Blowup and scattering problems for the nonlinear Schrödinger equations

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Abstract We consider L^2-supercritical and H^1-subcritical focusing nonlinear Schrödinger equations. We introduce a subset PW of $H^1(\mathbb{R}^d)$ for $d \geq 1$, and investigate behavior of the solutions with initial data in this set. To this end, we divide PW into two disjoint components PW_+ and PW_-. Then, it turns out that any solution starting from a datum in PW_+ behaves asymptotically free, and solution starting from a datum in PW_- blows up or grows up, from which we find that the ground state has two unstable directions. Our result is an extension of the one by Duyckaerts, Holmer, and Roudenko to the general powers and dimensions, and our argument mostly follows the idea of Kenig and Merle.

1. Introduction

In this paper, we consider the Cauchy problem for the nonlinear Schrödinger (NLS) equation

$$\text{(NLS)} \qquad 2i\frac{\partial \psi}{\partial t}(x,t) + \Delta\psi(x,t) + \left|\psi(x,t)\right|^{p-1}\psi(x,t) = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R},$$

where ψ is a complex-valued function on $\mathbb{R}^d \times \mathbb{R}$, Δ is the Laplace operator on $\mathbb{R}^d$, and p satisfies the so-called L^2-supercritical and H^1-subcritical condition

$$(1.1) \qquad 2 + \frac{4}{d} < p + 1 < 2^* := \begin{cases} \infty & \text{if } d = 1, 2, \\ \frac{2d}{d-2} & \text{if } d \geq 3. \end{cases}$$

Our (NLS) equation is invariant under the scaling

$$(1.2) \qquad \psi(x,t) \mapsto \psi_\lambda(x,t) := \lambda^{2/(p-1)}\psi(\lambda x, \lambda^2 t),$$

which determines a critical regularity

$$(1.3) \qquad s_p := \frac{d}{2} - \frac{2}{p-1}.$$

The condition (1.1) implies that $0 < s_p < 1$.

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We associate the equation (NLS) with the initial datum from the usual Sobolev space $H^1(\mathbb{R}^d)$:

$$\psi(\cdot,0)=\psi_0\in H^1(\mathbb{R}^d). \tag{1.4}$$

We summarize the basic properties of this Cauchy problem (NLS) and (1.4) (see, e.g., [4], [9], [13]–[15], [25]). The unique local existence of solutions is well known: for any $\psi_0\in H^1(\mathbb{R}^d)$, there exists a unique solution ψ in $C(I_{\max};H^1(\mathbb{R}^d))$ for some interval $I_{\max}=(-T_{\max}^-,T_{\max}^+)\subset\mathbb{R}$: maximal existence interval including 0; $T_{\max}^+$ $(-T_{\max}^-)$ is the maximal existence time for the future (the past). If $I_{\max}\subsetneqq\mathbb{R}$, then we have

$$\lim_{t\to *T_{\max}^*}\left\|\nabla\psi(t)\right\|_{L^2}=\infty \quad \text{(blowup)}, \tag{1.5}$$

provided that $T_{\max}^*<\infty$, where $*$ stands for $+$ or $-$. Besides, the solution ψ satisfies the following conservation laws of the mass $\mathcal{M}$, the Hamiltonian $\mathcal{H}$, and the momentum $\mathcal{P}$ in this order: for all $t\in I_{\max}$,

$$\mathcal{M}\big(\psi(t)\big):=\left\|\psi(t)\right\|_{L^2}^2=\mathcal{M}(\psi_0), \tag{1.6}$$

$$\mathcal{H}\big(\psi(t)\big):=\left\|\nabla\psi(t)\right\|_{L^2}^2-\frac{2}{p+1}\left\|\psi(t)\right\|_{L^{p+1}}^{p+1}=\mathcal{H}(\psi_0), \tag{1.7}$$

$$\mathcal{P}\big(\psi(t)\big):=\Im\int_{\mathbb{R}^d}\nabla\psi(x,t)\overline{\psi(x,t)}\,dx=\mathcal{P}(\psi_0). \tag{1.8}$$

