Estimates of the integral kernels arising from inverse problems for a three-dimensional heat equation in thermal imaging

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Abstract This paper studies precise estimates of integral kernels of some integral operators on the boundary ∂D of bounded and strictly convex domains with sufficiently regular boundary. Assume that an integral operator K_μ on ∂D has the integral kernel $K_\mu(x,y)$ with estimate $|K_\mu(x,y)| \le C\mu e^{-\mu|x-y|}$ $(x,y\in\partial D, \mu \gg 1)$. Then, from the Neumann series, the operator $K_\mu(I-K_\mu)^{-1}$ is also an integral operator. The problem is whether the integral kernel of $K_\mu(I-K_\mu)^{-1}$ can be estimated by the term $\mu e^{-\mu|x-y|}$ up to a constant or not. If the boundary ∂D is strictly convex, such types of estimates hold.

The most important point is that the obtained estimates have the same decaying behavior as $\mu\to\infty$ and the same exponential term as for the original kernel $K_\mu(x,y)$. These advantages are essentially needed to handle some inverse initial boundary value problems whose governing equation is the heat equation in three dimensions.

1. Introduction

Let D be a bounded domain of $\mathbf{R}^3$ with C^{2,α_0} $(0<\alpha_0\le 1)$ boundary and satisfy that $\mathbf{R}^3\setminus\overline{D}$ is connected. We denote by ν_x the unit outward normal vectors at $x\in\partial D$ on ∂D. Given $\delta_0>0$ we denote by $\mathbf{C}_{\delta_0}$ the set of all complex numbers λ such that $\operatorname{Re}\lambda\ge\delta_0|\operatorname{Im}\lambda|$. Throughout this paper, we always write $\mu=\operatorname{Re}\lambda$.

Let $K_\lambda(x,y)$ be a bounded measurable function on $\partial D\times\partial D$ with the parameter $\lambda\in\mathbf{C}_{\delta_0}$, continuous for all $x,y\in\partial D$, $x\ne y$ and satisfy

$$\big|K_\lambda(x,y)\big|\le C_0\mu e^{-\mu|x-y|},\quad x,y\in\partial D,\lambda\in\mathbf{C}_{\delta_0},\mu=\operatorname{Re}\lambda. \tag{1.1}$$

Let $\tilde{K}_\lambda(x,y)$ be a measurable function of $(x,y)\in\partial D\times\partial D$ with the parameter $\lambda\in\mathbf{C}_{\delta_0}$, continuous for all $x,y\in\partial D$, $x\ne y$ and satisfy

$$\big|\tilde{K}_\lambda(x,y)\big|\le\frac{C_1e^{-\mu|x-y|}}{|x-y|},\quad x,y\in\partial D,x\ne y,\lambda\in\mathbf{C}_{\delta_0},\mu=\operatorname{Re}\lambda. \tag{1.2}$$

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For these functions, we define the integral operators by

$$K_\lambda h(x) = \int_{\partial D} K_\lambda(x,y)h(y)\,dS_y \qquad \text{and} \qquad \tilde{K}_\lambda h(x) = \int_{\partial D} \tilde{K}_\lambda(x,y)h(y)\,dS_y.$$

We put $Y_\lambda = K_\lambda + \tilde{K}_\lambda$. In potential theory, the exterior problems for Laplacians with parameter can be reduced to integral equations on the boundary (cf. Section 7, and for detail, see Mizohata [3]). Note that the integral operators of these forms appear in such reduced integral equations.

It is well known that from (1.1) and (1.2) the operators K_λ and $\tilde{K}_\lambda$ are bounded on $C(\partial D)$ with bounds $\|K_\lambda\|_{B(C(\partial D))} + \|\tilde{K}_\lambda\|_{B(C(\partial D))} \le C(\mathrm{Re}\,\lambda)^{-1}$ ($\lambda \in \mathbf{C}_{\delta_0}$). Hence the Neumann series implies that for $\lambda \in \mathbf{C}_{\delta_0}$ with sufficiently large $\mu = \mathrm{Re}\,\lambda$, the operator $I - Y_\lambda$ is invertible and the inverse is given by $(I - Y_\lambda)^{-1} = \sum_{n=0}^{\infty}(Y_\lambda)^n$.

The purpose of this paper is to give estimates of the integral kernel for the operator $Y_\lambda(I - Y_\lambda)^{-1}$ with sufficiently large $\mu = \mathrm{Re}\,\lambda$. The main estimate is the following one.

THEOREM 1.1

Assume that ∂D is strictly convex. Then, there exist positive constants C and μ_0 depending only on C_0 in (1.1), C_1 in (1.2), and ∂D such that for all $\lambda \in \mathbf{C}_{\delta_0}$ and $\mu \ge \mu_0$, $Y_\lambda(I - Y_\lambda)^{-1}$ has the integral kernel $Y_\lambda^\infty(x,y)$ which is measurable for $(x,y) \in \partial D \times \partial D$, continuous for $x \ne y$, and has the estimate

$$\left|Y_\lambda^\infty(x,y)\right| \le C\Big(\mu + \frac{1}{|x-y|}\Big)e^{-\mu|x-y|}, \quad x,y \in \partial D. \tag{1.3}$$

The advantage of estimate (1.3) is in the form of the exponential term. Note that for any fixed $\delta > 0$, we can easily obtain the following estimate:

$$\left|Y_\lambda^\infty(x,y)\right| \le C_\delta\Big(\mu + \frac{1}{|x-y|}\Big)e^{-(1-\delta)\mu|x-y|}, \quad x,y \in \partial D. \tag{1.4}$$

To obtain (1.4) with $\delta > 0$, we do not need to assume strict convexity of ∂D. To take $\delta = 0$ in the above, that is, to obtain (1.3), however, we have to put the assumption that ∂D is strictly convex and to give more precise analysis on the boundary integrals. To compare differences between (1.3) and (1.4), we give a proof of (1.4) in the appendix.

Estimates of some integral kernels based on (1.3) are essentially needed to solve an inverse problem for a three-dimensional heat equation in thermal imaging. This is one of applications of Theorem 1.1 and the main motivation why we need to show Theorem 1.1. In Section 7, we introduce this application briefly and explain the reason why estimate (1.3) is needed to treat this inverse problem. The complete treatment about this inverse problem is given in another paper (cf. [1]).

Theorem 1.1 is also useful to obtain asymptotic behavior of the solution of the resolvent. Varadhan [4] considered the asymptotic behavior as $\lambda \longrightarrow \infty$ of the solution of the problem

$$\begin{cases}(\triangle-\lambda^2)v=0 & \text{in } \Omega,\\ v=1 & \text{on } \partial\Omega,\end{cases}$$

where Ω is a bounded domain. As is shown in Varadhan [4], this asymptotic behavior is very useful to establish the short time asymptotics of the *heat kernel*. When $v=0$ in some part of the boundary, for example, when Ω has a cavity (i.e., a domain D with $\overline{D}\subset\Omega$), the asymptotic behavior may change. Theorem 1.1 can be used to obtain the asymptotic behavior of the solution of the following problem:

$$\begin{cases}(\triangle-\lambda^2)w=0 & \text{in } \Omega\setminus\overline{D},\\ \frac{\partial w}{\partial\nu}+\rho_1 w=1 \quad \text{on } \partial\Omega, & \frac{\partial w}{\partial\nu}+\rho_2 w=0 \quad \text{on } \partial D,\end{cases}$$

where $\rho_1\in C(\partial\Omega)$ and $\rho_2\in C(\partial D)$, respectively. To keep this paper to an appropriate length, we only introduce this application here and do not give any detail. For the precise treatment, see the forthcoming article [2].

2. Properties of the function given by the length of broken paths

In this and following sections we always assume that ∂D is of class C^{2,α_0} with $0<\alpha_0\le 1$. We denote by $B(x,r)$ the open ball centered at x with radius r. The aim of this section is to study the behavior of the function $l_{(x,y)}(z)\equiv|x-z|+|z-y|$ with the independent variable $z\in\partial D$ and given $x,y\in\partial D$. These properties of $l_{(x,y)}(z)$ are essential to obtain Theorem 1.1.

We start with describing the following well-known facts.

LEMMA 2.1

(i) *There exists a positive constant C such that, for all $x,y\in\partial D$,*

$$|\nu_x-\nu_y|\le C|x-y|, \qquad \big|\nu_x\cdot(x-y)\big|\le C|x-y|^2.$$

(ii) *There exists $0<r_0$ such that, for all $x\in\partial D$, $\partial D\cap B(x,2r_0)$ can be represented as a graph of a function on the tangent plane of ∂D at x; that is, there exist an open neighborhood U_x of $(0,0)$ in $\mathbf{R}^2$ and a function $g=g_x\in C^{2,\alpha_0}(\mathbf{R}^2)$ with $g(0,0)=0$ and $\nabla g(0,0)=0$ such that the map*

$$U_x\ni\sigma=(\sigma_1,\sigma_2)\mapsto x+\sigma_1e_1+\sigma_2e_2-g(\sigma_1,\sigma_2)\nu_x\in\partial D\cap B(x,2r_0)$$

gives a system of local coordinates around x, where $\{e_1,e_2\}$ is an orthogonal basis for $T_x(\partial D)$. Moreover, the norm $\|g\|_{C^{2,\alpha_0}(\mathbf{R}^2)}$ has an upper bound independent of $x\in\partial D$.

In this paper we call this system of coordinates the standard system of local coordinates around x.

Let r_0 be the same constant as Lemma 2.1(ii). From Lemma 2.1(ii) we see that given $x\in\partial D$ and $y\in\partial D\cap B(x,2r_0)$ with $y\ne x$, the vectors $y-x$ and ν_x

are linearly independent. Thus one can choose $\{e_1, e_2\}$ in the standard system of local coordinates around x in such a way that $y-x$ is perpendicular to e_2 and $(y-x)\cdot e_1 > 0$. Therefore one can write

$$y = x + \sigma_1^0 e_1 - g(\sigma_1^0, 0)\nu_x$$

with $(\sigma_1^0)^2 + g(\sigma_1^0,0)^2 < (2r_0)^2$ and $\sigma_1^0 > 0$.

Let z be an arbitrary point in $\partial D \cap B(x, 2r_0)$; z has the expression

$$z = x + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma)\nu_x$$

with $\sigma_1^2 + \sigma_2^2 + g(\sigma)^2 < (2r_0)^2$. In the following proposition we denote by z' the point $x + \sigma_1 e_1 - g(\sigma)\nu_x$ which is the orthogonal projection of z onto the plane passing x and spanned by the vectors $y - x$ and ν_x.

PROPOSITION 2.1

Assume that ∂D is strictly convex.

(i) *For all $z \in \partial D \cap B(x, 2r_0)$ we have*

$$l_{(x,y)}(z) \geq |x-y| + \frac{1}{2}\frac{\sigma_2^2}{|z-x|}. \tag{2.1}$$

(ii) *One can choose r_0 in such a way that there exists $0 < r_1 < 2r_0$ such that for all $\sigma = (\sigma_1, \sigma_2)$ and $\sigma^0 = (\sigma_1^0, 0)$ with $\sigma_1 < 2\sigma_1^0/3$, $|\sigma| < r_1$, and $|\sigma^0| < r_1$,*

$$l_{(x,y)}(z) \geq |x-y| + \frac{c_0}{|z-x|}\big((\sigma_1^0)^2\sigma_1^2 + \sigma_2^2\big), \tag{2.2}$$

where c_0 is a positive constant depending only on ∂D.

REMARK 2.2

In this paper, we choose smaller $r_0 > 0$ if needed. This is always possibly since in Lemma 2.1(ii), r_0 can be arbitrarily small. Note also that $r_1 > 0$ in Proposition 2.1(ii) is determined by (2.8) in the proof of Proposition 2.1 for sufficiently small $r_0 > 0$.

Proof

First we give a proof of (2.1). Let $z \neq x$. Since

$$|y-z|^2 = \Big\{|y-x| - |z-x|\Big(\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)\Big\}^2 + |z-x|^2\Big\{1 - \Big(\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)^2\Big\},$$

it follows that

$$|y-z| \geq |y-x| - |z-x|\Big(\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big).$$

From this we obtain the estimate

$$l_{(x,y)}(z) \geq |y-x| + |z-x|\Big(1 - \frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big). \tag{2.3}$$

Since

$$2\Big(1-\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)\ge\Big(1+\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)\Big(1-\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)$$
$$=1-\Big(\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)^2=\Big|\frac{z-x}{|z-x|}\times\frac{y-x}{|y-x|}\Big|^2,$$

we have

$$|z-x|\Big(1-\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)\ge\frac{1}{2}\frac{|(z-x)\times(y-x)|^2}{|z-x||y-x|^2}. \tag{2.4}$$

From $z-z'=\sigma_2e_2$, it follows that $\{(z-z')\times(y-x)\}\cdot e_2=0$. On the other hand, $y-x=\sigma_1^0e_1-g(\sigma^0)\nu_x$ and $z'-x=\sigma_1e_1-g(\sigma)\nu_x$ imply that

$$(z'-x)\times(y-x)=-\big(\sigma_1g(\sigma^0)-\sigma_1^0g(\sigma)\big)e_1\times\nu_x=\alpha e_2$$

for some $\alpha\in\mathbf{R}$. Hence we have $\{(z-z')\times(y-x)\}\cdot\{(z'-x)\times(y-x)\}=0$. Moreover, since $z-z'$ and $y-x$ are perpendicular to each other, we have $|(z-z')\times(y-x)|=|z-z'||y-x|$. Thus this yields

$$\big|(z-x)\times(y-x)\big|^2=\big|(z-z')\times(y-x)\big|^2+\big|(z'-x)\times(y-x)\big|^2$$
$$+2\big\{(z-z')\times(y-x)\big\}\cdot\big\{(z'-x)\times(y-x)\big\}$$
$$=|z-z'|^2|y-x|^2+\big|(z'-x)\times(y-x)\big|^2\ge\sigma_2^2|y-x|^2.$$

Therefore (2.4) gives

$$|z-x|\Big(1-\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)\ge\frac{1}{2}\frac{\sigma_2^2}{|z-x|}. \tag{2.5}$$

Thus from this and (2.3) one gets (2.1).

Let $\sigma_1<2\sigma_1^0/3$. Here we prove another inequality,

$$|z-x|^2\Big(1-\frac{z-x}{|z-x|}\cdot\frac{y-x}{|y-x|}\Big)\ge c_1(\sigma_1^0)^2\sigma_1^2. \tag{2.6}$$

Note that (2.3), (2.5), and (2.6) imply (2.2) with $c_0=\min\{c_1/2,1/4\}$. Hence for finishing the proof of Proposition 2.1, it suffices to show (2.6).

By Lemma 2.1(ii) one can choose $R_1>0$ independent of $x\in\partial D$ in such a way that

$$\big|g(r\omega)\big|\le R_1r^2 \tag{2.7}$$

with arbitrary $r>0$ and unit vector ω in $\mathbf{R}^2$.

Write $(\sigma_1,\sigma_2)=r(\omega_1,\omega_2)$ with a unit vector $\omega=(\omega_1,\omega_2)$ in $\mathbf{R}^2$ and $r=|\sigma|$. Define

$$r_1=\frac{2r_0}{\sqrt{1+R_1^2(2r_0)^2}}<2r_0. \tag{2.8}$$

We know from (2.7) that if $r<r_1$, then it holds that $r^2+g(r\omega)^2<(2r_0)^2$ for all unit vectors ω. In this case $z=x+r\omega_1e_1+r\omega_2e_2-g(r\omega)\nu_x$ satisfies

$$r\le|z-x|\le r\sqrt{1+R_1^2r_1^2}\equiv rc_1\quad(0\le r\le r_1).$$

In what follows we will use this inequality without noticing.

Write

$$\tilde{g}(r\omega) = r^{-1}g(r\omega), \qquad \tilde{g}(\sigma_1^0, 0) = (\sigma_1^0)^{-1}g(\sigma_1^0, 0).$$

From (2.7) we have $\tilde{g}(r\omega) = O(r)$ and $\tilde{g}(\sigma_1^0, 0) = O(\sigma_1^0)$ uniformly in $x \in \partial D$ and $y \in \partial D \cap B(x, r_1)$. These yield

$$\begin{aligned}
\frac{z-x}{|z-x|} \cdot \frac{y-x}{|y-x|} &= \frac{\omega_1 + \tilde{g}(r\omega)\tilde{g}(\sigma_1^0, 0)}{\sqrt{1+\tilde{g}(r\omega)^2}\sqrt{1+\tilde{g}(\sigma_1^0, 0)^2}} \\
&= \omega_1\Big(1 - \frac{1}{2}\tilde{g}(r\omega)^2 + O(r^4)\Big)\Big(1 - \frac{1}{2}\tilde{g}(\sigma_1^0, 0)^2 + O\big((\sigma_1^0)^4\big)\Big) \\
&\quad + \tilde{g}(r\omega)\tilde{g}(\sigma_1^0, 0)\big(1 + O(r^2)\big)\big(1 + O\big((\sigma_1^0)^2\big)\big) \\
&= \omega_1 - \frac{1}{2}\omega_1\big(\tilde{g}(\sigma_1^0, 0)^2 + \tilde{g}(r\omega)^2\big) + \tilde{g}(r\omega)\tilde{g}(\sigma_1^0, 0) \\
&\quad + \sum_{j=0}^{4} O\big(r^{4-j}(\sigma_1^0)^j\big).
\end{aligned}$$

From this we obtain

$$\begin{aligned}
1 &- \frac{z-x}{|z-x|} \cdot \frac{y-x}{|y-x|} \\
&= (1-\omega_1)\Big(1 - \Big|\frac{\tilde{g}(\sigma_1^0, 0) + \tilde{g}(r\omega)}{2}\Big|^2\Big) + \Big|\frac{\tilde{g}(\sigma_1^0, 0) - \tilde{g}(r\omega)}{2}\Big|^2(1+\omega_1) \\
&\quad + \sum_{j=0}^{4} O\big(r^{4-j}(\sigma_1^0)^j\big) \qquad (2.9) \\
&= (1-\omega_1)\big(1 + O\big((\sigma_1^0)^2\big) + O(r^2)\big) + \Big|\frac{\tilde{g}(\sigma_1^0, 0) - \tilde{g}(r\omega)}{2}\Big|^2(1+\omega_1) \\
&\quad + \sum_{j=0}^{4} O\big(r^{4-j}(\sigma_1^0)^j\big).
\end{aligned}$$

Let $0 < \epsilon < 1$. Consider the case when $\omega_1 \geq 1 - \epsilon$. Since ∂D is C^{2,α_0} and ∂D is strictly convex, one has the expression

$$g(\sigma) = g_0(\sigma) + O\big(|\sigma|^{2+\alpha_0}\big),$$

where

$$g_0(\sigma) = a\sigma_1^2 + 2b\sigma_1\sigma_2 + c\sigma_2^2$$

with constants $a > 0$, $c > 0$, and $ac - b^2 > 0$. Note that a, $|b|$, c has a positive upper bound M_1 independent of x and r_0. Moreover, a has a positive lower bound M_2 independent of x and r_0.

Since $r(1-\epsilon) \leq r\omega_1 < 2\sigma_1^0/3$, we have $O(r) = O(\sigma_1^0)$. Then one can choose r_0 in such a way that

$$1 + O\big((\sigma_1^0)^2\big) + O(r^2) \geq 0 \quad (0 \leq r \leq r_1, \sigma_1^0 < r_1). \tag{2.10}$$

Further, using $|\omega_2| \le \sqrt{2\epsilon}$, $1-\epsilon \le \omega_1 \le 1$, and the assumption $r\omega_1 = \sigma_1 < 2\sigma_1^0/3$, we have

$$\begin{aligned}
\tilde{g}(\sigma_1^0, 0) - \tilde{g}(r\omega) &= a\sigma_1^0 - r(a\omega_1^2 + 2b\omega_1\omega_2 + c\omega_2^2) + O\big((\sigma_1^0)^{1+\alpha_0}\big) + O(r^{1+\alpha_0}) \\
&= \sigma_1^0\big(a + O\big((\sigma_1^0)^{\alpha_0}\big)\big) - ar\omega_1 \cdot \omega_1 - 2br\omega_1\omega_2 - cr\omega_1\frac{\omega_2^2}{\omega_1} \\
&\ge \sigma_1^0\big(a + O\big((\sigma_1^0)^{\alpha_0}\big)\big) - \frac{2}{3}a\sigma_1^0 - \frac{4}{3}|b|\sigma_1^0\sqrt{2\epsilon} - \frac{2}{3}c\sigma_1^0\frac{2\epsilon}{1-\epsilon} \\
&\ge \sigma_1^0\Big(\frac{M_2}{3} + O\big((\sigma_1^0)^{\alpha_0}\big)\Big) - \frac{4}{3}M_1\sigma_1^0\sqrt{2\epsilon} - \frac{2}{3}M_1\sigma_1^0\frac{2\epsilon}{1-\epsilon}.
\end{aligned}$$

Here we take a smaller r_0 in such a way that

$$\frac{M_2}{3} + O\big((\sigma_1^0)^{\alpha_0}\big) \ge \frac{M_2}{6} \quad (\sigma_1^0 < r_1). \tag{2.11}$$

Then we get

$$\tilde{g}(\sigma_1^0, 0) - \tilde{g}(r\omega) \ge \frac{M_2}{6}\sigma_1^0 - \frac{4}{3}M_1\sigma_1^0\sqrt{2\epsilon} - \frac{2}{3}M_1\sigma_1^0\frac{2\epsilon}{1-\epsilon}.$$

Therefore if one chooses a small ϵ in such a way that

$$\frac{M_2}{6} - \frac{4}{3}M_1\sqrt{2\epsilon} - \frac{2}{3}M_1\frac{2\epsilon}{1-\epsilon} > \frac{M_2}{24},$$

then one gets

$$\tilde{g}(\sigma_1^0, 0) - \tilde{g}(r\omega) \ge C\sigma_1^0 \quad (r < r_1, \sigma_1^0 < r_1, \sigma_1 < 2\sigma_1^0/3) \tag{2.12}$$

with $C = M_2/24$. Note that the choice of ϵ is independent of x and r_0. So one can choose r_0 satisfying (2.10) and (2.11) in such a way that

$$\Big(\frac{C}{2}\Big)^2(\sigma_1^0)^2 + \sum_{j=0}^{4} O\big(r^{4-j}(\sigma_1^0)^j\big) \ge \frac{1}{2}\Big(\frac{C}{2}\Big)^2(\sigma_1^0)^2. \tag{2.13}$$

Hereafter it is easy to see that, from (2.9), (2.10), (2.12), and (2.13) we obtain

$$1 - \frac{z-x}{|z-x|} \cdot \frac{y-x}{|y-x|} \ge c_2(\sigma_1^0)^2 \ge c_2(\sigma_1^0)^2\omega_1^2 \tag{2.14}$$

with $c_2 = (C/2)^2/2$.

