Duality theorem for inductive limit groups

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Abstract In this paper, we show the so-called weak duality theorem of Tannaka type for an inductive limit–type topological group $G = \lim_{n\to\infty} G_n$ in the case where each G_n is a locally compact group, and G_n is embedded into G_{n+1} homeomorphically as a closed subgroup. First, we explain what a weak duality theorem of Tannaka type is and explain the difference between the case of locally compact groups and the case of non-locally compact groups. Then we introduce the concept "separating system of unitary representations (SSUR)," which assures the existence of sufficiently many unitary representations. The present G has an SSUR. We prove that G is complete. We give semiregular representations and their extensions for G. Using them, we deduce a fundamental formula about "birepresentation" on G. Combining these results, we can prove the weak duality theorem of Tannaka type for G.

0. Introduction

Let $G = \lim_{n\to\infty} G_n$ be an inductive limit–type topological group. We treat the case where each G_n is a locally compact group and G_n is embedded in G_{n+1} homeomorphically as a closed subgroup. We say that such a group is a *closed type* (see Definition 3.1). About topologies and duality theorem of this group, some results were given in [1], [3], [5], and [6]. In the previous paper [4], we studied properties of such a group and obtained some family of unitary representations of it. Based on these results, we show in this paper that for these groups, the so-called weak duality theorem of Tannaka type is valid.

The weak Tannaka-type duality theorem for topological groups is stated as follows.

We take the set $\Omega \equiv \{D = (\mathcal{H}^D, T^D_g)\}$ of all unitary representations of a given topological group K, dimensions of which are bounded by $\max(\aleph_0, \#K)$. There exist operations between elements of Ω such as

(1) unitary equivalence: $D_1 \sim_W D_2$ (W: intertwining operator),
(2) direct sum: $D_1 \oplus D_2$,
(3) tensor product: $D_1 \otimes D_2$,
(4) contragradient representation: $D \to \overline{D}$.

We consider an operator field $\boldsymbol{U} \equiv \{U^D\}_{D\in\Omega}$ on Ω satisfying the following:

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(B-0) for each $D \in \Omega$, U^D is a unitary operator on the representation Hilbert space $\mathcal{H}^D$;

(B-1) $D_1 \sim_W D_2 \Longrightarrow WU^{D_1}W^{-1} = U^{D_2}$;

(B-2) $U^{D_1} \oplus U^{D_2} = U^{D_1 \oplus D_2}$;

(B-3) $U^{D_1} \otimes U^{D_2} = U^{D_1 \otimes D_2}$;

(B-4) $\overline{(U^D)} = U^{\overline{D}}$.

Here $\overline{D}$ is the contragradient representation of D, and $\overline{(U^D)}$ is the operator on $\mathcal{H}^{\overline{D}}$ "contragradient" to U^D (see Section 1).

We shall call such an operator field $\boldsymbol{U}$ a *birepresentation* of K.

On the space of all birepresentations, we give a topology which is the product of weak topologies on spaces of unitary operators on Hilbert spaces $\mathcal{H}^D$.

As it is shown in Section 1, Lemma 1.1 that for any g in K the operator field $\boldsymbol{T}_g \equiv \{T_g^D\}_{D\in\Omega}$ satisfies the conditions (B.1)–(B.4); therefore $\boldsymbol{T}_g$ gives a birepresentation.

The weak Tannaka-type duality theorem asserts the converse.

ASSERTION

For any birepresentation $\boldsymbol{U} \equiv \{U^D\}_{D\in\Omega}$,

$$\exists_1 g \in K \quad \textit{such that } U^D = T_g^D \quad (\forall D \in \Omega).$$

Moreover, the topology given above on the space of birepresentations coincides with the original topology of K under this correspondence.

As shown in our previous paper [2], for any locally compact group, a theorem of such type is valid. But there exist several differences for the definition of *birepresentation.*

In duality theory for locally compact groups, birepresentations are operator fields $\boldsymbol{U} \equiv \{U^D\}_D$ with values not only of unitary operators but also of nonzero bounded operators on $\mathcal{H}^D$ at $D \in \Omega$. However, for the case of nonlocally compact groups, the *unitary assumption* cannot be abridged.

The reason for this phenomenon is explained as follows.

In the case of a locally compact group, *norms of any component operators U^D of birepresentation $\boldsymbol{U}$ are bounded by* 1 (see [2, (2.16)]).

So, for the case where we take this bounded operator condition as the definition of birepresentation instead of unitarity, we shall call this type of duality for topological groups the B-type. Consider the space of operator fields

$$\boldsymbol{B} \equiv \left\{\mathcal{B} = \{B^D\}_{D\in\Omega} \,\middle|\, B^D \text{ bounded operator on } \mathcal{H}^D \text{ such that } \|B^D\| \leq 1\right\}$$

with product topology of weak ones. The ball $\boldsymbol{B}^D \equiv \{B^D; \|B^D\| \leq 1\}$ is weakly compact for each D. So the space $\boldsymbol{B}$, which is the direct product of these balls, is weakly compact too. The set K_U of all birepresentations is a subgroup of $\boldsymbol{B}$.

Suppose, for a topological group K, that the B-type duality theorem is valid. Then we ask what can we conclude on the topology of K.

It is easy to see that operations (1)–(4) are continuous with respect to the weak topology. The set $\boldsymbol{B}_0 \subset \boldsymbol{B}$ satisfying conditions (B-1)–(B-4) is closed in $\boldsymbol{B}$ and hence is weakly compact. So K is imbedded into $\boldsymbol{B}_0$ as $K \ni g \to \boldsymbol{T}_g \equiv \{T_g^D\}_{D\in\Omega} \in \boldsymbol{B}_0$ with the isomorphic image K_U.

The only difference between K_U and $\boldsymbol{B}_0$ is the *nonzero* condition. So we see $K_U = \boldsymbol{B}_0 \setminus \{0\}$. Now we conclude that K is locally compact as a set taken off one point from a compact set.

PROPOSITION 0.1

A topological group K for which the B-type duality is valid must be a locally compact group.

NOTATION

For a representation $D = \{\mathcal{H}^D, T_g^D\}$ of G, we take its cyclic subrepresentation on the closed subspace $(\mathcal{H}^D)$ of $\mathcal{H}^D$ spanned by $\{T_g^D v^D\}_{g\in G}$ ($v^D \in \mathcal{H}^D, \|v^D\| = 1$). We express it as $(D) = \{(\mathcal{H}^D), T_g^D, v^D\}$.

Hereafter, for two cyclic representations $D_j = \{\mathcal{H}^j, T_g^j, v^j\}$, $j = 1, 2$, we denote cyclic representations contained in $D_1 \oplus D_2$ and $D_1 \otimes D_2$, respectively, as

$$(D_1 \oplus D_2) \equiv \big\{(\mathcal{H}^1 \oplus \mathcal{H}^2), T_g^1 \oplus T_g^2, v^1 \oplus v^2\big\},$$
$$(D_1 \otimes D_2) \equiv \big\{(\mathcal{H}^1 \otimes \mathcal{H}^2), T_g^1 \otimes T_g^2, v^1 \otimes v^2\big\}.$$

This paper is organized as follows. In Section 1, we introduce the concept *separating system of unitary representations* (SSUR), which assures existence of sufficiently many unitary representations. G has an SSUR (see Section 1, Example 2).

To prove the completeness of G in Section 4, we prepare lemmata about G in Sections 2 and 3.

We give semiregular representations and their extensions of G in Section 5. Using results in Section 5, we deduce important properties about "birepresentations" on G in Section 6.

Combining results in Sections 4 and 6, we can prove the weak duality theorem of Tannaka type for G in Section 7.

In Section 8, we introduce a new notion, *isobirepresentation*, which gives a wider category than of birepresentation. We can prove an analogous duality theorem for these isobirepresentations.

1. Separating system of unitary representations

Consider a Hausdorff (i.e., T_2-) topological group K.

Let $D \equiv \{\mathcal{H}^D, T_g^D, v^D\}$ be a cyclic unitary representation of K. Here $\mathcal{H}^D$ is the representation space, T_g^D is the operator of representation, and v^D is the normalized cyclic vector $\|v^D\| = 1$. Then the function $\eta^D(g) \equiv \langle T_g^D v^D, v^D\rangle$ is a normalized continuous positive definite function on K.