If, in addition, $\psi_0\in L^2(\mathbb{R}^d,|x|^2\,dx)$, then the corresponding solution ψ also belongs to $C(I_{\max};L^2(\mathbb{R}^d,|x|^2\,dx))$ and satisfies the so-called virial identity (see [9]):

$$\begin{aligned}\int_{\mathbb{R}^d}|x|^2\big|\psi(x,t)\big|^2\,dx&=\int_{\mathbb{R}^d}|x|^2\big|\psi_0(x)\big|^2\,dx+2t\Im\int_{\mathbb{R}^d}x\cdot\nabla\psi_0(x)\overline{\psi_0(x)}\,dx\\&\quad+2\int_0^t\int_0^{t'}\mathcal{K}\big(\psi(t'')\big)\,dt''\,dt' \quad \text{for all } t\in I_{\max},\end{aligned} \tag{1.9}$$

where $\mathcal{K}$ is a functional defined by

$$\mathcal{K}(f):=\|\nabla f\|_{L^2}^2-\frac{d(p-1)}{2(p+1)}\|f\|_{L^{p+1}}^{p+1}, \quad f\in H^1(\mathbb{R}^d). \tag{1.10}$$

It is worthwhile to note that

$$\mathcal{K}(f)=\mathcal{H}(f)-\frac{(p-1)s_p}{p+1}\|f\|_{L^{p+1}}^{p+1}, \tag{1.11}$$

so that for any $f\in H^1(\mathbb{R}^d)\setminus\{0\}$, we have

$$\mathcal{K}(f)>\mathcal{H}(f) \quad \text{if } p<1+\frac{4}{d}, \tag{1.12}$$

$$\mathcal{K}(f)=\mathcal{H}(f) \quad \text{if } p=1+\frac{4}{d}, \tag{1.13}$$

$$\mathcal{K}(f)<\mathcal{H}(f) \quad \text{if } p>1+\frac{4}{d}. \tag{1.14}$$

The virial identity (1.9) tells us the behavior of the "variance" of a solution, from which we expect to obtain a kind of propagation or concentration estimates.

However, we cannot use (1.9) as it is, since we do not require the weight condition $\psi_0 \in L^2(\mathbb{R}^d, |x|^2\,dx)$. We will work in the pure energy space $H^1(\mathbb{R}^d)$, introducing a generalized version of the virial identity (see (A.9) in the appendix).

Our equation (NLS) has several kinds of solutions: standing waves, blowup solutions (see (1.5) above), and global-in-time solutions which asymptotically behave like free solutions in the distant future/distant past. Here, the standing wave is a nontrivial solution of the form

$$\psi(x,t) = e^{(i/2)\omega t} f(x), \quad \omega > 0, f \in H^1(\mathbb{R}^d) \setminus \{0\}. \tag{1.15}$$

Thus, f solves the following semilinear elliptic equation (nonlinear scalar field equation):

$$\Delta f - \omega f + |f|^{p-1} f = 0, \quad \omega > 0, f \in H^1(\mathbb{R}^d) \setminus \{0\}. \tag{1.16}$$

Here, we remark that every solution f to (1.16) satisfies $\mathcal{K}(f) = 0$. Indeed, since any solution f to (1.16) belongs to the space $H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, |x|^2\,dx)$, the standing wave $\psi = e^{(i/2)\omega t} f$ enjoys the virial identity (1.9), which immediately leads us to $\mathcal{K}(f) = 0$.