Fix ϵ above. Thus ϵ is independent of r_0. Next consider the case when $\omega_1 < 1 - \epsilon$.

In this case, from (2.9) we obtain

$$1 - \frac{z-x}{|z-x|} \cdot \frac{y-x}{|y-x|} \ge \epsilon\big(1 - O(r_0^2)\big) + O(r_0^4). \tag{2.15}$$

Thus if one chooses smaller r_0, then one gets from (2.15)

$$1 - \frac{z-x}{|z-x|} \cdot \frac{y-x}{|y-x|} \ge \frac{\epsilon}{2} = \frac{\epsilon}{2r_0^2}r_0^2 \ge \frac{\epsilon}{2r_0^2}(\sigma_1^0)^2 \ge \frac{\epsilon}{2r_0^2}(\sigma_1^0)^2\omega_1^2. \tag{2.16}$$

Finally choosing $c_1 = \min\{c_2, \epsilon/(2r_0^2)\}$, from (2.14) and (2.16) we obtain (2.6). $\square$

PROPOSITION 2.2

Assume that ∂D is strictly convex. Given $r_0 > 0$ there exists a positive constant c_0 such that

(i) *for all $x, y, z \in \partial D$ with $|x-y| \geq r_0$, $|x-z| \geq r_0/2$, and $|y-z| \geq r_0/2$ we have*

$$l_{(x,y)}(z) \geq |x-y| + c_0;$$

(ii) *for all $x, y, z \in \partial D$ with $|x-y| \geq r_0$, $|x-z| \leq r_0/2$ we have*

$$l_{(x,y)}(z) \geq |x-y| + c_0|z-x|.$$

Proof

First we give a proof of (i). Since ∂D is strictly convex, if $|x-z|+|z-y| = |x-y|$, then $z = x$ or $z = y$. Thus $|x-z|+|z-y|-|x-y| > 0$ for all $x, y, z \in \partial D$ with $|x-y| \geq r_0$, $|x-z| \geq r_0/2$, and $|y-z| \geq r_0/2$. Therefore (i) is a consequence of the compactness of the set $\{(x,y,z) \in \partial D^3 \mid |x-y| \geq r_0, |x-z| \geq r_0/2, |y-z| \geq r_0/2\}$ and the continuity of the function $(x,y,z) \longmapsto |x-z|+|z-y|-|x-y|$.

Second we give a proof of (ii). From (2.3) we see that it suffices to prove

$$\sup_{(x,y,z)\in X} \frac{z-x}{|z-x|} \cdot \frac{y-x}{|y-x|} < 1, \tag{2.17}$$

where $X = \{(x,y,z) \in \partial D^3 \mid |x-y| \geq r_0, 0 < |x-z| \leq r_0/2\}$.

Assume that (2.17) is not true. Then there exist sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ with $(x_n, y_n, z_n) \in X$ such that, as $n \longrightarrow \infty$,

$$\frac{z_n - x_n}{|z_n - x_n|} \cdot \frac{y_n - x_n}{|y_n - x_n|} \longrightarrow 1. \tag{2.18}$$

Moreover, one may assume that $x_n \longrightarrow x_0$, $y_n \longrightarrow y_0$, $z_n \longrightarrow z_0$, $(z_n - x_n)/|z_n - x_n| \longrightarrow \vartheta$ for a $(x_0, y_0, z_0) \in \partial D^3$ with $|x_0 - y_0| \geq r_0, |x_0 - z_0| \leq r_0/2$, and a unit vector ϑ.

Since $x_0 \neq y_0$, from (2.18) we have

$$\vartheta \cdot \frac{y_0 - x_0}{|y_0 - x_0|} = 1,$$

and thus this yields $\vartheta = (y_0 - x_0)/|y_0 - x_0|$. Since ∂D is strictly convex, we have $\vartheta \cdot \nu_{x_0} < 0$.

Consider the case when $z_0 = x_0$. From Lemma 2.1(i) one gets $\vartheta \cdot \nu_{x_0} = 0$, a contradiction.

Next consider the case when $z_0 \neq x_0$. Then we have $(z_0 - x_0)/|z_0 - x_0| = (y_0 - x_0)/|y_0 - x_0|$. From this one concludes that $z_0 \in \partial D$ is located on the line determined by x_0 and y_0. Since ∂D is strictly convex, we have $z_0 = y_0$. However, we have $|y_0 - z_0| \geq |x_0 - y_0| - |x_0 - z_0| \geq r_0/2$, a contradiction. $\square$

3. Basic estimates for integrals on the boundary

In this section, we prepare basic estimates for the boundary integrals appearing in this paper. We start with describing the following lemma.

LEMMA 3.1

Let r_0 be same as that of Lemma 2.1(ii). There exists a positive constant C depending only on ∂D such that

(i) *for all $x \in \partial D$, $0 < \rho_0' \le r_0$, $\mu > 0$, $0 \le k < 2$,*

$$\int_{B(x,\rho_0')\cap\partial D} \frac{e^{-\mu|x-z|}}{|x-z|^k}\, dS_z \le \frac{C}{2-k} \min\{\mu^{-2+k}, (\rho_0')^{2-k}\};$$

(ii) *for all $x \in \partial D$, $\mu > 0$, $0 \le k < 2$,*

$$\int_{\partial D} \frac{e^{-\mu|x-z|}}{|x-z|^k}\, dS_z \le \frac{C}{2-k}\mu^{-(2-k)}\Big(1 + \frac{\mu^{2-k}e^{-\mu r_0}}{r_0^k}\Big);$$

(iii) *there exists a constant $0 < c < 1$ such that for all $x \in \partial D$, $0 < \rho_0' \le r_0$, $\mu > 0$,*

$$\int_{(B(x,r_0)\setminus B(x,\rho_0'))\cap\partial D} \frac{e^{-\mu|x-z|}}{|x-z|^2}\, dS_z \le C \min\Big\{\log\frac{r_0}{c\rho_0'}, \frac{1}{c\rho_0'\mu}\Big\};$$

(iv) *for all $x \in \partial D$, $0 < \rho_0' \le r_0$, $\mu > 0$, $0 < \gamma < 1$,*

$$\int_{B(x,\rho_0')\cap\partial D} \frac{e^{-\mu|x-z|}}{|x-z|} \log\frac{r_0}{|x-z|}\, dS_z \le \min\big\{C\mu^{-1}\big(1 + \max\{0, \log\mu\}\big), C_\gamma \mu^{-1+\gamma}\big\}.$$

Lemma 3.1(i) and (ii) have already been given as [1, Lemma 6.1(i), (ii)]. Since the estimates in Lemma 3.1 frequently appear in this and the following sections, we present here all the proofs of (i)–(iv).

Proof of Lemma 3.1

Let $z = s(\sigma)$ be the standard system of local coordinates around x with $|\sigma|^2 + g(\sigma)^2 < (2r_0)^2$. We have

$$\int_{B(x,\rho_0')\cap\partial D} \frac{e^{-\mu|x-z|}}{|x-z|^k}\, dS_z = \int_{|\sigma|^2+g(\sigma)^2<(\rho_0')^2} \frac{e^{-\mu\sqrt{|\sigma|^2+g(\sigma)^2}}}{(|\sigma|^2+g(\sigma)^2)^{k/2}} \sqrt{1+\big|\nabla g(\sigma)\big|^2}\, d\sigma$$
$$\le C\int_0^{\rho_0'}\int_0^{2\pi} \frac{e^{-\mu r}}{r^k} r\, dr\, d\theta \le 2\pi C \int_0^{\rho_0'} e^{-\mu r} r^{1-k}\, dr.$$

Since

$$\int_0^{\rho_0'} e^{-\mu r} r^{1-k}\, dr \le \int_0^{\rho_0'} r^{1-k}\, dr = \frac{(\rho_0')^{2-k}}{2-k}$$

and

$$\int_0^{\rho_0'} e^{-\mu r} r^{1-k}\, dr = \mu^{k-2}\int_0^{\mu\rho_0'} e^{-r} r^{1-k}\, dr \le \mu^{k-2}\int_0^{\infty} e^{-r} r^{1-k}\, dr \le \frac{3}{2-k}\mu^{k-2},$$

we get (i). To verify (ii) we compute

$$\int_{\partial D\setminus B(x,r_0)} \frac{e^{-\mu|x-z|}}{|x-z|^k}\, dS_z \le e^{-r_0\mu}\int_{\partial D} \frac{1}{r_0^k}\, dS_z \le \frac{C}{r_0^k} e^{-r_0\mu}.$$

From this and (i) for $\rho_0' = r_0$ we obtain (ii).

From (2.7), we have $|g(\sigma)| \le R_1|\sigma|^2$ for $|\sigma|^2 + g(\sigma)^2 < (2r_0)^2$. Since $\rho_0' \le \sqrt{\sigma^2 + |g(\sigma)|^2} \le r_0$ implies $c\rho_0' \le |\sigma| \le r_0$ with $c = 1/\sqrt{1+R_1^2 r_0^2}$ (<1) independent of $x \in \partial D$, we get

$$\int_{(B(x,r_0)\setminus B(x,\rho_0'))\cap\partial D} \frac{e^{-\mu|x-z|}}{|x-z|^2}\, dS_z \le C\int_{c\rho_0'}^{r_0} \frac{e^{-\mu r}}{r}\, dr.$$

Note that

$$\int_{c\rho_0'}^{r_0} \frac{e^{-\mu r}}{r}\, dr \le \min\Big\{\int_{c\rho_0'}^{r_0} r^{-1}\, dr, \frac{1}{c\rho_0'}\int_{c\rho_0'}^{r_0} e^{-\mu r}\, dr\Big\} \le \min\Big\{\log\frac{r_0}{c\rho_0'}, \frac{1}{c\rho_0'\mu}\Big\}.$$

Thus we obtain (iii).

Finally it follows that

$$\int_{B(x,\rho_0')\cap\partial D} \frac{e^{-\mu|x-z|}}{|x-z|}\log\frac{r_0}{|x-z|}\, dS_z \le C\int_0^{\rho_0'} e^{-\mu r}\log\frac{r_0}{r}\, dr.$$

Since

$$0 \le \log\Big(\frac{r_0\mu}{r}\Big) \le \log r_0 + \max\{0, \log\mu\} + |\log r| \quad (0 < r \le r_0\mu),$$

one gets

$$\int_0^{\rho_0'} e^{-\mu r}\log\frac{r_0}{r}\, dr = \mu^{-1}\int_0^{\rho_0'\mu} e^{-r}\log\Big(\frac{r_0\mu}{r}\Big)\, dr \le C\mu^{-1}\big(1 + \max\{0, \log\mu\}\big).$$

Noting also that $\sup_{X\ge 1} X^{-\gamma}\log X < \infty$ for each fixed $\gamma > 0$, we obtain (iv) since

$$\int_0^{\rho_0'} e^{-\mu r}\log\frac{r_0}{r}\, dr \le C_\gamma\int_0^{\rho_0'} e^{-\mu r}\Big(\frac{r_0}{r}\Big)^{\gamma}\, dr \le C_\gamma\mu^{-1+\gamma}. \qquad \square$$

We choose $x, y \in \partial D$ arbitrary and set $\rho_0 = |x-y|$. Given $\epsilon > 0$ set $S_\epsilon^-(y) = \partial D \setminus B(y,\epsilon)$, $S_\epsilon(y) = \partial D \cap B(y,\epsilon)$ and also $S_\epsilon^-(x) = \partial D\setminus B(x,\epsilon)$, $S_\epsilon(x) = \partial D \cap B(x,\epsilon)$. We consider the following integral:

$$\int_{\partial D} e^{-\mu(|x-z|+|z-y|)}\, dS_z \le I_{-,1}(x,y) + I_{-,2}(x,y) + I_+(x,y), \tag{3.1}$$

where

$$I_{-,1}(x,y) = \int_{S_{\rho_0}^-(x)} e^{-\mu(|x-z|+|z-y|)}\, dS_z,$$

$$I_{-,2}(x,y) = \int_{S_{\rho_0}^-(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z,$$

$$I_+(x,y) = \int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z.$$

By Lemma 3.1(ii) for $k = 0$ and $\mu \ge 1$ we have

$$\int_{\partial D} e^{-\mu|y-z|}\, dS_z + \int_{\partial D} e^{-\mu|x-z|}\, dS_z \le C\mu^{-2}.$$

This gives

$$I_{-,1}(x,y) \le e^{-\mu\rho_0} \int_{S^-_{\rho_0}(x)} e^{-\mu|z-y|}\, dS_z \le C\mu^{-2}e^{-\mu\rho_0}; \tag{3.2}$$

similarly

$$I_{-,2}(x,y) \le C\mu^{-2}e^{-\mu\rho_0}. \tag{3.3}$$

Next we estimate $I_+(x,y)$, which is essential to obtain Theorem 1.1.

PROPOSITION 3.1

Assume that ∂D is strictly convex. Then, there exist positive constants $r_1 > 0$ and $C > 0$ depending only on ∂D such that, for $\mu \ge 1$, the following estimates are valid.

(i) *For $\rho_0 \le r_1$,*

$$I_+(x,y) \le Ce^{-\mu\rho_0} \min\Big\{\frac{\rho_0^{3/2}}{\sqrt{\mu}}, \frac{1}{\mu^2\rho_0^3}\Big\}. \tag{3.4}$$

(ii) *For $\rho_0 \ge r_1$,*

$$I_+(x,y) \le Ce^{-\mu\rho_0}\mu^{-2}. \tag{3.5}$$

REMARK 3.1

From (3.1)–(3.5), for all $x, y \in \partial D$ and $\mu \ge 1$ the following estimate is valid:

$$\int_{\partial D} e^{-\mu(|x-z|+|z-y|)}\, dS_z \le C\mu^{-2}e^{-\mu\rho_0}\Big(1 + \min\Big\{(\mu\rho_0)^{3/2}, \frac{1}{\rho_0^3}\Big\}\Big). \tag{3.6}$$

Hence noting that $1 + \min\{a,b\} \le 2\max\{1,a\}$, from (3.6) we obtain

$$\int_{\partial D} e^{-\mu(|x-z|+|z-y|)}\, dS_z \le C\mu^{-2}e^{-\mu\rho_0} \quad (\rho_0 \ge r_1/2). \tag{3.7}$$

Proof of Proposition 3.1

For $z, y \in \partial D \cap B(x, 2r_0)$ we use the same local coordinates σ, σ^0 with $\sigma_1^0 > 0$, respectively, around x as used in Proposition 2.1 and denote by $\tilde\sigma$, $\tilde\sigma^0$ with $\tilde\sigma_1^0 > 0$, respectively, the local coordinates for $z, x \in \partial D \cap B(y, 2r_0)$, respectively, obtained by changing the roles of x and y.

Set

$$\begin{aligned}
B'_x(0, 2\sigma_1^0/3) &= \big\{\sigma \in \mathbf{R}^2 \;\big|\; |\sigma| < \rho_0, \sigma_1 < 2\sigma_1^0/3\big\},\\
B'_y(0, 2\tilde\sigma_1^0/3) &= \big\{\tilde\sigma \in \mathbf{R}^2 \;\big|\; |\tilde\sigma| < \rho_0, \tilde\sigma_1 < 2\tilde\sigma_1^0/3\big\},\\
B_x(0, 2\sigma_1^0/3) &= \big\{z \in \partial D \;\big|\; \sigma \in B'_x(0, 2\sigma_1^0/3)\big\},\\
B_y(0, 2\tilde\sigma_1^0/3) &= \big\{z \in \partial D \;\big|\; \tilde\sigma \in B'_y(0, 2\tilde\sigma_1^0/3)\big\}.
\end{aligned}$$

Let r_1 be given by (2.8) in the proof of Proposition 2.1(ii). Here we claim that if r_0 is sufficiently small and $\rho_0 \le r_1$, then

$$S_{\rho_0}(x) \cap S_{\rho_0}(y) \subset B_x(0, 2\sigma_1^0/3) \cup B_y(0, 2\tilde\sigma_1^0/3). \tag{3.8}$$

This is proved as follows. Let $z \in S_{\rho_0}(x) \cap S_{\rho_0}(y)$. Then we have

(i) $(z-x)\cdot(y-x)/|y-x| \le \rho_0/2$ or
(ii) $-(z-y)\cdot(y-x)/|y-x| \le \rho_0/2$.

Consider the case (i). Since $(z-x)\cdot(y-x) = \sigma_1\sigma_1^0 + g(\sigma)g(\sigma_1^0,0)$ and the strict convexity of ∂D yields $g(\sigma)g(\sigma_1^0,0) \ge 0$, we get $\sigma_1\sigma_1^0 \le \rho_0^2/2$. It follows from this, (2.7), and (2.8) that

$$\sigma_1 \le \frac{\rho_0^2}{2\sigma_1^0} = \frac{\sigma_1^0}{2}\Big(\frac{\rho_0}{\sigma_1^0}\Big)^2 = \frac{\sigma_1^0}{2}\Big\{1+\Big(\frac{g(\sigma_1^0,0)}{\sigma_1^0}\Big)^2\Big\} \le \frac{\sigma_1^0}{2}\big(1+R_1^2(\sigma_1^0)^2\big),$$

which yields $\sigma_1 \le \sigma_1^0(1+R_1^2(2r_0)^2)/2$. Now choose r_0 in such a way that $R_1^2(2r_0)^2 < 1/3$. Then we get $z \in B_x(0, 2\sigma_1^0/3)$. Similarly, for case (ii) we get $z \in B_y(0,\tilde{\sigma}_1^0/3)$. This completes the proof of (3.8).

Now we fix r_0 as above. First we consider the case when $\rho_0 \le r_1$, where $r_1 > 0$ is given in Proposition 2.1(ii). Property (3.8) gives

$$I_+(x,y) \le I_{+,1}(x,y) + I_{+,2}(x,y), \tag{3.9}$$

where

$$I_{+,1}(x,y) = \int_{B_x(0,2\sigma_1^0/3)} e^{-\mu(|x-z|+|z-y|)}\,dS_z,$$

$$I_{+,2}(x,y) = \int_{B_y(0,2\tilde{\sigma}_1^0/3)} e^{-\mu(|x-z|+|z-y|)}\,dS_z.$$

For the estimation of $I_{+,1}(x,y)$ we introduce the polar coordinates $\sigma_1 = r\cos\theta$, $\sigma_2 = r\sin\theta$, $r>0$, $|\theta| \le \pi$. From (2.7) we have $|g(\sigma)| \le R_1 r r_1$ and $|g(\sigma_1^0,0)| \le R_1|\sigma_1^0| r_1$. These yield

$$\frac{r}{|z-x|} = \frac{1}{\sqrt{1+(g(\sigma)/r)^2}} \ge \frac{1}{\sqrt{1+(R_1r_1)^2}}, \qquad \frac{|\sigma_1^0|}{\rho_0} \ge \frac{1}{\sqrt{1+(R_1r_1)^2}}.$$

Thus it follows from Proposition 2.1(ii) that, for all $\sigma \in B_x'(0,2\sigma_1^0/3)$,

$$|x-z|+|z-y| \ge \rho_0 + \frac{c_1}{r}(\rho_0^2\sigma_1^2 + \sigma_2^2),$$

where $c_1 > 0$ depends only on ∂D.

Noting that

$$\rho_0^2\sigma_1^2+\sigma_2^2 = r^2(\rho_0^2\cos^2\theta+\sin^2\theta) = r^2\big(\rho_0^2+(1-\rho_0^2)\sin^2\theta\big)$$

and

$$|\sin\theta| \ge \frac{2}{\pi} f(\theta), \quad |\theta| \le \pi$$

with

$$f(\theta) = \begin{cases} |\theta| & \text{if } |\theta| \le \pi/2, \\ |\pi-\theta| & \text{if } \pi/2 \le \theta \le \pi, \\ |\pi+\theta| & \text{if } -\pi \le \theta \le -\pi/2, \end{cases}$$

one gets

$$|x-z|+|z-y| \geq \rho_0 + c_2 r\big(\rho_0^2 + f(\theta)^2\big), \qquad \sigma = (r\cos\theta, r\sin\theta) \in B'_x(0, 2\sigma_1^0/3)$$

provided r_0 is sufficiently small if necessary. Thus we obtain

$$\begin{aligned} I_{+,1}(x,y) &\leq Ce^{-\mu\rho_0}\int_0^{\rho_0}\int_{-\pi}^{\pi} e^{-\mu c_2 r(\rho_0^2+f(\theta)^2)} r\,dr\,d\theta \\ &\leq Ce^{-\mu\rho_0}\int_0^{\rho_0} e^{-\mu c_2 r\rho_0^2}\Big(\int_{-\pi}^{\pi} e^{-\mu c_2 r f(\theta)^2}\,d\theta\Big) r\,dr \\ &= 4Ce^{-\mu\rho_0}\int_0^{\rho_0} e^{-\mu c_2 r\rho_0^2}\Big(\int_0^{\pi/2} e^{-\mu c_2 r\theta^2}\,d\theta\Big) r\,dr. \end{aligned} \tag{3.10}$$

Since

$$\int_0^{\pi/2} e^{-\mu c_2 r\theta^2}\,d\theta \leq \frac{1}{\sqrt{\mu c_2 r}}\int_0^{\infty} e^{-\theta^2}\,d\theta = \frac{1}{2}\sqrt{\frac{\pi}{\mu c_2 r}},$$

it follows from (3.10) that

$$\begin{aligned} I_{+,1}(x,y) &\leq 2Ce^{-\mu\rho_0}\int_0^{\rho_0} e^{-\mu c_2 r\rho_0^2}\sqrt{\frac{\pi}{\mu c_2 r}}\,r\,dr \\ &= 2C\sqrt{\frac{\pi}{\mu c_2}}e^{-\mu\rho_0}\int_0^{\rho_0} e^{-\mu c_2 r\rho_0^2}\sqrt{r}\,dr. \end{aligned} \tag{3.11}$$

Since

$$\int_0^{\rho_0} e^{-\mu c_2 r\rho_0^2}\sqrt{r}\,dr \leq \min\Big\{\int_0^{\rho_0}\sqrt{r}\,dr, \frac{2}{(\mu c_2)^{3/2}\rho_0^3}\int_0^{\infty} e^{-s^2}s^2\,ds\Big\},$$

from (3.11) we obtain

$$I_{+,1}(x,y) \leq Ce^{-\mu\rho_0}\min\Big\{\frac{\rho_0^{3/2}}{\sqrt{\mu}}, \frac{1}{\mu^2\rho_0^3}\Big\}. \tag{3.12}$$

We see that $I_{+,2}(x,y)$ also has the same estimate as (3.12). Thus from (3.9) one gets (3.4).