For later arguments, we treat $\overline{D} \equiv \{\overline{\mathcal{H}}, \overline{T}_g, \overline{v}\}$, the so-called contragradient representation of $D \equiv \{\mathcal{H}, T_g, v\}$, where $\overline{\mathcal{H}}$ is the linear dual space of $\mathcal{H}$, and there exists an antilinear map from $\mathcal{H}$ to $\overline{\mathcal{H}}$ defined by

$$\mathcal{H} \ni u \to \overline{u} \in \overline{\mathcal{H}}: \quad \overline{u}(w) \equiv (w, \overline{u}) = \langle w, u \rangle. \tag{1.1}$$

In $\overline{\mathcal{H}}$, we introduce an inner product as $\langle \overline{w}, \overline{u} \rangle = \overline{\langle w, u \rangle}$. As is easily shown, $\overline{(\overline{u})} = u$. For a bounded operator A on $\mathcal{H}$, we define $\overline{A}\overline{u} \equiv \overline{(Au)}$; then $\overline{A}$ gives a bounded linear operator on $\overline{\mathcal{H}}$. By definition, $\overline{T_g}\overline{u} = \overline{(T_g u)}$, and if A and B are bounded operators on $\mathcal{H}$, then $\overline{AB}\overline{u} = \overline{A}(\overline{B}\overline{u}) = \overline{A}\overline{(Bu)} = \overline{(A(Bu))} = \overline{(ABu)}$.

Hence $g \to \overline{T}_g \equiv \overline{(T_g)}$ gives a unitary representation of K on $\overline{\mathcal{H}}$. This is by definition the contragradient representation of D. Let $\overline{v} \in \overline{\mathcal{H}}$ be the vector corresponding to $v \in \mathcal{H}$ as above; then

$$\forall g \in K, \quad \langle \overline{T}_g \overline{v}, \overline{v} \rangle = \overline{\langle T_g v, v \rangle}. \tag{1.2}$$

It is easy to see that $\overline{(\overline{D})}$ is equivalent to the original D.

LEMMA 1.1

We have $\forall g \in K, \forall D \in \Omega, \overline{(T_g^D)} = T_g^{\overline{D}}$.

Proof

It is just the definition of $\overline{D}$.

On the other hand, the representation $D^0 \equiv D \oplus \overline{D}$ is self-adjoint; that is, $\overline{(D^0)} \sim_W D^0$, with an intertwining operator W exchanging the first space with the second space, which maps a vector $u \oplus \overline{v}$ to $v \oplus \overline{u}$. □

LEMMA 1.2

Consider a representation $D^0 \equiv D \oplus \overline{D}$ *for a given representation* $D \equiv \{\mathcal{H}^D, T_g^D\}$. *Let* A *be a bounded operator on* $\mathcal{H}^D$.

Then, for $\forall u, v \in \mathcal{H}^D, \langle (A \oplus \overline{A})(u \oplus \overline{u}), v \oplus \overline{v} \rangle$ (≤ 1) *is real valued.*

Proof

We have $\langle (A \oplus \overline{A})(u \oplus \overline{u}), v \oplus v \rangle = \langle Au, v \rangle + \langle \overline{A}\overline{u}, \overline{v} \rangle = \langle Au, v \rangle + \langle \overline{(Au)}, \overline{v} \rangle = \langle Au, v \rangle + \overline{\langle Au, v \rangle} = 2\Re \langle Au, v \rangle$. ($\Re$ shows the real part.) □

Now for any given cyclic unitary representation $D \equiv \{\mathcal{H}, T_g, v\}$ $(\|v\| = 1)$, and the trivial representation $I = \{\boldsymbol{C}, I_g, v_0\}$, we consider unitary representation

$$D_p \equiv I \oplus D \oplus \overline{D}$$

and its cyclic part (D_p), whose representation space is spanned by the vector

$$v_p \equiv (2^{-1/2}) v_0 \oplus (1/2)(v \oplus \overline{v}).$$

COROLLARY 1.2.1

For the above cyclic representation (D_p), *the matrix element*

$$1 \geq \langle T_g^{D_p} v_p, v_p \rangle \geq 0.$$

Proof
At first, $\|v_p\|^2 = 2^{-1}\|v_0\|^2 + 2\times 2^{-2}\|v\|^2 = 2^{-1} + 2\times 2^{-2} = 1$.
Therefore $\langle T_g^{D_p} v_p, v_p\rangle \le 1$ and $|\langle T_g^D v, v\rangle| \le 1$,

$$\begin{aligned}1 &\ge \langle T_g^{D_p} v_p, v_p\rangle = (2^{-1})\langle I_g v_0, v_0\rangle + (4^{-1})\big\{\langle T_g^D v, v\rangle + \langle T_g^{\overline{D}}\overline{v}, \overline{v}\rangle\big\}\\ &= 2^{-1} + (2^{-1})\big(\Re\langle T_g^D v, v\rangle\big) \ge 0.\end{aligned}\tag{1.3}$$ □

Write $\eta(g) \equiv \langle T_g^D v, v\rangle$. Take $0 < \varepsilon < 1$, and put

$$F(D,\varepsilon) \equiv \big\{g\in K \mid \big|1-\eta(g)\big| < \varepsilon\big\}.$$

COROLLARY 1.2.2
Let D, and (D_p) be as in Corollary 1.2.1. Then

$$1 > \forall\varepsilon \ge 0, \exists\delta > 0, \quad F(D_p,\delta) \subset F(D,\varepsilon).\tag{1.4}$$

Proof

$$\big|1-\eta(g)\big| < \varepsilon < 1 \text{ shows that } \big|1-\Re\eta(g)\big| < 1; \quad \text{therefore } \Re\eta(g) > 0.$$

Since $\eta_p(g) = \langle T_g^{D_p} v_p, v_p\rangle = 2^{-1}(1 + (\Re\langle T_g^D v, v\rangle)$,

$$1-\eta_p(g) = 1-\Re\eta_p(g) = 1-2^{-1}-(2^{-1})\Re\eta(g) = 2^{-1}\big(1-\Re\eta(g)\big).$$

On the other hand, $1 \ge |\Re\eta(g)|^2 + |\Im\eta(g)|^2$. ($\Im$ is the imaginary part.)
So $|\Im\eta(g)|^2 \le 1-(\Re\eta(g))^2 = (1-\Re\eta(g))\times(1+\Re\eta(g)) \le 2(1-\Re\eta(g))$. Thus

$$\begin{aligned}\big|1-\eta(g)\big|^2 &= \big|1-\Re\eta(g)\big|^2 + \big|\Im\eta(g)\big|^2 \le \big(1-\Re\eta(g)\big)^2 + 2\big(1-\Re\eta(g)\big)\\ &= \big(1-\Re\eta(g)\big)\big(3-\Re\eta(g)\big) \le 3\big(1-\Re\eta(g)\big) = 6\big(1-\eta_p(g)\big).\end{aligned}$$

This shows that if $6\delta < \varepsilon^2$, then $\forall g \in F(D_p,\delta)$ leads us to $g \in F(D,\varepsilon)$. □

DEFINITION 1.1
We say that a set $\Omega_0 \equiv \{D_\alpha\}_{\alpha\in A}$ of cyclic unitary representations of K gives a *separating system of unitary representations (SSUR)* if, for any neighborhood V of the unit e in K, there exists a positive definite function $\eta^D(g) \equiv \langle T_g^D v^D, v^D\rangle$ $(D\in\Omega_0)$ and $\varepsilon > 0$ such that

$$\big\{g\in K \mid \big|1-\eta^D(g)\big| < \varepsilon\big\} \subset V.$$

EXAMPLE 1
Let K be a locally compact group. For any neighborhood V of e, we can take a continuous function f such that $[f^* * f] \subset V$, where $[f^* * f]$ denotes the carrier of the convolution function of f^* and f, where $f^*(g) = \Delta(g)\overline{f(g^{-1})}$ ($\Delta(g)$ shows the modular function on G).

As is well known, $f^* * f$ gives a positive definite function. This shows that the family of cyclic subrepresentations of the regular representation gives an SSUR of K.

EXAMPLE 2

In [4, Proposition 5.5], we have shown that, for a closed-type inductive limit group $G = \lim_{n\to\infty} G_n$ and a neighborhood V of e in G, we can construct a positive definite function F on G such that the carrier $[F] \subset V$. This shows that the family of representations given in the paper [4] gives an SSUR for G.

LEMMA 1.3

If a topological group K has an SSUR, then we can select a new SSUR $\Omega_1 \equiv \{D\}$, for which any $D = \{\mathcal{H}^D, T_g^D, v^D\} \in \Omega_1$ has a nonnegative valued positive definite function $\eta^D(g) \equiv \langle T_g^D v^D, v^D\rangle$.

Proof

Take, instead of the initially given SSUR, $\Omega \equiv \{D\}$, the new system $\Omega_1 \equiv \{(D_p)\}$ as above; Corollaries 1.2.1 and 1.2.2 show that Ω_1 is also an SSUR. □

For a given Hilbert space $\mathcal{H}$, we denote by $B(\mathcal{H})$ the space of all bounded operators, by $J(\mathcal{H})$ the space of all isometric operators, and by $U(\mathcal{H})$ the space of all unitary operators on $\mathcal{H}$. Put the weak topologies on $J(\mathcal{H})$'s.

For $U \in J(\mathcal{H})$ and $v \in \mathcal{H}$,

$$\begin{aligned} \|Uv - v\|^2 &= \|Uv\|^2 + \|v\|^2 - 2\Re\langle Uv, v\rangle \\ &= 2\big(\langle v, v\rangle - \Re\big(\langle Uv, v\rangle\big)\big) = 2\Re\big(\langle v - Uv, v\rangle\big). \end{aligned} \tag{1.5}$$

That is, the weak topology coincides with the strong topology.