The standing waves are one of the interesting objects in the study of NLSs for both mathematics and physics: standing waves are considered to be the states of Bose–Einstein condensations. In this paper, we are interested in the precise instability mechanism of the ground state. The ground state means a solution to (1.16) minimizing the action $\mathcal{S}_\omega$ among the solutions, where

$$\mathcal{S}_\omega(f) := \omega \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 - \frac{2}{p+1} \|f\|_{L^{p+1}}^{p+1}. \tag{1.17}$$

To this end, we employ the classical potential well theory traced back to Sattinger [23]. To define our potential well PW, we need to know some variational properties of the ground state. We shall give the precise definition of PW in (1.26) below. Anyway, our PW is divided into PW_+ and PW_- according to the sign of the functional $\mathcal{K}$, that is, $\mathrm{PW}_+ = \mathrm{PW} \cap [\mathcal{K} > 0]$, $\mathrm{PW}_- = \mathrm{PW} \cap [\mathcal{K} < 0]$ (see (1.29), (1.30)), and the ground state belongs to $\overline{\mathrm{PW}_+} \cap \overline{\mathrm{PW}_-}$. We show that any solution starting from PW_+ exists globally in time and asymptotically behaves like a free solution in the distant future and past (see Theorem 1.1); in contrast, PW_- gives rise to "singular" solutions (see Theorem 1.2). Thus, the ground state shows at least two types of instability, since it belongs to $\overline{\mathrm{PW}_+} \cap \overline{\mathrm{PW}_-}$.

To define our potential well PW, we shall investigate some properties of the ground states here. There is much literature concerning the elliptic equation (1.16) (see, e.g., [2], [7], [18], [24]). We know that if $d \geq 2$, there are infinitely many solutions (excited states) Q_ω^n $(n = 1, 2, \dots)$ such that

$$\mathcal{S}_\omega(Q_\omega^n) = \omega \|Q_\omega^n\|_{L^2}^2 + \|\nabla Q_\omega^n\|_{L^2}^2 - \frac{2}{p+1} \|Q_\omega^n\|_{L^{p+1}}^{p+1} \to \infty \quad (n \to \infty). \tag{1.18}$$

In the L^2-supercritical and H^1-subcritical case $(2 + (4/d) < p + 1 < 2^*)$, it is well known that for any $\omega > 0$, there exists a ground state Q_ω that is a unique positive radial function. Put $\mathcal{S} := \mathcal{S}_1$ and $Q := Q_1$. Then, we have the relation

$$Q_\omega(x) = \omega^{1/(p-1)} Q(\omega^{1/2} x). \tag{1.19}$$

Moreover, we can find that

$$
\begin{aligned}
\mathcal{S}(Q) &= \inf\{\mathcal{S}(f) \mid f \in H^1(\mathbb{R}^d)\setminus\{0\}, \mathcal{K}(f)=0\} \\
&= \inf\{\|f\|_{\widetilde{H}^1}^2 \mid f \in H^1(\mathbb{R}^d)\setminus\{0\}, \mathcal{K}(f)\le 0\} = \|Q\|_{\widetilde{H}^1}^2,
\end{aligned} \tag{1.20}
$$

where

$$
\|f\|_{\widetilde{H}^1}^2 := \mathcal{S}(f) - \frac{4}{d(p-1)}\mathcal{K}(f) = \frac{2s_p}{d}\|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2. \tag{1.21}
$$

For $\lambda > 0$, let T_λ be a scaling operator defined by

$$
(T_\lambda f)(x) := \lambda^{2/(p-1)} f(\lambda x). \tag{1.22}
$$

Then, we define

$$
\mathcal{N}(f) := \inf_{\lambda>0} \mathcal{S}(T_\lambda f) = \Big(\frac{\mathcal{M}(f)}{1-s_p}\Big)^{1-s_p}\Big(\frac{\mathcal{H}(f)}{s_p}\Big)^{s_p} \tag{1.23}
$$

for any function $f \in H^1(\mathbb{R}^d)$ with $\mathcal{H}(f) \ge 0$.* Moreover, we define

$$
\widetilde{\mathcal{N}}(f) := \inf_{\lambda>0} \|T_\lambda f\|_{\widetilde{H}^1}^2 = \Big(\frac{\mathcal{M}(f)}{1-s_p}\Big)^{1-s_p}\Big(\frac{2\|\nabla f\|_{L^2}^2}{d}\Big)^{s_p} \tag{1.24}
$$