Next consider the case when $\rho_0 > r_1$. Set

$$\begin{aligned} S_1 &= \big\{z \in S_{\rho_0}(x)\cap S_{\rho_0}(y) \,\big|\, |z-x| \leq r_1/2\big\}, \\ S_2 &= \big\{z \in S_{\rho_0}(x)\cap S_{\rho_0}(y) \,\big|\, |z-y| \leq r_1/2\big\}, \\ S_3 &= \big\{z \in S_{\rho_0}(x)\cap S_{\rho_0}(y) \,\big|\, |z-x| \geq r_1/2, |z-y| \geq r_1/2\big\}. \end{aligned}$$

Since $S_{\rho_0}(x)\cap S_{\rho_0}(y) \subset S_1\cup S_2\cup S_3$, we have

$$I_+(x,y) \leq \sum_{k=1}^{3} I_{+,k}(x,y),$$

where

$$I_{+,k}(x,y) = \int_{S_k} e^{-\mu(|x-z|+|z-y|)}\,dS_z \quad (k = 1, 2, 3).$$

From Proposition 2.2(ii) with $r_0 = r_1$ and Lemma 3.1(ii) with $k = 0$ we obtain

$$I_{+,1}(x,y) \le e^{-\mu\rho_0} \int_{\partial D} e^{-\mu c_0 |x-z|}\, dS_z \le C\mu^{-2} e^{-\mu\rho_0}$$

and also the same estimate for $I_{+,2}(x,y)$. For $I_{+,3}(x,y)$ we make use of Proposition 2.2(i) with $r_0 = r_1$ and get

$$I_{+,3}(x,y) \le e^{-\mu\rho_0} \int_{\partial D} e^{-\mu c_0}\, dS_z = C e^{-\mu\rho_0} e^{-\mu c_0}.$$

Therefore we obtain (3.5). This completes the proof of Proposition 3.1. $\square$

4. Estimates for repeated integral kernels, I

This and subsequent sections are essential for the study of the integral kernel of the operator $Y_\lambda(I - Y_\lambda)^{-1}$ as $|\lambda| \longrightarrow \infty$. Since one has the Neumann series expansion $(I - Y_\lambda)^{-1} = \sum_{n=0}^{\infty} (Y_\lambda)^n$ as $|\lambda| \longrightarrow \infty$, first we study the integral kernels of $(Y_\lambda)^n$, $n = 1, 2, \ldots,$ which are called the repeated integral kernels. Since it seems to be hard to treat $(Y_\lambda)^n = (K_\lambda + \tilde{K}_\lambda)^n$ directly, we first consider the repeated kernels of K_λ. This is the main subject of this section.

Using $K_\lambda(x,y)$ in Section 1, we define the functions $K_\lambda^{(n)}(x,y)$, $n = 1, 2, \ldots$ by the formula

$$K_\lambda^{(n+1)}(x,y) = \int_{\partial D} K_\lambda^{(n)}(x,z) K_\lambda(z,y)\, dS_z, \quad n = 1, 2, \ldots$$

and

$$K_\lambda^{(1)}(x,y) = K_\lambda(x,y).$$

We see that the integral kernel of the operator K_λ^n is given by function $K_\lambda^{(n)}(x,y)$, that is,

$$K_\lambda^n h(x) = \int_{\partial D} K_\lambda^{(n)}(x,y) h(y)\, dS_y, \quad n = 1, 2, \ldots.$$

In this section we always assume that ∂D is strictly convex and ∂D is of class C^{2,α_0} with $0 < \alpha_0 \le 1$. The main result of this section is the following one.

THEOREM 4.1

There exist positive constants C and μ_0 depending only on C_0 in (1.1) and ∂D such that, for all $\lambda \in \mathbf{C}_{\delta_0}$ and $\mu \ge \mu_0$ the operator $I - K_\lambda$ is invertible and $K_\lambda(I - K_\lambda)^{-1}$ has an integral kernel $K_\lambda^\infty(x,y)$ which is measurable for $(x,y) \in \partial D \times \partial D$, continuous for $x \ne y$, and has the estimate

$$\left| K_\lambda^\infty(x,y) \right| \le C\mu e^{-\mu|x-y|}, \quad x, y \in \partial D.$$

The estimate given in Theorem 4.1 is crucial for the study of the integral kernel of $Y_\lambda(I - Y_\lambda)^{-1}$. Note that Theorem 4.1 is immediately obtained by the following proposition.

PROPOSITION 4.1

There exist positive constants C and μ_0 depending only on C_0 in (1.1) and ∂D such that, for all $\lambda \in \mathbf{C}_{\delta_0}$ with $\mu \geq \mu_0$,

$$\left|K_\lambda^{(n+1)}(x,y)\right| \leq C\mu e^{-\mu|x-y|}\left(\frac{1}{2}\right)^n, \quad x,y \in \partial D, n = 0,1,2,\ldots.$$

The rest of this section is to devoted to obtaining Proposition 4.1.

REMARK 4.1

Using (3.1)–(3.5), we can immediately obtain the following estimates of the repeated kernel $K_\lambda^{(2)}(x,y)$: there exist $r_1 > 0$ and $C > 0$ depending only on C_0 in (1.1) and ∂D such that, for all $\lambda \in \mathbf{C}_{\delta_0}$ with $\mu = \operatorname{Re}\lambda \geq 1$,

(i) for all $x,y \in \partial D$ with $|x-y| \geq r_1$,

$$\left|K_\lambda^{(2)}(x,y)\right| \leq Ce^{-\mu|x-y|};$$

(ii) for all $x,y \in \partial D$ with $|x-y| \leq r_1$,

$$\left|K_\lambda^{(2)}(x,y)\right| \leq Ce^{-\mu|x-y|}\max\left\{1, \mu\left(\mu|x-y|^3\right)^{1/2}\right\};$$

(iii) for all $x,y \in \partial D$ with $|x-y| \leq r_1$,

$$\left|K_\lambda^{(2)}(x,y)\right| \leq Ce^{-\mu|x-y|}\max\left\{1, \frac{1}{|x-y|^3}\right\}.$$

However, note that these are not used to show Theorem 1.1.

4.1. Estimation of $K_\lambda^{(n+1)}(x,y)$

It seems to be hard to show Proposition 4.1 directly. In our proof, we need to divide two steps. In this subsection, as in the first step, we prove the following estimate of the repeated kernel.

PROPOSITION 4.2

Let K_λ be a bounded measurable function on $\partial D \times \partial D$ with the parameter $\lambda \in \mathbf{C}_{\delta_0}$, continuous for all $x,y \in \partial D$, $x \neq y$, and let it satisfy (1.1). Choose the r_0 in Lemma 2.1(ii) sufficiently small, and let $0 < r_1 < 2r_0$ be given by (2.8) in the proof of Proposition 2.1(ii). Then, there exists a positive constant C_2 such that, for all $x,y \in \partial D$, $\lambda \in \mathbf{C}_{\delta_0}$ with $\mu = \operatorname{Re}\lambda \geq 1$, and $n = 0,1,2,\ldots$,

$$\left|K_\lambda^{(n+1)}(x,y)\right| \leq C_2^n C_0^{n+1}\mu^{(2-n)/2}e^{-\mu|x-y|}\Phi_\mu^{(n)}\left(\min\left\{|x-y|^3, r_1^3\right\}\right),$$

where

$$\Phi_\mu^{(n)}(\alpha) = \sum_{p=0}^{n}\frac{1}{p!}\left(\frac{2\mu}{3}\right)^p\alpha^{p/2}, \quad \alpha \geq 0.$$

Proof

We employ an induction argument. It suffices to prove the following statement. If

$$\left|K_\lambda^{(n+1)}(x,y)\right| \le \tilde{C}_n C_0 \mu^{(2-n)/2} e^{-\mu|x-y|}\Phi_\mu^{(n)}\left(|x-y|^3\right), \quad |x-y|\le r_1 \tag{4.1}$$

and

$$\left|K_\lambda^{(n+1)}(x,y)\right| \le \tilde{C}_n C_0 \mu^{(2-n)/2} e^{-\mu|x-y|}\Phi_\mu^{(n)}(r_1^3), \quad |x-y|\ge r_1, \tag{4.2}$$

then there exists a positive constant C_2 independent of n such that

$$\begin{aligned}&\left|K_\lambda^{(n+2)}(x,y)\right| \\ &\quad\le C_2\tilde{C}_n C_0 \mu^{(2-n-1)/2} e^{-\mu|x-y|}\Phi_\mu^{(n+1)}\left(|x-y|^3\right), \quad |x-y|\le r_1,\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\left|K_\lambda^{(n+2)}(x,y)\right| \\ &\quad\le C_2\tilde{C}_n C_0 \mu^{(2-n-1)/2} e^{-\mu|x-y|}\Phi_\mu^{(n+1)}(r_1^3), \quad |x-y|\ge r_1.\end{aligned} \tag{4.4}$$

The size of r_0 which is independent of n (and $x,y\in\partial D$) will be clarified in this induction step. Since $\Phi_\mu^{(0)}(\alpha)=1$, from (1.1) one can see that for $n=0$, (4.1) and (4.2) are satisfied with $\tilde{C}_0=1$. Set $\rho_0=|x-y|$.

4.1.1. The case when $\rho_0\le r_1$: Proof of (4.3)
Since $S_{\rho_0}(x)\subset S_{2r_1}(x)$, we have

$$\partial D = S_{2r_1}^-(x)\cup\left(S_{\rho_0}^-(x)\cap S_{2r_1}(x)\right)\cup\left(S_{\rho_0}^-(y)\cap S_{2r_1}(x)\right)\cup\left(S_{\rho_0}(x)\cap S_{\rho_0}(y)\right).$$

From this and the definition of $K_\lambda^{(n+2)}(x,y)$ we get

$$\begin{aligned}\left|K_\lambda^{(n+2)}(x,y)\right| &\le \int_{\partial D}\left|K_\lambda^{(n+1)}(x,z)\right|\left|K_\lambda(z,y)\right| dS_z \\ &\le I_1+I_2+I_3+I_4,\end{aligned} \tag{4.5}$$

where I_1, I_2, I_3, and I_4 are the integrals of the function $|K_\lambda^{(n+1)}(x,z)||K_\lambda(z,y)|$ over the domains $S_{2r_1}^-(x)$, $S_{\rho_0}^-(x)\cap S_{2r_1}(x)$, $S_{\rho_0}^-(y)\cap S_{2r_1}(x)$, and $S_{\rho_0}(x)\cap S_{\rho_0}(y)$, respectively.

First we give an estimate for I_1. Since $|z-x|\ge 2r_1>r_1$ for $z\in S_{2r_1}^-(x)$, from (1.1) and (4.2) we get

$$\begin{aligned}I_1 &\le \tilde{C}_n C_0^2 \mu^{(2-n)/2+1}\Phi_\mu^{(n)}(r_1^3)\int_{S_{2r_1}^-(x)} e^{-\mu(|x-z|+|z-y|)}\,dS_z \\ &\le \tilde{C}_n C_0^2 \mu^{(2-n)/2+1} e^{-2\mu r_1}\Phi_\mu^{(n)}(r_1^3)\int_{\partial D} e^{-\mu|z-y|}\,dS_z.\end{aligned} \tag{4.6}$$

Note that

$$\begin{aligned}e^{-2\mu r_1}\Phi_\mu^{(n)}(r_1^3) &\le e^{-\mu r_1}e^{-\mu r_1}e^{2\mu r_1^{3/2}/3} \\ &\le e^{-\mu\rho_0}e^{-\mu r_1+2\mu r_1^{3/2}/3}\end{aligned}$$

and $2r_1^{3/2}/3-r_1=2r_1(\sqrt{r_1}-3/2)/3$. Thus choosing r_0 in such a way that

$$r_1\le\left(\frac{3}{2}\right)^2,$$

and using Lemma 3.1(ii), from (4.6) one obtains

$$
\begin{aligned}
I_1 &\leq \tilde{C}_n C_0^2 C \mu^{(2-n)/2+1-2} e^{-\mu\rho_0} = \tilde{C}_n C_0^2 C \mu^{-n/2} e^{-\mu\rho_0} \\
&= \tilde{C}_n C_0^2 C \mu^{(2-n-1)/2} \mu^{-1/2} e^{-\mu\rho_0} \leq \tilde{C}_n C_0^2 C \mu^{(2-n-1)/2} e^{-\mu\rho_0}.
\end{aligned}
\tag{4.7}
$$

Second we give an estimate for I_4. From (1.1), (4.1), and $\rho_0 \leq r_1$ we have

$$
\begin{aligned}
I_4 &\leq \tilde{C}_n C_0^2 \mu^{(2-n)/2+1} \int_{S_{\rho_0}(x) \cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)} \Phi_\mu^{(n)}\big(|x-z|^3\big)\, dS_z \\
&\leq \tilde{C}_n C_0^2 \mu^{(4-n)/2} \sum_{p=0}^{n} \frac{1}{p!} \Big(\frac{2\mu}{3}\Big)^p \\
&\quad \times \int_{S_{\rho_0}(x) \cap S_{\rho_0}(y)} |x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)}\, dS_z.
\end{aligned}
\tag{4.8}
$$

Here we claim that

$$
\begin{aligned}
&\int_{S_{\rho_0}(x) \cap S_{\rho_0}(y)} |x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)}\, dS_z \\
&\qquad \leq \tilde{C} \mu^{-3/2} e^{-\mu\rho_0} \frac{2\mu}{3(p+1)} \rho_0^{3(p+1)/2}.
\end{aligned}
\tag{4.9}
$$

This is proved as follows. Recalling the proof of (3.4) and Proposition 2.1(i), for $z = x + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma)\nu_x \in S_{\rho_0}(x)$ with $\sigma = (r\cos\theta, r\sin\theta)$ we have

$$
|x-z| + |z-y| \geq \rho_0 + c|x-z| f(\theta)^2 \tag{4.10}
$$

provided r_0 is sufficiently small if necessary. Thus one gets

$$
\begin{aligned}
&\int_{S_{\rho_0}(x) \cap S_{\rho_0}(y)} |x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)}\, dS_z \\
&\leq \int_{S_{\rho_0}(x)} |x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)}\, dS_z \\
&\leq \tilde{C} e^{-\mu\rho_0} \int_{-\pi}^{\pi} d\theta \int_0^{r(\theta)} e^{-\mu c\sqrt{r^2+g(r\cos\theta, r\sin\theta)^2} f(\theta)^2} \\
&\qquad \times \big(r^2 + g(r\cos\theta, r\sin\theta)^2\big)^{3p/4} r\, dr,
\end{aligned}
\tag{4.11}
$$

where $r(\theta) > 0$ satisfies $\rho_0 = \sqrt{r(\theta)^2 + g(r(\theta)\cos\theta, r(\theta)\sin\theta)^2}$.

For each fixed θ consider the change of variable $r \longrightarrow \rho$:

$$
\rho = \sqrt{r^2 + g(r\cos\theta, r\sin\theta)^2}, \quad 0 \leq r \leq r(\theta).
$$

Since

$$
\Big|\frac{d\rho}{dr}\Big| = \frac{1}{\rho}\big|r + g(\sigma)\big(\partial_{\sigma_1} g(\sigma)\cos\theta + \partial_{\sigma_2} g(\sigma)\sin\theta\big)\big| \geq \frac{r}{\rho}(1 - \tilde{C} r^2),
$$

we have

$$
\Big|\frac{d\rho}{dr}\Big| \geq \frac{\tilde{C} r}{\rho},
$$

that is,

$$\left|\frac{dr}{d\rho}\right| \le \frac{\tilde{C}\rho}{r} \tag{4.12}$$

provided that r_0 is sufficiently small if necessary. A combination of (4.11) and (4.12) gives

$$\begin{aligned}
&\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} |x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)}\, dS_z \\
&\quad \le \tilde{C}e^{-\mu\rho_0}\int_{-\pi}^{\pi} d\theta \int_0^{\rho_0} e^{-\mu c\rho f(\theta)^2}\rho^{3p/2}\rho\, d\rho \\
&\quad \le 4\tilde{C}e^{-\mu\rho_0}\int_0^{\rho_0}\rho^{3p/2+1}\Big(\int_0^{\pi/2} e^{-\mu c\rho\theta^2}\, d\theta\Big)\, d\rho \\
&\quad \le 4\tilde{C}\mu^{-1/2}\int_0^{\rho_0}\rho^{(3p+1)/2}\Big(\int_0^{\sqrt{\mu\rho}\pi/2} e^{-c\theta^2}\, d\theta\Big)\, d\rho \\
&\quad \le 4\tilde{C}\int_0^{\infty} e^{-c\theta^2}\, d\theta \mu^{-1/2}\int_0^{\rho_0}\rho^{(3p+1)/2}\, d\rho.
\end{aligned}$$

This yields (4.9). A combination of (4.8) and (4.9) gives

$$I_4 \le \tilde{C}_n\tilde{C}C_0^2\mu^{(2-n-1)/2}e^{-\mu\rho_0}\Phi_\mu^{(n+1)}(\rho_0^3). \tag{4.13}$$

Third we give an estimate for I_2. Since $\Phi_\mu^{(n)}$ is monotone increasing, it follows from (4.1) and (4.2) that (4.1) is valid for all $x, y \in \partial D$. This gives

$$\begin{aligned}
I_2 &\le \tilde{C}_n C_0^2\mu^{(2-n)/2+1} \\
&\quad \times \sum_{p=0}^{n}\frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p\int_{S_{\rho_0}^-(x)\cap S_{2r_1}(x)} |x-z|^{3p/2}e^{-\mu(|x-z|+|z-y|)}\, dS_z.
\end{aligned} \tag{4.14}$$

Here we describe a lemma concerned with the integral in the right-hand side of (4.14).

LEMMA 4.1

There exists a positive constant $\tilde{C}$ such that, for all $p = 0, 1, 2, \ldots,$ $x, y \in \partial D$ with $\rho_0 = |x-y| \le r_1$ and $\mu > 0$,

(i) *if $\mu\rho_0 \ge p$, then*

$$\begin{aligned}
&\int_{S_{\rho_0}^-(x)\cap S_{2r_1}(x)\cap S_{\rho_0}(y)} |x-z|^{3p/2}e^{-\mu(|x-z|+|z-y|)}\, dS_z \\
&\quad \le \tilde{C}\mu^{-1/2}e^{-\mu\rho_0}\frac{1}{p+1}\rho_0^{3(p+1)/2};
\end{aligned}$$

(ii) *if $\mu\rho_0 \le p$, then*

$$\begin{aligned}
&\int_{S_{\rho_0}^-(x)\cap S_{2r_1}(x)\cap S_{\rho_0}(y)} |x-z|^{3p/2}e^{-\mu(|x-z|+|z-y|)}\, dS_z \\
&\quad \le \tilde{C}e^{-\mu\rho_0}\mu^{-(p+3/2)}p!(2^{3/2}e)^p\rho_0^{(p+1)/2};
\end{aligned}$$

(iii) *we have*

$$\int_{S^-_{\rho_0}(x)\cap S_{2r_1}(x)\cap S^-_{\rho_0}(y)} |x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)}\, dS_z$$
$$\leq \tilde{C} e^{-\mu\rho_0} \mu^{-(p+2)} (p+1)!(2r_1)^{p/2}.$$

We give the proof of this lemma later and continue to estimate I_2. From (4.14) and Lemma 4.1 we obtain

$$\begin{aligned}
I_2 &\leq \tilde{C}_n \tilde{C} C_0^2 \mu^{(2-n-1)/2+1} e^{-\mu\rho_0} \sum_{p=0}^{n} \frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p \Big\{ \frac{1}{p+1} \rho_0^{3(p+1)/2} \\
&\quad + \mu^{-(p+1)} p!(2^{3/2}e)^p \rho_0^{(p+1)/2} + \mu^{-(p+3/2)}(p+1)!(2r_1)^{p/2} \Big\} \\
&\leq \tilde{C}_n \tilde{C} C_0^2 \mu^{(2-n-1)/2} e^{-\mu\rho_0} \Big\{ \sum_{p=0}^{n} \frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p \frac{\mu}{p+1} \rho_0^{3(p+1)/2} \\
&\quad + \sum_{p=0}^{n} \frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p \mu^{-p} p!(2^{3/2}e)^p \rho_0^{(p+1)/2} \\
&\quad + \sum_{p=0}^{n} \frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p \mu^{-(p+1/2)}(p+1)!(2r_1)^{p/2} \Big\}.
\end{aligned} \tag{4.15}$$

Here we estimate the right-hand side of (4.15) term by term. First we have

$$\begin{aligned}
\sum_{p=0}^{n} \frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p \frac{\mu}{p+1} \rho_0^{3(p+1)/2} &= \frac{3}{2} \sum_{p=1}^{n+1} \frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p \rho_0^{3p/2} \\
&\leq \frac{3}{2} \Phi_\mu^{(n+1)}(\rho_0^3).
\end{aligned} \tag{4.16}$$

Second we have

$$\begin{aligned}
\sum_{p=0}^{n} \frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p \mu^{-p} p!(2^{3/2}e)^p \rho_0^{(p+1)/2} &= \sum_{p=0}^{n} \Big(\frac{2^{5/2}e}{3}\Big)^p \rho_0^{(p+1)/2} \\
&\leq \sqrt{r_1} \sum_{p=0}^{n} \Big(\frac{2^{5/2}e\sqrt{r_1}}{3}\Big)^p.
\end{aligned}$$

Thus choosing r_0 in such a way that

$$\frac{2^{5/2}e\sqrt{r_1}}{3} \leq \frac{1}{2}, \tag{4.17}$$

we obtain

$$\sum_{p=0}^{n} \frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p \mu^{-p} p!(2^{3/2}e)^p \rho_0^{(p+1)/2} \leq \frac{3}{2^{5/2}e}. \tag{4.18}$$

Third for r_0 satisfying (4.17) we get

$$\begin{aligned}\sum_{p=0}^{n}\frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p\mu^{-(p+1/2)}(p+1)!(2r_1)^{p/2} &= \mu^{-1/2}\sum_{p=0}^{n}(p+1)\Big(\frac{2^{3/2}\sqrt{r_1}}{3}\Big)^p\\ &\leq 4.\end{aligned}\tag{4.19}$$

Now from (4.15), (4.16), (4.18) and (4.19) we obtain

$$I_2 \leq \tilde{C}_n\tilde{C}C_0^2\mu^{(2-n-1)/2}e^{-\mu\rho_0}\Phi_\mu^{(n+1)}(\rho_0^3)\tag{4.20}$$

for r_0 satisfying (4.17).

