on $V_{w,\mathbb{C}}^{\oplus 2}$ as follows: $F^r = V_{w,\mathbb{C}}^{\oplus 2}$, $F^{r+2} = 0$, and F^{r+1} is the $\mathbb{C}$-subspace of $V_{w,\mathbb{C}}^{\oplus 2}$ generated by $(i \otimes x, 1 \otimes x)$ $(x \in V_w)$.

LEMMA 2.3.2
This $(V^{\oplus 2}, W^{\oplus 2}, N_1^{\oplus 2}, \dots, N_n^{\oplus 2}, F)$ *is a DH system of* n *variables.*

This is checked easily.

2.3.3.
Thus we obtained the functor

$$\mathrm{D}_{n,\mathbb{R}} \to \mathrm{DH}_n, \qquad (V, W, N_1, \dots, N_n, \alpha) \mapsto (V^{\oplus 2}, W^{\oplus 2}, N_1^{\oplus 2}, \dots, N_n^{\oplus 2}, F).$$

The composition $\mathrm{D}_{n,\mathbb{R}} \to \mathrm{DH}_n \to \mathrm{D}_{n,\mathbb{R}}$ with the functor in Section 2.2 is

$$(V, W, N_1, \dots, N_n, \alpha) \mapsto (V^{\oplus 2}, W^{\oplus 2}, N_1^{\oplus 2}, \dots, N_n^{\oplus 2}, \alpha^{\oplus 2}).$$

On the other hand, the composition $\mathrm{DH}_n \to \mathrm{D}_{n,\mathbb{R}} \to \mathrm{DH}_n$ is a not so nice functor, for we forget the Hodge filtration of the original object.

2.3.4.
The functor $\mathrm{D}_{n,\mathbb{C}} \to \mathrm{DH}_n$ is defined as the composition

$$\mathrm{D}_{n,\mathbb{C}} \to \mathrm{D}_{n,\mathbb{R}} \to \mathrm{DH}_n,$$

where the first functor is to regard a $\mathbb{C}$-vector space as an $\mathbb{R}$-vector space by the restriction of scalars, and the second is the above functor from Section 2.3.3.

3. $\mathrm{SL}(2)$-orbits

3.1. Splittings of Deligne

We review two theorems of Deligne on splittings of weight filtrations of Deligne systems in Sections 3.1.3 and 3.1.4 below, which are introduced in [11, Theorems 1 and 2], respectively.

3.1.1.
First we review the notion of a primitive component. Let V be an abelian group, let W be a finite increasing filtration on V, and let $N : V \to V$ be a nilpotent endomorphism which respects W. Assume that the relative monodromy filtration W' of N with respect to W exists. Let $w \in \mathbb{Z}$, and let $m \geq 0$. Then $\mathrm{gr}^{W'}_{w+m}\mathrm{gr}^W_w = A \oplus B$, where A is the kernel of $\mathrm{gr}^{W'}_{w+m}\mathrm{gr}^W_w \xrightarrow{N^{m+1}} \mathrm{gr}^{W'}_{w-m-2}\mathrm{gr}^W_w$ and B is the image of $N : \mathrm{gr}^{W'}_{w+m+2}\mathrm{gr}^W_w \to \mathrm{gr}^{W'}_{w+m}\mathrm{gr}^W_w$. The component A is called the *primitive component* of $\mathrm{gr}^{W'}_{w+m}\mathrm{gr}^W_w$.

3.1.2.
Let V, W, N, W' be as in Section 3.1.1. Denote the filtration on $\mathrm{Hom}(V, V)$ induced by W (resp., W') by $W_\bullet \mathrm{Hom}(V, V)$ (resp., $W'_\bullet \mathrm{Hom}(V, V)$). Then $W'_\bullet \mathrm{Hom}(V, V)$

is the relative monodromy filtration of the nilpotent homomorphism $Ad(N)$: $\mathrm{Hom}(V,V) \to \mathrm{Hom}(V,V)$ with respect to $W_\bullet \mathrm{Hom}(V,V)$.

3.1.3.
Let (V,W,N,α) be a Deligne system of one variable over E. The first theorem of Deligne is that there is a unique action $\tau = (\tau_0, \tau_1)$ of $\mathbb{G}_m^2$ on V satisfying the following conditions (a)–(c).

(a) $\tau_1 = \alpha$.
(b) τ_0 splits $W^{(0)} = W$.
(c) For $k \geq 1$, let $N_{-k} \in \mathrm{gr}^W_{-k} \mathrm{Hom}(V,V)$ be the weight $-k$ part of N with respect to the action τ_0 of $\mathbb{G}_m$ on V. Then $N_{-1} = 0$, and for any $k \geq 2$, the class of N_{-k} in $\mathrm{gr}^{W'}_{-2} \mathrm{gr}^W_{-k} \mathrm{Hom}(V,V)$ belongs to the primitive component.

3.1.4.
The second theorem of Deligne is the following.

Let $(V,W,N_1,\dots,N_n,\alpha)$ be a Deligne system of n variables over E. Then there is a unique action of $\tau = (\tau_j)_{0\leq j\leq n}$ of $\mathbb{G}_m^{n+1}$ on V satisfying the following conditions (a) and (b).

(a) $\tau_n = \alpha$.
(b) For $1 \leq j \leq n$, $(V, W^{(j-1)}, N_j, \tau_j)$ is a Deligne system of one variable, and the action (τ_{j-1}, τ_j) of $\mathbb{G}_m^2$ coincides with the action of $\mathbb{G}_m^2$ in Section 3.1.3 associated to this Deligne system of one variable.

Furthermore, for this τ, we have the following (c), (d), and (e).

(c) For $0 \leq j \leq n$, τ_j splits $W^{(j)}$.
(d) For $1 \leq j \leq k \leq n$, N_j is of weight -2 for τ_k.
(e) Let $1 \leq j \leq n$, and let $\hat{N}_j$ be the component of N_j of weight 0 for τ_{j-1}. Then $\hat{N}_j$ is of weight 0 for τ_k for any $0 \leq k < j$.