for any $f \in H^1(\mathbb{R}^d)$.†

Since the sign of $\mathcal{K}$ is invariant under the scaling (1.22), we see from (1.20) that the ground state Q satisfies

$$
\begin{aligned}
\mathcal{S}(Q) = \mathcal{N}(Q) &= \inf\{\mathcal{N}(f) \mid f \in H^1(\mathbb{R}^d)\setminus\{0\}, \mathcal{K}(f)=0\} \\
&= \inf\{\widetilde{\mathcal{N}}(f) \mid f \in H^1(\mathbb{R}^d)\setminus\{0\}, \mathcal{K}(f)\le 0\} \\
&= \widetilde{\mathcal{N}}(Q) = \|Q\|_{\widetilde{H}^1}^2.
\end{aligned} \tag{1.25}
$$

Now, we define our "potential well" PW by

$$
\begin{aligned}
\mathrm{PW} &= \bigcup_{\omega>0} \{f \in H^1(\mathbb{R}^d) \mid \mathcal{S}_\omega(f) < \mathcal{S}_\omega(Q_\omega)\} \\
&= \{f \in H^1(\mathbb{R}^d) \mid \mathcal{S}_\omega(f) < \mathcal{S}_\omega(Q_\omega) \text{ for some } \omega > 0\}.
\end{aligned} \tag{1.26}
$$

Then, we see from an elementary calculus that

$$
\mathrm{PW} = \{f \in H^1(\mathbb{R}^d) \mid \mathcal{H}(f) < \max_{\omega>0}[\mathcal{S}_\omega(Q_\omega) - \omega\mathcal{M}(f)] =: \mathcal{B}(f)\}. \tag{1.27}
$$

Using (1.19) and (1.25), we obtain the explicit form of $\mathcal{B}(f)$ for any $f \in H^1(\mathbb{R}^d)\setminus\{0\}$:

$$
\mathcal{B}(f) = \frac{2s_p}{d}\Big(\frac{\mathcal{M}(Q)}{\mathcal{M}(f)}\Big)^{\frac{1-s_p}{s_p}} \|\nabla Q\|_{L^2}^2. \tag{1.28}
$$

*We have $\inf_{\lambda>0}\mathcal{S}(T_\lambda f) = \mathcal{S}(T_\lambda f)\big|_{\lambda=\sqrt{\frac{s_p\mathcal{M}(f)}{(1-s_p)\mathcal{H}(f)}}}$ for any $f \in H^1(\mathbb{R}^d)$ with $\mathcal{H}(f) > 0$.

†We have $\inf_{\lambda>0}\|T_\lambda f\|_{\widetilde{H}^1}^2 = \|T_\lambda f\|_{\widetilde{H}^1}^2\big|_{\lambda=\sqrt{\frac{d\mathcal{M}(f)}{2(1-s_p)\|\nabla f\|_{L^2}^2}}}$ for any $f \in H^1(\mathbb{R}^d)\setminus\{0\}$.

We divide PW into two components according to the sign of $\mathcal{K}$:

$$\mathrm{PW}_{+} = \{ f \in \mathrm{PW} \mid \mathcal{K}(f) > 0 \}, \tag{1.29}$$

$$\mathrm{PW}_{-} = \{ f \in \mathrm{PW} \mid \mathcal{K}(f) < 0 \}. \tag{1.30}$$

It is worthwhile noting the following facts.

1. PW_{+} and PW_{-} are unbounded open sets in $H^1(\mathbb{R}^d)$: indeed, one can easily verify this fact by considering the scaled functions $T_\lambda f$ for $f \in H^1(\mathbb{R}^d)$ and $\lambda > 0$.