Moreover, $U(\mathcal{H})$ becomes a topological group with the multiplication of operators and this topology. As a group topology, this topology gives a uniform structure on $U(\mathcal{H})$.

Now, for a topological group K, we consider any unitary representation $D \equiv \{\mathcal{H}^D, T_g^D\}$ and the map $K \ni g \to T_g^D \in U(\mathcal{H}^D)$. Of course this map is continuous for each D.

Construct $\boldsymbol{U}(\Omega) \equiv \prod_{D\in\Omega} U(\mathcal{H}^D)$ with natural product topology. Then $\boldsymbol{U}(\Omega)$ is a topological group too, by the componentwise multiplication. The map

$$K \ni g \longmapsto (T_g^D)_{D\in\Omega} \in \prod_{D\in\Omega} U(\mathcal{H}^D) = \boldsymbol{U}(\Omega) \tag{1.6}$$

is an injective homomorphism as topological groups.

Now let K be a T_2-topological group with an SSUR.

LEMMA 1.4

For this group K the map (1.6) is an injective isomorphism, so by this map, K is embedded as a topological group in $\boldsymbol{U}(\Omega)$.

Proof

According to the above discussions, this map is continuous.

Conversely the T_2-property and existence of an SSUR Ω_0 show that this map must be injective, and for any neighborhood V of $e \in K$, we can select elements

$D \in \Omega_0$ and $\varepsilon > 0$ such that

$$\{g \in K \mid \|T_g^D v - v\| < \varepsilon\} \subset V.$$

This shows the continuity of the inverse map of (1.6). □

Hereafter we denote by K_U the image of K under (1.6) in $\boldsymbol{U}(\Omega)$.

2. Cauchy filter base

DEFINITION 2.1

Take a filter base $\mathcal{F} \equiv \{F_\alpha\}_{\alpha\in\Gamma}$ on a T_2-topological group K, where Γ is a partially ordered set. We say that $\mathcal{F}$ is *l-Cauchy* (resp., *r-Cauchy*) if for any neighborhood V of $e \in K$, there exists $\alpha \in \Gamma$ such that

$$\forall \beta, \gamma \succ \alpha \ (\beta, \gamma \in \Gamma), \quad F_\beta^{-1} F_\gamma \subset V \quad (\text{resp., } F_\beta F_\gamma^{-1} \subset V).$$

And we say that $\mathcal{F}$ is *b-Cauchy* (both Cauchy) when $\mathcal{F}$ is l-Cauchy and at the same time $\mathcal{F}^{-1} \equiv \{F_\alpha^{-1}\}_{\alpha\in\Gamma}$ is l-Cauchy.

If any l-Cauchy (resp., r-Cauchy, b-Cauchy) filter bases have limit points in K, we say that K is *l-complete* (resp., *r-complete*, *b-complete*).

As is well known that an l-complete group is also r-complete.

For simplicity, hereafter we use the word "Cauchy" (resp., complete) for "l-Cauchy" (resp., l-complete).

A b-Cauchy filter base is also l-Cauchy, so an l-complete group is b-complete too.

If a filter base $\mathcal{F}$ converges to a point g_0 in K, then $\mathcal{F}^{-1}$ converges to g_0^{-1}.

LEMMA 2.1

For an arbitrary given Cauchy filter base $\mathcal{F} \equiv \{F_\alpha\}_{\alpha\in\Gamma}$ on a T_2-topological group K, the set $\overline{\mathcal{F}} \equiv \{\overline{F_\alpha}\}_{\alpha\in\Gamma}$ gives a base of a Cauchy filter on K too, where $\overline{F_\alpha}$ denotes the closure of F_α.

If one of $\mathcal{F}$ and $\overline{\mathcal{F}}$ converges, then the other one converges to the same limit point. (We say that this property is cofinal.*)*

Proof

Since $\overline{F_\alpha} \cap \overline{F_\beta} \supset \overline{F_\alpha \cap F_\beta}$, $\overline{\mathcal{F}} \equiv \{\overline{F_\alpha}\}_{\alpha\in\Gamma}$ gives a base of a filter on K.

We show that it is Cauchy.

For any given neighborhood W of e, take a symmetric neighborhood V (i.e., $V = V^{-1}$) of e as $V^3 \subset W$.

Because $\mathcal{F}$ is Cauchy, $\exists \alpha \in \Gamma$ such that $\forall \beta, \gamma \succ \alpha$, $F_\beta^{-1} F_\gamma \subset V$.

But $\overline{F_\alpha} \subset F_\alpha V, \overline{F_\beta} \subset F_\beta V$. Therefore $\overline{F_\alpha}^{-1} \overline{F_\beta} \subset V F_\alpha^{-1} F_\beta V \subset V^3 \subset W$.

We have $\forall \alpha, \overline{F_\alpha} \supset F_\alpha$, so if $\overline{\mathcal{F}}$ converges, then $\mathcal{F}$ converges to the same limit.

Conversely if $\mathcal{F}$ converges, then $\mathcal{V}\mathcal{F} \equiv \{V F_\alpha\}_{\alpha\in\Gamma, V\in\mathcal{V}}$ ($\mathcal{V}$ is the set of all neighborhoods of e) converges to the same limit.

Since $\mathcal{V}\mathcal{F} \supset \overline{\mathcal{F}}$, this concludes the proof. □

DEFINITION 2.2

We call a filter base $\mathcal{F} \equiv \{F_\alpha\}$ such that all F_α are closed, as C-*filter base.*

3. Partially compact set in an inductive limit group

DEFINITION 3.1

We say that an inductive limit group $G = \lim_{n\to\infty} G_n$ *of closed type* if each G_n is a locally compact group, and G_n is imbedded in G_{n+1} homeomorphically as a closed subgroup.

REMARK

Here we can assume without any loss of generality that each G_n is not locally isomorphic to G_{n+1}.

If there is an n such that $\forall m \geq n, G_m$ is locally isomorphic to G_n, then G is itself locally compact group which is locally isomorphic to G_n. So we have no need to discuss this case.

If not, we can take a subsequence $\{G_{n(m)} \ (n(m) < n(m+1))\}$ such that $G_{n(m)}$ is not locally isomorphic to $G_{n(m+1)}$, and G is isomorphic to $\lim_{n\to\infty} G_{n(m)}$, which satisfies the above condition.

In this section, we consider a closed-type inductive limit group $G = \lim_{n\to\infty} G_n$.

We quote some results from our previous paper [4].

DEFINITION 3.2

Take for each n a neighborhood W_n of e in G_n. Then

$$W \equiv \bigcup_{1\leq k<\infty} W_1 \cdot W_2 \cdots W_k \tag{3.1}$$

is a neighborhood of e in G. We call a neighborhood of this type *a BS (bamboo shoot) neighborhood.*

In [4, Proposition 2.3], the next proposition was shown.

PROPOSITION 3.1

Let G be a closed-type inductive limit group. Then the family of BS neighborhoods gives a fundamental system of neighborhoods of e in G.

DEFINITION 3.3

A subset E $(\subset G)$ is called a *partially compact set*, or a PC set, if for any $n, E \cap G_n$ is a compact set (may be vacant).

The next is obvious.

LEMMA 3.1

For any PC set E and $\forall g \in G$, the sets Eg, gE are also PC sets.

LEMMA 3.2

Let E be a PC set in G such that $\exists n, E \cap G_n = \emptyset$. Then there exists a neighborhood W of e in G such that $E \cap G_n W = \emptyset$.

Proof

Since G_n is closed in G_{n+1}, for the compact set $E_{n+1} \equiv E \cap G_{n+1}$, we can take a compact neighborhood W_{n+1} of e in G_{n+1} such that $E_{n+1} \cap G_n W_{n+1} = \emptyset$.

Obviously $G_n W_{n+1} \subset G_{n+1}$, G_n is closed, and W_{n+1} is compact, so its product $G_n W_{n+1}$ is closed.