3.1.5.
If $(V,W,N_1,\dots,N_n,F)$ is a DH system of n variables, then we have the associated action $\tau = (\tau_j)_{0\leq j\leq n}$ of $\mathbb{G}_m^{n+1}$ on V defined by the corresponding Deligne system $(V,W,N_1,\dots,N_n,\alpha)$ (see Section 2.2).

3.2. SL(2)-orbits

3.2.1.
We say that a Deligne system $(V,W,N_1,\dots,N_n,\alpha)$ of n variables is an SL(2)-orbit if

$$\tau_k(a) N_j \tau_k(a)^{-1} = N_j \quad \text{for } 0 \leq k < j \leq n$$

for any $a \in \mathbb{G}_m$, where $\tau = (\tau_j)_{0\leq j\leq n}$ is as in Section 3.1.4 (i.e., N_j is of weight 0 for τ_k for $0 \leq k < j \leq n$).

Recall that N_j is of weight -2 for τ_k if $k \geq j$.

We denote the full subcategory of $\mathrm{D}_{n,E}$ consisting of SL(2)-orbits by $\hat{\mathrm{D}}_{n,E}$.

We say that a DH system $H=(V,N_1,\dots,N_n,F)$ of n variables is an SL(2)-orbit if

$$\tau_k(a)N_j\tau_k(a)^{-1}=N_j \quad \text{for } 0\le k<j\le n \qquad \text{and} \qquad \tau_k(a)F=F \quad \text{for } 0\le k\le n$$

for any $a\in\mathbb{G}_m$, where $\tau=(\tau_j)_{0\le j\le n}$ is as in Section 3.1.5.

We denote the full subcategory of DH_n consisting of SL(2)-orbits by $\hat{\mathrm{DH}}_n$.

LEMMA 3.2.2

In the category $\mathrm{D}_{n,E}$ (resp., DH_n), $\hat{\mathrm{D}}_{n,E}$ (resp., $\hat{\mathrm{DH}}_n$) is stable under taking direct sums, tensor products, symmetric powers, exterior powers, duals, and Tate twists.

3.2.3.

As is easily seen, we have the following equivalence of categories between $\hat{\mathrm{D}}_{n,E}$ and the category of finite-dimensional representations of $\mathbb{G}_m\times\mathrm{SL}(2)^n$ over E. For an object $(V,W,N_1,\dots,N_n,\alpha)$ of $\mathrm{D}_{n,E}$ with the associated $(\tau_j)_{0\le j\le n}$, the corresponding representation is (V,ρ) where ρ is the action of $\mathbb{G}_m\times\mathrm{SL}(2)^n$ on V characterized by the following properties (a), (b), and (c).

(a) The action of $\mathbb{G}_m=\mathbb{G}_m\times\{1\}\subset\mathbb{G}_m\times\mathrm{SL}(2)^n$ is τ_0.

(b) For $1\le j\le n$ and $a\in\mathbb{G}_m$, the action of $\left(\begin{smallmatrix}1/a & 0\\ 0 & a\end{smallmatrix}\right)$ in the jth SL(2) is $\tau_j(a)/\tau_{j-1}(a)$.

(c) In the action of $\mathfrak{sl}(2)$ on V induced by the action of the jth SL(2), $\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)\in\mathfrak{sl}(2)$ acts as N_j.

We have the following.

(d) For $0\le j\le n$ and $a\in\mathbb{G}_m$, $\tau_j(a)=\rho(a,g)$ where $g=(g_k)_{1\le k\le n}\in\mathrm{SL}(2)^n$ with $g_k=\left(\begin{smallmatrix}1/a & 1\\ 0 & a\end{smallmatrix}\right)$ if $k\le j$, and $g_k=1$ if $k>j$.

Conversely, for a finite-dimensional representation (V,ρ) of $\mathbb{G}_m\times\mathrm{SL}(2)^n$, the corresponding object $(V,W,N_1,\dots,N_n,\alpha)$ of $\hat{\mathrm{DH}}_n$ is given as follows: W is defined by τ_0, the N_j's are given by the above (c), and $\alpha=\tau_n$ is given by the case $j=n$ of the above (d).

3.2.4.

We next consider $\hat{\mathrm{DH}}_n$.

Let (V,ρ) be a finite-dimensional representation of $\mathbb{G}_m\times\mathrm{SL}(2)^n$ over $\mathbb{R}$ such that the action τ_n of $\mathbb{G}_m$ on V defined by the case $j=n$ of Section 3.2.3(d) has only even weights. Then we have an object $[\rho]$ of $\hat{\mathrm{DH}}_n$ defined as follows. Let $(V,W,N_1,\dots,N_n,\alpha)$ be the object of $\hat{\mathrm{D}}_{n,\mathbb{R}}$ corresponding to (V,ρ) as in Section 3.2.3 (so $\alpha=\tau_n$ has only even weights), and let V_{2r} ($r\in\mathbb{Z}$) be the weight $2r$ part of V with respect to α. Let $[\rho]=(V,W,N_1,\dots,N_n,F)$ where F is the direct sum over r of the decreasing filtrations on $V_{2r,\mathbb{C}}$ defined by $F^rV_{2r,\mathbb{C}}=V_{2r,\mathbb{C}}$ and $F^{r+1}V_{2r,\mathbb{C}}=0$.

Then a general object of $\hat{\mathrm{DH}}_n$ is isomorphic to a direct sum of objects of the form $[\rho]\otimes H$, where H is a pure Hodge structure which we regard as an object of $\hat{\mathrm{DH}}_n$ in the trivial way: $N_j=0$ on H for all j, and W is the pure weight filtration of the weight of H. More precisely, we have the description from Proposition 3.2.5(b) of $\hat{\mathrm{DH}}_n$ below.

The following Proposition 3.2.5(a) is a consequence of Section 3.2.3 and the well-known classification of representations of $\mathrm{SL}(2)^n$. Proposition 3.2.5(b) is deduced from Proposition 3.2.5(a) by using the functor $\mathrm{DH}_n\to \mathrm{D}_{n,\mathbb{R}}$ from Section 2.2.

