Mapping properties of the discrete fractional maximal operator in metric measure spaces

Toni Heikkinen, Juha Kinnunen, Juho Nuutinen, and Heli Tuominen

Abstract This work studies boundedness properties of the fractional maximal operator on metric measure spaces under standard assumptions on the measure. The main motivation is to show that the fractional maximal operator has similar smoothing and mapping properties as the Riesz potential. Instead of the usual fractional maximal operator, we also consider a so-called discrete maximal operator which has better regularity. We study the boundedness of the discrete fractional maximal operator in Sobolev, Hölder, Morrey, and Campanato spaces. We also prove a version of the Coifman–Rochberg lemma for the fractional maximal function.

1. Introduction

The fractional maximal function is a standard tool in partial differential equations, potential theory and harmonic analysis (see [2]–[4]). It is also closely related to the definition of Morrey spaces. This class of functions can be used, for example, to show that weak solutions to certain partial differential equations are locally Hölder continuous. Hölder continuity can also be characterized through the Campanato spaces. For some values of parameters, Morrey and Campanato spaces coincide (see [6], [22], [25]). However, the main difference is that the Morrey-type condition gives a bound for the growth of the integral average of a function, but the Campanato-type condition gives a similar bound for the mean oscillation. Boundedness of the classical operators in harmonic analysis in Morrey and Campanato spaces has been studied in [6], [9], and [27].

This work studies boundedness properties of the fractional maximal operator in Sobolev, Hölder, Morrey, and Campanato spaces on metric measure spaces. The main motivation is to show that the fractional maximal operator has similar smoothing and mapping properties to those of the Riesz potential (see [2], [3], [12]–[14], [24]–[26]). Note that the Campanato estimates for the Riesz potentials do not immediately imply the corresponding oscillation estimates for the fractional maximal function. The Morrey estimates are probably known by the

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experts at least in special cases, but the main contribution of this work is to provide results in Sobolev, Hölder, and Campanato spaces. There is also an unexpected obstruction in the metric case, as the examples in [8] show. Indeed, it may happen that even the standard Hardy–Littlewood maximal function of a Lipschitz continuous function may fail to be continuous. For this reason, we consider a so-called discrete maximal function, which is constructed in terms of coverings and partitions of unities as in [1], [18], and [20]. The discrete fractional maximal function is comparable to the standard fractional maximal function provided the measure is doubling. Hence for all practical purposes, it does not matter which one we choose. However, the discrete maximal function seems to behave better as far as regularity is concerned.

The main purpose of this work is to extend the Euclidean result with the Lebesgue measure in [19] to metric measure spaces. We show that under relatively mild conditions on the measure, the discrete fractional maximal function of an L^p-function belongs to a Sobolev space. Another example of a smoothing property is shown by the result that the discrete fractional maximal operator maps Sobolev, Morrey, and Campanato spaces to a slightly better similar space. As a special case, we obtain a result which implies that the discrete fractional maximal operator maps Hölder continuous functions to Hölder continuous functions with a better exponent. The example in [8] can be modified to show that corresponding results do not hold for the standard fractional maximal function. Our arguments also apply in a more general context of spaces of homogeneous type (see [11], [13]–[15], [21], [22], [29]), but we have chosen to work in the metric space setting for expository purposes.

We discuss L^p-estimates for the fractional maximal function also in the case when the measure is not necessarily doubling. This is closely related to [28], [29], and [30]. The new aspects in our work compared with earlier results, for example, in [6] and [25], are that our main focus is on the fractional maximal function instead of the standard Hardy–Littlewood maximal function, and we also consider Sobolev and Campanato spaces. In addition, we prove a version of a result of Coifman and Rochberg [10] for the fractional maximal function. In the classical case the result states that the Hardy–Littlewood maximal function raised to the power γ, with $0 < \gamma < 1$, is the so-called Muckenhoupt's A_1-weight provided it is finite almost everywhere. We show that the same result holds true for the fractional maximal function even without taking the power.

2. The fractional maximal function

We assume that $X = (X, \mathrm{d}, \mu)$ is a separable metric measure space equipped with a metric d and a Borel regular outer measure μ, which satisfies $0 < \mu(U) < \infty$ whenever U is nonempty, open, and bounded.

The measure is doubling if there is a fixed constant $c_d > 0$, called a doubling constant of μ, such that

$$\mu\big(B(x,2r)\big) \le c_d \mu\big(B(x,r)\big) \tag{2.1}$$

for every ball $B(x,r) = \{y \in X : \mathrm{d}(y,x) < r\}$.

The doubling condition implies that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge C\Big(\frac{r}{R}\Big)^Q \tag{2.2}$$

for every $0 < r \le R$ and $y \in B(x,R)$ for some C and $Q > 1$ that only depend on c_d. In fact, we may take $Q = \log_2 c_d$.

Throughout the paper, the characteristic function of a set $E \subset X$ is denoted as χ_E. In general, C will denote a positive constant whose value is not necessarily the same at each occurrence. The integral average of a function $u \in L^1(A)$ over a μ-measurable set A with finite and positive measure is denoted by

$$u_A = \fint_A u\, d\mu = \frac{1}{\mu(A)} \int_A u\, d\mu.$$

Let $0 \le \alpha \le Q$. The fractional maximal function of $u \in L^1_{\mathrm{loc}}(X)$ is

$$\mathcal{M}_\alpha u(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |u|\, d\mu. \tag{2.3}$$

For $\alpha = 0$, we have the usual Hardy–Littlewood maximal function

$$\mathcal{M} u(x) = \sup_{r>0} \fint_{B(x,r)} |u|\, d\mu.$$

By the Hardy–Littlewood maximal function theorem for doubling measures (see [11]), we see that the Hardy–Littlewood maximal operator is bounded on $L^p(X)$ when $1 < p \le \infty$ and maps $L^1(X)$ to the weak $L^1(X)$. In our definition, we consider balls that are centered at x, but we obtain a noncentered maximal function by taking the supremum over all balls containing x. For doubling measures, these maximal functions are comparable, and it does not matter which one we choose.

Another way to define the fractional maximal function is

$$\widetilde{\mathcal{M}}_\alpha u(x) = \sup_{r>0} \mu\big(B(x,r)\big)^\alpha \fint_{B(x,r)} |u|\, d\mu, \tag{2.4}$$

where $0 \le \alpha \le 1$. If the measure is Ahlfors Q-regular, that is,

$$C^{-1} r^Q \le \mu\big(B(x,r)\big) \le C r^Q$$

for every $x \in X$ and $r > 0$, then $\mathcal{M}_\alpha u$ and $\widetilde{\mathcal{M}}_{\alpha/Q} u$ are comparable in the sense that there exists a constant $C \ge 1$, depending only on the doubling constant, such that

$$C^{-1} \mathcal{M}_\alpha u \le \widetilde{\mathcal{M}}_{\alpha/Q} u \le C \mathcal{M}_\alpha u.$$

In the case when only the lower bound holds in the Alhfors regularity condition, then we say that the measure satisfies the measure lower bound condition.