Finally we estimate I_3. Similarly to (4.14) we have

$$\begin{aligned}I_3 \leq \tilde{C}_nC_0^2\mu^{(2-n)/2+1}&\sum_{p=0}^{n}\frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p\\ &\times\int_{S^-_{\rho_0}(y)\cap S_{2r_1}(x)}|x-z|^{3p/2}e^{-\mu(|x-z|+|z-y|)}\,dS_z.\end{aligned}$$

Since $|z-y|\geq\rho_0$ for $z\in S^-_{\rho_0}(y)$, we get

$$\begin{aligned}I_3 \leq \tilde{C}_nC_0^2\mu^{(2-n)/2+1}e^{-\mu\rho_0}&\sum_{p=0}^{n}\frac{1}{p!}\Big(\frac{2\mu}{3}\Big)^p\\ &\times\int_{S_{2r_1}(x)}|x-z|^{3p/2}e^{-\mu|x-z|}\,dS_z.\end{aligned}\tag{4.21}$$

Here we claim a lemma concerned with an estimate for the integral in the right-hand side of (4.21). The proof is almost identical with that of Lemma 4.1(iii).

LEMMA 4.2

There exists a positive constant $\tilde{C}$ such that, for all $p=0,1,2,\dots$, $x\in\partial D$, $a>0$, and $\mu>0$,

$$\int_{S_{2r_1}(x)}|x-z|^{3p/2}e^{-\mu|x-z|}\,dS_z \leq \tilde{C}(2r_1)^{p/2}\mu^{-(p+2)}(p+1)!.$$

The proof is given at the end of this subsection, and we continue to estimate I_3.

From Lemma 4.2 and (4.21) we get

$$\begin{aligned}I_3 &\leq \tilde{C}_n\tilde{C}C_0^2\mu^{(2-n-1)/2}\mu^{-1/2}e^{-\mu\rho_0}\sum_{p=0}^{n}\Big(\frac{2^{3/2}\sqrt{r_1}}{3}\Big)^p(p+1)\\ &\leq \tilde{C}_n4\tilde{C}C_0^2\mu^{(2-n-1)/2}\mu^{-1/2}e^{-\mu\rho_0}\\ &\leq \tilde{C}_n4\tilde{C}C_0^2\mu^{(2-n-1)/2}e^{-\mu\rho_0}\end{aligned}\tag{4.22}$$

provided that r_0 satisfies (4.17). Now from (4.5), (4.7), (4.13), (4.20), and (4.22) we conclude that (4.3) is valid.

4.1.2. The case when $\rho_0 \geq r_1$: Proof of (4.4)

Since $\Phi_\mu^{(n)}$ is monotone increasing, it follows from (4.1) and (4.2) that (4.2) is valid for all $x, y \in \partial D$. Since $\rho_0 \geq r_1/2$, it follows from the definition of $K_\lambda^{(n+2)}$ and (3.7) that

$$\begin{aligned}\left|K_\lambda^{(n+2)}(x,y)\right| &\leq \tilde{C}_n C_0^2 \mu^{(2-n)/2} \mu \Phi_\mu^{(n)}(r_1^3) \int_{\partial D} e^{-\mu(|x-z|+|z-y|)}\, dS_z \\ &\leq \tilde{C}_n C C_0^2 \mu^{(2-n-1)/2} \mu^{-1/2} e^{-\mu\rho_0} \Phi_\mu^{(n)}(r_1^3) \\ &\leq \tilde{C}_n C C_0^2 \mu^{(2-n-1)/2} e^{-\mu\rho_0} \Phi_\mu^{(n+1)}(r_1^3).\end{aligned}$$

Thus (4.4) is valid. This completes the proof of Proposition 4.2. □

4.1.3. Proof of Lemma 4.1

Set $\rho = |x - z|$. Since $|z - y| \geq |x - z| - |x - y|$, we have

$$|x-z| + |z-y| \geq 2\rho - \rho_0 = \rho_0 + 2\rho_0\Big(\frac{\rho}{\rho_0} - 1\Big).$$

A combination of this and (4.10) gives

$$\begin{aligned}|x-z| + |y-z| &\geq \frac{7}{8}\Big\{\rho_0 + 2\rho_0\Big(\frac{\rho}{\rho_0} - 1\Big)\Big\} + \frac{1}{8}\big(\rho_0 + c\rho f(\theta)^2\big) \\ &= \rho_0 + \frac{7}{4}\rho_0\Big(\frac{\rho}{\rho_0} - 1\Big) + \tilde{c}\rho f(\theta)^2.\end{aligned} \tag{4.23}$$

Set

$$s = \frac{\rho}{\rho_0} - 1.$$

We have $\rho = \rho_0(1+s)$, and this together with (4.23) gives

$$\begin{aligned}|x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)} &\leq \rho^{3p/2} e^{-\mu\rho_0} e^{-\mu\frac{7}{4}\rho_0(\frac{\rho}{\rho_0}-1)} e^{-\tilde{c}\mu\rho f(\theta)^2} \\ &= e^{-\mu\rho_0} \rho_0^{3p/2} (1+s)^{3p/2} e^{-\mu\frac{7}{4}\rho_0 s} e^{-\tilde{c}\mu\rho f(\theta)^2}.\end{aligned} \tag{4.24}$$

Let $z \in S_{\rho_0}^-(x) \cap S_{2r_1}(x) \cap S_{\rho_0}(y)$. Since $|z-x| \geq \rho_0$ and $|z-y| \leq \rho_0$, we have $\rho_0 \leq \rho \leq 2\rho_0$. Thus s in (4.24) has the bound $0 \leq s \leq 1$.

First we give a proof of (i). Let $\mu\rho_0 \geq p$. Since $e^r \geq 1 + r$ for $r \geq 0$, we have $(1+r)^{3/2} e^{-\frac{7}{4}r} \leq (1+r)^{-1/4}$. Combining this with (4.24), we get

$$\begin{aligned}|x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)} &\leq e^{-\mu\rho_0} \rho_0^{3p/2} (1+s)^{3\mu\rho_0/2} e^{-\frac{7}{4}\mu\rho_0 s} e^{-\tilde{c}\mu\rho f(\theta)^2} \\ &= e^{-\mu\rho_0} \rho_0^{3p/2} \big((1+s)^{3/2} e^{-\frac{7}{4}s}\big)^{\mu\rho_0} e^{-\tilde{c}\mu\rho f(\theta)^2} \\ &\leq e^{-\mu\rho_0} \rho_0^{3p/2} \big((1+s)^{-1/4}\big)^{\mu\rho_0} e^{-\tilde{c}\mu\rho f(\theta)^2} \\ &= e^{-\mu\rho_0} \rho_0^{3p/2} \Big(\frac{\rho}{\rho_0}\Big)^{-\mu\rho_0/4} e^{-\tilde{c}\mu\rho f(\theta)^2}.\end{aligned} \tag{4.25}$$

Using the polar coordinate $z = x + r\cos\theta e_1 + r\sin\theta e_2 - g(r\cos\theta, r\sin\theta)\nu_x$, one gets from (4.25),

$$
\begin{aligned}
&\int_{S^-_{\rho_0}(x)\cap S_{2r_1}(x)\cap S_{\rho_0}(y)} |x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)}\,dS_z \\
&\le \int_{S_{2\rho_0}(x)\setminus S_{\rho_0}(x)} e^{-\mu\rho_0}\rho_0^{3p/2}\Big(\frac{|x-z|}{\rho_0}\Big)^{-\mu\rho_0/4} e^{-\tilde{c}\mu|x-z|f(\theta)^2}\,dS_z \qquad (4.26)\\
&\le Ce^{-\mu\rho_0}\rho_0^{3p/2}\int_{-\pi}^{\pi} d\theta \int_{R_{\rho_0}(\theta)}^{R_{2\rho_0}(\theta)} \Big(\frac{\rho(r,\theta)}{\rho_0}\Big)^{-\mu\rho_0/4} e^{-\tilde{c}\mu\rho(r,\theta)f(\theta)^2} r\,dr,
\end{aligned}
$$

where $\rho(r,\theta) = \sqrt{r^2 + (g(r\cos\theta, r\sin\theta))^2}$; $\rho(R_{2\rho_0}(\theta),\theta) = 2\rho_0$; and $\rho(R_{\rho_0}(\theta), \theta) = \rho_0$. For each fixed θ consider the change of variable $r \longrightarrow \rho(r,\theta)$, $R_{\rho_0}(\theta) \le r \le R_{2\rho_0}(\theta)$. Using (4.12), we obtain

$$
\begin{aligned}
&\int_{-\pi}^{\pi} d\theta \int_{R_{\rho_0}(\theta)}^{R_{2\rho_0}(\theta)} \Big(\frac{\rho(r,\theta)}{\rho_0}\Big)^{-\mu\rho_0/4} e^{-\tilde{c}\mu\rho(r,\theta)f(\theta)^2} r\,dr \\
&\le C\int_{-\pi}^{\pi} d\theta \int_{\rho_0}^{2\rho_0} \Big(\frac{\rho}{\rho_0}\Big)^{-\frac{\mu\rho_0}{4}} e^{-\tilde{c}\mu\rho f(\theta)^2}\rho\,d\rho \\
&= 4C\int_0^{\pi/2} d\theta \int_{\rho_0}^{2\rho_0} \Big(\frac{\rho}{\rho_0}\Big)^{-\mu\rho_0/4} e^{-\tilde{c}\mu\rho\theta^2}\rho\,d\rho \\
&= 4C\int_{\rho_0}^{2\rho_0} \Big(\frac{\rho}{\rho_0}\Big)^{-\mu\rho_0/4} \Big(\int_0^{\pi/2} e^{-\tilde{c}\mu\rho\theta^2}\,d\theta\Big)\rho\,d\rho \qquad (4.27)\\
&\le \tilde{C}\int_{\rho_0}^{2\rho_0} \Big(\frac{\rho}{\rho_0}\Big)^{-\mu\rho_0/4} \frac{\rho}{\sqrt{\mu\rho}}\,d\rho = \frac{\tilde{C}}{\sqrt{\mu}}\rho_0^{3/2}\int_1^2 t^{-\frac{\mu\rho_0}{4}+\frac{1}{2}}\,dt \\
&\le \frac{C}{\sqrt{\mu}\mu\rho_0}\rho_0^{3/2}.
\end{aligned}
$$

Note that we made use of the estimate

$$
\int_1^2 t^{-\frac{\mu\rho_0}{4}+\frac{1}{2}}\,dt = \int_1^2 t^{-\frac{\mu\rho_0}{4}-1} t^{3/2}\,dt \le 2^{3/2}\int_1^2 t^{-\frac{\mu\rho_0}{4}-1}\,dt \le \frac{2^{7/2}}{\mu\rho_0}.
$$

Hence for $p \ge 1$, a combination of (4.26) and (4.27) gives

$$
\begin{aligned}
&\int_{S^-_{\rho_0}(x)\cap S_{2r_1}(x)\cap S_{\rho_0}(y)} |x-z|^{3p/2} e^{-\mu(|x-z|+|z-y|)}\,dS_z \\
&\le Ce^{-\mu\rho_0}\rho_0^{3p/2}\frac{\rho_0^{3/2}}{\sqrt{\mu}\mu\rho_0} \le C\mu^{-1/2}e^{-\mu\rho_0}\rho_0^{3(p+1)/2}\frac{1}{p} \\
&\le 2C\mu^{-1/2}e^{-\mu\rho_0}\rho_0^{3(p+1)/2}\frac{1}{p+1}.
\end{aligned}
$$

For $p = 0$, note that (4.23) implies also that

$$
e^{-\mu(|x-z|+|z-y|))} \le e^{-\mu\rho_0}e^{-\mu\tilde{c}\rho f(\theta)^2}.
$$

From this estimate, the same argument as for obtaining (4.26) and (4.27) yields

$$
\begin{aligned}
&\int_{S^-_{\rho_0}(x)\cap S_{2r_1}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\,dS_z \\
&\le Ce^{-\mu\rho_0}\int_{-\pi}^{\pi}d\theta\int_{R_{\rho_0}(\theta)}^{R_{2\rho_0}(\theta)} e^{-\mu\tilde{c}\rho f(\theta)^2}r\,dr \\
&\le Ce^{-\mu\rho_0}\int_{\rho_0}^{2\rho_0}\frac{\rho}{\sqrt{\mu\rho}}\,d\rho \le C\mu^{-1/2}e^{-\mu\rho_0}(2\rho_0)^{3/2}.
\end{aligned}
$$

This completes the proof of (i).

Second we give a proof of (ii). Let $\mu\rho_0 \le p$. Since $|x-z|+|z-y|\ge\rho_0$, similarly to (4.26) and (4.27), it follows that

$$
\begin{aligned}
&\int_{S^-_{\rho_0}(x)\cap S_{2r_1}(x)\cap S_{\rho_0}(y)} |x-z|^{3p/2}e^{-\mu(|x-z|+|z-y|)}\,dS_z \\
&\le \tilde{C}e^{-\mu\rho_0}\rho_0^{3p/2}\int_{-\pi}^{\pi}d\theta\int_{\rho_0}^{2\rho_0}\Big(\frac{\rho}{\rho_0}\Big)^{3p/2}\rho\,d\rho \\
&\le \tilde{C}e^{-\mu\rho_0}\rho_0^{p+1}\rho_0^{p/2+1}\frac{2^{\frac{3p}{2}}}{p+1} \\
&\le \tilde{C}e^{-\mu\rho_0}\mu^{-(p+1)}\rho_0^{p/2+1}(p+1)^{p+1}\frac{2^{\frac{3p}{2}}}{p+1}.
\end{aligned}
\tag{4.28}
$$

The Stirling formula $p!\sim\sqrt{2\pi p}p^pe^{-p}$ tells us

$$
(p+1)^{p+1}e^{-(p+1)} \le C(p+1)!(p+1)^{-1/2}, \quad p=0,1,2,\dots.
$$

Applying this to the right-hand side of (4.28), we obtain

$$
\begin{aligned}
&\int_{S^-_{\rho_0}(x)\cap S_{2r_1}(x)\cap S_{\rho_0}(y)} |x-z|^{3p/2}e^{-\mu(|x-z|+|z-y|)}\,dS_z \\
&\le \tilde{C}e^{-\mu\rho_0}\mu^{-(p+1)}\rho_0^{p/2+1}e^{p+1}(p+1)!(p+1)^{-1/2}\frac{2^{\frac{3p}{2}}}{p+1} \\
&\le \tilde{C}e^{-\mu\rho_0}\mu^{-(p+1)}\rho_0^{p/2+1}(2^{3/2}e)^p p!p^{-1/2} \\
&\le \tilde{C}e^{-\mu\rho_0}\mu^{-(p+1)}\rho_0^{p/2+1}(2^{3/2}e)^p p!(\mu\rho_0)^{-1/2}.
\end{aligned}
$$

This completes the proof of (ii).

Finally we give a proof of (iii). Since $|z-y|\ge\rho_0$ for $z\in S^-_{\rho_0}(y)$, we have

$$
\begin{aligned}
&\int_{S^-_{\rho_0}(x)\cap S_{2r_1}(x)\cap S^-_{\rho_0}(y)} |x-z|^{3p/2}e^{-\mu(|x-z|+|z-y|)}\,dS_z \\
&\le e^{-\mu\rho_0}\rho_0^{3p/2}\int_{S_{2r_1}(x)\setminus S_{\rho_0}(x)}\Big(\frac{|x-z|}{\rho_0}\Big)^{3p/2}e^{-\mu|x-z|}\,dS_z \\
&\le \tilde{C}e^{-\mu\rho_0}\rho_0^{3p/2}\int_{-\pi}^{\pi}d\theta\int_{R_{\rho_0}(\theta)}^{R_{2r_1}(\theta)}\Big(\frac{\rho(r,\theta)}{\rho_0}\Big)^{3p/2}e^{-\mu\rho(r,\theta)}r\,dr
\end{aligned}
$$

$$\le \tilde{C}e^{-\mu\rho_0}\rho_0^{3p/2}\int_{\rho_0}^{2r_1}\Big(\frac{\rho}{\rho_0}\Big)^{3p/2}e^{-\mu\rho}\rho\,d\rho$$
$$\le \tilde{C}e^{-\mu\rho_0}\rho_0^{3p/2}\Big(\frac{2r_1}{\rho_0}\Big)^{p/2}\int_{\rho_0}^{2r_1}\Big(\frac{\rho}{\rho_0}\Big)^{p}e^{-\mu\rho}\rho\,d\rho$$
$$\le \tilde{C}e^{-\mu\rho_0}\rho_0^{3p/2}\Big(\frac{2r_1}{\rho_0}\Big)^{p/2}\frac{1}{(\mu\rho_0)^p\mu^2}\int_{\mu\rho_0}^{2\mu r_1}s^{p+1}e^{-s}\,ds$$
$$\le \tilde{C}e^{-\mu\rho_0}(2r_1)^{p/2}\mu^{-(p+2)}(p+1)!.$$

This completes the proof of (iii).

4.2. Proof of Proposition 4.1

Let $R>0$. Consider the case when $|x-y|\le r_1$ and $\mu|x-y|^3\le R$. From Proposition 4.2 we have

$$\big|K_\lambda^{(n+1)}(x,y)\big|\le C_2^nC_0^{n+1}\mu^{(2-n)/2}e^{-\mu|x-y|}\Phi_\mu^{(n)}\big(|x-y|^3\big)$$
$$\le C_0\mu e^{-\mu|x-y|}\sum_{p=0}^{n}\Big(\frac{(C_0C_2)^2}{\mu}\Big)^{(n-p)/2}\frac{1}{p!}\Big(\frac{2}{3}\Big)^p(C_0C_2)^pR^{p/2}.$$

Thus if $\mu\ge R(C_0C_2)^2$, then one gets

$$\big|K_\lambda^{(n+1)}(x,y)\big|\le C_0\mu e^{-\mu|x-y|}\sum_{p=0}^{n}\Big(\frac{1}{R}\Big)^{(n-p)/2}\frac{1}{p!}\Big(\frac{2}{3}\Big)^p(C_0C_2)^pR^{p/2}$$
$$\le C_0\mu e^{-\mu|x-y|}\Big(\frac{1}{R}\Big)^{n/2}\sum_{p=0}^{n}\frac{1}{p!}\Big(\frac{2C_0C_2R}{3}\Big)^p$$
$$\le C_0\mu e^{-\mu|x-y|}\Big(\frac{1}{R}\Big)^{n/2}\exp\Big(\frac{2C_0C_2R}{3}\Big).$$

Set $C_2'=(2C_0C_2)/3$ and $\mu_1=(C_0C_2)^2+1$. As a simple consequence of the argument done above we have the following: if $|x-y|\le r_1$, $\mu|x-y|^3\le R$, $\lambda\in\mathbf{C}_{\delta_0}$, $\mu=\mathrm{Re}\,\lambda\ge\max\{\mu_1R,1\}$, and $R>0$, then

$$\big|K_\lambda^{(n+1)}(x,y)\big|\le C_0\mu e^{-\mu|x-y|}\Big(\frac{1}{R}\Big)^{n/2}e^{C_2'R}\quad(n=0,1,2,\ldots). \tag{4.29}$$

Here we prepare a lemma that covers all the remaining cases for x,y.

LEMMA 4.3

Under all the assumptions made in Theorem 4.1 there exists a positive constant C_3 such that, for all $\lambda\in\mathbf{C}_{\delta_0}$ with $\mu\ge\mu_1R$ and $n=0,1,2,\ldots$,

$$\big|K_\lambda^{(n+1)}(x,y)\big|\le C_0C_3^n\mu e^{-\mu|x-y|}\Big(\frac{1}{R}\Big)^{n/2}e^{C_2'R},\quad x,y\in\partial D, R\ge 1.$$

Choosing $R\ge1$ in Lemma 4.3 in such a way that $C_3^2/R\le 1/4$ and setting $\mu_0=\mu_1R$, we immediately obtain the desired estimate in Proposition 4.1. Thus in the following we present a proof of Lemma 4.3.

Proof of Lemma 4.3
We employ an induction argument. First we prove that one can find a suitable positive constant C_3 independent of n, x, y, μ, and R such that, if we have, for all $x, y \in \partial D$,

$$\begin{aligned} &\left|K_\lambda^{(n+1)}(x,y)\right| \le C_0 \tilde{C}_n \mu e^{-\mu|x-y|} \left(\frac{1}{R}\right)^{n/2} e^{C_2' R}, \\ &\lambda \in \mathbf{C}_{\delta_0}, \operatorname{Re}\lambda = \mu \ge \mu_1 R, R \ge 1, \end{aligned} \tag{4.30}$$

then we have, for all $x, y \in \partial D$,

$$\begin{aligned} &\left|K_\lambda^{(n+2)}(x,y)\right| \le C_0 \frac{C_3}{2}(1+\tilde{C}_n) \mu e^{-\mu|x-y|} \left(\frac{1}{R}\right)^{(n+1)/2} e^{C_2' R}, \\ &\lambda \in \mathbf{C}_{\delta_0}, \operatorname{Re}\lambda = \mu \ge \mu_1 R, R \ge 1. \end{aligned} \tag{4.31}$$

We divide the proof into three cases: (a) $\rho_0 = |x-y| \le r_1$ and $\mu\rho_0^3 \le R$; (b) $\rho_0 \le r_1$ and $\mu\rho_0^3 \ge R$; (c) $\rho_0 \ge r_1$.