PROPOSITION 3.2.5

(a) *For $1\le j\le n$, let P_j be the object of $\hat{\mathrm{D}}_{n,E}$ corresponding to the two-dimensional representation of $\mathbb{G}_m\times \mathrm{SL}(2)^n$ given by the projection to the jth $\mathrm{SL}(2)$. For $k\in\mathbb{Z}$, let S_k be the object of $\hat{\mathrm{D}}_{n,E}$ corresponding to the one-dimensional representation of $\mathbb{G}_m\times \mathrm{SL}(2)^n$ defined as $(a,g)\mapsto a^k$ ($a\in\mathbb{G}_m$, $g\in \mathrm{SL}(2)^n$).*

Then the category $\hat{\mathrm{D}}_{n,E}$ is equivalent to the category of families $(H_{m,k})_{m\in\mathbb{N}^n,k\in\mathbb{Z}}$, where $H_{m,k}$ is a finite-dimensional E-vector space for each m,k, satisfying $H_{m,k}=0$ for almost all (m,k). The functor from the latter category to the former category

$$(H_{m,k})_{m,k}\mapsto \bigoplus_{m,k}\mathrm{Sym}^{m(1)}(P_1)\otimes\cdots\otimes \mathrm{Sym}^{m(n)}(P_n)\otimes S_k\otimes H_{m,k}$$

gives an equivalence of categories. Here $H_{m,k}$ is regarded as an object of $\hat{\mathrm{D}}_{n,E}$ in the following simple way: $V=H_{m,k}$, $W_0=V$, $W_{-1}=0$, $N_j=0$ for all j, and α is trivial.

The inverse functor sends an object $(V,W,N_1,\dots,N_n,\alpha)$ to $(H_{m,k})_{m,k}$, where

$$H_{m,k}=\left\{x\in V\;\middle|\; N_j(x)=0,\tau_j(a)x=a^k\prod_{\ell=1}^{j}a^{-m(\ell)}x\ \ (1\le j\le n, a\in\mathbb{G}_m)\right\}.$$

(b) *For $1\le j\le n$, let P_j be the object $[\rho]$ of $\hat{\mathrm{DH}}_n$ corresponding to the two-dimensional representation ρ of $\mathbb{G}_m\times \mathrm{SL}(2)^n$ given by $(a,g)\mapsto ag_j$ ($a\in\mathbb{G}_m$, $g=(g_k)_k\in \mathrm{SL}(2)^n$).*

Then the category $\hat{\mathrm{DH}}_n$ is equivalent to the category of families $(H_{m,k})_{m\in\mathbb{N}^n,k\in\mathbb{Z}}$, where $H_{m,k}$ is a pure Hodge structure of weight k for each m,k satisfying $H_{m,k}=0$ for almost all (m,k). The functor from the latter category to the former category

$$(H_{m,k})_{m,k}\mapsto \bigoplus_{m,k}\mathrm{Sym}^{m(1)}(P_1)\otimes\cdots\otimes \mathrm{Sym}^{m(n)}(P_n)\otimes H_{m,k}$$

gives an equivalence of categories. Here $H_{m,k}$ is regarded as an object of $\hat{\mathrm{DH}}_n$ in the trivial way explained as H in Section 3.2.4.

The inverse functor sends an object $(V,W,N_1,\dots,N_n,F)$ to $(H_{m,k})_{m,k}$, where

$$H_{m,k} = \{x \in V \mid N_j(x) = 0, \tau_0(a)x = a^k x,$$
$$\tau_j(a)\tau_{j-1}(a)^{-1}x = a^{-m(j)}x \ (1 \le j \le n, a \in \mathbb{G}_m)\},$$

whose Hodge filtration is the restriction of F.

PROPOSITION 3.2.6

Let $(V, W, N_1, \dots, N_n, \alpha)$ be an object of $\hat{\mathrm{D}}_{n,E}$. Fix $0 \le \ell \le j < k \le n$. Then for any nonzero elements y_t of E ($\ell + 1 \le t \le k$), $W^{(k)}$ is the relative monodromy filtration of $\sum_{t=\ell+1}^{k} y_t N_t$ with respect to $W^{(j)}$. In other words, $(V, W^{(j)}, \sum_{t=\ell+1}^{k} y_t N_t, \tau_k)$ is a Deligne system of one variable.

Proof

By Proposition 3.2.5(a), it is sufficient to check this in the cases of the objects P_s $(1 \le s \le n)$ and S_w $(w \in \mathbb{Z})$ in Proposition 3.2.5(a). These are checked easily. □

PROPOSITION 3.2.7

An object of $\hat{\mathrm{DH}}_n$ is an IMHM. Moreover, if $H = (V, W, N_1, \dots, N_n, F)$ is an object of $\hat{\mathrm{DH}}_n$, then for each $w \in \mathbb{Z}$, there is a nondegenerate $\mathbb{R}$-bilinear form $\langle\cdot,\cdot\rangle_w$ on gr^W_w such that, for any $y_j > 0$ ($1 \le j \le n$), $(\mathrm{gr}^W_w, \langle\cdot,\cdot\rangle_w, \exp(\sum_{j=1}^{n} iy_j N_j)F(\mathrm{gr}^W_w))$ is a polarized Hodge structure of weight w and such that

$$\langle \tau_j(a)u, \tau_j(a)v \rangle_w = a^{2w}\langle u, v\rangle_w, \qquad \langle N_j u, v\rangle_w + \langle u, N_j v\rangle_w = 0$$

for any $u, v \in \mathrm{gr}^W_w$, $a \in \mathbb{G}_m$, and $1 \le j \le n$.

Proof

By Proposition 3.2.5(b), it is sufficient to prove this for the objects P_j of $\hat{\mathrm{DH}}_n$ $(1 \le j \le n)$ in Proposition 3.2.5(b) and for a pure Hodge structure regarded as an object of $\hat{\mathrm{DH}}_n$ in the trivial way as H in Section 3.2.4.