3. Lebesgue spaces

In this section, we study the action of fractional maximal operators on L^p-spaces. We do not assume that μ is doubling. In this generality, the Hardy–Littlewood maximal function theorem does not hold for the standard maximal operator. Therefore, we consider a modified version of the fractional maximal operator as in [28] and [30]. For $\kappa \geq 1$, define

$$\mathcal{M}_\alpha^\kappa u(x) = \sup_{r>0} \frac{r^\alpha}{\mu(B(x,\kappa r))} \int_{B(x,r)} |u|\,d\mu \tag{3.1}$$

and

$$\widetilde{\mathcal{M}}_\alpha^\kappa u(x) = \sup_{r>0} \mu\big(B(x,\kappa r)\big)^{\alpha-1} \int_{B(x,r)} |u|\,d\mu. \tag{3.2}$$

When $\alpha = 0$, we denote $\mathcal{M}^\kappa = \mathcal{M}_\alpha^\kappa = \widetilde{\mathcal{M}}_\alpha^\kappa$. Sawano [28] proved that the estimates

$$\mu\big(\{x \in X : \mathcal{M}^\kappa u(x) > \lambda\}\big) \leq \lambda^{-1} \|u\|_{L^1(X)} \tag{3.3}$$

for every $\lambda > 0$ and

$$\|\mathcal{M}^\kappa u\|_{L^p(X)} \leq C \|u\|_{L^p(X)}, \tag{3.4}$$

$1 < p \leq \infty$, hold if $\kappa \geq 2$. He also showed that they are not true, in general, if $1 \leq \kappa < 2$.

Using these estimates and some simple pointwise inequalities, we obtain Sobolev-type theorems for modified fractional maximal operators (3.1) and (3.2).

THEOREM 3.1

Let $0 \leq \alpha < 1$. Then

$$\mu\big(\{x \in X : \widetilde{\mathcal{M}}_\alpha^2 u(x) > \lambda\}\big) \leq \big(\lambda^{-1} \|u\|_{L^1(X)}\big)^{1/(1-\alpha)} \tag{3.5}$$

for every $\lambda > 0$ and $u \in L^1(X)$.

Proof

Fix $x \in X$. Then for every ball $B(x,r)$, we have

$$\begin{aligned}
&\mu\big(B(x,2r)\big)^{\alpha-1} \int_{B(x,r)} |u|\,d\mu \\
&\quad = \Big(\frac{1}{\mu(B(x,2r))} \int_{B(x,r)} |u|\,d\mu\Big)^{1-\alpha} \Big(\int_{B(x,r)} |u|\,d\mu\Big)^\alpha \\
&\quad \leq \big(\mathcal{M}^2 u(x)\big)^{1-\alpha} \|u\|_{L^1(X)}^\alpha,
\end{aligned}$$

which implies that

$$\widetilde{\mathcal{M}}_\alpha^2 u(x) \leq \big(\mathcal{M}^2 u(x)\big)^{1-\alpha} \|u\|_{L^1(X)}^\alpha.$$

Using this and (3.3), we obtain (3.5). $\square$

The proof of the following bound for the modified fractional maximal function is similar to one in [16].

THEOREM 3.2

Let $p > 1$ and $\alpha p \le 1$. Then

$$\|\widetilde{\mathcal{M}}_\alpha^2 u\|_{L^{p/(1-\alpha p)}(X)} \le C\|u\|_{L^p(X)}$$

for every $u \in L^p(X)$.

Proof

Let $x \in X$. Using Hölder's inequality, we have

$$\begin{aligned}
&\mu\big(B(x,2r)\big)^{\alpha-1}\int_{B(x,r)}|u|\,d\mu \\
&= \mu\big(B(x,2r)\big)^{\alpha-1}\Big(\int_{B(x,r)}|u|\,d\mu\Big)^{\alpha p}\Big(\int_{B(x,r)}|u|\,d\mu\Big)^{1-\alpha p} \\
&\le \mu\big(B(x,2r)\big)^{\alpha-1}\mu\big(B(x,r)\big)^{(1-1/p)\alpha p}\|u\|_{L^p(X)}^{\alpha p}\Big(\int_{B(x,r)}|u|\,d\mu\Big)^{1-\alpha p} \\
&\le \|u\|_{L^p(X)}^{\alpha p}\Big(\mu\big(B(x,2r)\big)^{-1}\int_{B(x,r)}|u|\,d\mu\Big)^{1-\alpha p} \\
&\le \|u\|_{L^p(X)}^{\alpha p}\big(\mathcal{M}^2 u(x)\big)^{1-\alpha p},
\end{aligned}$$

for every ball $B(x,r)$, which implies that

$$\widetilde{\mathcal{M}}_\alpha^2 u(x) \le \|u\|_{L^p(X)}^{\alpha p}\big(\mathcal{M}^2 u(x)\big)^{1-\alpha p}.$$

Using this and (3.4), we obtain

$$\begin{aligned}
\|\widetilde{\mathcal{M}}_\alpha^2 u\|_{L^{p/(1-\alpha p)}(X)} &\le \|u\|_{L^p(X)}^{\alpha p}\big\|(\mathcal{M}^2 u)^{1-\alpha p}\big\|_{L^{p/(1-\alpha p)}(X)} \\
&= \|u\|_{L^p(X)}^{\alpha p}\|\mathcal{M}^2 u\|_{L^p(X)}^{1-\alpha p} \\
&\le C\|u\|_{L^p(X)}.
\end{aligned}$$

□

If the measure lower bound condition holds, then

$$\mathcal{M}_\alpha^\kappa u \le C\widetilde{\mathcal{M}}_{\alpha/Q}^\kappa u,$$

where the constant C depends on α, κ and on the constant of the lower bound condition. Thus, Theorems 3.1 and 3.2 imply the following results.

THEOREM 3.3

Assume that the measure lower bound condition holds. Let $0 < \alpha < Q$. Then there is a constant $C > 0$, depending only on the constant in the measure lower bound and α, such that

$$\mu\big(\{\mathcal{M}_\alpha^2 u > \lambda\}\big) \le C\big(\lambda^{-1}\|u\|_{L^1(X)}\big)^{Q/(Q-\alpha)},$$

for every $\lambda > 0$ and $u \in L^1(X)$.

THEOREM 3.4

Assume that the measure lower bound condition holds. Let $p > 1$, and assume that

$0 < \alpha \le Q/p$. Then there is a constant $C > 0$, depending only on the constant of the measure lower bound condition, p, and α, such that

$$\|\mathcal{M}_\alpha^2 u\|_{L^{p^*}(X)} \le C\|u\|_{L^p(X)},$$

for every $u \in L^p(X)$ with $p^* = Qp/(Q-\alpha p)$.