For a closed set $G_n W_{n+1}$ in G_{n+2} and a compact set $E_{n+2} \equiv E \cap G_{n+2}$,

$$\begin{aligned} E_{n+2} \cap G_n W_{n+1} &= E \cap G_{n+2} \cap G_n W_{n+1} = E \cap G_{n+1} \cap G_n W_{n+1} \\ &= E_{n+1} \cap G_n W_{n+1} = \emptyset. \end{aligned}$$

So we can take a compact neighborhood W_{n+2} of e in G_{n+2} such that

$$E_{n+2} \cap G_n W_{n+1} W_{n+2} = \emptyset.$$

Similarly for the closed set $G_n W_{n+1} W_{n+2}$ in G_{n+3} and a compact set $E_{n+3} \equiv E \cap G_{n+3}$, define a compact neighborhood W_{n+3} of e in G_{n+3} such that

$$E_{n+3} \cap G_n W_{n+1} W_{n+2} W_{n+3} = \emptyset.$$

Repeating these steps, by induction on k, we can obtain a compact neighborhood W_{n+k} such that

$$E_{n+k} \cap G_n W_{n+1} W_{n+2} \cdots W_{n+k} = \emptyset.$$

We have $\forall k > m, E_{n+k} \supset E_{n+m}, E_{n+m} \cap G_n W_{n+1} W_{n+2} \cdots W_{n+k} = \emptyset$, that is,

$$E_{n+m} \cap G_n \Big(\bigcup_{k \geq 1} W_{n+1} W_{n+2} \cdots W_{n+k} \Big) = \emptyset.$$

The set $W \equiv \bigcup_{k \geq 1} W_{n+1} W_{n+2} \cdots W_{n+k}$ is a BS neighborhood of e in G, and

$$\forall m, \quad E_{n+m} \cap G_n W = \emptyset.$$

But $E = \bigcup_{k \geq 1} E_{n+k}$, and so $E \cap G_n W = \emptyset$. □

LEMMA 3.3

If there exists a family $\{F_m\}_{m \geq 1}$ of PC sets in G satisfying

$$(1) \quad \forall m, \quad F_m \supset F_{m+1}, \qquad (2) \quad \forall m, \quad F_m \cap G_m = \emptyset,$$

then we can take a neighborhood V of $e \in G$ such that

$$\forall m, \quad F_{m+1} \cap G_m V = \emptyset.$$

Proof

Take F_{m+1} as E in Lemma 3.2; then there exists a neighborhood V_{m+1} of $e \in G$ such that $F_{m+1} \cap G_m V_{m+1} = \emptyset$. Here $G_m V_{m+1}$ is a neighborhood of $e \in G$.

Put $V \equiv \bigcap_{m\geq 1} G_m V_{m+1}$. We show that V is also a neighborhood of $e \in G$.

For this it is enough to show that V contains an open neighborhood of $e \in G$. Since V_m is a neighborhood in G, there exists an open neighborhood $O_m \subset V_m$ of $e \in G$. Put

$$V = \bigcap_{m\geq 1} G_m V_{m+1} \supset O \equiv \bigcap_{m\geq 1} G_m O_{m+1} \ni e.$$

To prove that O is open in G, it is enough to see that $\forall k, O \cap G_k$ is open in G_k. But $\forall m \geq k, G_k \subset G_m O_{m+1}$. So

$$\begin{aligned} O \cap G_k &= \Big(\bigcap_{m<k} G_m O_{m+1}\Big) \cap \Big(\bigcap_{m\geq k} G_m O_{m+1}\Big) \cap G_k \\ &= \Big(\bigcap_{m<k} G_m O_{m+1}\Big) \cap \Big(\Big(\bigcap_{m\geq k} G_m O_{m+1}\Big) \cap G_k\Big) \\ &= \bigcap_{m<k} G_m O_{m+1} \cap G_k, \end{aligned}$$

which is open in G_k. And

$$\forall m, \quad F_{m+1} \cap G_m V = F_{m+1} \cap G_m \Big(\bigcap_{k\geq 1} G_k V_{k+1}\Big) \subset F_{m+1} \cap G_m V_{m+1} = \emptyset. \qquad \square$$

LEMMA 3.4

For any neighborhood V of $e \in G$, there exists a PC neighborhood of $e \in G$ contained in V.

Proof

Without any loss of generality, we can assume that V is a BS neighborhood; that is, we can consider it as $V = \bigcup_{1\leq k<\infty} V_1 \cdot V_2 \cdots V_k$.

We will take W_n in (3.1) inductively as $\forall n, W_n \subset V_n$.

At first, take an open relatively compact neighborhood of e in G_1 as $W_1 \subset V_1$.

Next, we select W_2 a relatively compact neighborhood of e in G_2 as $W_2 \subset V_2$ and $(W_2)^2 \cap G_1 \subset W_1$.

Now, if we can determine W_{j-1}, then we select W_j as a relatively compact neighborhood of e in G_j satisfying $W_j \subset V_j$ and $(W_j)^2 \cap G_{j-1} \subset W_{j-1}$.

In this situation $W_1 \cdot W_2 \cdots W_{j-1} \cdot (W_j)^2$ is a relatively compact neighborhood of e in G_j. Now put $W \equiv \bigcup_{k\geq 1} W_1 \cdot W_2 \cdots W_k$ and $E(k,j) \equiv W_1 \cdot W_2 \cdots W_k \cap G_j$.

If $k \leq j$, then $G_j \supset W_1 \cdot W_2 \cdots W_k \supset E(k,j)$.

When $k > j$, then

$$\begin{aligned} E(k,j) &\equiv W_1 \cdot W_2 \cdots W_k \cap G_j \subset W_1 \cdot W_2 \cdots W_{k-1}(W_k)^2 \cap G_j \\ &= W_1 \cdot W_2 \cdots W_{k-1}(W_k)^2 \cap G_{k-1} \cap G_j \\ &= \big(W_1 \cdot W_2 \cdots W_{k-1}(W_k)^2 \cap G_{k-1}\big) \cap G_j \\ &\subset \big((W_1 \cdot W_2 \cdots W_{k-1})\big((W_k)^2 \cap G_{k-1}\big)\big) \cap G_j \end{aligned}$$

$$\subset W_1 \cdot W_2 \cdots W_{k-2}(W_{k-1})^2 \cap G_{k-2} \cap G_j$$
$$\cdots$$
$$\subset W_1 \cdot W_2 \cdots W_{j-1}(W_j)^2 \cap G_j = W_1 \cdot W_2 \cdots W_{j-1}(W_j)^2.$$

In both cases, $\forall k, E(k,j) \subset W_1 \cdot W_2 \cdots W_{j-1}(W_j)^2$, that is,

$$W \cap G_j \equiv \bigcup_{k\geq 1} W_1 \cdot W_2 \cdots W_k \cap G_j \subset W_1 \cdot W_2 \cdots W_{j-1}(W_j)^2$$

is a relatively compact set in G.

Since W is a neighborhood of e in the topological group G, there exists a neighborhood V_0 of e such that $(V_0)^2 \subset W$ and $\overline{V_0} \subset (V_0)^2 \subset W$.

Thus $\overline{V_0}$ is closed, and $\forall n, \overline{V_0} \cap G_n$ $(\subset W \cap G_n)$ is compact.

$\overline{V_0}$ is a PC neighborhood of $e \in G$ contained in V. □

COROLLARY 3.4.1

If $\mathcal{F} = \{F_\alpha\}_{\alpha\in A}$ is a Cauchy C-filter base, then there exists an α such that for all $\beta \succ \alpha, F_\beta$ are PC sets.

Proof

Take a PC neighborhood W of $e \in G$. Since $\mathcal{F} = \{F_\alpha\}_{\alpha\in A}$ is Cauchy, $\exists\alpha, \forall\beta > \alpha, F_\alpha^{-1}F_\beta \subset W$. So, for some $g \in F_\alpha, F_\beta \subset gW$. This shows that $\forall\beta > \alpha$, F_β is a PC set. □

LEMMA 3.5

A σ-compact set in a Hilbert space is contained in a closed separable subspace.

Proof

Any compact set C in a metric vector space has a countable dense subset Ξ.

For a given σ-compact set $B \equiv \bigcup_{n\geq 1} C_n$ with C_n compact, take Ξ_n a countable dense set in C_n. Then $\bigcup_{n\geq 1} \Xi_n$ spans a closed separable subspace containing B. □

COROLLARY 3.5.1

Let E be a partially compact set in G.

For any unitary representation $D = \{\mathcal{H}^D, T_g^D\}$ of G and $v \in \mathcal{H}^D$, the set $T_E^D v \equiv \{T_g^D v \mid g \in E\}$ is contained in a closed separable subspace.

Proof

The set E is σ-compact, so its continuous image $T_E^D v$ is the same. □

4. Completeness of $G = \lim_{n\to\infty} G_n$

THEOREM 1

A closed-type inductive limit group $G = \lim_{n\to\infty} G_n$ is complete.

Proof
We show that any Cauchy filter base $\mathcal{F} \equiv \{F_\alpha\}_{\alpha\in A}$ on G converges to a point in G.

There exist two cases.

Case (1) We have $\exists n$ such that $\forall\alpha, F_\alpha \cap G_n \neq \emptyset$;

Case (2) We have $\forall n$, $\exists\alpha$ such that $F_\alpha \cap G_n = \emptyset$.

In case (1), the set $\mathcal{F}_n \equiv \{F_{\alpha,n} \equiv F_\alpha \cap G_n\}_{\alpha\in A}$ gives a Cauchy filter base in the locally compact group G_n. In fact,

$$\forall F_{\alpha,n}, F_{\beta,n} \in \mathcal{F}_n, \quad F_{\alpha,n} \cap F_{\beta,n} = (F_\alpha \cap G_n) \cap (F_\beta \cap G_n) = (F_\alpha \cap F_\beta) \cap G_n$$

contains an element of $\mathcal{F}_n$. So $\mathcal{F}_n$ gives a filter base. The "Cauchy property" for $\mathcal{F}$ assures the same property for $\mathcal{F}_n$.