For a pure Hodge structure, what we have to show is that any pure Hodge structure is polarizable. (Note that we consider only $\mathbb{R}$-Hodge structures in this paper.) This is a well-known fact. In fact, any pure Hodge structure is a finite direct sum of pure Hodge structure of the following forms: (a) pure Hodge structure of rank 1 of even weight, and (b) pure Hodge structure (V, F) such that V has an $\mathbb{R}$-basis (e_1, e_2) with the property that, for some $p \ne q$, $e_1 + ie_2$ is of Hodge type (p, q) and $e_1 - ie_2$ is of Hodge type (q, p). It is easy to see that the pure Hodge structures in these (a) and (b) are polarizable.

Next we consider the case of P_j. It is a two-dimensional vector space V over $\mathbb{R}$ with basis (e_1, e_2), $W_1 = V$, $W_0 = 0$, $N_j e_2 = e_1$, $N_j e_1 = 0$, and $N_k = 0$ for any $k \ne j$, and the Hodge filtration on $V_{\mathbb{C}}$ is defined by $F^0 = V_{\mathbb{C}} \supset F^1 = \mathbb{C}e_2 \supset F^2 = 0$. The condition (g) in Section 2.1.9 is satisfied because the antisymmetric bilinear form on gr^W_1 defined by $\langle e_2, e_1\rangle_1 = 1$ satisfies the condition (g). The condition (h) in Section 2.1.9 is satisfied because $W^{(n)}$ is the relative monodromy filtration of

N_j with respect to W and, for $k \neq j$, W is the relative monodromy filtration of $N_k = 0$ with respect to W. □

3.3. Associated $\mathrm{SL}(2)$-orbits

3.3.1.
For an object $(V, W, N_1, \ldots, N_n, F)$ of DH_n, let

$$\hat{F} = s\bigl(F(\mathrm{gr}^{W^{(n)}})\bigr),$$

where $s : \mathrm{gr}^{W^{(n)}} \overset{\cong}{\to} V$ is the canonical splitting of $W^{(n)}$ associated to the mixed Hodge structure $(W^{(n)}, F)$ (see Section 2.2.1).

PROPOSITION 3.3.2

(a) *Let $(V, W, N_1, \ldots, N_n, \alpha)$ be an object of $\mathrm{D}_{n,E}$. Then $(V, W, \hat{N}_1, \ldots, \hat{N}_n, \alpha)$, where the $\hat{N}_j$'s are as in Section 3.1.4(e), is an object of $\hat{\mathrm{D}}_{n,E}$.*

(b) *Let $(V, W, N_1, \ldots, N_n, F)$ be an object of DH_n. Then $(V, W, \hat{N}_1, \ldots, \hat{N}_n, \hat{F})$, where the $\hat{N}_j$'s are as in Section 3.1.4(e) and $\hat{F}$ is as in Section 3.3.1, is an object of $\hat{\mathrm{DH}}_n$.*

We call the object of $\hat{\mathrm{D}}_{n,E}$ (resp., $\hat{\mathrm{DH}}_n$) associated to an object of $\mathrm{D}_{n,E}$ (resp., DH_n) in Proposition 3.3.2 the *associated* $\mathrm{SL}(2)$-*orbit*.

The proof of Proposition 3.3.2(a) is easy. (The key point is Section 3.1.4(e).)

The following counterpart of Section 3.1.4(e) for $\hat{F}$ proves Proposition 3.3.2(b).

PROPOSITION 3.3.3

Let $(V, W, N_1, \ldots, N_n, F)$ be an object of DH_n with the associated $\tau = (\tau_j)_{0 \le j \le n}$ (see Section 3.1.5). Let $\hat{F}$ be as in Section 3.3.1. Then we have $\tau_j(a)\hat{F} = \hat{F}$ for any $0 \le j \le n$ and any $a \in \mathbb{G}_m$.

In the case of an IMHM, this is [2, Lemma 5.5]. We give the proof in the general case in Sections 3.3.7 and 3.3.8 below after preparations.

3.3.4.
Let (V, W, N, F) be an IMHM of one variable. Then $(V, W, \exp(iN)\hat{F})$ is a mixed Hodge structure. Let τ_0' be the representation of $\mathbb{G}_m$ on V defined by the canonical splitting of W associated to this mixed Hodge structure. On the other hand, let (V, W, N, α) be the Deligne system of one variable associated to (V, W, N, F) (see Section 2.2), and consider its τ_0.

(a) An important theorem of Deligne is that

$$\tau_0' = \tau_0.$$

This is introduced in [2, Lemma 2.2], and the proof is given in that paper.

(b) On the other hand, in [7], it is proved that $\tau_0'(a)\hat{F} = \hat{F}$ for $a \in \mathbb{G}_m$.

By (a) and (b), we have that $\tau_0(a)\hat{F} = \hat{F}$.

LEMMA 3.3.5

Let (V, W, N, F) be a DH system of one variable. Assume that W is pure, and assume that $\hat{F} = F$. Then this object is an $\mathrm{SL}(2)$*-orbit, that is,* $(V, W, N, F) \in \hat{\mathrm{DH}}_1$.

This is evident.

The following is a special case of Theorem 1.4. (This theorem shows that the assumption $\hat{F} = F$ is not necessary in the following lemma.)

LEMMA 3.3.6

Let (V, W, N, F) be a DH system of one variable. Assume that $\hat{F} = F$. Then this object is an IMHM.

This follows from Lemma 3.3.5 and Proposition 3.2.7.

3.3.7.

We prove Proposition 3.3.3 in the case where $n = 1$. Let (V, W, N, F) be a DH system of one variable. Then $(V, W, N, \hat{F})$ is a DH system of one variable and satisfies the assumption of Lemma 3.3.6. Hence it is an IMHM. Hence by Section 3.3.4, we have that $\tau_0(a)\hat{F} = \hat{F}$.

3.3.8.

We prove Proposition 3.3.3 in general by induction on n. Assume that $n \geq 2$. Note that $(V, W^{(1)}, N_2, \ldots, N_n, F)$ is an object of DH_{n-1} and the associated action $(\tau'_j)_{0 \leq j \leq n-1}$ of $\mathbb{G}_m^n$ is given by $\tau'_j = \tau_{j+1}$. By the hypothesis of induction, $(V, W^{(1)}, \hat{N}_2, \ldots, \hat{N}_n, \hat{F})$ is an SL(2)-orbit. From this and from Proposition 3.2.7, we have the following.