Observe, that if the measure is doubling, then the results in this section hold for the standard maximal functions with $\kappa = 1$.

4. Morrey spaces

In this section, we study the behavior of the fractional maximal operator on Morrey spaces. Let $1 \le p < \infty$ and $\beta \in \mathbb{R}$. A function $u \in L^1_{\mathrm{loc}}(X)$ belongs to the Morrey space $\mathcal{M}^{p,\beta,\kappa}(X)$, if

$$\|u\|_{\mathcal{M}^{p,\beta,\kappa}(X)} = \sup r^{-\beta}\Big(\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)}|u|^p\,d\mu\Big)^{1/p} < \infty,$$

where the supremum is taken over all $x \in X$ and $r > 0$ (see [24]). Observe that for $\beta \le 0$, this is equivalent to the requirement

$$\mathcal{M}^\kappa_{-\beta p}\big(|u|^p\big) \in L^\infty(X).$$

A result of Chiarenza and Frasca [9] says that the Hardy–Littlewood maximal operator is bounded on $\mathcal{M}^{p,\beta,1}(\mathbf{R}^n)$, when $p > 1$. This was extended to the nondoubling metric space setting in [24], where it was shown that

$$\|\mathcal{M}^2 u\|_{\mathcal{M}^{p,\beta,4}(X)} \le C\|u\|_{\mathcal{M}^{p,\beta,2}(X)}, \tag{4.1}$$

for $p > 1$.

Our next result is a Sobolev-type inequality for the modified fractional maximal operator acting on Morrey spaces. This could be deduced from the corresponding result for the Riesz potential (see [24]), but we provide a simple direct proof.

THEOREM 4.1

Let $\alpha > 0$ and $\alpha + \beta < 0$. Let $u \in \mathcal{M}^{p,\beta,2}(X)$ with $1 < p < \infty$. Then there is a constant $C > 0$, depending only p, α, and β, such that

$$\|\mathcal{M}_\alpha^2 u\|_{\mathcal{M}^{p/(1+\alpha/\beta),\alpha+\beta,4}(X)} \le C\|u\|_{\mathcal{M}^{p,\beta,2}(X)}.$$

Proof

Let $\alpha > 0$. Let $x \in X$ and $r > 0$. Using Hölder's inequality, we have

$$\begin{aligned}
&\frac{r^\alpha}{\mu(B(x,2r))}\int_{B(x,r)}|u|\,d\mu \\
&\quad= \Big(\frac{1}{\mu(B(x,2r))}\int_{B(x,r)}|u|\,d\mu\Big)^{1+\alpha/\beta}\Big(\frac{r^{-\beta}}{\mu(B(x,2r))}\int_{B(x,r)}|u|\,d\mu\Big)^{-\alpha/\beta} \\
&\quad\le \big(\mathcal{M}^2 u(x)\big)^{1+\alpha/\beta}\|u\|^{-\alpha/\beta}_{\mathcal{M}^{p,\beta,2}(X)}.
\end{aligned}$$

Because the right-hand side above does not depend on r, we obtain

$$\mathcal{M}_\alpha^2 u(x) \le \big(\mathcal{M}^2 u(x)\big)^{1+\alpha/\beta} \|u\|_{\mathcal{M}^{p,\beta,2}(X)}^{-\alpha/\beta}.$$

Using this and (4.1), we obtain

$$\begin{aligned} &r^{-(\alpha+\beta)}\Big(\frac{1}{\mu(B(x,4r))}\int_{B(x,r)} (\mathcal{M}_\alpha^2 u)^{p/(1+\alpha/\beta)}\,d\mu\Big)^{(1+\alpha/\beta)/p} \\ &\quad\le \Big(r^{-\beta}\Big(\frac{1}{\mu(B(x,4r))}\int_{B(x,r)} (\mathcal{M}^2 u)^p\,d\mu\Big)^{1/p}\Big)^{1+\alpha/\beta} \|u\|_{\mathcal{M}^{p,\beta,2}(X)}^{-\alpha/\beta} \\ &\quad\le \|\mathcal{M}^2 u\|_{\mathcal{M}^{p,\beta,4}(X)}^{1+\alpha/\beta} \|u\|_{\mathcal{M}^{p,\beta,2}(X)}^{-\alpha/\beta} \le C\|u\|_{\mathcal{M}^{p,\beta,2}(X)}. \end{aligned}$$

□

REMARK 4.2

If we define the Morrey space with the norm

$$\|u\|_{\widetilde{\mathcal{M}}^{p,\beta,\kappa}(X)} = \sup \mu\big(B(x,\kappa r)\big)^{-\beta}\Big(\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)} |u|^p\,d\mu\Big)^{1/p},$$

where the supremum is taken over all $x \in X$ and $r > 0$, then the same proof as above gives

$$\|\widetilde{\mathcal{M}}_\alpha^2 u\|_{\widetilde{\mathcal{M}}^{p/(1+\alpha/\beta),\alpha+\beta,4}(X)} \le C\|u\|_{\widetilde{\mathcal{M}}^{p,\beta,2}(X)}.$$

5. The discrete fractional maximal function

From now on, we assume that the measure is doubling. We begin the construction of the discrete maximal function with a covering of the space. Let $r > 0$. Since the measure is doubling, there are balls $B(x_i, r)$, $i = 1, 2, \dots$, such that

$$X = \bigcup_{i=1}^{\infty} B(x_i, r) \quad \text{and} \quad \sum_{i=1}^{\infty} \chi_{B(x_i,6r)} \le N < \infty.$$

This means that the dilated balls $B(x_i, 6r)$, $i = 1, 2, \dots$, are of bounded overlap. The constant N depends only on the doubling constant, and, in particular, it is independent of r.

Then we construct a partition of unity subordinate to the covering $B(x_i, r)$, $i = 1, 2, \dots$, of X. Indeed, there is a family of functions φ_i, $i = 1, 2, \dots$, such that $0 \le \varphi_i \le 1$, $\varphi_i = 0$ in $X \setminus B(x_i, 6r)$, $\varphi_i \ge \nu$ in $B(x_i, 3r)$, φ_i is Lipschitz with constant L/r with ν and L depending only on the doubling constant, and

$$\sum_{i=1}^{\infty} \varphi_i(x) = 1$$

for every $x \in X$.

The discrete convolution of $u \in L^1_{\mathrm{loc}}(X)$ at the scale $3r$ is

$$u_r(x) = \sum_{i=1}^{\infty} \varphi_i(x) u_{B(x_i,3r)}$$

for every $x \in X$, and we write $u_r^\alpha = r^\alpha u_r$. Observe that the kernel of the integral operator in the definition of the discrete convolution is not symmetric. Coverings, partitions of unity, and discrete convolutions are standard tools in harmonic analysis on metric measure spaces (see [11], [21]).