Since a locally compact group is complete, $\mathcal{F}_n$ converges to a point in G_n. This point is also the limit of $\mathcal{F}$. So $\mathcal{F}$ converges to a point of G_n.

Now we shall show that case (2) does not exist.

Assume that $\mathcal{F}$ is a C-filter base as in case (2). By Corollary 3.4.1, all elements of $\mathcal{F}$ can be taken as PC sets. Since $G = \bigcup G_n$, for $\forall F_\alpha, \exists n$ such that

$$F_\alpha \cap G_n \neq \emptyset. \tag{4.1}$$

Now take $F_1 \in \mathcal{F}$ as $F_1 \cap G_1 = \emptyset$ and $n(1)$ as $F_1 \cap G_{n(1)} \neq \emptyset$.

Next take $F_2 \in \mathcal{F}$ as $F_2 \subset F_1, F_2 \cap G_{n(1)} = \emptyset$, and $n(2)$ as $F_2 \cap G_{n(2)} \neq \emptyset$.

Repeating these steps inductively, after determining F_{k-1} and $n(k-1)$, we take

$$F_k \in \mathcal{F} \quad \text{as } F_k \subset F_{k-1}, F_k \cap G_{n(k-1)} = \emptyset, \qquad \text{and} \qquad n(k) \quad \text{as } F_k \cap G_{n(k)} \neq \emptyset.$$

Thus we obtain a sequence $\{F_m, n(m)\}_{m\geq 1}$ of pairs as

$$\forall m, \quad F_{m+1} \subset F_m, \qquad F_{m+1} \cap G_{n(m)} = \emptyset, \qquad F_{m+1} \cap G_{n(m+1)} \neq \emptyset. \tag{4.2}$$

For any inductive limit group $G = \lim_{n\to\infty} G_n$, we can omit components in the middle of the sequence. So we rewrite $G_{n(m)}$ to G_m, and apply Lemma 3.3. Then we obtain a neighborhood V of $e \in G$ satisfying $\forall m, F_{m+1} \cap G_m V = \emptyset$. In other words,

$$G_m F_{m+1} \cap V = \emptyset \qquad \text{and} \qquad F_{m+1} \cap G_{m+1} \neq \emptyset. \tag{4.3}$$

Now, we quote here the result of [5, Proposition 5.5, Theorem 5.10].

PROPOSITION
Let $G = \lim_{n\to\infty} G_n$ be a closed-type inductive limit group. Then for any neighborhood V of e in G, there exists a continuous positive definite function η such that its support is contained in V:

$$[\eta] \subset V. \tag{4.4}$$

Recall that the GNS-construction method gives a cyclic unitary representation $D \equiv \{\mathcal{H}, T_g, v\}$ such that $\eta(g) = \langle T_g v, v\rangle$, and (4.4) means that, if g_0 is not contained in V, then $\langle T_{g_0}v, v\rangle = 0$; that is the same as

$$T_{g_0}v \perp v. \tag{4.5}$$

Combining with (4.3), we get

$$\forall h_m \in G_m, \forall g_{m+1} \in F_{m+1}, \quad T_{h_m^{-1} g_{m+1}} v \perp v, \tag{4.6}$$

that is,

$$\forall h_m \in G_m, \forall g_{m+1} \in F_{m+1}, \quad T_{g_{m+1}} v \perp T_{h_m} v. \tag{4.7}$$

On the other hand, the family of sets of vectors $\{T_F v\}_{F\in\mathcal{F}}$ must be a Cauchy filter base in the representation Hilbert space $\mathcal{H}$. Let u be the limit of $\{T_F v\}_{F\in\mathcal{F}}$. Now put

$$D_m \equiv \{g \in G \mid \|T_g v - u\| < 1/m\}.$$

Since u is the limit of monotone filter base $\{T_{F_m} v\}_{m\geq 1}$, we have $\forall m, n$, $D_m \cap F_n \neq \emptyset$.

Put $E(m) \equiv D_m \cap F_m$. Then these sets are nonempty and monotone decreasing with respect to m. By (4.2), $G_m \cap E(m+1) = \emptyset$.

At first, we take the minimal $n(1)$ such that $G_{n(1)} \cap E(1) \neq \emptyset$ and take $g_1 \in G_{n(1)} \cap E(1)$.

Next take the minimal $n(2)$ such that $G_{n(2)} \cap E(n(1)+1) \neq \emptyset$ and $g_2 \in G_{n(2)} \cap E(n(1)+1)$.

Repeating this process, take the minimal $n(k)$ such that $G_{n(k)} \cap E(n(k-1)+1) \neq \emptyset$, we obtain a sequence of pairs $\{(n(k), g_k)\}_{k\geq 1}$, where $g_k \in G_{n(k)} \cap E(n(k-1)+1)$. Since G_n are monotone increasing,

$$\forall m < k, \quad g_m \in G_{n(k)-1}, \qquad \text{but} \qquad g_k \in E\big(n(k-1)+1\big) \subset F\big(n(k-1)+1\big).$$

The equation (4.7) claims $\forall m < k, T_{g_m} v \perp T_{g_k} v$; that is, all the elements of the sequence $\{T_{g_k} v\}_{k\geq 1}$ are mutually orthogonal. But we say that the family of sets of vectors $\{T_F v\}_{F\in\mathcal{F}}$ must be a Cauchy filter base. This is a contradiction. □

5. Semiregular representation

In the previous paper [5, Sections 5.1–5.3, Theorem 5.10], for any given PC neighborhood E of $e \in G$, we construct explicitly a cyclic unitary representation of G corresponding to a positive definite function $\eta(g) \equiv \langle R_g f^\sim, f^\sim\rangle$ such that the support of η satisfies $[\eta] \subset E$. We write this unitary representation as $\mathfrak{R} \equiv \{\mathfrak{H}, R_g, f^\sim\}$, and review its construction.

First we select a sequence $\{(f_n^\sim, \mu_n)\}_{n\geq 1}$ of pairs of positive-valued continuous function $f_n^\sim$ and right Haar measure μ_n on G_n, inductively.

The vector $f^\sim$ is given as a continuous positive-valued function on G whose restriction on each G_n is in $L^2(\mu_n)$, and $f^\sim$ is the uniform limit and at the same time the limit in $L^2(\mu_n)$ of $f_n^\sim, n \geq 1$.

We show

$$\begin{aligned}\|f^{\sim}\| &\equiv \lim_{n\to\infty}\|f_n^{\sim}\|_{L^2(\mu_n)} \\ &= \lim_{n\to\infty}\Big(\int_{G_n}\big|f_n^{\sim}(g)\big|^2\,d\mu_n(g)\Big)^{1/2} = 1,\end{aligned} \tag{5.1}$$

$$\|R_g f^{\sim}\| \equiv \lim_{n\to\infty}\|R_g f_n^{\sim}\|_{L^2(\mu_n)} = \|f^{\sim}\| \quad (\forall g\in G), \tag{5.2}$$

$$\big(\|R_{g_1} f_n^{\sim}\|_{L^2(\mu_n)}\big)^2 = \int_{G_n}\big|R_{g_1} f_n^{\sim}(g)\big|^2\,d\mu_n(g) \quad (\forall g_1\in G_n). \tag{5.3}$$

Here R_g denotes the right translation by g on a function.

Consider the space $\boldsymbol{H}$ linearly spanned by $\{R_g f^{\sim}\}_{g\in G}$, that is, the space of functions $\{\sum_j c_j R_{g_j} f^{\sim}(g)\}$ on G. The norm $\|*\|$ gives a pre-Hilbert space structure on $\boldsymbol{H}$, and its completion $\mathfrak{H}$ is a Hilbert space, and $\mathfrak{R}\equiv\{\mathfrak{H}, R_g, f^{\sim}\}$ is a unitary representation of G such that $[\eta]\subset E$ for $\eta(g)\equiv\langle R_g f^{\sim}, f^{\sim}\rangle$. Thus the construction is reviewed.

DEFINITION 5.1

We call the representation $\mathfrak{R}\equiv\{\mathfrak{H}, R_g, f^{\sim}\}$, a *semiregular representation.*

However, in the following, we will be forced to treat other unitary representations.