(a) $(V, W^{(1)}, F')$ with $F' = \exp(\sum_{j=2}^n i\hat{N}_j)\hat{F}$ is a mixed Hodge structure.
(b) $\tau_1(a)F' = F'$ for any $a \in \mathbb{G}_m$.

For each $w \in \mathbb{Z}$, (a) and (b) also hold when we replace V by $U := W_w$ $(w \in \mathbb{Z})$ and replace W, N_j, and F by their restrictions to U. From this, we see that (V, W, N_1, F') is a DH system of one variable. By the case $n = 1$ of Proposition 3.3.3 proved in Section 3.3.7 and by (b), which shows that the functor $F \mapsto \hat{F}$ applied to F' does not change F', we have that $\tau_0(a)F' = F'$ for any $a \in \mathbb{G}_m$. Since $\tau_0(a)\hat{N}_j\tau_0(a)^{-1} = \hat{N}_j$ for any j, this proves that $\tau_0(a)\hat{F} = \hat{F}$. This completes the proof of Proposition 3.3.3 and hence the proof of Proposition 3.3.2.

PROPOSITION 3.3.9

Let $H = (V, W, N_1, \ldots, N_n, \alpha)$ (resp., $H = (V, W, N_1, \ldots, N_n, F)$) be an object of $\mathrm{D}_{n,E}$ *(resp.,* DH_n*), and let $\phi(H)$ be the associated* $\mathrm{SL}(2)$*-orbit. Then we have the following.*

(a) $\phi(\phi(H)) = \phi(H)$.

(b) *H is an $\mathrm{SL}(2)$-orbit if and only if $\phi(H)=H$.*

(c) *$(\tau_j)_{0\le j\le n}$ associated to H is the same as that associated to $\phi(H)$.*

Proof

To start, (c) is reduced to the case of Deligne systems of one variable (see Section 3.1.3) and is seen easily in that case. Then, (a) and (b) follow from (c). □

PROPOSITION 3.3.10

For an object of $\mathrm{D}_{n,E}$ with $E=\mathbb{R}$ or $\mathbb{C}$, or for an object of DH_n, we use the notation

$$\beta(y)=\prod_{j=0}^{n}\tau_j\big((y_j/y_{j+1})^{1/2}\big)\quad \text{for } y=(y_0,\dots,y_n)\in\mathbb{R}^{n+1}_{>0},$$

where y_{n+1} denotes 1.

(a) *For an object of $\mathrm{D}_{n,E}$ with $E=\mathbb{R},\mathbb{C}$ or of DH_n and for $y=(y_j)_{0\le j\le n}\in\mathbb{R}^{n+1}_{>0}$ such that $y_j/y_{j+1}\to\infty$ $(0\le j<n)$, we have the convergences*

$$\beta(y)y_kN_k\beta(y)^{-1}\to\hat{N}_k,\qquad \beta(y)\Big(\sum_{j\in I}y_jN_j\Big)\beta(y)^{-1}\to\sum_{j\in I}\hat{N}_j$$

for $1\le k\le n$ and for any subset I of $\{1,\dots,n\}$.

(b) *For any object of DH_n and for $y=(y_j)_{0\le j\le n}\in\mathbb{R}^{n+1}_{>0}$ such that $y_j/y_{j+1}\to\infty$ $(0\le j\le n$, y_{n+1} denotes 1$)$, we have the convergences*

$$\beta(y)F\to\hat{F},\qquad \beta(y)\exp\Big(\sum_{j\in I}iy_jN_j\Big)F\to\exp\Big(\sum_{j\in I}i\hat{N}_j\Big)\hat{F}$$

for any subset I of $\{1,\dots,n\}$.

Proof

We prove (a). Write $N_k=\sum_{m\in\mathbb{Z}^n}N_k^{[m]}$, and write $\tau_j(a)N_k^{[m]}\tau_j(a)^{-1}=a^{m(j)}N_k^{[m]}$ $(1\le j\le n,\ a\in\mathbb{G}_m)$. Then $N_k^{[m]}=0$ unless m satisfies the following.

(1) $m(j)=-2$ for any j such that $k\le j\le n$.

(2) $m(j)\le 0$ for $1\le j<k$.

For m satisfying (1), we have that $\beta(y)y_kN_k\beta(y)^{-1}=(\prod_{j=0}^{k-1}(y_j/y_{j+1})^{m(j)/2})\cdot N_k^{[m]}$. When $y_j/y_{j+1}\to\infty$ for $0\le j<n$, this converges to $N_k^{[m]}$ if $m(j)=0$ for $0\le j<k$, and to 0 otherwise, and hence converges to $\hat{N}_k^{[m]}$ for any m.

We prove (b). It is sufficient to prove that $\beta(y)F\to\hat{F}$, because the rest follows from this and from (a).

Let $s:\mathrm{gr}^{W^{(n)}}\xrightarrow{\cong}V$ be the canonical splitting of $W^{(n)}$ associated to the mixed Hodge structure $(V,W^{(n)},F)$, and let $\delta,\zeta:\mathrm{gr}^{W^{(n)}}\to\mathrm{gr}^{W^{(n)}}$ be the maps associated to this mixed Hodge structure (see Section 2.2.1). We have $F=s(\exp(-\zeta)\exp(i\delta)F(\mathrm{gr}^{W^{(n)}}))=\exp(\nu)\hat{F}$, where $\nu:V\to V$ is the nilpotent linear map characterized by $\exp(\nu)=s\exp(-\zeta)\exp(i\delta)s^{-1}$. For any $0\le j<n$, $k\in\mathbb{Z}$,

and $U := W_k^{(j)}$, the restriction $(U, W^{(n)}|_U, F|_U)$ to U is a mixed Hodge structure (see Section 2.1.2(f.2)), and δ and ζ associated to the last mixed Hodge structure are compatible with the above δ and ζ, respectively (see Section 2.2.1). This shows that ν is of weight less than or equal to 0 for τ_j for any $0 \le j < n$. Furthermore, ν is of weight less than or equal to -2 for τ_n. Hence $\beta(y)\nu\beta(y)^{-1} \to 0$. Furthermore, $\beta(y)\hat{F} = \hat{F}$ by Proposition 3.3.3. Hence

$$\beta(y)F = \big(\beta(y)\exp(\nu)\beta(y)^{-1}\big)\beta(y)\hat{F} \to \hat{F}. \qquad \square$$

REMARK 3.3.11

The terminology SL(2)-orbit in the present paper is different from that in [8]. In [8], we called an IMHM $H = (V, W, N_1, \dots, N_n, F)$ an SL(2)-orbit if

$$\tau_k(a)N_j\tau_k(a)^{-1} = N_j \quad (1 \le k < j \le n) \qquad \text{and} \qquad \tau_k(a)F = F \quad (1 \le k \le n)$$

for any $a \in \mathbb{G}_m$. The difference is that τ_0 does not appear in this formulation of [8].