Let r_j, $j = 1, 2, \dots$, be an enumeration of the positive rationals, and let balls $B(x_i, r_j)$, $i = 1, 2, \dots$, be a covering of X as above. The discrete fractional maximal function of u in X is

$$\mathcal{M}^*_\alpha u(x) = \sup_j |u|^\alpha_{r_j}(x)$$

for every $x \in X$. For $\alpha = 0$, we obtain the Hardy–Littlewood-type discrete maximal function studied in [1], [18], and [20]. Observe that the construction depends on the choice of the coverings, but our goal is to derive estimates that are independent of the chosen coverings.

The discrete fractional maximal function is comparable to the standard fractional maximal function. The proof is similar to that for the discrete maximal function and Hardy–Littlewood maximal function in [18, Lemma 3.1].

LEMMA 5.1
Assume that the measure is doubling. Let $u \in L^1_{\mathrm{loc}}(X)$. Then there is a constant $C \geq 1$, depending only on the doubling constant, such that

$$C^{-1}\mathcal{M}_\alpha u(x) \leq \mathcal{M}^*_\alpha u(x) \leq C\mathcal{M}_\alpha u(x),$$

for every $x \in X$.

Proof
We begin by proving the second inequality. Let $x \in X$, and let r_j be a positive rational number. Since $\varphi_i = 0$ on $X \setminus B(x_i, 6r_j)$ and $B(x_i, 3r_j) \subset B(x, 9r_j)$ for every $x \in B(x_i, 6r_j)$, we have by the doubling condition that

$$\begin{aligned} |u|^\alpha_{r_j}(x) &= r_j^\alpha \sum_{i=1}^\infty \varphi_i(x) |u|_{B(x_i,3r_j)} \\ &\leq r_j^\alpha \sum_{i=1}^\infty \varphi_i(x) \frac{\mu(B(x,9r_j))}{\mu(B(x_i,3r_j))} \fint_{B(x,9r_j)} |u| \, d\mu \leq C\mathcal{M}_\alpha u(x), \end{aligned}$$

where C depends only on the doubling constant. The required inequality follows by taking the supremum on the left-hand side.

To prove the first inequality, we observe that for each $x \in X$ there exists $i = i_x$ such that $x \in B(x_i, r_j)$. This implies that $B(x, r_j) \subset B(x_i, 2r_j)$ and hence

$$\begin{aligned} r_j^\alpha \fint_{B(x,r_j)} |u| \, d\mu &\leq C r_j^\alpha \fint_{B(x_i,3r_j)} |u| \, d\mu \\ &\leq C r_j^\alpha \varphi_i(x) \fint_{B(x_i,3r_j)} |u| \, d\mu \leq C\mathcal{M}^*_\alpha u(x), \end{aligned}$$

where C depends only on the doubling constant. In the second inequality, we used the fact that $\varphi_i \geq \nu$ on $B(x_i, r_j)$. Again the claim follows by taking the supremum on the left-hand side. $\square$

Since the discrete and the standard maximal functions are comparable, the Sobolev and the weak-type estimates hold for the discrete fractional maximal function as well (see Theorems 3.4, 3.3).

6. Sobolev spaces

A nonnegative Borel function g on X is said to be an upper gradient of a function $u\colon X \to [-\infty, \infty]$ if for all rectifiable paths $\gamma\colon [0,1] \to X$, we have

$$|u(\gamma(0)) - u(\gamma(1))| \le \int_\gamma g\, ds, \tag{6.1}$$

whenever both $u(\gamma(0))$ and $u(\gamma(1))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. The assumption that g is a Borel function is needed in the definition of the path integral. If g is merely a μ-measurable function and (6.1) holds for p-almost every path, then g is said to be a p-weak upper gradient of u. By saying that (6.1) holds for p-almost every path we mean that it fails only for a path family with zero p-modulus. A family Γ of curves is of zero p-modulus if there is a nonnegative Borel measurable function $\rho \in L^p(X)$ such that for all curves $\gamma \in \Gamma$, the path integral $\int_\gamma \rho\, ds$ is infinite. If we redefine a p-weak upper gradient on a set of measure zero we obtain an upper gradient of the same function. If g is a p-weak upper gradient of u, then there is a sequence g_i, $i = 1, 2, \dots$, of upper gradients of u such that

$$\int_X |g_i - g|^p \, d\mu \to 0$$

as $i \to \infty$. Hence every p-weak upper gradient can be approximated by upper gradients in the $L^p(X)$-norm. If u has an upper gradient that belongs to $L^p(X)$ with $p \ge 1$, then it has a minimal p-weak upper gradient g_u in the sense that for every p-weak upper gradient g of u, $g_u \le g$ almost everywhere.

We define the first-order Sobolev spaces on the metric space X by using the p-weak upper gradients. These spaces are called Newtonian spaces. For $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \Big(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \Big)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The Newtonian space on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$. The same definition applies to subsets of X as well. The notion of a p-weak upper gradient is used to prove that $N^{1,p}(X)$ is a Banach space. For the properties of Newtonian spaces, we refer to [7], [31], and [32].

We say that X supports a (weak) $(1,p)$-Poincaré inequality if there exist constants $c > 0$ and $\tau \ge 1$ such that for all balls $B(x,r) \subset X$, for all locally integrable functions u on X, and for all p-weak upper gradients g of u,

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le cr \Big(\fint_{B(x,\tau r)} g^p \, d\mu \Big)^{1/p}. \tag{6.2}$$

Note that since p-weak upper gradients can be approximated by upper gradients in the $L^p(X)$-norm, it would be enough to require the Poincaré inequality for upper gradients only.

By Hölder's inequality it is easy to see that if X supports a $(1,p)$-Poincaré inequality, then it supports a $(1,q)$-Poincaré inequality for every $q > p$. It is shown in [17] that if X is complete and μ doubling, then a $(1,p)$-Poincaré inequality implies a $(1,p')$-Poincaré inequality for some $p' < p$. Hence the $(1,p)$-Poincaré inequality is a self-improving condition.

The following Sobolev-type theorem is a generalization of the main result of [19] to the metric setting. It shows that the discrete fractional maximal operator is a smoothing operator. More precisely, the discrete fractional maximal function of an L^p-function has a weak upper gradient, and both u and the weak upper gradient belong to a higher Lebesgue space than u.

We use the following simple fact in the proof. Suppose that u_i, $i = 1, 2, \ldots,$ are functions and g_i, $i = 1, 2, \ldots,$ are p-weak upper gradients of u_i, respectively. Let $u = \sup_i u_i$ and $g = \sup_i g_i$. If u is finite almost everywhere, then g is a p-weak upper gradient of u. For the proof, we refer to [7].