Case (a). In this case it follows from (4.29) that

$$\left|K_\lambda^{(n+2)}(x,y)\right| \le C_0 \mu e^{-\mu|x-y|} \left(\frac{1}{R}\right)^{(n+1)/2} e^{C_2' R}. \tag{4.32}$$

Case (b). From the definition of $K_\lambda^{(n+2)}(x,y)$ we get

$$\left|K_\lambda^{(n+2)}(x,y)\right| \le J_1 + J_2 + J_3, \tag{4.33}$$

where J_1, J_2, and J_3 are the integrals of the function $|K_\lambda^{(n+1)}(x,z)||K_\lambda(z,y)|$ in z over the domains $S_{\rho_0}^-(x)$, $S_{\rho_0}(x) \cap S_{\rho_0}^-(y)$, and $S_{\rho_0}(x) \cap S_{\rho_0}(y)$, respectively.

First we estimate J_1. Since $|z-x| \ge \rho_0$ for $z \in S_{\rho_0}^-(x)$, from (1.1) and (4.30) we obtain

$$\begin{aligned} J_1 &\le \int_{S_{\rho_0}^-(x)} C_0 \tilde{C}_n \mu e^{-\mu|x-z|} \left(\frac{1}{R}\right)^{n/2} e^{C_2' R} C_0 \mu e^{-\mu|z-y|}\, dS_z \\ &= C_0^2 \tilde{C}_n e^{C_2' R} \left(\frac{1}{R}\right)^{n/2} \mu^2 \int_{S_{\rho_0}^-(x)} e^{-\mu(|x-z|+|z-y|)}\, dS_z \\ &\le C_0^2 \tilde{C}_n e^{C_2' R} \left(\frac{1}{R}\right)^{n/2} \mu^2 e^{-\mu\rho_0} \int_{S_{\rho_0}^-(x)} e^{-\mu|z-y|}\, dS_z. \end{aligned}$$

From this, Lemma 3.1(ii), $\mu_1 \ge 1$, and $R \ge 1$ one gets

$$\begin{aligned} J_1 &\le C C_0^2 \tilde{C}_n e^{C_2' R} \left(\frac{1}{R}\right)^{n/2} e^{-\mu\rho_0} = C C_0^2 \tilde{C}_n e^{C_2' R} \mu e^{-\mu\rho_0} \left(\frac{1}{R}\right)^{n/2} \left(\frac{1}{R}\right)^{1/2} \frac{R^{1/2}}{\mu} \\ &= C C_0^2 \tilde{C}_n e^{C_2' R} \mu e^{-\mu\rho_0} \left(\frac{1}{R}\right)^{(n+1)/2} \frac{R^{1/2}}{\mu_1 R} \le C C_0^2 \tilde{C}_n e^{C_2' R} \mu e^{-\mu\rho_0} \left(\frac{1}{R}\right)^{(n+1)/2}. \end{aligned}$$

The estimation for J_2 is the same as J_1 since $S_{\rho_0}(x) \cap S_{\rho_0}^-(y) \subset S_{\rho_0}^-(y)$.

Finally we give an estimation for J_3. Since $\rho_0 \le r_1$, one has (3.4). This together with (1.1) and (4.30) gives

$$J_3 \le C_0^2 \tilde{C}_n \mu^2 e^{C_2' R} \Big(\frac{1}{R}\Big)^{n/2} \int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z$$
$$\le C C_0^2 \tilde{C}_n e^{C_2' R} \mu e^{-\mu\rho_0} \Big(\frac{1}{R}\Big)^{n/2} \frac{1}{\mu\rho_0^3} \le C C_0^2 \tilde{C}_n e^{C_2' R} \mu e^{-\mu\rho_0} \Big(\frac{1}{R}\Big)^{(n+1)/2}$$

for $R \ge 1$. Summing these up, we obtain, for all $x, y \in \partial D$ with $\mu|x-y|^3 \ge R$ and $|x-y| \le r_1$,

$$\big|K_\lambda^{(n+2)}(x,y)\big| \le C C_0^2 \tilde{C}_n \mu e^{-\mu|x-y|} \Big(\frac{1}{R}\Big)^{(n+1)/2} e^{C_2' R}. \tag{4.34}$$

Case (c). In this case we choose the integral domains of J_1, J_2, and J_3 in (4.33) as $S_{r_1/2}(x)$, $S_{r_1/2}(y)$, and $S^-_{r_1/2}(x) \cap S^-_{r_1/2}(y)$, respectively.

From (4.30) we have

$$J_1 \le C_0^2 \tilde{C}_n \mu^2 e^{C_2' R} \Big(\frac{1}{R}\Big)^{n/2} \int_{S_{r_1/2}(x)} e^{-\mu(|x-z|+|z-y|)}\, dS_z. \tag{4.35}$$

Since $\rho_0 \ge r_1$ and $|x-z| \le r_1/2$ for $z \in S_{r_1/2}(x)$ it follows from Proposition 2.2(ii) for the case $r_0 = r_1$ that $|x-z| + |z-y| \ge \rho_0 + c_0|z-x|$. A combination of Lemma 3.1(ii) and (4.35) yields

$$J_1 \le C_0^2 \tilde{C}_n \mu^2 e^{C_2' R} \Big(\frac{1}{R}\Big)^{n/2} e^{-\mu\rho_0} \int_{S_{r_1/2}(x)} e^{-\mu c_0|z-y|}\, dS_z$$
$$\le C_0^2 \tilde{C}_n \mu^2 e^{C_2' R} \Big(\frac{1}{R}\Big)^{n/2} e^{-\mu\rho_0} \tilde{C} \mu^{-2}$$
$$= C C_0^2 \tilde{C}_n e^{C_2' R} \mu e^{-\mu\rho_0} \Big(\frac{1}{R}\Big)^{(n+1)/2} \frac{R^{1/2}}{\mu}.$$

Combining this with $\mu \ge \mu_1 R \ge R \ge R^{1/2}$, we get

$$J_1 \le C C_0^2 \tilde{C}_n e^{C_2' R} \mu e^{-\mu\rho_0} \Big(\frac{1}{R}\Big)^{(n+1)/2}.$$

Similarly, for J_2 we get the same estimate.

It follows from (4.30) that

$$J_3 \le C_0^2 \tilde{C}_n \mu^2 e^{C_2' R} \Big(\frac{1}{R}\Big)^{n/2} \int_{S^-_{r_1/2}(x)\cap S^-_{r_1/2}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z.$$

From Proposition 2.2(i) for the case $r_0 = r_1$ we have $|x-z| + |z-y| \ge \rho_0 + c_0$. This yields

$$J_3 \le C_0^2 \tilde{C}_n \mu^2 e^{C_2' R} \Big(\frac{1}{R}\Big)^{n/2} e^{-\mu\rho_0} e^{-\mu c_0} |\partial D| \le C_0^2 \tilde{C}_n \mu^2 e^{C_2' R} \Big(\frac{1}{R}\Big)^{n/2} e^{-\mu\rho_0} C \mu^{-2}$$
$$\le C C_0^2 \tilde{C}_n e^{C_2' R} \mu e^{-\mu\rho_0} \Big(\frac{1}{R}\Big)^{(n+1)/2}.$$

In summary,

$$\left|K^{(n+2)}(x,y)\right| \le CC_0^2\tilde{C}_n\mu e^{-\mu\rho_0}\left(\frac{1}{R}\right)^{(n+1)/2}e^{C_2'R} \tag{4.36}$$

for $x, y, \partial D$ with $|x-y| \ge r_1$ and $\lambda \in \mathbf{C}_{\delta_0}$ with $\operatorname{Re}\lambda = \mu \ge \mu_1 R, R \ge 1$.

From (4.32), (4.34), and (4.36) one concludes that if one chooses

$$C_3 = 2\max\{1, CC_0\}$$

in advance, then (4.30) for all $x, y \in \partial D$ implies (4.31) for all $x, y \in \partial D$.

Moreover, define

$$\tilde{C}_{n+1} = \frac{C_3}{2}(1+\tilde{C}_n), \quad n = 0, 1, \ldots, \tilde{C}_0 = 1. \tag{4.37}$$

Then we obtain, for all $\lambda \in \mathbf{C}_{\delta_0}$ with $\mu \ge \mu_1 R$ and $n = 0, 1, 2, \ldots,$

$$\left|K_\lambda^{(n+1)}(x,y)\right| \le C_0\tilde{C}_n\mu e^{-\mu|x-y|}\left(\frac{1}{R}\right)^{n/2}e^{C_2'R}, \quad x, y \in \partial D, R \ge 1.$$

It follows from (4.37) and $C_3 \ge 1$ that $\tilde{C}_n \le C_3^n$. Thus this completes the proof of Lemma 4.3. □

5. Estimates for repeated integral kernels, II

Continuing from Section 4 we give an estimate for another repeated integral kernel which is necessary for the estimation of the integral kernel of the operator $Y_\lambda(I - Y_\lambda)^{-1}$.

For the kernels $K_\lambda(x,y)$ and $\tilde{K}_\lambda(x,y)$ given in Section 1, define

$$L_\lambda(x,y) = \tilde{K}_\lambda(x,y) + \int_{\partial D}\tilde{K}_\lambda(x,z)K_\lambda(z,y)\,dS_z. \tag{5.1}$$

This is the integral kernel of the operator $L_\lambda = \tilde{K}_\lambda(I + K_\lambda)$. We introduce the sequence of functions

$$L_\lambda^{(n+1)}(x,y) = \int_{\partial D} L_\lambda^{(n)}(x,z)L_\lambda^{(1)}(z,y)\,dS_z, \quad n = 1, 2, \ldots,$$
$$L_\lambda^{(1)}(x,y) = L_\lambda(x,y).$$

This is the integral kernel of the operator $(L_\lambda)^n = (\tilde{K}_\lambda(I + K_\lambda))^n$.

In this section we always assume that ∂D is strictly convex and ∂D is of class C^{2,α_0} with $0 < \alpha_0 \le 1$, and the kernels $K_\lambda(x,y)$ and $\tilde{K}_\lambda(x,y)$ satisfy all conditions described in Section 1. The main result of this section is the following.

THEOREM 5.1

There exist positive constants C and $\mu_0 \ge 1$ depending only on C_0 in (1.1), C_1 in (1.2), and ∂D such that

(i) *the operator $L_\lambda(I - L_\lambda)^{-1}$ with $\lambda \in \mathbf{C}_{\delta_0}$ and $\operatorname{Re}\lambda = \mu \ge \mu_0$ has an integral kernel $L_\lambda^\infty(x,y)$;*

(ii) *the integral kernel* $\tilde{L}_\lambda^\infty(x,y)$ *of the operator* $L_\lambda^3(I-L_\lambda)^{-1}$ *is continuous on* $\partial D\times\partial D$ *and satisfies*

$$\left|\tilde{L}_\lambda^\infty(x,y)\right|\le C\mu^{-2/3}(1+\log\mu)e^{-\mu|x-y|},\quad x,y\in\partial D,\lambda\in\mathbf{C}_{\delta_0},\mu=\mathrm{Re}\,\lambda\ge\mu_0;$$

(iii) $L_\lambda^\infty(x,y)$ *in (i) is measurable on* $\partial D\times\partial D$*, continuous for* $(x,y)\in\partial D\times\partial D$ *with* $x\ne y$*, and has the estimate*

$$\left|L_\lambda^\infty(x,y)\right|\le\frac{Ce^{-\mu|x-y|}}{|x-y|},\quad x,y\in\partial D,x\ne y,\lambda\in\mathbf{C}_{\delta_0},\mu=\mathrm{Re}\,\lambda\ge\mu_0.$$

Theorem 5.1 is based on the following estimates for $L_\lambda^{(n)}(x,y)$ as derived in Section 5.1.

PROPOSITION 5.2
There exists a positive constant C *depending only on* ∂D*,* C_0 *in (1.1), and* C_1 *in (1.2) such that the following holds for all* $x,y\in\partial D$ *with* $x\ne y$ *and all* $\lambda\in\mathbf{C}_{\delta_0}$ *with* $\mu=\mathrm{Re}\,\lambda\ge 1$*:*

$$\left|L_\lambda^{(1)}(x,y)\right|\le\frac{Ce^{-\mu|x-y|}}{|x-y|},\tag{5.2}$$

$$\left|L_\lambda^{(2)}(x,y)\right|\le Ce^{-\mu|x-y|}\left(1+\max\left\{0,\log\frac{r_0}{|x-y|}\right\}\right),\tag{5.3}$$

$$\left|L_\lambda^{(3)}(x,y)\right|\le C\mu^{-2/3}(1+\log\mu)e^{-\mu|x-y|}.\tag{5.4}$$

PROPOSITION 5.3
There exists a positive constant C *depending only on* ∂D*,* C_0 *in (1.1), and* C_1 *in (1.2) such that, for all* $x,y\in\partial D$ *with* $x\ne y$ *and all* $\lambda\in\mathbf{C}_{\delta_0}$ *with* $\mu=\mathrm{Re}\,\lambda\ge 1$*,*

$$\left|L_\lambda^{(n+2)}(x,y)\right|\le C^n\mu^{-2n/3}(1+\log\mu)e^{-\mu|x-y|},\quad n=1,2,3,\ldots.\tag{5.5}$$

The proof of Propositions 5.2 and 5.3 are given in the next subsection.

5.1. Two lemmas and proof of Propositions 5.2 and 5.3

First we describe two lemmas needed for the proof of Propositions 5.2 and 5.3. In the following we set $\rho_0=|x-y|$.

LEMMA 5.1
There exists a positive constant $\tilde{C}$ *such that the following holds for any* $\mu\ge 1$*:*

$$\int_{S_{\rho_0}^-(x)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\,dS_z\le\tilde{C}\mu^{-1}e^{-\mu\rho_0},\quad x,y\in\partial D,\tag{5.6}$$

$$\begin{aligned}&\int_{S_{\rho_0}^-(x)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\,dS_z\\&\quad\le\tilde{C}e^{-\mu\rho_0}\left(1+\max\left\{0,\log\frac{r_0}{\rho_0}\right\}\right),\quad x,y\in\partial D,x\ne y,\end{aligned}\tag{5.7}$$

$$\int_{S_{\rho_0}(x)\cap S^-_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\,dS_z \le \tilde{C}\mu^{-1}e^{-\mu\rho_0}, \quad x,y\in\partial D, \tag{5.8}$$

$$\begin{aligned}&\int_{S_{\rho_0}(x)\cap S^-_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\,dS_z \\ &\quad\le \tilde{C}e^{-\mu\rho_0}\Big(1+\max\Big\{0,\log\frac{r_0}{\rho_0}\Big\}\Big), \quad x,y\in\partial D, x\ne y,\end{aligned} \tag{5.9}$$

$$\begin{aligned}&\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\,dS_z \\ &\quad\le \tilde{C}e^{-\mu\rho_0}\min\Big\{\frac{1}{\mu\rho_0},\mu^{-2/3}\Big\}, \quad x,y\in\partial D,\end{aligned} \tag{5.10}$$

$$\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\,dS_z \le \tilde{C}e^{-\mu\rho_0}, \quad x,y\in\partial D, x\ne y. \tag{5.11}$$

LEMMA 5.2

There exists a positive constant $\tilde{C}$ such that the following holds for all $x,y\in\partial D$ with $x\ne y$ and $\mu\ge 1$:

$$\int_{S^-_{\rho_0}(x)\cap S_{r_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z \le \tilde{C}\mu^{-2/3}e^{-\mu\rho_0}, \tag{5.12}$$

$$\int_{S_{\rho_0}(x)\cap S^-_{\rho_0}(y)\cap S_{r_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z \le \tilde{C}\mu^{-2/3}e^{-\mu\rho_0}, \tag{5.13}$$

$$\begin{aligned}&\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)\cap S_{r_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z \\ &\quad\le \tilde{C}\mu^{-2/3}(1+\log\mu)e^{-\mu\rho_0}.\end{aligned} \tag{5.14}$$

The proofs of those lemmas are given in Sections 5.2 and 5.3.

Proof of Proposition 5.2

From (1.1) and (1.2) we have

$$\big|L^{(1)}_\lambda(x,y)\big| \le C_1\Big(\frac{e^{-\mu|x-y|}}{|x-y|}+C_0\mu\int_{\partial D}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\,dS_z\Big). \tag{5.15}$$

Noting that $\partial D = S^-_{\rho_0}(x)\cup(S_{\rho_0}(x)\cap S^-_{\rho_0}(y))\cup(S_{\rho_0}(x)\cap S_{\rho_0}(y))$ and $\rho_0\le \operatorname{diam} D$ we obtain from (5.6), (5.8), and (5.10),

$$\int_{\partial D}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\,dS_z \le 2\tilde{C}\mu^{-1}e^{-\mu\rho_0}+\tilde{C}e^{-\mu\rho_0}\frac{1}{\mu\rho_0}\le\frac{Ce^{-\mu\rho_0}}{\mu\rho_0}. \tag{5.16}$$

A combination of (5.15) and (5.16) yields (5.2).

From (5.2) we get

$$\big|L^{(2)}_\lambda(x,y)\big| \le C^2\int_{\partial D}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\,dS_z. \tag{5.17}$$

Using the same decomposition of ∂D as above, we obtain from (5.7), (5.9), and (5.11),

$$\begin{aligned}
&\int_{\partial D} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\,dS_z \\
&\le 2\tilde{C}e^{-\mu\rho_0}\Big(1+\max\Big\{0,\log\frac{r_0}{\rho_0}\Big\}\Big)+\tilde{C}e^{-\mu\rho_0} \\
&\le 3\tilde{C}e^{-\mu\rho_0}\Big(1+\max\Big\{0,\log\frac{r_0}{\rho_0}\Big\}\Big).
\end{aligned} \tag{5.18}$$

This together with (5.17) gives (5.3).

Using (5.2) and (5.3), we obtain

$$\begin{aligned}
&\big|L_\lambda^{(3)}(x,y)\big| \\
&\le C\Big(\int_{\partial D} \frac{e^{-\mu(|x-z|+|z-y|)}}{|z-y|}\,dS_z \\
&\quad+\int_{\partial D} \frac{e^{-\mu(|x-z|+|z-y|)}}{|z-y|}\max\Big\{0,\log\frac{r_0}{|x-z|}\Big\}\,dS_z\Big).
\end{aligned} \tag{5.19}$$

It follows from (5.6), (5.8), and (5.10) that, for all $\mu \ge 1$,

$$\begin{aligned}
\int_{\partial D} \frac{e^{-\mu(|x-z|+|z-y|)}}{|z-y|}\,dS_z &\le 2\tilde{C}\mu^{-1}e^{-\mu\rho_0}+\tilde{C}\mu^{-2/3}e^{-\mu\rho_0} \\
&\le 3\tilde{C}\mu^{-2/3}e^{-\mu\rho_0}.
\end{aligned} \tag{5.20}$$

Noting that $\log(r_0/|x-z|)\ge 0$ on $S_{r_0}(x)$ and $\log(r_0/|x-z|)\le 0$ outside $S_{r_0}(x)$ and using the decomposition $S_{r_0}(x)=(S^-_{\rho_0}(y)\cap S_{r_0}(x))\cup(S_{\rho_0}(y)\cap S^-_{\rho_0}(x)\cap S_{r_0}(x))\cup(S_{\rho_0}(y)\cap S_{\rho_0}(x)\cap S_{r_0}(x))$, we obtain from (5.12)–(5.14) by changing the roles of x and y therein,

$$\begin{aligned}
&\int_{\partial D} \frac{e^{-\mu(|x-z|+|z-y|)}}{|z-y|}\max\Big\{0,\log\frac{r_0}{|x-z|}\Big\}\,dS_z \\
&=\int_{S_{r_0}(x)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|z-y|}\log\frac{r_0}{|x-z|}\,dS_z \\
&\le \tilde{C}e^{-\mu\rho_0}\{2\mu^{-2/3}+\mu^{-2/3}(1+\log\mu)\}.
\end{aligned}$$

This together with (5.19) and (5.20) yields (5.4). This completes the proof of Proposition 5.2. □

Proof of Proposition 5.3

We denote by C_4 and $\tilde{C}$ the constants appearing in Proposition 5.2 and Lemma 5.1, respectively. Set $C=C_4\max\{3\tilde{C},1\}$. Note that we have (5.20) with the constant $\tilde{C}$ in Lemma 5.1. We prove (5.5) by an induction argument. In the case when $n=1$ it follows from (5.4) that (5.5) is valid. Assume that (5.5) is valid for some n. From (5.2) we get

$$\left|L_\lambda^{(n+3)}(x,y)\right| \le C^n C_4 \mu^{-2n/3}(1+\log\mu)\int_{\partial D}\frac{e^{-\mu(|x-z|+|z-y|)}}{|z-y|}\,dS_z.$$

This together with (5.20) yields

$$\left|L_\lambda^{(n+3)}(x,y)\right| \le C^{n+1}\mu^{-2(n+1)/3}(1+\log\mu)e^{-\mu|x-y|}.$$

This completes the proof of Proposition 5.3. □

5.2. Proof of Lemma 5.1

Proof of (5.6)

Since $|x-z|\ge\rho_0$ for $z\in S^-_{\rho_0}(x)$, we have

$$\begin{aligned}\int_{S^-_{\rho_0}(x)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\,dS_z &\le e^{-\mu\rho_0}\int_{S^-_{\rho_0}(x)}\frac{e^{-\mu|z-y|}}{|x-z|}\,dS_z\\ &\le e^{-\mu\rho_0}\int_{\partial D}\frac{e^{-\mu|z-y|}}{|x-z|}\,dS_z.\end{aligned}\tag{5.21}$$

Note that ∂D has the decomposition

$$\partial D = S_1\cup S_2,\tag{5.22}$$

where

$$S_1=\left\{z\in\partial D\,\middle|\,|x-z|\le|y-z|\right\},\qquad S_2=\left\{z\in\partial D\,\middle|\,|x-z|\ge|y-z|\right\}.$$

On S_1,

$$\frac{e^{-\mu|z-y|}}{|y-z|}\le\frac{e^{-\mu|z-y|}}{|x-z|}\le\frac{e^{-\mu|x-z|}}{|x-z|}.\tag{5.23}$$

On S_2,

$$\frac{e^{-\mu|z-y|}}{|x-z|}\le\frac{e^{-\mu|z-y|}}{|y-z|}.\tag{5.24}$$

Thus this together with Lemma 3.1(ii) yields

$$\begin{aligned}\int_{\partial D}\frac{e^{-\mu|z-y|}}{|x-z|}\,dS_z &\le \int_{\partial D}\frac{e^{-\mu|x-z|}}{|x-z|}\,dS_z+\int_{\partial D}\frac{e^{-\mu|y-z|}}{|y-z|}\,dS_z\\ &\le 2C\mu^{-1}\Big(1+\frac{\mu e^{-\mu r_0}}{r_0}\Big).\end{aligned}\tag{5.25}$$

Now from this and (5.21) we obtain (5.6). □

Proof of (5.7)

We have

$$\int_{S^-_{\rho_0}(x)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\,dS_z\le e^{-\mu\rho_0}(I_1+I_2+I_3),$$

where $\rho_0' = \min\{\rho_0, r_0\}$ and

$$I_1 = \int_{S^-_{\rho_0}(x)\cap S_{\rho_0'}(y)} \frac{e^{-\mu|z-y|}}{|x-z||z-y|}\,dS_z,$$

$$I_2 = \int_{S^-_{\rho_0}(x)\cap (S_{r_0}(y)\setminus S_{\rho_0'}(y))} \frac{e^{-\mu|z-y|}}{|x-z||z-y|}\,dS_z,$$

$$I_3 = \int_{S^-_{r_0}(y)} \frac{e^{-\mu|z-y|}}{|x-z||z-y|}\,dS_z.$$

Since we have

$$I_1 \le \frac{1}{\rho_0'} \int_{S_{\rho_0'}(y)} \frac{e^{-\mu|z-y|}}{|z-y|}\,dS_z,$$

it follows from Lemma 3.1(i) that $I_1 \le C$.