We have the following.

(a) Let $n = 0$. Then H is an SL(2)-orbit in the sense of [8]. On the other hand, H is an SL(2)-orbit in the sense of the present paper if and only if $\hat{F} = F$.

(b) For $n \ge 1$, H is an SL(2)-orbit in the sense of [8] if and only if $\hat{N}_j = N_j$ for $2 \le j \le n$ and $\hat{F} = F$. The difference is that in [8] there is no condition on N_1.

(c) In the pure case, there is no difference between the formulation in [8] and that in the present paper.

Thus there are more SL(2)-orbits in [8] than in the present paper. The SL(2)-orbits in this paper are very simple objects and are useful by their simplicity. On the other hand, the formulation of the SL(2)-orbit in [8] is useful for the study of classifying spaces of degenerating mixed Hodge structures. In fact, in [8, Part 2], the classifying space $\{\text{MHS}\}$ of mixed Hodge structures is enlarged as

$$\begin{aligned}\{\text{MHS}\} &= \{\text{SL(2)-orbit of zero variable}\} \subset \{\text{SL(2)-orbit}\} \\ &= \{\text{degenerating MHS}\}.\end{aligned}$$

4. Proofs of the main results

4.1. DH systems and IMHM

We prove Theorem 1.4 from the introduction. We also prove the following.

THEOREM 4.1.1

Let $E = \mathbb{R}$ or $\mathbb{C}$, and let $(V, W, N_1, \dots, N_n, \alpha)$ be an object of $\mathrm{D}_{n,E}$. Take $0 \le \ell \le j < k \le n$. Then for $y_t > 0$ $(\ell + 1 \le t \le k)$ such that $y_t/y_{t+1} \gg 0$ $(\ell + 1 \le t < k)$, $W^{(k)}$ is the relative monodromy filtration of $\sum_{t=\ell+1}^k y_t N_t$ with respect to $W^{(j)}$. In other words, $(V, W^{(j)}, \sum_{t=\ell+1}^k y_t N_t, \tau_k)$ is a Deligne system of one variable.

Proof

For $y = (y_t)_{0 \le t \le n} \in \mathbb{R}_{>0}^{n+1}$, let $N_y = \beta(y)(\sum_{t=\ell+1}^k y_t N_t)\beta(y)^{-1}$ where $\beta(y)$ is as in

Proposition 3.3.10. Let $\hat{N}=\sum_{t=\ell+1}^{k}\hat{N}_t$. Then by Proposition 3.3.10(a), N_y converges to $\hat{N}$ when $y_t/y_{t+1}\to\infty$ $(0\le t<n)$. By Propositions 3.3.2(a) and 3.2.6, the map $\hat{N}^m:\mathrm{gr}^{W^{(k)}}_{w+m}\mathrm{gr}^{W^{(j)}}_w\to\mathrm{gr}^{W^{(k)}}_{w-m}\mathrm{gr}^{W^{(j)}}_w$ is an isomorphism for any $w\in\mathbb{Z}$ and any $m\ge 0$. It follows that if $y_t/y_{t+1}\gg 0$ $(0\le t<n)$, then the map $N_y^m:\mathrm{gr}^{W^{(k)}}_{w+m}\mathrm{gr}^{W^{(j)}}_w\to\mathrm{gr}^{W^{(k)}}_{w-m}\mathrm{gr}^{W^{(j)}}_w$ is an isomorphism and hence the map $(\sum_{t=\ell+1}^{k}y_tN_t)^m:\mathrm{gr}^{W^{(k)}}_{w+m}\mathrm{gr}^{W^{(j)}}_w\to\mathrm{gr}^{W^{(k)}}_{w-m}\mathrm{gr}^{W^{(j)}}_w$ is an isomorphism. □

4.1.2.
We prove Theorem 1.4.

By Theorem 4.1.1, the condition (h) in the definition of an IMHM (Section 2.1.9) is satisfied. In fact, for $N'_j=\sum_{k=1}^{j}a_{j,k}N_k$ with $a_{j,k}>0$ such that $a_{j,k}/a_{j,k+1}\gg 0$ $(1\le k<j)$, by Theorem 4.1.1, $W^{(j)}$ is the relative monodromy filtration of N'_j with respect to W.

It remains to consider the condition (g) in Section 2.1.9, that is, the polarizability of gr^W. On gr^W_w, put the bilinear form in Proposition 3.2.7. For $y=(y_j)_{0\le j\le n}\in\mathbb{R}^{n+1}_{>0}$, let $F(y)=\exp(\sum_{j=1}^{n}iy_jN_j)F$, and let $I=\exp(\sum_{j=1}^{n}i\hat{N}_j)\hat{F}$. Let $\beta(y)$ be as in Proposition 3.3.10. Then by Proposition 3.3.10(b), $\beta(y)F(y)$ converges to I when $y_j/y_{j+1}\to\infty$ $(0\le j\le n$, y_{n+1} denotes 1). Since (V,W,I) is a mixed Hodge structure (see Proposition 3.2.7), we have that $(V,W,\beta(y)F(y))$ is a mixed Hodge structure when $y_j/y_{j+1}\gg 0$. Hence we can consider the Hermitian form associated to $(\langle\cdot,\cdot\rangle_w,\beta(y)F(y)(\mathrm{gr}^W_w))$. This Hermitian form converges to the Hermitian form associated to $(\langle\cdot,\cdot\rangle_w,I(\mathrm{gr}^W_w))$ which is positive definite. Hence the former Hermitian form is positive definite if $y_j/y_{j+1}\gg 0$. Hence when $y_j/y_{j+1}\gg 0$, $(V,W,F(y))$ is a mixed Hodge structure and $(\mathrm{gr}^W_w,\langle\cdot,\cdot\rangle_w,F(y)(\mathrm{gr}^W_w))$ is a polarized Hodge structure of weight w for each w. This proves Theorem 1.4.