THEOREM 6.1

Assume that the measure is doubling and that the measure lower bound condition holds. Assume that $u \in L^p(X)$ with $1 < p < Q$. Let

$$1 \le \alpha < Q/p, \quad p^* = Qp/(Q - \alpha p), \qquad \text{and} \qquad q = Qp/\big(Q - (\alpha - 1)p\big).$$

Then $\mathcal{M}^*_{\alpha-1} u$ is a weak upper gradient of $\mathcal{M}^*_\alpha u$. Moreover, there is a constant $C > 0$, depending only on the doubling constant, the constant in the measure lower bound, p, and α, such that

$$\|C\mathcal{M}^*_\alpha u\|_{L^{p^*}(X)} \le C\|u\|_{L^p(X)} \qquad \text{and} \qquad \|\mathcal{M}^*_{\alpha-1} u\|_{L^q(X)} \le C\|u\|_{L^p(X)}.$$

Proof

We begin by considering $|u|^\alpha_r$. By Lemma 5.1, we have

$$|u|^\alpha_r(x) = r^\alpha |u|_r(x) \le \mathcal{M}^*_\alpha u(x) \le C\mathcal{M}_\alpha u(x)$$

for every $x \in X$. Then we consider the weak upper gradient of $|u|^\alpha_r$. Since

$$|u|^\alpha_r(x) = r^\alpha \sum_{i=1}^\infty \varphi_i(x) |u|_{B(x_i,3r)},$$

each φ_i is L/r-Lipschitz continuous and has a support in $B(x_i, 6r)$, the function

$$g_r(x) = Lr^{\alpha-1} \sum_{i=1}^\infty |u|_{B(x_i,3r)} \chi_{B(x_i,6r)}(x)$$

is a weak upper gradient of $|u|_r^\alpha$. If $x \in B(x_i,r)$, then $B(x_i,3r) \subset B(x,9r) \subset B(x_i,15r)$ and

$$|u|_{B(x_i,3r)} \leq C \fint_{B(x,9r)} |u|\,d\mu.$$

The bounded overlap property of the balls $B(x_i,6r)$, $i = 1,2,\dots$, implies that

$$g_r(x) \leq C r^{\alpha-1} \fint_{B(x,9r)} |u|\,d\mu \leq C\mathcal{M}_{\alpha-1}u(x) \leq C\mathcal{M}^*_{\alpha-1}u(x),$$

and consequently $C\mathcal{M}^*_{\alpha-1}u$ is a weak upper gradient of $|u|_r^\alpha$ as well.

By Lemma 5.1 and Theorem 3.4, $\mathcal{M}^*_\alpha u$ belongs to $L^{p^*}(X)$, and hence $\mathcal{M}^*_\alpha u$ is finite almost everywhere. As

$$\mathcal{M}^*_\alpha u(x) = \sup_j |u|^\alpha_{r_j}(x),$$

and because $C\mathcal{M}^*_{\alpha-1}u$ is an upper gradient of $|u|^\alpha_{r_j}$ for every $j = 1,2,\dots$, we conclude that it is an upper gradient of $\mathcal{M}^*_\alpha u$ as well. The norm bounds follow from Theorem 3.4. □

REMARK 6.2

With the assumptions of Theorem 6.1, $\mathcal{M}^*_\alpha u \in N^{1,q}_{\mathrm{loc}}(X)$ and

$$\|\mathcal{M}^*_\alpha u\|_{N^{1,q}(A)} \leq \mu(A)^{1/q-1/p^*}\|u\|_{L^p(A)}$$

for all open sets $A \subset X$ with $\mu(A) < \infty$.

Next we study the behavior of the discrete fractional maximal function in Newtonian spaces. The first result shows that if the function u is a Sobolev function, then its discrete fractional maximal function belongs to a Sobolev space with the Sobolev conjugate exponent.

THEOREM 6.3

Assume that the measure is doubling and that the measure lower bound condition holds and that X is a complete metric space which supports a $(1,p)$-Poincaré inequality with $1 < p < \infty$. Assume that $u \in N^{1,p}(X)$ and that $0 < \alpha < Q/p$. Then $\mathcal{M}^*_\alpha u \in N^{1,p^*}(X)$ with $p^* = Qp/(Q-\alpha p)$. Moreover, there is a constant $C > 0$, depending only on the doubling constant, the constant in the measure lower bound, p, and α, such that

$$\|\mathcal{M}^*_\alpha u\|_{N^{1,p^*}(X)} \leq C\|u\|_{N^{1,p}(X)}.$$

Proof

Let $u \in N^{1,p}(X)$, and let $g \in L^p(X)$ be a weak upper gradient of u. By Theorem 3.4, we have

$$\|\mathcal{M}^*_\alpha u\|_{L^{p^*}(X)} \leq C\|u\|_{L^p(X)}.$$

For the weak upper gradient, let $x, y \in B(x_j, r)$, and let

$$I_j = \{i : B(x_i, 6r) \cap B(x_j, r) \neq \emptyset\}.$$

By the bounded overlap of the balls $B(x_i, 6r)$, the set I_j is finite and the cardinality does not depend on j. By the (L/r)-Lipschitz continuity of functions φ_i and by the $(1, p')$-Poincaré inequality, which follows from the $(1, p)$-Poincaré inequality for some $1 < p' < p$, we have

$$\begin{aligned}
\big||u|_r^\alpha(x) - |u|_r^\alpha(y)\big| &= r^\alpha \Big|\sum_{i=1}^{\infty} \big(|u|_{B(x_i,3r)} - |u|_{B(x_j,3r)}\big)\big(\varphi_i(x) - \varphi_i(y)\big)\Big| \\
&\le C r^{\alpha-1} \mathrm{d}(x, y) \sum_{i \in I_j} \big||u|_{B(x_i,3r)} - |u|_{B(x_j,3r)}\big| \\
&\le C r^{\alpha-1} \mathrm{d}(x, y) \fint_{B(x_j,10r)} \big||u| - |u|_{B(x_j,10r)}\big| \, d\mu \\
&\le C r^{\alpha} \mathrm{d}(x, y) \Big(\fint_{B(x_j,10\lambda r)} g^{p'} \, d\mu\Big)^{1/p'}.
\end{aligned}$$

Since the pointwise Lipschitz constant of a function is a weak upper gradient, we see that

$$g_r(x) = C r^\alpha \sum_{j=1}^{\infty} \Big(\fint_{B(x_j,10\lambda r)} g^{p'} \, d\mu\Big)^{1/p'} \chi_{B(x_j,6r)}(x)$$

is a weak upper gradient of $|u|_r^\alpha$. Moreover, by the bounded overlap of the balls,

$$\begin{aligned}
g_r(x) &\le C \sum_{j=1}^{\infty} \Big(r^{\alpha p'} \fint_{B(x_j,10\lambda r)} g^{p'} \, d\mu\Big)^{1/p'} \chi_{B(x_j,6r)}(x) \\
&\le C \big(\mathcal{M}_{\alpha p'}^* g^{p'}(x)\big)^{1/p'}.
\end{aligned}$$