Using the decomposition (5.22) and estimates (5.23) and (5.24), we get

$$\begin{aligned} I_2 \le &\int_{S^-_{\rho_0}(x)\cap (S_{r_0}(y)\setminus S_{\rho_0'}(y))\cap S_1} \frac{e^{-\mu|x-z|}}{|x-z|^2}\,dS_z \\ &+ \int_{S^-_{\rho_0}(x)\cap (S_{r_0}(y)\setminus S_{\rho_0'}(y))\cap S_2} \frac{e^{-\mu|y-z|}}{|y-z|^2}\,dS_z. \end{aligned} \tag{5.26}$$

Note that the sets $S^-_{\rho_0}(x)\cap (S_{r_0}(y)\setminus S_{\rho_0'}(y))\cap S_1$ and $S^-_{\rho_0}(x)\cap (S_{r_0}(y)\setminus S_{\rho_0'}(y))\cap S_2$ are contained in the sets $S_{r_0}(x)\cap S^-_{\rho_0'}(x)$ and $S_{r_0}(y)\cap S^-_{\rho_0'}(y)$, respectively. Therefore one can apply Lemma 3.1(iii) to the right-hand side of (5.26) and get

$$I_2 \le C\Big(1+\log\frac{r_0}{c\rho_0'}\Big) \le \tilde{C}\Big(1+\log\frac{r_0}{\rho_0'}\Big) = \tilde{C}\Big(1+\max\Big\{0, \log\frac{r_0}{\rho_0}\Big\}\Big).$$

Finally, using (5.25) and $\mu \ge 1$, we get

$$I_3 \le \frac{1}{r_0}\int_{\partial D} \frac{e^{-\mu|z-y|}}{|x-z|}\,dS_z \le \frac{2C}{r_0}\Big(1+\frac{1}{r_0}\Big).$$

Summing up the estimates for I_1, I_2, and I_3 above, we obtain (5.7). □

Proof of (5.8)

Since $|z-y| \ge \rho_0$ for $z \in S_{\rho_0}(x)\cap S^-_{\rho_0}(y)$, we have

$$\int_{S_{\rho_0}(x)\cap S^-_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\,dS_z \le e^{-\mu\rho_0}\int_{\partial D} \frac{e^{-\mu|x-z|}}{|x-z|}\,dS_z.$$

From this and Lemma 3.1(ii) we obtain (5.8). □

Proof of (5.9)

Since $S_{\rho_0}(x)\cap S^-_{\rho_0}(y)$ is contained in $S^-_{\rho_0}(y)$ and the integrand of (5.9) is invariant under the change of the role of x and y, (5.9) is a direct consequence of (5.7). □

Proof of (5.10)
Let r_1 be the same as in Proposition 3.1. For $\rho_0' = \min\{\rho_0, r_1\}$ we have

$$\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\, dS_z \le J_1 + J_2,$$

where

$$J_1 = \int_{(S_{\rho_0}(x)\setminus S_{\rho_0'/2}(x))\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\, dS_z,$$

$$J_2 = \int_{S_{\rho_0'/2}(x)\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\, dS_z.$$

Consider first J_1. One has

$$J_1 \le \frac{2}{\rho_0'} \int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z.$$

Note that if $\rho_0 \le r_1$, then $\rho_0' = \rho_0$ and it follows from (3.4) that

$$\begin{aligned}\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z &\le \tilde{C} e^{-\mu\rho_0} \Big(\frac{\rho_0^{3/2}}{\sqrt{\mu}}\Big)^{8/9} \Big(\frac{1}{\mu^2\rho_0^3}\Big)^{1/9}\\ &= \tilde{C} e^{-\mu\rho_0} \rho_0' \mu^{-2/3}\end{aligned}$$

and

$$\begin{aligned}\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z &\le \tilde{C} e^{-\mu\rho_0} \Big(\frac{\rho_0^{3/2}}{\sqrt{\mu}}\Big)^{2/3} \Big(\frac{1}{\mu^2\rho_0^3}\Big)^{1/3}\\ &= \tilde{C} e^{-\mu\rho_0} \mu^{-1};\end{aligned}$$

if $\rho_0 \ge r_1$, then $\rho_0' = r_1$ and from (3.5) we have

$$\begin{aligned}\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z &\le \tilde{C} e^{-\mu\rho_0} \mu^{-2} = \tilde{C} e^{-\mu\rho_0} \mu^{-2} \frac{r_1}{r_1}\\ &\le \tilde{C} r_1^{-1} e^{-\mu\rho_0} \rho_0' \mu^{-2/3}.\end{aligned}$$

Thus one gets

$$\begin{aligned}J_1 &\le \frac{2}{\rho_0'} C e^{-\mu\rho_0} \min\{\mu^{-1}, \rho_0'\mu^{-2/3}\} = 2C e^{-\mu\rho_0} \min\Big\{\frac{1}{\mu\rho_0'}, \mu^{-2/3}\Big\}\\ &\le C' e^{-\mu\rho_0} \min\Big\{\frac{1}{\mu\rho_0}, \mu^{-2/3}\Big\}.\end{aligned}$$

Note that we have used the fact that if $\rho_0 \ge r_1$, then

$$(\rho_0')^{-1} = r_1^{-1} = \rho_0^{-1}\rho_0 r_1^{-1} \le \big(r_1^{-1} \max_{x,y\in\partial D} |x-y|\big)\rho_0^{-1}.$$

Now consider J_2. Recalling the argument in the proof of (3.8), we see that if $\rho_0 \le r_1$, then $\rho_0' = \rho_0$ and $S_{\rho_0'/2}(x) \subset B_x(0, 2\sigma_1^0/3)$. Moreover, in the proof of Proposition 3.1, we have already known that, for all $z = x + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma)\nu_x$

with $\sigma=(r\cos\theta,r\sin\theta)\in B'_x(0,2\sigma_1^0/3)$,

$$|x-z|+|z-y|\geq\rho_0+c_2r\big(\rho_0^2+f(\theta)^2\big).$$

Therefore we obtain

$$\begin{aligned} J_2&\leq\tilde{C}e^{-\mu\rho_0}\int_0^{\rho_0}\int_{-\pi}^{\pi}\frac{e^{-\mu c_2r(\rho_0^2+f(\theta)^2)}}{r}r\,dr\,d\theta\\ &\leq 2\tilde{C}e^{-\mu\rho_0}\int_0^{\rho_0}\int_{-\pi/2}^{\pi/2}e^{-\mu c_2r(\rho_0^2+\theta^2)}\,dr\,d\theta \qquad (5.27)\\ &\leq Ce^{-\mu\rho_0}\mu^{-1/2}\int_0^{\rho_0}\frac{e^{-\mu c_2r\rho_0^2}}{\sqrt{r}}\,dr. \end{aligned}$$

Since

$$\int_0^{\rho_0}\frac{e^{-\mu c_2r\rho_0^2}}{\sqrt{r}}\,dr\leq\frac{C'}{\sqrt{\mu}\rho_0}$$

and

$$\int_0^{\rho_0}\frac{e^{-\mu c_2r\rho_0^2}}{\sqrt{r}}\,dr\leq C''\sqrt{\rho_0},$$

we have

$$\begin{aligned} \int_0^{\rho_0}\frac{e^{-\mu c_2r\rho_0^2}}{\sqrt{r}}\,dr&\leq C\min\Big\{\frac{1}{\sqrt{\mu}\rho_0},\Big(\frac{1}{\sqrt{\mu}\rho_0}\Big)^{1/3}(\sqrt{\rho_0})^{2/3}\Big\}\\ &=C\min\{\mu^{-1/2}\rho_0^{-1},\mu^{-1/6}\}. \end{aligned}$$

This together with (5.27) yields

$$J_2\leq\tilde{C}e^{-\mu\rho_0}\min\{\mu^{-1}\rho_0^{-1},\mu^{-2/3}\}. \qquad (5.28)$$

If $\rho_0\geq r_1$, then $\rho_0'=r_1$ and from Proposition 2.2(ii) we obtain, for all $z\in S_{\rho_0'/2}(x)$,

$$|x-z|+|z-y|\geq\rho_0+c_0|z-x|.$$

This together with Lemma 3.1(ii) yields

$$\begin{aligned} J_2&\leq e^{-\mu\rho_0}\int_{S_{\rho_0'/2}(x)}\frac{e^{-\mu c_0|z-x|}}{|z-x|}\,dS_z\\ &\leq e^{-\mu\rho_0}\int_{\partial D}\frac{e^{-\mu c_0|z-x|}}{|z-x|}\,dS_z\\ &\leq Ce^{-\mu\rho_0}\mu^{-1}. \end{aligned}$$

Since

$$\frac{1}{\mu}=\frac{\rho_0}{\mu\rho_0}\leq\frac{C}{\mu\rho_0}$$

and $\mu^{-1}\leq\mu^{-2/3}$ for $\mu\geq 1$ we conclude that (5.28) is true also in the case when $\rho_0\geq r_1$. Thus, we have (5.10). $\square$

Proof of (5.11)

Consider first the case when $\rho_0 \geq r_1$. We have

$$\begin{aligned} I &\equiv \int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\, dS_z \\ &= \int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)} \Big(\frac{1}{|x-z|} + \frac{1}{|z-y|}\Big) \frac{dS_z}{|x-z|+|z-y|} \\ &\leq \frac{1}{\rho_0} \int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)} \Big(\frac{1}{|x-z|} + \frac{1}{|z-y|}\Big)\, dS_z. \end{aligned}$$

Then a combination of this and (5.10) ensures that integral I has a bound $\tilde{C}e^{-\mu\rho_0}$.

Now consider the case when $\rho_0 \leq r_1$. Decompose

$$\begin{aligned} I &= \int_{S_{\rho_0}(x)\cap(S_{\rho_0}(y)\setminus S_{\rho_0/2}(y))} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\, dS_z \\ &\quad + \int_{S_{\rho_0}(x)\cap S_{\rho_0/2}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\, dS_z \\ &\equiv I_1(x,y) + I_2(x,y). \end{aligned}$$

Using the same local coordinates as in the proof of (4.9), and noting (4.12), we obtain

$$\begin{aligned} I_1(x,y) &\leq \frac{2}{\rho_0} \int_{S_{\rho_0}(x)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\, dS_z \\ &\leq \frac{\tilde{C}e^{-\mu\rho_0}}{\rho_0} \int_{-\pi}^{\pi} d\theta \int_0^{r(\theta)} \frac{e^{-\mu c\sqrt{r^2+g(r\cos\theta, r\sin\theta)^2}f(\theta)^2}}{\sqrt{r^2+g(r\cos\theta, r\sin\theta)^2}}\, r\,dr \\ &\leq \frac{\tilde{C}e^{-\mu\rho_0}}{\rho_0} \int_{-\pi}^{\pi} d\theta \int_0^{\rho_0} e^{-\mu c\rho f(\theta)^2}\, d\rho \leq Ce^{-\mu\rho_0}. \end{aligned}$$

Since it holds that $|z-x| \geq |x-y| - |y-z| \geq \rho_0 - \rho_0/2 = \rho_0/2$ for all $z \in S_{\rho_0/2}(y)$, we have

$$I_2(x,y) \leq \int_{(S_{\rho_0}(x)\setminus S_{\rho_0/2}(x))\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\, dS_z = I_1(y,x) \leq Ce^{-\mu\rho_0}.$$

Therefore I has a bound $\tilde{C}e^{-\mu\rho_0}$ also in the case when $\rho_0 \leq r_1$. This completes the proof of (5.11). □

5.3. Proof of Lemma 5.2

Proof of (5.12)

Let S_1 and S_2 be the same as in (5.22). We have $S_{r_0}(y)\cap S_1 \subset S_{r_0}(x)$ and the following estimates hold: on $S_{r_0}(y)\cap S_1$,

$$\frac{e^{-\mu|z-y|}}{|x-z|}\log\frac{r_0}{|z-y|} \leq \frac{e^{-\mu|x-z|}}{|x-z|}\log\frac{r_0}{|x-z|};$$

on $S_{r_0}(y)\cap S_2$,

$$\frac{e^{-\mu|z-y|}}{|x-z|}\log\frac{r_0}{|z-y|}\le\frac{e^{-\mu|y-z|}}{|y-z|}\log\frac{r_0}{|y-z|}.$$

These together with Lemma 3.1(iv) yield

$$\begin{aligned}
&\int_{S_{r_0}(y)}\frac{e^{-\mu|z-y|}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z\\
&=\int_{S_{r_0}(y)\cap S_1}\frac{e^{-\mu|z-y|}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z+\int_{S_{r_0}(y)\cap S_2}\frac{e^{-\mu|z-y|}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z\\
&\le\int_{S_{r_0}(x)}\frac{e^{-\mu|x-z|}}{|x-z|}\log\frac{r_0}{|x-z|}\,dS_z+\int_{S_{r_0}(y)}\frac{e^{-\mu|y-z|}}{|y-z|}\log\frac{r_0}{|y-z|}\,dS_z\le\tilde{C}\mu^{-2/3}.
\end{aligned}$$

Now (5.12) is a direct consequence of this and the estimate

$$\begin{aligned}
&\int_{S^-_{\rho_0}(x)\cap S_{r_0}(y)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z\\
&\quad\le e^{-\mu\rho_0}\int_{S_{r_0}(y)}\frac{e^{-\mu|z-y|}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z.
\end{aligned}$$

□

Proof of (5.13)
If $r_0<\rho_0$, then $S_{\rho_0}(x)\cap S^-_{\rho_0}(y)\cap S_{r_0}(y)=\emptyset$. Thus it suffices to consider only the case $\rho_0\le r_0$. Since $S_{\rho_0}(x)\cap S^-_{\rho_0}(y)\cap S_{r_0}(y)\subset S_1$, we have

$$\begin{aligned}
&\int_{S_{\rho_0}(x)\cap S^-_{\rho_0}(y)\cap S_{r_0}(y)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z\\
&\quad\le e^{-\mu\rho_0}\int_{S_{\rho_0}(x)}\frac{e^{-\mu|x-z|}}{|x-z|}\log\frac{r_0}{|z-x|}\,dS_z.
\end{aligned}$$

Now from Lemma 3.1(iv) one gets (5.13). □

Proof of (5.14)
Set $\rho_0'=\min\{\rho_0,r_0\}$. Decompose

$$\begin{aligned}
&\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)\cap S_{r_0}(y)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z\\
&=\int_{S_{\rho_0'}(x)\cap S_{\rho_0'}(y)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z\\
&\quad+\int_{(S_{\rho_0}(x)\setminus S_{\rho_0'}(x))\cap S_{\rho_0'}(y)}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-y|}\,dS_z\\
&\equiv J_1+J_2.
\end{aligned}$$

Consider first J_2. If $\rho_0\le r_0$, then $J_2=0$. If $\rho_0>r_0$, then $\rho_0'=r_0$. In this case we have

$$
\begin{aligned}
J_2 &\le \frac{1}{r_0}\int_{(S_{\rho_0}(x)\setminus S_{r_0}(x))\cap S_{r_0}(y)} e^{-\mu(|x-z|+|z-y|)}\log\frac{r_0}{|z-y|}\,dS_z \\
&= \int_{(S_{\rho_0}(x)\setminus S_{r_0}(x))\cap S_{r_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|z-y|}\frac{|z-y|}{r_0}\log\frac{r_0}{|z-y|}\,dS_z \qquad (5.29)\\
&\le \tilde{C}\int_{S_{\rho_0}(x)\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|z-y|}\,dS_z \le Ce^{-\mu\rho_0}\mu^{-2/3}.
\end{aligned}
$$

Note that we made use of the fact $\sup_{x\ge 1} x^{-1}\log x<\infty$ and (5.10) of Lemma 5.1.

Now consider J_1. Using the decomposition $S_{\rho_0'}(x)\cap S_{\rho_0'}(y)=(S_{\rho_0'}(x)\cap S_{\rho_0'}(y)\cap S_1)\cup(S_{\rho_0'}(x)\cap S_{\rho_0'}(y)\cap S_2)$, we have

$$
\begin{aligned}
J_1 &\le \int_{S_{\rho_0'}(x)\cap S_{\rho_0'}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-x|}\,dS_z \\
&\quad + \int_{S_{\rho_0'}(x)\cap S_{\rho_0'}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|y-z|}\log\frac{r_0}{|z-y|}\,dS_z \\
&\equiv J_1'(x,y)+J_1'(y,x).
\end{aligned}
$$

Thus if we have the estimate

$$
J_1'(x,y)\le \tilde{C}e^{-\mu\rho_0}\mu^{-2/3}(1+\log\mu),\quad x,y\in\partial D, \qquad (5.30)
$$

then for $J_1'(y,x)$, hence, for J_1, the same estimate as (5.30) is also valid. A combination of this and (5.29) yields (5.14).

Estimate (5.30) is proved as follows. We have

$$
\begin{aligned}
J_1'(x,y) &= \int_{(S_{\rho_0'}(x)\setminus S_{\rho_0'/2}(x))\cap S_{\rho_0'}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-x|}\,dS_z \\
&\quad + \int_{S_{\rho_0'/2}(x)\cap S_{\rho_0'}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-x|}\,dS_z \\
&\equiv J_1''+J_2''.
\end{aligned}
$$

Consider first J_1''. We have

$$
J_1''\le\frac{2}{\rho_0'}\log\frac{2r_0}{\rho_0'}\int_{(S_{\rho_0'}(x)\setminus S_{\rho_0'/2}(x))\cap S_{\rho_0'}(y)} e^{-\mu(|x-z|+|z-y|)}\,dS_z. \qquad (5.31)
$$

If $\rho_0\ge r_0$, then $\rho_0'=r_0$ and from (3.6) and (5.31) one gets

$$
\begin{aligned}
J_1'' &\le \frac{2}{r_0}\log\frac{2r_0}{r_0}\int_{\partial D} e^{-\mu(|x-z|+|z-y|)}\,dS_z \qquad (5.32)\\
&\le C\mu^{-2}e^{-\mu\rho_0}\Big(1+\frac{1}{\rho_0^3}\Big)\le Ce^{-\mu\rho_0}\mu^{-2}\Big(1+\frac{1}{r_0^3}\Big).
\end{aligned}
$$

If $\rho_0<r_0$, then $\rho_0'=\rho_0$. Let r_1 be the same as in the proof of Proposition 3.1. We further divide this case into two: (a) $\rho_0>r_1$; (b) $\rho_0\le r_1$.

Case (a). In this case $\rho_0' = \rho_0 > r_1$. A combination of (5.31) and (3.5) gives

$$J_1'' \le \frac{2}{r_1} \log \frac{2r_0}{r_1} \int_{S_{\rho_0}(x) \cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z \le C e^{-\mu\rho_0} \mu^{-2}. \tag{5.33}$$

Case (b). From (3.4) and (5.31) one gets

$$\begin{aligned} J_1'' &\le \frac{2}{\rho_0} \log \frac{2r_0}{\rho_0} \int_{(S_{\rho_0}(x) \setminus S_{\rho_0/2}(x)) \cap S_{\rho_0}(y)} e^{-\mu(|x-z|+|z-y|)}\, dS_z \\ &\le \tilde{C} e^{-\mu\rho_0} \frac{2}{\rho_0} \log \frac{2r_0}{\rho_0} \min\Big\{ \frac{\rho_0^{3/2}}{\sqrt{\mu}}, \frac{1}{\mu^2 \rho_0^3} \Big\} \\ &= \frac{C e^{-\mu\rho_0}}{r_0} \min\Big\{ \frac{r_0^{3/2}}{\sqrt{\mu}} X^{-1/2} \log 2X, \frac{1}{r_0^3 \mu^2} X^4 \log 2X \Big\}, \end{aligned} \tag{5.34}$$

where $X = r_0/\rho_0$ (≥ 1).

Let μ (≥ 1) satisfy $\mu\rho_0^3 \le 2^3 e^{-6} r_0^3$. Since $e^2/2 \le (e^2/2)\mu^{1/3} \le X$ and the function $\xi \longmapsto \xi^{-1/2} \log 2\xi$ is monotone decreasing on $[e^2/2, \infty[$, it follows from (5.34) that

$$\begin{aligned} J_1'' &\le \frac{C e^{-\mu\rho_0}}{r_0} \frac{r_0^{3/2}}{\sqrt{\mu}} X^{-1/2} \log 2X \\ &\le \frac{C e^{-\mu\rho_0}}{r_0} \frac{r_0^{3/2}}{\sqrt{\mu}} \big((e^2/2)\mu^{1/3} \big)^{-1/2} \log\big\{ 2(e^2/2)\mu^{1/3} \big\} \\ &\le \tilde{C} e^{-\mu\rho_0} \mu^{-2/3} (1 + \log\mu). \end{aligned} \tag{5.35}$$

Let μ (≥ 1) satisfy $\mu\rho_0^3 > 2^3 e^{-6} r_0^3$. We have $(e^2/2)\mu^{1/3} > X$. Since the function $\xi \longmapsto \xi^4 \log 2\xi$ is monotone increasing on $[1, \infty[$, it follows from (5.34) that

$$\begin{aligned} J_1'' &\le \frac{C e^{-\mu\rho_0}}{r_0} \frac{1}{r_0^3 \mu^2} X^4 \log 2X \\ &\le \frac{C e^{-\mu\rho_0}}{r_0} \frac{1}{r_0^3 \mu^2} \big\{ (e^2/2)\mu^{1/3} \big\}^4 \log 2\big\{ (e^2/2)\mu^{1/3} \big\} \\ &\le \tilde{C} e^{-\mu\rho_0} \mu^{-2/3} (1 + \log\mu). \end{aligned} \tag{5.36}$$

Therefore from (5.32), (5.33), (5.35), and (5.36) one gets

$$J_1'' \le \tilde{C} e^{-\mu\rho_0} \mu^{-2/3} (1 + \log\mu). \tag{5.37}$$

Next we consider J_2''.