By Theorem 1.4, SL(2)-orbit theorems for IMHM in [10], [3], [9], and [7] are generalized to DH_n. For example from [3, Theorem 4.20(vii)], we have the following.

THEOREM 4.1.3
Let $(V,W,N_1,\dots,N_n,F)$ be an object of DH_n. Assume that W is pure. Then there is a convergent series $g(T_1,\dots,T_n)\in\mathrm{End}_{\mathbb{R}}(V)[[T_1,\dots,T_n]]$ with constant term 1 such that, when $y_j/y_{j+1}\gg 0$ $(1\le j\le n$, y_{n+1} denotes 1), we have

$$\exp\Big(\sum_{j=1}^{n}iy_jN_j\Big)F=g(y_2/y_1,y_3/y_2,\dots,y_{n+1}/y_n)\cdot\prod_{j=1}^{n}\tau_j\big((y_{j+1}/y_j)^{1/2}\big)\cdot\exp\Big(\sum_{j=1}^{n}i\hat{N}_j\Big)\hat{F}.$$

4.2. On Theorem 1.7 and Proposition 1.8

Concerning Theorem 1.7, we give a more precise statement about the convergence of the splitting of W.

THEOREM 4.2.1

Let $n \geq 1$, and let $(V, W, N_1, \ldots, N_n, F)$ (resp., $(V, W, N_1, \ldots, N_n, \alpha)$) be an object of DH_n (resp., $\mathrm{D}_{n,E}$ with $E = \mathbb{R}$ or $\mathbb{C}$). For $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_{>0}$, let $H(y) = (V, W, \exp(\sum_{j=1}^n iy_jN_j)F)$ (resp., $H(y) = (V, W, \sum_{j=1}^n y_jN_j, \alpha)$). Let $\hat{H} = (V, W, \exp(\sum_{j=1}^n i\hat{N}_j)\hat{F})$ (resp., $\hat{H} = (V, W, \sum_{j=1}^n \hat{N}_j, \alpha)$). Let $h = n$ (resp., $h = n - 1$). Then there is a family $(u_m)_{m \in \mathbb{N}^h}$ of $\mathbb{R}$-linear (resp., E-linear) maps $u_m : V \to V$ having the following properties.

(a) *$u_0 = 1$, and $u_m W_w \subset W_{w-1}$ for any $m \neq 0$ and any $w \in \mathbb{Z}$. For $1 \leq j \leq n$, $u_m W_w^{(j)} \subset W_{w+m(j)}^{(j)}$ for any m and any w.*

(b) *Let $u(T_1, \ldots, T_h) = \sum_{m \in \mathbb{N}^h} u_m T_1^{m(1)} \cdots T_n^{m(h)}$. Then there is $c > 0$ such that $u(T_1, \ldots, T_n)$ absolutely converges if $|T_j| < c$ for all j.*

(c) *For $y_j > 0$ $(1 \leq j \leq n)$ such that $y_j/y_{j+1} \gg 0$ $(1 \leq j \leq n$ where y_{n+1} denotes $1)$ (resp., $(1 \leq j < n)$), let $s(y) : \mathrm{gr}^W \overset{\cong}{\to} V$ be the canonical splitting (resp., the splitting by τ_0) of W in Section 2.2.1 (resp., Section 3.1.3) associated to the mixed Hodge structure (resp., Deligne system of one variable) $H(y)$. Let $\hat{s} : \mathrm{gr}^W \overset{\cong}{\to} V$ be the one associated to the mixed Hodge structure (resp., Deligne system of one variable) $\hat{H}$. Then*

$$s(y) = u(y_2/y_1, \ldots, y_{h+1}/y_h)\hat{s}$$

when $y_j/y_{j+1} \gg 0$ $(1 \leq j \leq h)$.

By Theorem 1.4, Theorems 1.7 and 4.2.1 for DH_n follow from the corresponding result in [7, Theorem 0.5] for IMHMs.

We will prove in Section 4.2.3 the parts concerning $\mathrm{D}_{n,E}$ $(E = \mathbb{R}, \mathbb{C})$ by reducing them to the part of DH_n.

LEMMA 4.2.2

Let $E = \mathbb{R}$, let (V, W, N, α) be a Deligne system of one variable with the associated $(\tau_j)_{j=0,1}$, and let $(V^{\oplus 2}, W^{\oplus 2}, N^{\oplus 2}, F)$ be the corresponding object of DH_1 (see Section 2.3). Then $(V^{\oplus 2}, W^{\oplus 2}, \exp(iN^{\oplus 2})F)$ is a mixed Hodge structure, and we have $\tau_0' = \tau_0^{\oplus 2}$ where τ_0' denotes the canonical splitting of $W^{\oplus 2}$ (see Section 2.2.1) associated to this mixed Hodge structure.

Proof

By Theorem 1.4, any object of DH_1 is an IMHM. Hence $(V^{\oplus 2}, W^{\oplus 2}, N^{\oplus 2}, F)$ is an IMHM. It is easy to see that $\hat{F} = F$ by construction in Section 2.3. Hence the result follows from Lemma 3.3.6 and Section 3.3.4(a). □

4.2.3.

We prove the parts of $\mathrm{D}_{n,E}$ in Theorems 1.7 and 4.2.1.