Take a cyclic unitary representation $D\equiv\{\mathcal{H}^D, T_g^D, v^D\}$ of G, and take a cyclic part of the tensor product

$$D^{\sim}\equiv(D\otimes\mathfrak{R}) = \big\{(\mathcal{H}^D\otimes\mathfrak{H}), T_g^D\otimes R_g, \boldsymbol{f}^{\sim}\equiv v^D\otimes f^{\sim}\big\},$$

where $(D\otimes\mathfrak{R})$ means the subrepresentation of $D\otimes\mathfrak{R}$ on the subspace $(\mathcal{H}^D\otimes\mathfrak{H})$ of $\mathcal{H}^D\otimes\mathfrak{H}$ spanned by $\boldsymbol{f}^{\sim}$. As is easily shown, an element of $\mathcal{H}^D\otimes\boldsymbol{H}$ is considered as a vector-valued function $\boldsymbol{f}(g)\equiv\sum_j c_j R_{g_j} f^{\sim}(g) v_j$ $(v_j\in\mathcal{H}^D)$ on G, and for $\boldsymbol{f},\boldsymbol{k}\in\mathcal{H}^D\otimes\boldsymbol{H}$,

$$\|\boldsymbol{f}\|^2 = \lim_{n\to\infty}\int_{G_n}\big\|\boldsymbol{f}(g)\big\|^2_{\mathcal{H}^D}\,d\mu_n(g), \tag{5.4}$$

$$\langle\boldsymbol{f},\boldsymbol{k}\rangle = \lim_{n\to\infty}\int_{G_n}\big\langle\boldsymbol{f}(g),\boldsymbol{k}(g)\big\rangle_{\mathcal{H}^D}\,d\mu_n(g). \tag{5.5}$$

Then the representation $D^{\sim}$ belongs to the positive definite function

$$\begin{aligned}\langle T_g^{D^{\sim}}\boldsymbol{f}^{\sim},\boldsymbol{f}^{\sim}\rangle &= \big\langle(T_g^D\otimes R_g)(v^D\otimes f^{\sim}),(v^D\otimes f^{\sim})\big\rangle \\ &= \langle T_g^D v^D, v^D\rangle\cdot\langle R_g f^{\sim}, f^{\sim}\rangle.\end{aligned} \tag{5.6}$$

As a product of two continuous functions, $\langle T_g^{D^{\sim}}\boldsymbol{f}^{\sim},\boldsymbol{f}^{\sim}\rangle$ is continuous. So the "direct product" representation $D^{\sim}$ is also continuous.

Elements of $\mathcal{H}^D\otimes\mathfrak{H}$ are considered to be vector-valued functions on G.

We consider an operator $(T_g^0\boldsymbol{f})(*)\equiv T_g^D\boldsymbol{f}(*g)$ on this space; then

$$D^{(0)}\equiv\{\mathcal{H}^D\otimes\mathfrak{H}, T_g^0\}$$

gives a unitary representation of G. As the restriction to the subspace $(\mathcal{H}^D \otimes \mathfrak{H})$ of $\mathcal{H}^D \otimes \mathfrak{H}$, $D^{\sim}$ is a subrepresentation of $D^{(0)}$.

The vector $\boldsymbol{f}^{\sim} = v^D \otimes f^{\sim}$ is represented as $f^{\sim}(g)v^D$, and $T^0_g(v^D \otimes f^{\sim}) = f^{\sim}(*g)(T^D_g v^D)$.

Consider an operator

$$W : \boldsymbol{f}(g) \to T^D_g \boldsymbol{f}(g) \tag{5.7}$$

on $\mathcal{H}^D \otimes \mathfrak{H}$, that is, for $\boldsymbol{f}^{\sim} \equiv v^D \otimes f^{\sim}, W\boldsymbol{f}^{\sim}(g) = T^D_g v^D \otimes f^{\sim}(g)$, and

$$\begin{aligned} \|W\boldsymbol{f}^{\sim}\|_{\mathcal{H}^D\otimes\mathfrak{H}} &= \left\|T^D_* \boldsymbol{f}^{\sim}(*)\right\|_{\mathcal{H}^D\otimes\mathfrak{H}} \\ &= \lim_{n\to\infty}\Big(\int_{G_n} \left\|f^{\sim}_n(g)T^D_g v^D\right\|^2 d\mu(g)\Big)^{1/2} \\ &= \lim_{n\to\infty}\Big(\int_{G_n} \left|f^{\sim}_n(g)\right|^2 \|v^D\|^2 \, d\mu(g)\Big)^{1/2} \\ &= \lim_{n\to\infty}\Big(\int_{G_n} \left\|f^{\sim}_n(g)v^D\right\|^2 d\mu(g)\Big)^{1/2} = \|\boldsymbol{f}^{\sim}\|_{\mathcal{H}^D\otimes\mathfrak{H}}. \end{aligned} \tag{5.8}$$

Moreover, $T^D_{g^{-1}} = (T^D_g)^{-1}$, so W gives a unitary operator. Therefore $D^1 \equiv \{\mathcal{H}^D \otimes \mathfrak{H}, WT^{D^0}_g W^{-1}\}$ is a unitary representation of G and is equivalent to D^0. The relation

$$\begin{aligned} WT^{D^0}_{g_0} W^{-1}\big(T^D_*(v^D) \otimes f^{\sim}(*)\big) &= W\big(T^{D^0}_{g_0} v^D f^{\sim}(*g_0)\big) \\ &= W\big((T^D_{*g_0} v^D) f(*g_0)\big) = \boldsymbol{f}(*g_0) \end{aligned} \tag{5.9}$$

shows that the operator $T^{D^1}_g \equiv WT^{D^0}_g W^{-1}$ is the right translation operator by g in this representation $D^1 \equiv \{\mathcal{H}^D \otimes \mathfrak{H}, T^1_g\}$.

We take E as a PC set in G. By Corollary 3.5.1, $T^D_E v^D$ is contained in a closed separable subspace $\mathcal{H}^D_0$ in $\mathcal{H}^D$. We fix a CONS$\{v_j\}$ in $\mathcal{H}^D_0$ such as $v_1 = v^D$.

Now expand with respect to this CONS$\{v_j\}$, $\boldsymbol{f}$ in $\mathcal{H}^D$ as

$$\boldsymbol{f}(*) = \sum_{j\geq 1} \big\langle \boldsymbol{f}(*), v_j \big\rangle v_j; \tag{5.10}$$

then,

$$(T^1_g \boldsymbol{f})(*) = \sum_{j\geq 1} \big\langle \boldsymbol{f}(*g), v_j \big\rangle v_j \quad (g \in E). \tag{5.11}$$

This means that for each j the space $\boldsymbol{H}_j \equiv \{\langle \boldsymbol{f}(*), v_j\rangle v_j\}_{\boldsymbol{f}\in(\mathcal{H}^D\otimes\mathfrak{H})}$ is an invariant subspace in $\mathcal{H}^D$ for any f such that $[f] \subset E$.

We return to our $D^{\sim}$. According to the above arguments, $W(f^{\sim}(g)v^D) = f^{\sim}(g)T^D_g v^D$ for any $g \in G$, and $f^{\sim}(g) = 0$ if $g \notin E$, so the components in (5.10) are

$$\big\langle \boldsymbol{f}(*), v_j \big\rangle = f^{\sim}(*)\langle T^D_* v^D, v_j\rangle. \tag{5.12}$$

Especially in the case $j = 1$, we have $\langle \boldsymbol{f}(*), v^D\rangle = f^{\sim}(*)\langle T^D_* v^D, v^D\rangle$.

The subrepresentation corresponding to this component is realized on a function space $\mathfrak{H}_D$ on G, spanned by $\{R_g(f^{\sim}(*)\langle T_*^D v^D, v^D\rangle)\}_{g\in G}$, and the operators of representation are the right translation R_g on this function space.

DEFINITION 5.2

We call the representation

$$D^{\sim}(D) \equiv \{(\mathfrak{H}_D), R_g, f^{\sim}(*)\langle T_*^D v^D, v^D\rangle\} \tag{5.13}$$

of G a *generalized semiregular representation.*

6. Birepresentation of G

Now we remark on some elementary properties of birepresentations.

From the condition (B-4), for any birepresentation $\boldsymbol{U} \equiv \{U^D\}$,

$$U^{\overline{D}} = \overline{(U^D)}. \tag{6.1}$$

LEMMA 6.1

For $D^0 \equiv D \oplus \overline{D}$, $\langle U^{D^0}(u \oplus \overline{u}), v \oplus \overline{v}\rangle$ is real valued.