Case (i): $\rho_0 \ge r_0$. In this case $\rho_0' = r_0$. Since $|x - y| = \rho_0 \ge r_0$ and $|z - x| \le r_0/2$ for $z \in S_{r_0/2}(x)$, using Proposition 2.2(ii), we obtain

$$\begin{aligned} J_2'' &\le e^{-\mu\rho_0} \int_{S_{r_0/2}(x) \cap S_{r_0}(y)} \frac{e^{-\mu c_0 |z-x|}}{|x-z|} \log \frac{r_0}{|z-x|}\, dS_z \\ &\le e^{-\mu\rho_0} \int_{S_{r_0}(x)} \frac{e^{-\mu c_0 |z-x|}}{|x-z|} \log \frac{r_0}{|z-x|}\, dS_z. \end{aligned}$$

Now Lemma 3.1(iv) yields

$$J_2'' \le Ce^{-\mu\rho_0}\mu^{-2/3}. \tag{5.38}$$

Case (ii): $\rho_0 < r_0$. In this case, $\rho_0' = \rho_0$. We divide this case into two subcases: (a) $r_1 < \rho_0$; (b) $\rho_0 \le r_1$:

(a) Divide J_2'' into two parts:

$$\begin{aligned} J_2'' = &\int_{S_{r_1/2}(x)\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-x|}\,dS_z \\ &+ \int_{(S_{\rho_0/2}(x)\setminus S_{r_1/2}(x))\cap S_{\rho_0}(y)} \frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\log\frac{r_0}{|z-x|}\,dS_z. \end{aligned} \tag{5.39}$$

One can apply Proposition 2.2(ii) to the first term in the right-hand side of (5.39) and a similar argument done in case (i) above together with Lemma 3.1(iv) yields the bound $Ce^{-\mu\rho_0}\mu^{-2/3}$. For the second term in the right-hand of (5.39) we employ a similar argument done for the derivation of (5.32) and get the same bound. Thus we obtain $J_2'' \le \tilde{C}e^{-\mu\rho_0}\mu^{-2/3}$.

(b) Since $z \in S_{\rho_0/2}(x)\cap S_{\rho_0}(y)$ implies $(z-x)\cdot(y-x)/|y-x| \le \rho_0/2$, using the local coordinates used in the proof of (3.4), we have

$$\begin{aligned} J_2'' &\le \tilde{C}e^{-\mu\rho_0}\int_0^{\rho_0}\int_0^{\pi/2}\frac{e^{-\mu c_2 r(\rho_0^2+\theta^2)}}{r}\Big(\log\frac{r_0}{r}\Big) r\,dr\,d\theta \\ &= \tilde{C}e^{-\mu\rho_0}\int_0^{\rho_0} e^{-\mu c_2 r\rho_0^2}\log\frac{r_0}{r}\Big(\int_0^{\pi/2} e^{-\mu c_2 r\theta^2}\,d\theta\Big)\,dr \\ &\le Ce^{-\mu\rho_0}\mu^{-1/2}\int_0^{\rho_0}\frac{e^{-\mu c_2 r\rho_0^2}}{\sqrt{r}}\log\frac{r_0}{r}\,dr. \end{aligned}$$

Note that we have

$$\begin{aligned} \int_0^{\rho_0}\frac{e^{-\mu c_2 r\rho_0^2}}{\sqrt{r}}\log\frac{r_0}{r}\,dr &\le \int_0^{\rho_0}\frac{1}{\sqrt{r}}\Big(\log\frac{r_0}{r}\Big)\,dr \le \tilde{C}\sqrt{\rho_0}\Big(1+\log\frac{r_0}{\rho_0}\Big) \\ &\le \tilde{C}\sqrt{\rho_0}\Big(\log 3+\log\frac{r_0}{\rho_0}\Big) \le C'\Big(\frac{r_0}{\rho_0}\Big)^{-1/2}\log\frac{3r_0}{\rho_0} \end{aligned}$$

and also

$$\begin{aligned} \int_0^{\rho_0}\frac{e^{-\mu c_2 r\rho_0^2}}{\sqrt{r}}\log\frac{r_0}{r}\,dr &= \frac{1}{\sqrt{\mu}\rho_0}\int_0^{\mu\rho_0^3}\frac{e^{-c_2 r}}{\sqrt{r}}\Big(\log\frac{\mu\rho_0^2 r_0}{r}\Big)\,dr \\ &\le \frac{C}{\sqrt{\mu}\rho_0}\big(1+\big|\log(\mu\rho_0^2 r_0)\big|\big) \\ &\le \frac{C' r_0}{\sqrt{\mu}\rho_0}\Big(1+\log\frac{r_0}{\rho_0}+\big|\log(\mu\rho_0^3)\big|\Big). \end{aligned}$$

Therefore setting $X = r_0/\rho_0$, we obtain

$$J_2'' \le \tilde{C}e^{-\mu\rho_0}\min\Big\{\mu^{-1/2}X^{-1/2}\log 3X, \frac{1}{\mu}X\big(1+\log X+\big|\log(\mu\rho_0^3)\big|\big)\Big\}. \tag{5.40}$$

Let μ (≥ 1) satisfy $\mu\rho_0^3 \leq 3^3 e^{-6} r_0^3$. Since $(e^2/3) \leq (e^2/3)\mu^{1/3} \leq X$ and the function $\xi \longmapsto \xi^{-1/2}\log 3\xi$ is monotone decreasing on $[e^2/3, \infty[$, from (5.40) we obtain

$$J_2'' \leq \tilde{C} e^{-\mu\rho_0}\mu^{-1/2}\big((e^2/3)\mu^{1/3}\big)^{-1/2}\log\big(3(e^2/3)\mu^{1/3}\big) \leq C e^{-\mu\rho_0}\mu^{-2/3}(1+\log\mu).$$

Let μ (≥ 1) satisfy $\mu\rho_0^3 \geq 3^3 e^{-6} r_0^3$. Since $(e^2/3)\mu^{1/3} \geq X \geq 1$ and the function $\xi \longmapsto \xi\log\xi$ is monotone increasing on $]e^{-1}, \infty[$, it follows from (5.40) that

$$\begin{aligned} J_2'' &\leq \tilde{C} e^{-\mu\rho_0}\frac{1}{\mu}(e^2/3)\mu^{1/3}\big(1+\log\big((e^2/3)\mu^{1/3}\big) + \big|\log(\mu\rho_0^3)\big|\big) \\ &\leq C e^{-\mu\rho_0}\mu^{-2/3}(1+\log\mu), \end{aligned}$$

where we used $3^3 e^{-6} r_0^3 \leq \mu\rho_0^3 \leq \mu r_0^3$.

Thus in any subcase of case (ii) we have

$$J_2'' \leq \tilde{C} e^{-\mu\rho_0}\mu^{-2/3}(1+\log\mu), \quad \mu \geq 1.$$

This together with (5.38) yields that J_2'' has the same bound as (5.37) for J_1''. This completes the proof of (5.30). □

5.4. Proof of Theorem 5.1

If $\lambda \in \mathbf{C}_{\delta_0}$ satisfies $\operatorname{Re}\lambda = \mu \gg 1$, then the operator $L_\lambda(I - L_\lambda)^{-1}$ is given by the Neumann series

$$L_\lambda(I-L_\lambda)^{-1} = \sum_{n=1}^{\infty} L_\lambda^n.$$

The integral kernel of the operator L_λ^n is given by $L_\lambda^{(n)}(x,y)$.

Let C be the constant in Proposition 5.3. Choose $\mu_0 \geq 1$ in such a way that $\max\{C,1\}\mu_0^{-2/3} \leq 1/2$. Let $\mu \geq \mu_0$. Then

$$\Big(\frac{C}{\mu^{2/3}}\Big)^n \leq \frac{C}{\mu^{2/3}}\Big(\frac{1}{2}\Big)^{n-1}$$

and from (5.5) we have, for all $x, y \in \partial D$ with $x \neq y$, all $\lambda \in \mathbf{C}_{\delta_0}$ with $\operatorname{Re}\lambda = \mu \geq \mu_0$ and $n = 1, 2, \ldots,$

$$\big|L_\lambda^{(n+2)}(x,y)\big| \leq C\mu^{-2/3}\Big(\frac{1}{2}\Big)^{n-1}(1+\log\mu)e^{-\mu|x-y|}.$$

Therefore the series $\sum_{n=3}^{\infty} L_\lambda^{(n)}(x,y)$ is uniformly convergent and satisfies

$$\sum_{n=3}^{\infty}\big|L_\lambda^{(n)}(x,y)\big| \leq C\mu^{-2/3}(1+\log\mu)e^{-\mu|x-y|}.$$

These yield Theorem 5.1(i) and (ii). Moreover, from this, (5.2), and (5.3) we obtain

$$\begin{aligned} \sum_{n=1}^{\infty}\big|L_\lambda^{(n)}(x,y)\big| \leq{}& \frac{Ce^{-\mu|x-y|}}{|x-y|} + Ce^{-\mu|x-y|}\Big(1+\max\Big\{0, \log\frac{r_0}{|x-y|}\Big\}\Big) \\ &+ C\mu^{-2/3}(1+\log\mu)e^{-\mu|x-y|} \end{aligned}$$

$$\leq \frac{C' e^{-\mu|x-y|}}{|x-y|}\Big(1 + r_0 \frac{|x-y|}{r_0} \max\Big\{0, \log \frac{r_0}{|x-y|}\Big\} + |x-y|\Big)$$
$$\leq \frac{\tilde{C} e^{-\mu|x-y|}}{|x-y|}.$$

This gives Theorem 5.1(iii). □

6. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Since $Y_\lambda = K_\lambda + \tilde{K}_\lambda$, it follows that for $\lambda \in \mathbf{C}_{\delta_0}$ with sufficiently large $\mu = \mathrm{Re}\,\lambda$,

$$(I - Y_\lambda)(I - K_\lambda)^{-1} = I - \tilde{K}_\lambda (I - K_\lambda)^{-1}.$$

Set $L_\lambda = \tilde{K}_\lambda (I - K_\lambda)^{-1}$. The above equality yields

$$\begin{aligned}(I - Y_\lambda)^{-1} &= (I - K_\lambda)^{-1}\big(I - \tilde{K}_\lambda (I - K_\lambda)^{-1}\big)^{-1}\\ &= (I - K_\lambda)^{-1}(I - L_\lambda)^{-1},\end{aligned}$$

which implies that

$$\begin{aligned}(I - Y_\lambda)^{-1} &= \big\{I + K_\lambda (I - K_\lambda)^{-1}\big\}\big\{I + L_\lambda (I - L_\lambda)^{-1}\big\}\\ &= I + K_\lambda (I - K_\lambda)^{-1} + L_\lambda (I - L_\lambda)^{-1} + K_\lambda (I - K_\lambda)^{-1} L_\lambda (I - L_\lambda)^{-1}.\end{aligned}$$

Hence $Y_\lambda (I - Y_\lambda)^{-1}$ can be represented as

$$\begin{aligned}Y_\lambda (I - Y_\lambda)^{-1} &= K_\lambda (I - K_\lambda)^{-1} + L_\lambda (I - L_\lambda)^{-1}\\ &\quad + K_\lambda (I - K_\lambda)^{-1} L_\lambda (I - L_\lambda)^{-1}.\end{aligned} \tag{6.1}$$

Noting that $L_\lambda = \tilde{K}_\lambda (I - K_\lambda)^{-1} = \tilde{K}_\lambda + \tilde{K}_\lambda K_\lambda (I - K_\lambda)^{-1}$, from Theorem 4.1 and (5.1), we know that for $\lambda \in \mathbf{C}_{\delta_0}$ with sufficiently large $\mu = \mathrm{Re}\,\lambda$, L_λ has an integral kernel $L_\lambda(x,y)$ given by the formula

$$L_\lambda(x,y) = \tilde{K}_\lambda(x,y) + \int_{\partial D} \tilde{K}_\lambda(x,z) K_\lambda^\infty(z,y)\, dS_z,$$

where $K_\lambda^\infty(x,y)$ is the integral kernel of $K_\lambda (I - K_\lambda)^{-1}$ in Theorem 4.1.

Choosing larger μ_0 if necessary, we conclude from Theorems 4.1 and 5.1 that for all $\mu \geq \mu_0$ the operator $L_\lambda (I - L_\lambda)^{-1}$ has an integral kernel $L_\lambda^\infty(x,y)$ that is measurable on $\partial D \times \partial D$, continuous for $x \neq y$, and satisfies

$$\big|L_\lambda^\infty(x,y)\big| \leq \frac{C e^{-\mu|x-y|}}{|x-y|}, \quad x, y \in \partial D, \lambda \in \mathbf{C}_{\delta_0}, \mu = \mathrm{Re}\,\lambda \geq \mu_0. \tag{6.2}$$

From (6.1) we know that the integral kernel $Y_\lambda^\infty(x,y)$ of the operator $M_\lambda = Y_\lambda (I - Y_\lambda)^{-1}$ is given by the formula

$$Y_\lambda^\infty(x,y) = K_\lambda^\infty(x,y) + L_\lambda^\infty(x,y) + \int_{\partial D} K_\lambda^\infty(x,z) L_\lambda^\infty(z,y)\, dS_z. \tag{6.3}$$

Using Theorem 4.1, (6.2), and (5.16), one gets

$$\Big|\int_{\partial D} K_\lambda^\infty(x,z)L_\lambda^\infty(z,y)\,dS_z\Big| \le \frac{Ce^{-\mu|x-y|}}{|x-y|}.$$

This together with Theorem 4.1, (6.2), and (6.3) yields

$$\big|Y_\lambda^\infty(x,y)\big| \le C\Big(\mu+\frac{1}{|x-y|}\Big)e^{-\mu|x-y|},\quad x,y\in\partial D, \lambda\in\mathbf{C}_{\delta_0}, \mu=\operatorname{Re}\lambda\ge\mu_0,$$

for some positive constants C and μ_0 depending only on ∂D, C_0 in (1.1), and C_1 in (1.2). This completes the proof of Theorem 1.1. □

Put $N_\lambda = \tilde{K}_\lambda + (Y_\lambda)^2(I-Y_\lambda)^{-1}$. We denote the integral kernel of N_λ by $N_\lambda(x,y)$. We can also give estimates of the integral kernel $N_\lambda(x,y)$ which also we need to study the inverse problem described in Section 1.

THEOREM 6.1
There exist positive constants C and $\mu_0\ge 1$ such that for all $\lambda\in\mathbf{C}_{\delta_0}$ with $\mu=\operatorname{Re}\lambda\ge\mu_0$ the operator N_λ has an integral kernel $N_\lambda(x,y)$ which is measurable for $(x,y)\in\partial D\times\partial D$, continuous for $x\ne y$, and has the estimate

$$\big|N_\lambda(x,y)\big| \le Ce^{-\mu|x-y|}\Big(1+\frac{1}{|x-y|}+\min\Big\{\mu\big(\mu|x-y|^3\big)^{1/2},\frac{1}{|x-y|^3}\Big\}\Big). \tag{6.4}$$

REMARK 6.1
Since $\min\{\sqrt{a},a^{-1}\}\le 1$ for all $a>0$, from (6.4) we get

$$\big|N_\lambda(x,y)\big| \le C\Big(\mu+\frac{1}{|x-y|}\Big)e^{-\mu|x-y|}. \tag{6.5}$$

Proof of Theorem 6.1
Since $N_\lambda=\tilde{K}_\lambda+K_\lambda Y_\lambda(I-Y_\lambda)^{-1}+\tilde{K}_\lambda Y_\lambda(I-Y_\lambda)^{-1}$, the integral kernel of N_λ is given by the formula

$$N_\lambda(x,y)=\tilde{K}_\lambda(x,y)+\int_{\partial D}\big(K_\lambda(x,z)+\tilde{K}_\lambda(x,z)\big)Y_\lambda^\infty(z,y)\,dS_z. \tag{6.6}$$

Hence from (1.1), (1.2), and Theorem 1.1, and from (3.6), (5.16), and (5.18), it follows that

$$\begin{aligned}
&\Big|\int_{\partial D}\big(K_\lambda(x,z)+\tilde{K}_\lambda(x,z)\big)Y_\lambda^\infty(z,y)\,dS_z\Big|\\
&\quad\le C^2\Big\{\mu^2\int_{\partial D}e^{-\mu(|x-z|+|z-y|)}\,dS_z+\mu\int_{\partial D}\frac{e^{-\mu(|x-z|+|z-y|)}}{|y-z|}\,dS_z\\
&\qquad+\mu\int_{\partial D}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z|}\,dS_z+\int_{\partial D}\frac{e^{-\mu(|x-z|+|z-y|)}}{|x-z||z-y|}\,dS_z\Big\}\\
&\quad\le Ce^{-\mu|x-y|}\Big[1+\min\Big\{(\mu|x-y|)^{3/2},\frac{1}{|x-y|^3}\Big\}\\
&\qquad+\frac{1}{|x-y|}+\max\Big\{0,\log\frac{r_0}{|x-y|}\Big\}\Big].
\end{aligned}\tag{6.7}$$

Since

$$\max\Big\{0, \log\frac{r_0}{|x-y|}\Big\} \leq \frac{C}{|x-y|},$$

from (1.2), (6.6), and (6.7) we obtain (6.4). This completes the proof of Theorem 6.1. □

7. The reason why Theorem 1.1 is needed

For $\rho = \rho(x) \in C^{0,\alpha_0}(\partial D)$, we consider the following elliptic problem in the exterior domain $\mathbf{R}^3 \setminus \overline{D}$:

$$\begin{aligned} &(\triangle - \lambda^2)w = 0 \quad \text{in } \mathbf{R}^3 \setminus \overline{D}, \\ &\frac{\partial w}{\partial \nu} + \rho(x)w = g(x) \quad \text{on } \partial D. \end{aligned} \tag{7.1}$$

It is well known that for any $g \in C(\partial D)$ and $\lambda \in \mathbf{C}$ with $|\arg\lambda| < \pi/2$, the L^2-solution $w(x;\lambda)$ of (7.1) is given by

$$w(x;\lambda) = V_D(\lambda)\psi(x) = \int_{\partial D} E_\lambda(x,y)\psi(y)\, dS_y, \quad x \in \mathbf{R}^3 \setminus \partial D,$$

where $E_\lambda(x,y)$ is a fundamental solution of $(\triangle_x - \lambda^2)E_\lambda(x,y) = -2\delta(x-y)$ of the form

$$E_\lambda(x,y) = \frac{e^{-\lambda|x-y|}}{2\pi|x-y|}, \quad x \neq y, |\arg\lambda| < \frac{\pi}{2}.$$

As is in Mizohata [3], for example, this is a famous approach in potential theory. Problem (7.1) can be reduced to the following equation on the boundary:

$$\big(I - Y(\lambda)\big)\psi = g \quad \text{in } C(\partial D).$$

In the above, $Y(\lambda)$ is the integral operator on ∂D with the integral kernel $Y(x,y;\lambda)$ defined by

$$Y(x,y;\lambda) = M_0(y,x;\lambda) + \tilde{M}(y,x;\lambda), \tag{7.2}$$

where

$$M_0(y,x;\lambda) = \frac{\lambda}{2\pi} e^{-\lambda|x-y|} \frac{\nu_x \cdot (y-x)}{|x-y|^2} \tag{7.3}$$

and

$$\tilde{M}(y,x;\lambda) = \frac{1}{2\pi}\frac{e^{-\lambda|x-y|}}{|x-y|}\Big(\frac{\nu_x\cdot(y-x)}{|x-y|^2} + \rho(y)\Big). \tag{7.4}$$

As is described in Section 1, an inverse problem for a three-dimensional heat equation in thermal imaging is considered in [1]. We recall this problem briefly. Let Ω be a bounded domain of $\mathbf{R}^3$ with C^{2,α_0} boundary and $0 < \alpha_0 \leq 1$. Assume that the domain D satisfies $\overline{D} \subset \Omega$ and has all the properties described in Section 1. We take a function $f \in L^2(\partial\Omega \times]0,T[)$ for some fixed $T > 0$ as an input datum of the inverse problem. We consider the solution $u(x,t)$ of the following problem:

$$\begin{cases} u_t - \triangle u = 0 & \text{in } (\Omega \setminus \overline{D}) \times]0,T[, \\ (\partial_\nu + \rho)u = 0 & \text{on } \partial D \times]0,T[, \\ \partial_\nu u = f & \text{on } \partial\Omega \times]0,T[, \\ u(x,0) = 0 & \text{in } \Omega \setminus \overline{D}, \end{cases}$$

where $\partial_\nu = \nu \cdot \nabla_x$. The original inverse problem studied in [1] is to find information of D from the one measurement, that is, a pair of input and output data $(f(x,t), \partial_\nu u(x,t))$ on $\partial\Omega \times]0,T[$.

For any fixed $p \in \mathbf{R}^3 \setminus \overline{\Omega}$, we put $l_p(x,z) = |p-x| + |x-z|$ $(x \in \partial D, z \in \partial\Omega)$. The essential problem of the approach presented in [1] is to obtain the asymptotic behavior of the following type of integral:

$$J(\lambda, p) = \int_{\partial\Omega} dS_z \varphi(z;\lambda) \int_{\partial D} e^{-\lambda l_p(x,z)} G(x,z,p;\lambda)\, dS_x,$$

where $\varphi(\cdot;\lambda) \in C(\partial\Omega)$ is bounded in $\lambda \in \mathbf{C}_{\delta_0}$, $\mu = \mathrm{Re}\,\lambda \geq 1$, and

$$G(x,z,p;\lambda) = a(x,z) + b(x,z)\big(F_0(x,p;\lambda) + F_1(x,p;\lambda)\big)$$

for some continuous functions $a(x,z)$ and $b(x,z)$ of $(x,z) \in \partial D \times \partial\Omega$, and continuous functions $F_j(x,p;\lambda)$ $(j=0,1)$ of $x \in \partial D$ with the parameter λ. Thus, to obtain asymptotic behavior of $J(\lambda,p)$ as $|\lambda| \to \infty$, we need to know how $F_j(x,p;\lambda)$ $(j=0,1)$ behave as $|\lambda| \to \infty$.