By Lemma 4.2.2, these theorems for $\mathrm{D}_{n,E}$ are reduced to the parts for DH_n by using the functor $\mathrm{D}_{n,E} \to \mathrm{DH}_n$ (see Section 2.3). In fact, we have the result that $(s(y)\circ \hat{s}^{-1})^{\oplus 2}: V^{\oplus 2}\to V^{\oplus 2}$ for $y_j/y_{j+1}\gg 0$ (y_{n+1} denotes 1) is a convergent series in $y_2/y_1,\dots,y_{n+1}/y_n$ with constant term 1 satisfying the conditions in Theorem 4.2.1(a) with W and $W^{(j)}$ $(1\le j\le n)$ replaced by $W^{\oplus 2}$ and $(W^{(j)})^{\oplus 2}$, respectively. This shows that $u(y):=s(y)\circ\hat{s}^{-1}: V\to V$ is a convergent series in $y_2/y_1,\dots,y_{n+1}/y_n$ with constant term 1 satisfying the conditions in Theorem 4.2.1(a). Since $s(y)$ depends only on the ratio $(y_1:\cdots:y_n)$, $u(y)$ is actually a series in $y_2/y_1,\dots,y_n/y_{n-1}$.

4.2.4.

For $a\in\mathbb{R}$, define the functor $\theta^a:\mathrm{DH}_n\to\mathrm{DH}_n$ as

$$(V,W,N_1,\dots,N_n,F)\mapsto(V,N'_1,\dots,N'_n,F)\quad\text{where } N'_j=\sum_{k=0}^{j-1}(a^k/k!)N_{j-k}.$$

That is, $(N'_1,\dots,N'_n)^t=\exp(aR)(N_1,\dots,N_n)^t$ where R is the (n,n) matrix whose (j,k)th entry is 1 if $k=j-1$ and 0 otherwise, and $(\cdot)^t$ denotes the transpose.

For an object H of DH_n, we have the following.

(a) $\theta^{a+b}H=\theta^a(\theta^b H)$.

Since $\theta^a\theta^{-a}$ is the identity functor by (a), we see that the following holds.

(b) $\theta^a:\mathrm{DH}_n\to\mathrm{DH}_n$ is an equivalence of categories.

By Theorem 1.4, we have that the following holds.

(c) If H is an object of DH_n, then $\theta^a H$ is an IMHM if $a\gg 0$.

For $a\in\mathbb{R}$, let $\mathrm{DH}_n^{(a)}$ be the full subcategory of DH_n consisting of all objects H such that $\theta^a H$ is an IMHM. By (c), we have the following.

(d) $\mathrm{DH}_n=\bigcup_a \mathrm{DH}_n^{(a)}$. Note that $\mathrm{DH}_n^{(a)}\subset\mathrm{DH}_n^{(b)}$ if $a\le b$.

4.2.5.

We prove Proposition 1.8.

First we prove the part concerning DH_n. Proposition 1.8 is true if DH_n is replaced by the category of IMHMs of n variables (see [5]). For $a\in\mathbb{R}$, the category $\mathrm{DH}_n^{(a)}$ in Section 4.2.4 is equivalent to the category of IMHMs of n variables by the functor θ^a. This shows that $\mathrm{DH}_n^{(a)}$ is an abelian category and the kernel and the cokernel are described as in Proposition 1.8. By Section 4.2.4(d), this proves Proposition 1.8 for DH_n.

We prove the part concerning $\mathrm{D}_{n,E}$. First we show that we can assume that $E=\mathbb{C}$. This is because an object of $\mathrm{D}_{n,E}$ or a morphism of $\mathrm{D}_{n,E}$ comes from $\mathrm{D}_{n,E'}$ for some subfield E' of E which is finitely generated over $\mathbb{Q}$. Then we have an embedding of E' into $\mathbb{C}$ as a subfield. Hence by Lemma 2.1.7(a), we are reduced to the case $E=\mathbb{C}$.

Next by Lemma 2.1.7(b), we can assume that $E=\mathbb{R}$.

We prove Proposition 1.8 in the case in which $E=\mathbb{R}$. We denote the functor $\mathrm{DH}_n\to\mathrm{D}_{n,\mathbb{R}}$ in Section 2.2 by a and the functor $\mathrm{D}_{n,\mathbb{R}}\to\mathrm{DH}_n$ in Section 2.3 by b. Let $f:A=(V_A,W_A,N_{1,A},\dots,N_{n,A},\alpha_A)\to B=(V_B,W_B,N_{1,B},\dots,N_{n,B},\alpha_B)$ be a morphism of $\mathrm{D}_{n,\mathbb{R}}$, let V_K (resp., V_C) be the kernel (resp., cokernel) of $f:V_A\to V_B$, and let $W_K,N_{j,K},\alpha_K$ on V_K (resp., $W_C,N_{j,C},\alpha_C$ on V_C) be the ones induced from those of A (resp., B). Then f induces a morphism $b(f) : b(A) \to b(B)$ of DH_n, and the kernel (resp., cokernel) of $b(f)$ is described as in the part of Proposition 1.8 concerning DH_n, which we have proved. By applying the functor a, we see that $(V_K^{\oplus 2}, W_K^{\oplus 2}, N_{1,K}^{\oplus 2}, \dots, N_{n,K}^{\oplus 2}, \alpha_K^{\oplus 2})$ (resp., $(V_C^{\oplus 2}, W_C^{\oplus 2}, N_{1,C}^{\oplus 2}, \dots, N_{n,C}^{\oplus 2}, \alpha_C^{\oplus 2})$) is an object of $\mathrm{D}_{n,\mathbb{R}}$. This shows that $K := (V_K, W_K, N_{1,K}, \dots, N_{n,K}, \alpha_K)$ (resp., $C := (V_C, W_C, N_{1,C}, \dots, N_{n,C}, \alpha_C)$) is an object of $\mathrm{D}_{n,\mathbb{R}}$. We have shown that the kernel and the cokernel of a morphism in $\mathrm{D}_{n,\mathbb{R}}$ exist and are described as in Proposition 1.8. Let I be the cokernel of $K \to A$ (resp., J be the kernel of $B \to C$). Since DH_n is an abelian category, the canonical morphism $b(I) \to b(J)$ is an isomorphism. By applying the functor a, we see that the canonical morphism $I^{\oplus 2} \to J^{\oplus 2}$ is an isomorphism. Hence the canonical morphism $I \to J$ is an isomorphism. This proves Proposition 1.8 for $\mathrm{D}_{n,\mathbb{R}}$.

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