By the same argument as in the proof of Theorem 6.1, we conclude that $(\mathcal{M}_{\alpha p'}^* g^{p'})^{1/p'}$ is a weak upper gradient of $\mathcal{M}_\alpha^* u$. Since $g^{p'} \in L^{p/p'}(X)$ and $p/p' > 1$, Theorem 3.4 implies that

$$\big\|(\mathcal{M}_{\alpha p'}^* g^{p'})^{1/p'}\big\|_{L^{p^*}(X)} \le C \|g\|_{L^p(X)},$$

and the claim follows. □

7. Campanato spaces

In this section, we study the behavior of the discrete fractional maximal operator on Campanato spaces. Let $1 \le p < \infty$ and $\beta \in \mathbb{R}$. A function $u \in L^1_{\mathrm{loc}}(X)$ belongs to the Campanato space $\mathcal{L}^{p,\beta}(X)$ if

$$\|u\|_{\mathcal{L}^{p,\beta}(X)} = \sup r^{-\beta} \Big(\fint_{B(x,r)} |u - u_{B(x,r)}|^p \, d\mu\Big)^{1/p} < \infty.$$

Here the supremum is taken over all $x \in X$ and $r > 0$. We denote the standard Morrey space as $\mathcal{M}^{p,\beta}(X) = \mathcal{M}^{p,\beta,1}(X)$. Observe, that $\|\cdot\|_{\mathcal{M}^{p,\beta}(X)}$ is a norm in the Morrey space, but $\|\cdot\|_{\mathcal{L}^{p,\beta}(X)}$ is merely a seminorm in the Campanato space.

Morrey spaces, Campanato spaces, functions of bounded mean oscillation (BMO), and functions in $C^{0,\beta}(X)$ have the following connections (see [5], [6], [22], [23], [25], [27]):

- $\mathcal{M}^{p,\beta}(X) \subset \mathcal{L}^{p,\beta}(X)$,
- $\mathcal{L}^{p,\beta}(X) = \mathcal{M}^{p,\beta}(X)$ if $-Q/p < \beta < 0$ (here we identify functions that differ only by an additive constant),
- $\mathcal{L}^{1,0}(X) = \mathrm{BMO}(X)$, and
- $\mathcal{L}^{p,\beta}(X) = C^{0,\beta}(X)$ if $0 < \beta \le 1$.

Recall that $u \in C^{0,\beta}(X)$ means that u is a Hölder continuous function with exponent $0 < \beta \le 1$, that is,

$$\big|u(x) - u(y)\big| \le C\mathrm{d}(x,y)^{\beta}$$

for all $x, y \in X$.

The following technical lemma will be useful for us.

LEMMA 7.1

Assume that the measure is doubling. Assume that $u \in \mathcal{L}^{p,\beta}(X)$. Let $x \in X$, $0 < 2r < R$, and $y \in B(x, 2R)$. If $\beta < 0$, then

$$|u_{B(y,r)} - u_{B(x,R)}| \le Cr^{\beta}\|u\|_{\mathcal{L}^{p,\beta}(X)}. \tag{7.1}$$

If $\beta = 0$, then

$$|u_{B(y,r)} - u_{B(x,R)}| \le C\log\frac{6R}{r}\|u\|_{\mathcal{L}^{p,0}(X)}. \tag{7.2}$$

The constant C depends only on the doubling constant.

Proof

Let k be the smallest index such that $2^k r \ge 3R$. Then $B(x,R) \subset B(y, 2^k r)$ and

$$\begin{aligned}
&|u_{B(y,r)} - u_{B(x,R)}| \\
&\quad \le \sum_{i=1}^{k} |u_{B(y,2^i r)} - u_{B(y,2^{i-1}r)}| + |u_{B(y,2^k r)} - u_{B(x,R)}| \\
&\quad \le \sum_{i=1}^{k} \fint_{B(y,2^{i-1}r)} |u - u_{B(y,2^i r)}|\, d\mu + \fint_{B(x,R)} |u - u_{B(y,2^k r)}|\, d\mu \\
&\quad \le C\sum_{i=1}^{k} \fint_{B(y,2^{i}r)} |u - u_{B(y,2^i r)}|\, d\mu + C\fint_{B(y,2^k r)} |u - u_{B(y,2^k r)}|\, d\mu \\
&\quad \le Cr^{\beta}\|u\|_{\mathcal{L}^{p,\beta}(X)}\Big(\sum_{i=1}^{\infty} 2^{i\beta} + 2^{k\beta}\Big) \le Cr^{\beta}\|u\|_{\mathcal{L}^{p,\beta}(X)},
\end{aligned}$$

where C depends only on the doubling constant and the sum converges since $\beta < 0$. This proves (7.1).

The proof of (7.2) is quite similar. Indeed, by the choice of k, we have $2^k r \leq 6R$ and consequently

$$\begin{aligned}
&|u_{B(y,r)} - u_{B(x,R)}| \\
&\quad \leq C \sum_{i=1}^{k} \fint_{B(y,2^i r)} |u - u_{B(y,2^i r)}|\, d\mu + C \fint_{B(y,2^k r)} |u - u_{B(y,2^k r)}|\, d\mu \\
&\quad \leq Ck \|u\|_{\mathcal{L}^{p,0}(X)} \leq C \log \frac{6R}{r} \|u\|_{\mathcal{L}^{p,0}(X)}. \qquad \square
\end{aligned}$$

The next results show that the fractional maximal function of a Hölder continuous function is Hölder continuous with a better exponent or a Lipschitz function. A similar result for the fractional integral operator can be found in [13] and [14]. Recall that $\mathcal{L}^{p,\beta}(X) = C^{0,\beta}(X)$ for $0 < \beta \leq 1$.

THEOREM 7.2

Assume that the measure is doubling. Let $u \in C^{0,\beta}(X)$ with $0 < \beta \leq 1$. If $\alpha + \beta \leq 1$, then $\mathcal{M}^*_\alpha u \in C^{0,\alpha+\beta}(X)$.

Proof

Let $r > 0$. We begin by proving the claim for $|u|^\alpha_r$. Let $x, y \in X$. Assume first that $\mathrm{d}(x,y) > r$. Then

$$\begin{aligned}
\big||u|^\alpha_r(x) - |u|^\alpha_r(y)\big| \leq r^\alpha \Big(\big|u(x) - u(y)\big| &+ \sum_{i=1}^{\infty} \varphi_i(x) \big||u|_{B(x_i,3r)} - |u(x)|\big| \\
&+ \sum_{i=1}^{\infty} \varphi_i(y) \big||u|_{B(x_i,3r)} - |u(y)|\big| \Big).
\end{aligned}$$

In the first sum, $\varphi_i(x) \neq 0$ only if $x \in B(x_i, 6r)$. For such i, by the Hölder continuity of u, we have

$$\big||u|_{B(x_i,3r)} - |u(x)|\big| \leq \fint_{B(x_i,3r)} \big|u(z) - u(x)\big|\, d\mu \leq C r^\beta.$$