We define $M_0(\lambda)$ and $\tilde{M}(\lambda)$ by

$$M_0(\lambda)h(x) = \int_{\partial D} M_0(x,y;\lambda)h(y)\, dS_y$$

and

$$\tilde{M}(\lambda)h(x) = \int_{\partial D} \tilde{M}(x,y;\lambda)h(y)\, dS_y.$$

Put $M_1(\lambda) = \tilde{M}(\lambda) + ({}^tY(\lambda))^2(I - {}^tY(\lambda))^{-1}$, where ${}^tY(\lambda)$ is the integral operator defined by

$${}^tY(\lambda)h(x) = \int_{\partial D} Y(y,x;\lambda)h(y)\, dS_y.$$

As is in [1, (3.18), (3.20), (3.21)], $F_j(x,p;\lambda)$ $(j=0,1)$ are given by

$$F_j(x,p;\lambda) = e^{\lambda|x-p|}\Big(M_j(\lambda)\Big(\frac{e^{-\lambda|\cdot-p|}}{|\cdot-p|}\Big)\Big)(x), \quad j=0,1.$$

For $F_0(x,p;\lambda)$, it follows that

$$F_0(x,p;\lambda) = \frac{\lambda}{2\pi}\int_{\partial D} e^{-\lambda(|x-y|+|y-p|-|x-p|)} \frac{\nu_y \cdot (x-y)}{|x-y|^2} \frac{1}{|y-p|}\, dS_y,$$

which implies that $F_0(x,p;\lambda) = O(|\lambda|)$ at worst. Thus we can see that there is no exponentially growing factor in $\mu = \mathrm{Re}\,\lambda$. For $F_1(x,p;\lambda)$, we also have

$$F_1(x,p;\lambda) = \int_{\partial D} e^{-\lambda(|y-p|-|x-p|)} M_1(x,y;\lambda) \frac{1}{|y-p|}\, dS_y,$$

where $M_1(x,y;\lambda)$ is the integral kernel of $M_1(\lambda)$. Hence to obtain the asymptotic behavior of $J(\lambda,p)$, it is important to determine the exponential term of the estimate of $M_1(x,y;\lambda)$. From Lemma 2.1(i) and (7.2)–(7.4), ${}^tY(\lambda)=M_0(\lambda)+\tilde{M}(\lambda)$ satisfies all assumptions in Theorems 1.1 and 6.1. Hence (6.5) implies that

$$\big|M_1(x,y;\lambda)\big|\le C\Big(\mu+\frac{1}{|x-y|}\Big)e^{-\mu|x-y|},$$

which yields

$$\big|F_1(x,p;\lambda)\big|\le C\int_{\partial D}e^{-\mu(|x-y|+|y-p|-|x-p|)}\Big(\mu+\frac{1}{|x-y|}\Big)\frac{1}{|y-p|}\,dS_y.$$

This implies that $F_1(x,p;\lambda)$ does also not contain exponentially growing factors in $\mu=\operatorname{Re}\lambda$. Thus we can handle the term containing $F_1(x,p;\lambda)$ in the same way as the other ones. To ensure this, we need to obtain the estimate of the integral kernel introduced in Theorem 1.1. This is why we have to get Theorem 1.1.

Appendix

Here, we show estimate (1.4) for $\delta>0$ when we do not assume that ∂D is strictly convex. In what follows, we assume only that D is a bounded domain of $\mathbf{R}^3$ with C^{2,α_0} $(0<\alpha_0\le 1)$-boundary. Even in this case, Lemma 3.1 holds since in the proof of Lemma 3.1, the convexity assumption for ∂D does not used.

LEMMA A.1

There exists a constant $C>0$ such that

$$\int_{\partial D}\frac{e^{-\mu|z-y|}}{|x-z|}\,dS_z\le C\mu^{-1},\quad x,y\in\partial D,\mu>0,$$

and

$$\int_{\partial D}\frac{e^{-\mu|z-y|}}{|x-z||y-z|}\,dS_z\le\frac{C\mu^{-1}}{|x-y|},\quad x,y\in\partial D,x\ne y,\mu>0.$$

Proof

From Lemma 3.1(ii), it follows that

$$\int_{\partial D}\frac{e^{-\mu|z-y|}}{|y-z|^k}\,dS_z\le C\mu^{-2+k},\quad y\in\partial D,\mu>0,k=0,1. \tag{A.1}$$

Note that decomposition (5.22) of ∂D and estimates (5.23) and (5.24) imply

$$\int_{\partial D}\frac{e^{-\mu|z-y|}}{|x-z|}\,dS_z\le\int_{\partial D\cap S_1}\frac{e^{-\mu|z-x|}}{|x-z|}\,dS_z+\int_{\partial D\cap S_2}\frac{e^{-\mu|z-y|}}{|y-z|}\,dS_z.$$

Combining this estimate with (A.1), we obtain the first estimate of Lemma A.1.

For the second estimate of Lemma A.1, we note that

$$|x-y|\frac{e^{-\mu|z-y|}}{|x-z||y-z|}\le\frac{|x-z|+|y-z|}{|x-z||y-z|}e^{-\mu|z-y|}=\frac{e^{-\mu|z-y|}}{|x-z|}+\frac{e^{-\mu|z-y|}}{|y-z|}. \tag{A.2}$$

Combining this estimate with (A.1) and the first estimate of Lemma A.1, we obtain the second estimate in Lemma A.1. $\square$

LEMMA A.2

There exists a positive constant C depending only on ∂D such that for all $\mu > 0$ and $0 < \delta \le 1$,

$$\int_{\partial D} e^{-\mu(1-\delta)|x-z|} e^{-\mu|z-y|} \Big(\delta\mu + \frac{1}{|x-z|}\Big)\Big(\delta\mu + \frac{1}{|y-z|}\Big)\, dS_z$$
$$\le C\delta^{-1}\mu^{-1}\Big(\delta\mu + \frac{1}{|x-y|}\Big) e^{-(1-\delta)\mu|x-y|}, \quad x, y \in \partial D, x \ne y.$$

Proof

From the fact that

$$(1-\delta)|x-z| + |z-y| = (1-\delta)\big(|x-z| + |z-y|\big) + \delta|z-y|$$
$$\ge (1-\delta)|x-y| + \delta|z-y|,$$

it follows that

$$\int_{\partial D} e^{-(1-\delta)\mu|x-z|} e^{-\mu|z-y|} \Big(\delta\mu + \frac{1}{|x-z|}\Big)\Big(\delta\mu + \frac{1}{|y-z|}\Big)\, dS_z$$
$$\le e^{-(1-\delta)\mu|x-y|} \int_{\partial D} e^{-\delta\mu|z-y|} \Big(\delta\mu + \frac{1}{|x-z|}\Big)\Big(\delta\mu + \frac{1}{|y-z|}\Big)\, dS_z.$$

Hence, we obtain Lemma A.2 since Lemma A.1 and (A.1) imply

$$\int_{\partial D} e^{-\mu|z-y|} \Big(\mu + \frac{1}{|x-z|}\Big)\Big(\mu + \frac{1}{|y-z|}\Big)\, dS_z \le C\mu^{-1}\Big(\mu + \frac{1}{|x-y|}\Big)$$

for all $x, y \in \partial D$, $x \ne y$, and $\mu > 0$. □

As is in the beginning of Section 4, the Neumann series expansion implies $Y_\lambda(I - Y_\lambda)^{-1} = \sum_{n=1}^{\infty} (Y_\lambda)^n$, where operators $(Y_\lambda)^n$ are the integral operator with the integral kernel $Y_\lambda^{(n)}(x, y)$ $(n = 1, 2, \ldots)$ defined by

$$Y_\lambda^{(n+1)}(x, y) = \int_{\partial D} Y_\lambda^{(n)}(x, z) Y_\lambda^{(1)}(z, y)\, dS_z, \quad n = 1, 2, \ldots$$

and

$$Y_\lambda^{(1)}(x, y) = K_\lambda(x, y) + \tilde{K}_\lambda(x, y).$$

For the constants C_0 and C_1 in (1.1) and (1.2), respectively, we put $C_5 = \max\{C_0, C_1\} > 0$. Note that it follows that

$$\big|Y_\lambda^{(1)}(x, y)\big| \le C_5\Big(\mu + \frac{1}{|x-y|}\Big) e^{-\mu|x-y|},$$
$$x, y \in \partial D, x \ne y, \lambda \in \mathbf{C}_{\delta_0}, \mu = \mathrm{Re}\,\lambda. \tag{A.3}$$

Now, we state the following theorem describing (1.4).

THEOREM A.1

Let D be a bounded domain of $\mathbf{R}^3$ with C^{2,α_0} $(0 < \alpha_0 \le 1)$-boundary. Then, there exists a constant $\mu_0 > 0$ such that $\lambda \in \mathbf{C}_{\delta_0}$ with $\mu \ge \mu_0$, the operator $I - Y_\lambda$ is *invertible, and* $Y_\lambda(I-Y_\lambda)^{-1}$ *has an integral kernel* $Y_\lambda^\infty(x,y)$ *which is measurable for* $(x,y)\in\partial D\times\partial D$*, continuous for* $x\neq y$*, and has the estimate*

$$\big|Y_\lambda^\infty(x,y)\big|\le 2C_5\delta^{-1}\Big(\delta\mu+\frac{1}{|x-y|}\Big)e^{-(1-\delta)\mu|x-y|},$$
$$x,y\in\partial D, x\neq y, \lambda\in\mathbf{C}_{\delta_0}, \mu\neq\operatorname{Re}\lambda\ge\delta^{-2}\mu_0,$$

for all $0<\delta\le 1$*, where* C_5 *is the constant in (A.3).*

Proof
We give the following estimates for $Y_\lambda^{(n+1)}(x,y)$ $(n=0,1,2,\ldots)$:

$$\begin{aligned}&\big|Y_\lambda^{(n+1)}(x,y)\big|\le C_5\delta^{-1}(CC_5\delta^{-2})^n\mu^{-n}\Big(\delta\mu+\frac{1}{|x-y|}\Big)e^{-(1-\delta)\mu|x-y|},\\ &x,y\in\partial D, x\neq y, \lambda\in\mathbf{C}_{\delta_0}, 0<\delta\le 1,\end{aligned}\tag{A.4}$$

where the constant C in the above is just given in Lemma A.2. From (A.3), it follows that

$$\big|Y_\lambda^{(1)}(x,y)\big|\le C_5\delta^{-1}\Big(\delta\mu+\frac{1}{|x-y|}\Big)e^{-\mu|x-y|}\quad(0<\delta\le 1),$$

which means that (A.4) holds for $n=0$ for $0<\delta\le 1$. Assume that (A.4) holds for some nonnegative integer n. Then the definition of $Y_\lambda^{(n+2)}(x,y)$ implies that

$$\begin{aligned}\big|Y_\lambda^{(n+2)}(x,y)\big|&\le\int_{\partial D}\big|Y_\lambda^{(n+1)}(x,z)\big|\big|Y_\lambda^{(1)}(z,y)\big|\,dS_z\\ &\le C_5\delta^{-1}(CC_5\delta^{-2})^n\mu^{-n}C_5\delta^{-1}\\ &\quad\times\int_{\partial D}e^{-(1-\delta)\mu|x-z|}e^{-\mu|z-y|}\Big(\delta\mu+\frac{1}{|x-z|}\Big)\Big(\delta\mu+\frac{1}{|z-y|}\Big)\,dS_z.\end{aligned}\tag{A.5}$$

Combining this estimate with Lemma A.2, we obtain

$$\big|Y_\lambda^{(n+2)}(x,y)\big|\le C_5\delta^{-1}(CC_5\delta^{-2})^{n+1}\mu^{-(n+1)}\Big(\delta\mu+\frac{1}{|x-y|}\Big)e^{-(1-\delta)\mu|x-y|},$$

that is, (A.4) for $n+1$. Thus, (A.4) holds for any $n=0,1,2,\ldots$.

We put $\mu_0=2CC_5>0$. For $\lambda\in\mathbf{C}_{\delta_0}$ with $\mu\ge\mu_0\delta^{-2}$, (A.4) implies

$$\big|Y_\lambda^{(n+1)}(x,y)\big|\le C_5\delta^{-1}\Big(\frac{1}{2}\Big)^n\Big(\delta\mu+\frac{1}{|x-y|}\Big)e^{-(1-\delta)\mu|x-y|},$$
$$x,y\in\partial D, x\neq y, 0<\delta\le 1.$$

Noting this estimate and $|Y_\lambda^\infty(x,y)|\le\sum_{n=0}^\infty|Y_\lambda^{(n+1)}(x,y)|$, we obtain the estimate of $Y_\lambda^\infty(x,y)$ in Theorem A.1, which completes the proof of Theorem A.1. □

As is in Theorem 5.1 and Proposition 5.2, we can expect that $Y_\lambda^{(n)}(x,y)$ $(n\ge 2)$ are more regular than $Y_\lambda^{(1)}(x,y)$. We can also show the following estimates.

PROPOSITION A.1

Let D be a bounded domain of $\mathbf{R}^3$ with C^{2,α_0} $(0<\alpha_0\le 1)$-boundary. Then there exist constants $C>0$ and $\mu_0>0$ such that

(i) $Y_\lambda^{(2)}(x,y)$ *satisfies*

$$\big|Y_\lambda^{(2)}(x,y)\big|\le C\delta^{-2}e^{-(1-\delta)\mu|x-y|}\Big(1+\max\Big\{0,\log\frac{r_0}{|x-y|}\Big\}\Big)$$

for all $x,y\in\partial D$, $x\ne y$, $\lambda\in\mathbf{C}_{\delta_0}$, $\mu=\operatorname{Re}\lambda\ge\delta^{-2}\mu_0$, and $0<\delta\le 1$;

(ii) *the integral kernel $\tilde{Y}_\lambda^\infty(x,y)$ of the operator $Y_\lambda^3(I-Y_\lambda)^{-1}$ is continuous on $\partial D\times\partial D$ and satisfies*

$$\big|\tilde{Y}_\lambda^\infty(x,y)\big|\le C\delta^{-4}\mu^{-1}\big(1+|\log\delta|+\log\mu\big)e^{-(1-\delta)\mu|x-y|}$$

for all $x,y\in\partial D$, $x\ne y$, $\lambda\in\mathbf{C}_{\delta_0}$, $\mu=\operatorname{Re}\lambda\ge\delta^{-2}\mu_0$, and $0<\delta\le 1$.

Proof

First we show that there exists a constant $C>0$ such that for any $\mu>0$,

$$\text{(A.6)}\quad \int_{\partial D}\frac{e^{-\mu|z-y|}}{|x-z||y-z|}\,dS_z\le C\Big(1+\max\Big\{0,\log\frac{r_0}{|x-y|}\Big\}\Big),\quad x,y\in\partial D, x\ne y,$$

and

$$\text{(A.7)}\quad \begin{aligned}&\int_{\partial D}\frac{e^{-\mu|z-y|}}{|y-z|}\max\Big\{0,\log\frac{r_0}{|x-z|}\Big\}\,dS_z\\ &\quad\le C\mu^{-1}\big(1+\max\{0,\log\mu\}\big),\quad x,y\in\partial D,\end{aligned}$$

where $r_0>0$ is the constant described in Lemma 2.1.

For $x,y\in\partial D$, we put $\rho_0=|x-y|$. First consider the case when $\rho_0>r_0$. Estimate (A.2) implies that

$$\frac{e^{-\mu|z-y|}}{|x-z||y-z|}\le\frac{1}{r_0}\Big(\frac{e^{-\mu|z-y|}}{|x-z|}+\frac{e^{-\mu|z-y|}}{|y-z|}\Big)\le\frac{1}{r_0}\Big(\frac{1}{|x-z|}+\frac{1}{|y-z|}\Big),$$

which yields (A.6) since $\int_{\partial D}|x-z|^{-1}\,dS_z\le C$. For (A.7), since $|x-z|\le r_0/2<\rho_0/2$ implies that $|y-z|\ge|y-x|-|x-z|=\rho_0-r_0/2\ge r_0/2$, from (A.1) and the argument of the proof of Lemma 3.1(iv), the integral in (A.7) is estimated by

$$\int_{S_{r_0/2}(x)}\frac{e^{-\mu r_0/2}}{r_0/2}\log\frac{r_0}{|x-z|}\,dS_z+\log 2\int_{S_{r_0}(x)\setminus S_{r_0/2}(x)}\frac{e^{-\mu|z-y|}}{|y-z|}\le C\mu^{-1}.$$

Next we consider the case of $\rho_0\le r_0$. For (A.6), we put $D_1(x,y)=S_{\rho_0}^-(x)$, $D_2(x,y)=S_{\rho_0}(x)\cap S_{\rho_0}^-(y)$, $D_3(x,y)=S_{\rho_0}(x)\cap(S_{\rho_0}(y)\setminus S_{\rho_0/2}(y))$, $D_4(x,y)=S_{\rho_0}(x)\cap S_{\rho_0/2}(y)$, and

$$I_j(x,y)=\int_{D_j(x,y)}\frac{e^{-\mu|z-y|}}{|x-z||y-z|}\,dS_z\quad(j=1,2,3,4).$$

To show (A.6), it suffices to give estimates of each $I_j(x,y)$.

Since $|x-z| \le \rho_0 \le |y-z|$ for $z \in D_2(x,y)$, it follows that

$$I_2(x,y) \le \int_{S^-_{\rho_0}(y)} \frac{e^{-\mu|z-x|}}{|x-z||y-z|}\, dS_z = I_1(y,x).$$

Combining this fact with the proof of (5.7), we obtain

$$I_1(x,y)+I_2(x,y) \le I_1(x,y)+I_1(y,x) \le C\Big(1+\max\Big\{0, \log\frac{r_0}{|x-y|}\Big\}\Big).$$

Since $\rho_0 \ge |y-z| \ge \rho_0/2$ and $|x-z|/2 \le \rho_0/2 \le |y-z|$ for $z \in D_3(x,y)$, from Lemma 3.1(i), it follows that

$$I_3(x,y) \le \frac{2}{\rho_0}\int_{S_{\rho_0}(x)} \frac{e^{-\mu|x-z|/2}}{|x-z|}\, dS_z \le C.$$

If $z \in D_4(x,y)$, it follows that $|x-z| \ge |x-y|-|y-z| \ge \rho_0-\rho_0/2=\rho_0/2$. This fact and Lemma 3.1(i) yields

$$I_4(x,y) \le \frac{2}{\rho_0}\int_{S_{\rho_0/2}(y)} \frac{e^{-\mu|y-z|}}{|y-z|}\, dS_z \le C.$$

Combining all estimates for $I_j(x,y)$, we obtain (A.6).

Next we show (A.7) for the case $\rho_0 \le r_0$. Note that the integral domain of the integral in (A.7) is $S_{r_0}(x)$. From (5.22), (5.23), and the fact that $1 \le r_0/|x-z| \le r_0/|y-z|$ for all $z \in S_{r_0}(x) \cap S_2 \subset S_{r_0}(y)$, the integral in (A.7) is estimated by

$$\int_{S_{r_0}(x)} \frac{e^{-\mu|z-x|}}{|x-z|}\log\frac{r_0}{|x-z|}\, dS_z + \int_{S_{r_0}(y)} \frac{e^{-\mu|z-y|}}{|y-z|}\log\frac{r_0}{|y-z|}\, dS_z.$$

Form the above estimate and Lemma 3.1(iv), we obtain (A.7).

Using (A.5) with $n=0$ and the argument obtaining Lemma A.2, we obtain

$$\begin{aligned}\big|Y^{(2)}_\lambda(x,y)\big| &\le (C_5\delta^{-1})^2 e^{-(1-\delta)\mu|x-y|}\\ &\qquad\times\int_{\partial D} e^{-\delta\mu|z-y|}\Big(\delta\mu+\frac{1}{|x-z|}\Big)\Big(\delta\mu+\frac{1}{|z-y|}\Big)\, dS_z.\end{aligned}$$

Hence, the first estimate of Lemma A.1, (A.1), and (A.6) imply Proposition A.1(i).

For (ii), note that

$$\big|\tilde{Y}^\infty_\lambda(x,y)\big| \le \int_{\partial D}\big|Y^{(2)}_\lambda(x,z)\big|\big|Y^\infty_\lambda(z,y)\big|\, dS_z$$

since $Y^3_\lambda(I-Y_\lambda)^{-1}=Y^2_\lambda\cdot Y_\lambda(I-Y_\lambda)^{-1}$. For $0<\delta\le 1$, we put $\delta'=\delta/4$. It yields that

$$\begin{aligned}(1-\delta)|x-z|+(1-2\delta')|z-y| &= (1-\delta)\big(|x-z|+|z-y|\big)+2\delta'|z-y|\\ &\ge (1-\delta)|x-y|+2\delta'|z-y|,\end{aligned}$$

which implies $e^{-(1-\delta)\mu|x-z|}e^{-(1-2\delta')\mu|z-y|} \le e^{-(1-\delta)\mu|x-y|}e^{-2\delta'\mu|z-y|}$. Hence, from Theorem A.1 and Proposition A.1(i) it follows that

$$\left|\tilde{Y}_\lambda^\infty(x,y)\right| \le 4CC_5\delta^{-3}e^{-(1-\delta)\mu|x-y|}\int_{\partial D} e^{-2\delta'\mu|z-y|} \times \left(1+\max\left\{0, \log\frac{r_0}{|x-z|}\right\}\right)\left(2\delta'\mu+\frac{1}{|z-y|}\right)dS_z,$$

where $C>0$ is the constant in Proposition A.1(i). Noting that

$$\delta'\mu e^{-2\delta'\mu|z-y|} \le \frac{e^{-\delta'\mu|z-y|}}{2|z-y|},$$

we obtain

$$\left|\tilde{Y}_\lambda^\infty(x,y)\right| \le 8CC_5\delta^{-3}e^{-(1-\delta)\mu|x-y|} \times \int_{\partial D} e^{-\delta'\mu|z-y|}\left(1+\max\left\{0, \log\frac{r_0}{|x-z|}\right\}\right)\frac{1}{|z-y|}\,dS_z.$$

From the above estimate, (A.7), and Lemma 3.1(ii), the estimate in Proposition A.1(ii) holds. This completes the proof of Proposition A.1. □

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