Proof

We have

$$\begin{aligned}\big\langle U^{D^0}(u \oplus \overline{u}), v \oplus \overline{v}\big\rangle &= \langle U^D u, v\rangle + \langle U^{\overline{D}}\overline{u}, \overline{v}\rangle \\ &= \langle U^D u, v\rangle + \big\langle \overline{(U^D u)}, \overline{v}\big\rangle = \langle U^D u, v\rangle + \overline{\langle U^D u, v\rangle} \in \boldsymbol{R}. \quad \square\end{aligned}$$

COROLLARY 6.1.1

Put $D_p \equiv I \oplus D \oplus \overline{D}$. Take vectors $w_0 \in \mathcal{H}^I, w \in \mathcal{H}^D$ such that $2^{1/2}\|w_0\| = 2\|w\| = 1$, and put $v_p \equiv w_0 \oplus w \oplus \overline{w}$. Then

$$\langle U^{D_p} v_p, v_p\rangle = \big\langle U^{D_p}(w_0 \oplus w \oplus \overline{w}), w_0 \oplus w \oplus \overline{w}\big\rangle \geq 0. \tag{6.2}$$

Proof

The proof of this corollary is completely similar to one of Corollary 1.2.1. We substitute in (1.3), U^{D_p} as $T_g^{D_p}$, U^D as T_g^D and w as v. And get

$$\langle U^{D_p} v_p, v_p\rangle = 2^{-1} + 2^{-1}\Re\langle U^D w, w\rangle.$$

But

$$\big|\langle U^D w, w\rangle\big| \leq \|w\|^2 = 2^{-2}.$$

So $-2^{-1} \leq 2\Re\langle U^D w, w\rangle \leq 2^{-1}$, whence $\langle U^{D_p} w_p, w_p\rangle \geq 0$. $\square$

COROLLARY 6.1.2

As in the case of Corollary 6.1.1, for $D_p \equiv I \oplus D \oplus \overline{D}$,

$$\forall g \in G, \quad \langle T_g^{D_p} U^{D_p} v_p, v_p\rangle \geq 0 \quad (v_p = w_0 \oplus w \oplus \overline{w}). \tag{6.3}$$

Proof
For any birepresentation $\boldsymbol{U} \equiv \{U^D\}$ and $\boldsymbol{T}_g \equiv \{T_g^D\}, \boldsymbol{T}_g\boldsymbol{U} \equiv \{T_g^D U^D\}$ is also a birepresentation. So we can apply the result of Corollary 6.1.1. □

Now we consider a birepresentation for a closed-type inductive limit group G.

Let $D = \{\mathcal{H}^D, T_g^D, v^D\}$ be a cyclic unitary representation of G. Denote by $\eta^D(g) \equiv \langle T_g^D v^D, v^D\rangle$ the positive definite function to which D belongs, and put

$$K^D(g) \equiv \langle T_g^D U^D v^D, v^D\rangle.$$

LEMMA 6.2
We have

$$\sup_{g\in G}\bigl|K^D(g)\bigr| = \sup_{g\in G}\bigl|\eta^D(g)\bigr| = \eta^D(e) = \|v^D\|^2 = 1. \tag{6.4}$$

Proof
Since $\|v^D\| = 1$ and U^D, T_g^D are unitary, $|K^D(g)| \le 1$.

If there exists a $\delta > 0$ such that $a \equiv \sup_{g\in G}|K^D(g)| < 1-\delta$, then using the continuity of $\eta^D(g)$, we can select a neighborhood V of e in G in such a way that if $g \in V$, then $\Re(\eta^D(g)) > 1-\delta$.

By Section 5, there exists a semiregular representation

$$\mathfrak{R} \equiv \{\mathfrak{H}, R_g, f^\sim\} \quad \text{as } \bigl[\langle R_g f^\sim, f^\sim\rangle\bigr] \subset V.$$

On the tensor product $D^0 \equiv (D \otimes \mathfrak{R}) = \{(\mathcal{H}^D \otimes \mathfrak{H}), T_g^D \otimes R_g, v^D \otimes f^\sim\}$,

$$W\bigl(U^{D^0}(v^D \otimes f^\sim)\bigr) = U^{\mathfrak{R}} W(v^D \otimes f^\sim) = U^{\mathfrak{R}}\bigl(\langle T_*^D v^D, v^D\rangle f^\sim(*)\bigr), \tag{6.5}$$

where W is given in (5.7). On the other hand,

$$\begin{aligned} W\bigl(U^{D^0}(v^D \otimes f^\sim)\bigr) &= W(U^D v^D \otimes U^{\mathfrak{R}} f^\sim) \\ &= v^D \otimes \bigl(\langle T_*^D U^D v^D, v^D\rangle U^{\mathfrak{R}} f^\sim(*)\bigr). \end{aligned} \tag{6.6}$$

Take the norm of both sides; then

$$\begin{aligned} \bigl\|U^{\mathfrak{R}}\bigl(\langle T_*^D v^D, v^D\rangle f^\sim(*)\bigr)\bigr\| &= \bigl\|\langle T_*^D v^D, v^D\rangle f^\sim(*)\bigr\| \\ &> (1-\delta)\|f^\sim\| = 1-\delta, \end{aligned} \tag{6.7}$$

$$\begin{aligned} \bigl\|\langle T_*^D U^D v^D, v^D\rangle U^{\mathfrak{R}} f(*)\bigr\| &= \bigl\|K(*) U^{\mathfrak{R}} f(*)\bigr\| \\ &< (1-\delta)\|U^{\mathfrak{R}} f^\sim\| = (1-\delta)\|f^\sim\| = 1-\delta. \end{aligned} \tag{6.8}$$

This is a contradiction. □

REMARK 6.1
By just analogous arguments, using the regular representation instead of the semiregular representation, we obtain the same assertion for any locally compact group.

7. Duality theorem for well-behaved group

DEFINITION 7.1

We call a topological group G a *well-behaved group* if

(W-1) G has an SSUR,
(W-2) G is b-complete,
(W-3) for any cyclic unitary representation $D \equiv \{\mathcal{H}^D, T_g^D, v^D\}$ ($\|v^D\| = 1$) and any birepresentation $\boldsymbol{U} \equiv \{U^D\}_D$, there holds

$$\sup_{g \in G} \left| \langle T_g^D U^D v^D, v^D \rangle \right| = 1.$$

The next lemma has been shown in the arguments in Lemma 6.2 and Remark 6.1.

LEMMA 7.1

Any locally compact groups and closed-type inductive limit groups are all well behaved.

Now we fix a birepresentation $\boldsymbol{U} \equiv \{U^D\}_D$ of a well-behaved group G.

As in Section 6, we will use the notation $K^D(g) \equiv \langle T_g^D U^D v^D, v^D \rangle$.

In the same section, we gave $D_p \equiv I \oplus D \oplus \overline{D}$, which contains a cyclic subrepresentation such that associated positive definite functions satisfy $K^{D_p}(g) \geq 0$.

By Corollary 6.1.2, we take actually the following as cyclic subrepresentation

$$(D_p) = \left\{ (\boldsymbol{C} \oplus \mathcal{H}^D \oplus \mathcal{H}^{\overline{D}}), (I \oplus T_g^D \oplus T_g^{\overline{D}}), v_p \equiv w_0 \oplus w \oplus \overline{w} \right\}. \tag{7.1}$$

LEMMA 7.2

If for any $g \in G, K^D(g) = \langle T_g^D U^D v^D, v^D \rangle \geq 0$, then

$$\inf_{g \in G} \left(1 - K^D(g) \right) = 0. \tag{7.2}$$

Proof

From the assumption, U^D is a unitary operator and $\|v^D\| = 1$.

Hence $1 \geq K^D(g) \geq 0$, and so $|K^D(g)| = K^D(g)$.

Then condition (W-3) gives $\sup_{g \in G} K^D(g) = 1$. This is the result. □

We denote by Ω_+ the set of all cyclic representations $(\mathcal{H}^D, T_g^D, v^D)$ satisfying

$$K^D(g) = \langle T_g^D U^D v^D, v^D \rangle \geq 0.$$

As was shown, Ω_+ contains cyclic representations of types as (D_p).

If $K^{D^1}(g), K^{D^2}(g) \geq 0$, then $K^{D^1}(g) \times K^{D^2}(g) \geq 0$. That is, the following holds.

LEMMA 7.3

We have $D^1, D^2 \in \Omega_+ \Rightarrow (D^1 \otimes D^2) \in \Omega_+$, and the corresponding function is

$$K^{D^1 \otimes D^2}(g) = K^{D^1}(g) \times K^{D^2}(g).$$

LEMMA 7.4

For a birepresentation $\boldsymbol{U} \equiv \{U^D\}$*, put*

$$F(D,\varepsilon) \equiv \{g \in G \mid 1 - K^D(g) < \varepsilon\}$$

for $\varepsilon > 0, D \in \Omega_+$*, and consider the family of sets*

$$\mathrm{Z} \equiv \{F(D,\varepsilon)\}_{D\in\Omega_+,\varepsilon>0}. \tag{7.3}$$

Then, with the order of set inclusion, Z *gives an* l*-Cauchy filter base on* G*.*

Proof

Condition (W-3) in Definition 7.1 shows $F(D,\varepsilon) \neq \emptyset$. Evidently

$$\varepsilon_1 > \varepsilon_2 \Longrightarrow F(D,\varepsilon_1) \supseteq F(D,\varepsilon_2). \tag{7.4}$$

Next, for given two $D^j \equiv \{\mathcal{H}^j, T_g^j, v^j\}$ $(j = 1,2)$, we consider $D^0 \equiv (D^1 \otimes D^2)$. By Lemma 7.3, and since $0 \le K^{D^1}(g), K^{D^2}(g) \le 1$, we have

$$K^{D^0}(g) = K^{D^1}(g) \times K^{D^2}(g) \le K^{D^1}(g), K^{D^2}(g).$$

So

$$1 - K^{D^0}(g) \ge 1 - K^{D^1}(g), 1 - K^{D^2}(g). \tag{7.5}$$

This means that

$$F(D^1,\varepsilon) \cap F(D^2,\varepsilon) \supseteq F(D^0,\varepsilon) \neq \phi. \tag{7.6}$$

Hence Z gives a filter base.