A similar estimate holds for terms of the second sum when $y \in B(x_i, 6r)$. The bounded overlap of the balls $B(x_i, 6r)$, $i = 1, 2, \ldots$, and the Hölder continuity of u imply that

$$\big||u|^\alpha_r(x) - |u|^\alpha_r(y)\big| \leq C r^\alpha \big(\mathrm{d}(x,y)^\beta + r^\beta\big) \leq C \mathrm{d}(x,y)^{\alpha+\beta}.$$

Assume then that $\mathrm{d}(x,y) \leq r$. Now

$$\big||u|^\alpha_r(x) - |u|^\alpha_r(y)\big| \leq r^\alpha \Big(\sum_{i=1}^{\infty} \big|\varphi_i(x) - \varphi_i(y)\big| \big||u|_{B(x_i,3r)} - |u(x)|\big| \Big),$$

where $\varphi_i(x) - \varphi_i(y) \neq 0$ only if $x \in B(x_i, 6r)$ or $y \in B(x_i, 6r)$. If $y \in B(x_i, 6r)$, then the assumption $\mathrm{d}(x,y) \leq r$ implies that $x \in B(x_i, 7r)$. Hence for such i, as above,

$$\big||u|_{B(x_i,3r)} - |u(x)|\big| \le Cr^{\beta}.$$

By the L/r-Lipschitz continuity of the functions φ_i and the bounded overlap of the balls $B(x_i, 6r)$, we have

$$\big||u|_r^{\alpha}(x) - |u|_r^{\alpha}(y)\big| \le Cr^{\alpha}\mathrm{d}(x,y)r^{\beta-1},$$

where, if $\alpha+\beta \le 1$,

$$r^{\alpha}\mathrm{d}(x,y)r^{\beta-1} \le \mathrm{d}(x,y)^{\alpha+\beta}.$$

The claim for $|u|_r^{\alpha}$ follows from this.

Then we prove the claim for $\mathcal{M}_{\alpha}^{*}u$. We may assume that $\mathcal{M}_{\alpha}^{*}u(x) \ge \mathcal{M}_{\alpha}^{*}u(y)$. Let $\varepsilon > 0$, and let $r_\varepsilon > 0$ be such that

$$|u|_{r_\varepsilon}^{\alpha}(x) > \mathcal{M}_{\alpha}^{*}u(x) - \varepsilon.$$

Then, by the first part of the proof,

$$\mathcal{M}_{\alpha}^{*}u(x) - \mathcal{M}_{\alpha}^{*}u(y) \le |u|_{r_\varepsilon}^{\alpha}(x) - |u|_{r_\varepsilon}^{\alpha}(y) + \varepsilon \le C\mathrm{d}(x,y)^{\alpha+\beta} + \varepsilon,$$

if $\alpha+\beta < 1$. By letting $\varepsilon \to 0$, we obtain

$$\big|\mathcal{M}_{\alpha}^{*}u(x) - \mathcal{M}_{\alpha}^{*}u(y)\big| \le C\mathrm{d}(x,y)^{\alpha+\beta}. \qquad \square$$

According to the next result, the fractional maximal operator maps functions in Campanato spaces to Hölder continuous functions. For a related result concerning the fractional integral operator (see [26]).

THEOREM 7.3

Assume that the measure is doubling. Let $\alpha > 0$, $0 \le \alpha+\beta \le 1$, and let $u \in \mathcal{L}^{p,\beta}(X)$. Then there is a constant $C > 0$, depending only on the doubling constant p and α and β, such that

$$\|\mathcal{M}_{\alpha}^{*}u\|_{C^{0,\alpha+\beta}(X)} \le C\|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

Proof

Let $r > 0$. We begin by proving the claim for $|u|_r^{\alpha}$. Let $x, y \in X$. Assume first that $r < \mathrm{d}(x,y)$. Let $B = B(x, 4\mathrm{d}(x,y))$. Then

$$\begin{aligned} &\big||u|_r^{\alpha}(x) - |u|_r^{\alpha}(y)\big| \\ &\quad \le \big||u|_r^{\alpha}(x) - r^{\alpha}|u|_B\big| + \big|r^{\alpha}|u|_B - |u|_r^{\alpha}(y)\big| \\ &\quad \le r^{\alpha}\Big(\sum_{i=1}^{\infty}\varphi_i(x)\big||u|_{B(x_i,3r)} - |u|_B\big| + \sum_{i=1}^{\infty}\varphi_i(y)\big||u|_{B(x_i,3r)} - |u|_B\big|\Big). \end{aligned}$$

In the first sum, $\varphi_i(x) \ne 0$ only if $x \in B(x_i, 6r)$ and in the second sum, only if $y \in B(x_i, 6r)$. If $\beta < 0$, we use the bounded overlap of the balls $B(x_i, 6r)$, $i = 1, 2, \ldots$, and (7.1), and we have

$$\big||u|_r^{\alpha}(x) - |u|_r^{\alpha}(y)\big| \le Cr^{\alpha+\beta}\|u\|_{\mathcal{L}^{p,\beta}(X)} \le C\mathrm{d}(x,y)^{\alpha+\beta}\|u\|_{\mathcal{L}^{p,\beta}(X)}.$$

Similarly, if $\beta = 0$, estimate (7.2) implies that

$$\begin{aligned}\big||u|_r^\alpha(x) - |u|_r^\alpha(y)\big| &\le Cr^\alpha \log \frac{C\mathrm{d}(x,y)}{r}\|u\|_{\mathcal{L}^{p,\beta}(X)} \\ &= C\mathrm{d}(x,y)^\alpha \Big(\frac{r}{C\mathrm{d}(x,y)}\Big)^\alpha \log \frac{C\mathrm{d}(x,y)}{r}\|u\|_{\mathcal{L}^{p,\beta}(X)} \\ &\le C\mathrm{d}(x,y)^\alpha \|u\|_{\mathcal{L}^{p,\beta}(X)}.\end{aligned}$$

If $r \ge \mathrm{d}(x,y)$, then

$$\begin{aligned}\big||u|_r^\alpha(x) - |u|_r^\alpha(y)\big| &\le r^\alpha \Big(\sum_{i=1}^\infty \big|\varphi_i(x) - \varphi_i(y)\big|\big||u|_{B(x_i,3r)} - |u|_{B(x,10r)}\big|\Big) \\ &\le Cr^{\alpha+\beta-1}\mathrm{d}(x,y)\|u\|_{\mathcal{L}^{p,\beta}(X)} \\ &\le C\mathrm{d}(x,y)^{\alpha+\beta}\|u\|_{\mathcal{L}^{p,\beta}(X)}.\end{aligned}$$

The claim for $\mathcal{M}_\alpha^* u$ follows as in the proof of Theorem 7.2.

If $\beta > 0$, then $\mathcal{L}^{p,\beta}(X) = C^{0,\beta}(X)$ and the result follows from Theorem 7.2. This completes the proof. □

8. The Coifman–Rochberg lemma

By the classical theorem by Coifman and Rochberg [10], $(\mathcal{M}u)^\gamma$, the Hardy–Littlewood maximal function of u raised to any power $0 < \gamma < 1$, is a Muckenhoupt A_1-weight whenever $\mathcal{M}u$ is finite almost everywhere. This means that there exists a constant C such that

$$\fint_{B(x,r)} (\mathcal{M}u)^\gamma \, d\mu \le C \operatorname*{ess\,inf}_{B(x,r)} (\mathcal{M}u)^\gamma$$

for every ball $B(x,r)$ in X (see also [6], [33] for the corresponding result in the metric setting with a doubling measure). For the fractional maximal function, we obtain the result even without taking the power. In this section, we consider the uncentered fractional maximal function, which is comparable to the centered fractional maximal function.

THEOREM 8.1

Let $0 < \alpha < Q$. Assume that $u \in L^1_{\mathrm{loc}}(X)$ is such that $\mathcal{M}_\alpha u$ is finite almost everywhere. Then $\mathcal{M}_\alpha u$ is a Muckenhoupt A_1-weight; that is,

$$\fint_{B(x,r)} \mathcal{M}_\alpha u \, d\mu \le C \operatorname*{ess\,inf}_{B(x,r)} \mathcal{M}_\alpha u$$

for every ball $B(x,r)$ in X. The constant C does not depend on u.

Proof

Let $B(x_0,r) \subset X$ be a ball. We have to show that

$$\fint_{B(x_0,r)} \mathcal{M}_\alpha u \, d\mu \le C\mathcal{M}_\alpha u(x) \tag{8.1}$$

for almost all $x \in B(x_0, r)$. We divide $|u|$ in two parts by setting $v_1 = |u|\chi_{B(x_0,3r)}$ and $v_2 = |u|\chi_{X\setminus B(x_0,3r)}$. Then, for each $x \in B(x_0, r)$, we have

$$\mathcal{M}_\alpha u(x) \le \mathcal{M}_\alpha v_1(x) + \mathcal{M}_\alpha v_2(x). \tag{8.2}$$

Since we also have

$$\mathcal{M}_\alpha v_i(x) \le \mathcal{M}_\alpha u(x) \tag{8.3}$$

for $i = 1, 2$, it suffices to prove inequality (8.1) for v_1 and v_2.

Let $x \in B(x_0, r)$. Then

$$\begin{aligned}
&\fint_{B(x_0,r)} \mathcal{M}_\alpha v_1 \, d\mu \\
&\quad = \frac{1}{\mu(B(x_0,r))} \int_0^\infty \mu\big(\{y \in B(x_0,r) : \mathcal{M}_\alpha v_1(x) > \lambda\}\big) \, d\lambda \\
&\quad = \frac{1}{\mu(B(x_0,r))} \Big(\int_0^a + \int_a^\infty\Big),
\end{aligned}$$

where $a > 0$ will be determined later. For the first integral, we use a trivial estimate

$$\frac{1}{\mu(B(x_0,r))} \int_0^a \mu\big(\{x \in B(x_0,r) : \mathcal{M}_\alpha v_1(x) > \lambda\}\big) \, d\lambda \le a.$$

For the second integral, we use Theorem 3.3 and obtain

$$\begin{aligned}
&\int_a^\infty \mu\big(\{x \in B(x_0,r) : \mathcal{M}_\alpha v_1(x) > \lambda\}\big) \, d\lambda \\
&\quad \le C \int_a^\infty \Big(\frac{\|v_1\|_1}{\lambda}\Big)^{Q/(Q-\alpha)} d\lambda \\
&\quad \le C \|v_1\|_1^{Q/(Q-\alpha)} \frac{Q-\alpha}{\alpha} \lambda^{-\alpha/(Q-\alpha)},
\end{aligned}$$

and hence

$$\fint_{B(x_0,r)} \mathcal{M}_\alpha v_1 \, d\mu \le a + C \frac{\|v_1\|_1^{Q/(Q-\alpha)}}{\mu(B(x_0,r))} \lambda^{-\alpha/(Q-\alpha)}.$$

By choosing

$$a = \frac{\|v_1\|_1}{\mu(B(x_0,r))^{1-\alpha/Q}},$$

we obtain

$$\begin{aligned}
\fint_{B(x_0,r)} \mathcal{M}_\alpha v_1 \, d\mu &\le C \frac{\|v_1\|_1}{\mu(B(x_0,r))^{1-\alpha/Q}} \\
&= \frac{C}{\mu(B(x_0,r))^{1-\alpha/Q}} \int_{B(x_0,3r)} v_1 \, d\mu \le C \mathcal{M}_\alpha v_1(x).
\end{aligned}$$

Inequality (8.1) for v_2 follows immediately if we can show that

$$\mathcal{M}_\alpha v_2(y) \le C \mathcal{M}_\alpha v_2(x)$$

for all $y \in B(x_0, r)$. Let $y \in B(x_0, r)$, and let $B(x', r')$ be a ball such that $y \in B(x', r')$ and $B(x', r') \cap (X \setminus B(x_0, 3r)) \neq \emptyset$. Then $B(x_0, r) \subset B(x', 3r')$. Using the doubling property of μ and the fact that $x \in B(x', 3r')$, we obtain

$$\begin{aligned}\frac{1}{\mu(B(x',r'))^{1-\alpha/Q}} \int_{B(x',r')} v_2 \, d\mu &\leq C \frac{1}{\mu(B(x',3r'))^{1-\alpha/Q}} \int_{B(x',3r')} v_2 \, d\mu \\ &\leq C \mathcal{M}_\alpha v_2(x).\end{aligned}$$

The claim follows because the right-hand side does not depend on y.

To complete the proof, we use (8.2), the estimates above, and (8.3) to obtain

$$\fint_{B(x_0,r)} \mathcal{M}_\alpha u \, d\mu \leq C \mathcal{M}_\alpha v_1(x) + C \mathcal{M}_\alpha v_2(x) \leq C \mathcal{M}_\alpha u(x). \qquad \square$$

REMARK 8.2

Under the assumptions of the previous theorem, we also have

$$\fint_{B(x,r)} (\mathcal{M}_\alpha u)^\gamma \, d\mu \leq C \operatorname*{ess\,inf}_{B(x,r)} (\mathcal{M}_\alpha u)^\gamma$$

for $0 < \gamma \leq 1$ by Hölder's inequality.

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Heikkinen: Department of Mathematics, Aalto University, Finland; toni.heikkinen@aalto.fi

Kinnunen: Department of Mathematics, Aalto University, Finland; juha.k.kinnunen@aalto.fi

Nuutinen: Department of Mathematics and Statistics, University of Jyväskylä, Finland; juho.nuutinen@jyu.fi

Tuominen: Department of Mathematics and Statistics, University of Jyväskylä, Finland; heli.m.tuominen@jyu.fi