Next, the inequality $1 - K^D(g) < \varepsilon$ gives

$$\begin{aligned} \|T_g^D U^D v^D - v^D\|^2 &= \|T_g^D U^D v^D\|^2 + \|v^D\|^2 - 2K^D(g) \\ &= 2\big(1 - K^D(g)\big) \le 2\varepsilon. \end{aligned} \tag{7.7}$$

Therefore,

$$\|U^D v^D - T_{g^{-1}}^D v^D\| = \|T_g^D U^D v^D - v^D\| \le (2\varepsilon)^{1/2}. \tag{7.8}$$

So, for $g, h \in F(D,\varepsilon)$,

$$\begin{aligned} \|T_{hg^{-1}}^D v^D - v^D\| &= \|T_{g^{-1}}^D v^D - T_{h^{-1}}^D v^D\| \\ &\le \|T_{g^{-1}}^D v^D - U^D v^D\| + \|U^D v^D - T_{h^{-1}}^D v^D\| \le 2(2\varepsilon)^{1/2}. \end{aligned} \tag{7.9}$$

From separating condition (W-1) in Definition 7.1, for any neighborhood V of e in G, there exists a $D \in \Omega_+$ and $\delta > 0$ such that $\{g \in G \mid |\langle T_g^D v^D - v^D, v^D\rangle| < \delta\} \subset V$.

The equation

$$\begin{aligned} \|T_g^D v^D - v^D\|^2 &= 2\big(1 - \Re(\langle T_g^D v^D, v^D\rangle)\big) \\ &= 2\big(1 - \langle T_g^D v^D, v^D\rangle\big) \\ &= 2\langle v^D - T_g^D v^D, v^D\rangle \end{aligned}$$

means that, if we take $\zeta > 0$ as $\zeta^2 < 2\delta$, then

$$\|T_g^D v^D - v^D\| < \zeta \Rightarrow g \in V.$$

Consequently when $2(2\varepsilon)^{1/2} < \zeta$ and $\zeta^2 < 2\delta$, that is, when $4\varepsilon < \delta$,

$$g, h \in F(D,\varepsilon) \quad \text{deduces } \|T^D_{hg^{-1}} v^D - v^D\| < \zeta \Longrightarrow hg^{-1} \in V; \tag{7.10}$$

that is,

$$F(D,\varepsilon)F(D,\varepsilon)^{-1} \subset V. \tag{7.11}$$

This shows that Z gives a Cauchy filter base. □

LEMMA 7.5

Z is a b-Cauchy filter base.

Proof

It is enough to show that $\mathrm{Z}^{-1} \equiv \{F(D,\varepsilon)^{-1}\}_{D\in\Omega_+,\varepsilon>0}$ is Cauchy.

Here $F(D,\varepsilon) \equiv \{g \mid 1 - K^D(g) < \varepsilon\}$ for $\varepsilon > 0, D \in \Omega_+$.

The condition (7.2) does not change if we take g^{-1} instead of g. So we exchange g and h to g^{-1}, h^{-1} in the proof of Lemma 7.4; that is, $F(D,\varepsilon)$ becomes $F(D,\varepsilon)^{-1}$, and lastly Z becomes Z^{-1}.

This shows that Z^{-1} is Cauchy; therefore Z is b-Cauchy. □

LEMMA 7.6

There exists a unique element $g_{\boldsymbol{U}} \in G$, such that

$$\forall D \in \Omega_+, \quad U^D v^D = T^D_{g_{\boldsymbol{U}}} v^D. \tag{7.12}$$

Proof

By condition (W-2) in Definition 7.1, G is b-complete. The b-Cauchy filter base Z converges to a unique element $(g_{\boldsymbol{U}})^{-1}$ in G; that is, $\bigcap_{(D,\varepsilon)} \overline{F(D,\varepsilon)} = \{(g_{\boldsymbol{U}})^{-1}\}$. For any $D \in \Omega_+$,

$$\bigcap_{\varepsilon} \overline{F(D,\varepsilon)} = \bigcap_{\varepsilon} \overline{\{g \mid 1 - \langle T_g^D U^D v^D, v^D\rangle < \varepsilon\}} \ni (g_{\boldsymbol{U}})^{-1}.$$

This means that $1 = \langle T^D_{(g_{\boldsymbol{U}})^{-1}} U^D v^D, v^D\rangle$, that is,

$$T^D_{(g_{\boldsymbol{U}})^{-1}} U^D v^D = v^D \qquad \text{or} \qquad U^D v^D = T^D_{g_{\boldsymbol{U}}} v^D.$$ □

LEMMA 7.7

For any $D \in \Omega, U^D = T^D_{g_{\boldsymbol{U}}}$.

Proof

For any $v \in \mathcal{H}^D$ ($\|v\| = 1$), consider a cyclic representation $(D) \equiv \{(\mathcal{H}^D), T_g^D, v^D\}$. Then

$$(D_p) = \left\{(\boldsymbol{C} \oplus \mathcal{H}^D \oplus \mathcal{H}^{\overline{D}}), I \oplus T_g^D \oplus T_g^{\overline{D}}, u \equiv w_0 \oplus w \oplus \overline{w}\right\} \in \Omega_+. \tag{7.13}$$

By Lemma 7.5, $U^{D_p}u = T^{D_p}_{g_U}u$, that is,

$$Iw_0 \oplus U^D w \oplus U^{\overline{D}}\overline{w} = Iw_0 \oplus T^D_{g_U} w \oplus T^{\overline{D}}_{g_U}\overline{w}. \tag{7.14}$$

Thus $U^D w = T^D_{g_U} w$. Here we can select as $v \equiv 2w$ any normalized vector in $\mathcal{H}^D$.

This shows that $U^D = T^D_{g_U}$ on the whole space $\mathcal{H}^D$. □

In this way, we get the weak duality theorem for a well-behaved group.

THEOREM 2

For any well-behaved group G, a Tannaka-type weak duality theorem holds.

COROLLARY A

For locally compact groups, a Tannaka-type weak duality theorem holds.

COROLLARY B

For closed type inductive limit topological groups, Tannaka-type weak duality theorem holds.

8. Isobirepresentation

We remark that we can give some modification to the notion of birepresentation in the category of associative algebras over $\boldsymbol{C}$.

In the introduction, we defined "birepresentation" as an operator field $\boldsymbol{U} \equiv \{U^D\}_{D\in\Omega}$ on Ω satisfying the following:

(B-0) for each $D \in \Omega$, U^D is a unitary operator on the representation Hilbert spaces $\mathcal{H}^D$;

(B-1) $D_1 \sim_W D_2 \Longrightarrow WU^{D_1}W^{-1} = U^{D_2}$;

(B-2) $U^{D_1} \oplus U^{D_2} = U^{D_1\oplus D_2}$;

(B-3) $U^{D_1} \otimes U^{D_2} = U^{D_1\otimes D_2}$;

(B-4) $\overline{(U^D)} = U^{\overline{D}}$.

Now we set the condition (B-0′) below instead of (B-0) above.

(B-0′) For each $D \in \Omega$, U^D is an isometric operator on the representation Hilbert spaces $\mathcal{H}^D$.

And consider operator field $\boldsymbol{J} \equiv \{J^D\}_{D\in\Omega}$ on Ω satisfying (B-0′), (B-1), (B-2), (B-3), and (B-4).

We call these operator fields *isobirepresentations* of G.

Through this paper, the arguments in Sections 1, 6, and 7 used only isometric properties of U^D, not necessarily unitary one. This shows that we can obtain a somewhat wider duality theorem for well-behaved groups.

THEOREM 3

For well-behaved groups G, for any isobirepresentation $\boldsymbol{J} \equiv \{J^D\}_{D\in\Omega}$,

$$\exists_1 g \in G \quad \textit{such that } J^D = T_g^D \quad (\forall D \in \Omega).$$

A unitary operator is isometric. So any birepresentation is also an isobirepresentation. This shows that the result of Theorem 3 is just stronger than the one of Theorem 2.

Moreover, any complete group is also b-complete. That is, if we replace condition (2) in Definition 7.1 of "well-behaved group" with the condition

(W-2′) G is complete,

we get a more narrow category of groups.

We say a group G is *strongly well behaved* if it satisfies the following conditions:

(W-1) G has an SSUR;
(W-2′) G is complete;
(W-3) for any cyclic unitary representation $D \equiv \{\mathcal{H}^D, T_g^D, v^D\}$ ($\|v^D\| = 1$) and any birepresentation $\boldsymbol{U} \equiv \{U^D\}_D$, there holds

$$\sup_{g\in G} \left|\langle T_g^D U^D v^D, v^D\rangle\right| = 1.$$

Of course the weak duality theorem is valid for these groups.

REMARK

When G is a locally compact group, it is remarkable that to prove the weak duality theorem for G we do not need the condition (B-4) in the definitions of birepresentation and isobirepresentation (cf. [2]).

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