2. We have

$$\{ f \in H^1(\mathbb{R}^d) \mid \mathcal{H}(f) < 0 \} \subset \mathrm{PW}_{-}. \tag{1.31}$$

3. We have $\mathrm{PW} = \mathrm{PW}_{+} \cup \mathrm{PW}_{-} \cup \{0\}$ (see Lemma 2.1) and $\mathrm{PW}_{+} \cap \mathrm{PW}_{-} = \emptyset$.

4. PW_{+} and PW_{-} are invariant under the flow defined by (NLS) (see Lemma 2.2).

5. The ground state Q_ω belongs to $\overline{\mathrm{PW}_{+}} \cap \overline{\mathrm{PW}_{-}}$ and $Q_\omega \notin \mathrm{PW}_{+} \cup \mathrm{PW}_{-}$ for any $\omega > 0$, where $\overline{\mathrm{PW}_{+}}$ and $\overline{\mathrm{PW}_{-}}$ are the closures of PW_{+} and PW_{-} in the H^1-topology, respectively (see Corollary 1.3). Moreover, the orbit

$$\{ e^{i\theta} Q_\omega(\cdot - a) \mid \omega > 0, a \in \mathbb{R}^d, \theta \in \mathbb{R} \} \tag{1.32}$$

is contained in $\overline{\mathrm{PW}_{+}} \cap \overline{\mathrm{PW}_{-}}$.

Here, the last fact above is the key to show the instability of the ground state. We will prove these facts in Section 2.

We can see from (1.25) that PW_{+} is rewritten in the form

$$\mathrm{PW}_{+} = \{ f \in H^1(\mathbb{R}^d) \mid \mathcal{K}(f) > 0, \mathcal{N}(f) < \mathcal{N}(Q) \}. \tag{1.33}$$

To consider the wave operators, we introduce a set Ω which is a subset of PW_{+}:

$$\Omega := \Big\{ f \in H^1(\mathbb{R}^d) \setminus \{0\} \,\Big|\, \widetilde{\mathcal{N}}(f) < \Big(\frac{2s_p}{d}\Big)^{s_p} \mathcal{N}(Q) \Big\}. \tag{1.34}$$

Now, we are in a position to state our main results. When symbols with $\pm$ appear in the following theorems and propositions, we always take both upper signs or both lower signs in the double signs.

The first theorem below is concerned with the behavior of the solutions with initial data from PW_{+}.

THEOREM 1.1 (GLOBAL EXISTENCE AND SCATTERING)

Assume that $d \geq 1$, $2 + (4/d) < p + 1 < 2^*$ and $\psi_0 \in \mathrm{PW}_{+}$. Then, the corresponding solution ψ to the equation (NLS) exists globally in time, that is, $I_{\max} = \mathbb{R}$, and has the following properties:

(i) *ψ stays in PW_{+} for all time and satisfies*

$$\inf_{t \in \mathbb{R}} \mathcal{K}(\psi(t)) > 0; \tag{1.35}$$

(ii) *we have*

$$\sup_{t\in\mathbb{R}}\big\|\nabla\psi(t)\big\|_{L^2}^2<\infty; \tag{1.36}$$

(iii) *there exist unique* $\phi_+\in\Omega$ *and* $\phi_-\in\Omega$ *such that*

$$\lim_{t\to\pm\infty}\big\|\psi(t)-e^{(i/2)t\Delta}\phi_\pm\big\|_{H^1}=\lim_{t\to\pm\infty}\big\|e^{-(i/2)t\Delta}\psi(t)-\phi_\pm\big\|_{H^1}=0. \tag{1.37}$$

This formula defines the operators $U_\pm\colon\psi_0\mapsto\phi_\pm=\lim_{t\to\pm\infty}e^{-(i/2)t\Delta}\psi(t)$. *These operators become homeomorphisms from* PW_+ *to* Ω*, so that we can define the scattering operator* $S:=U_+^{-1}W_-$ *from* Ω *into itself, where* $W_-:=(U_-)^{-1}$.

REMARK 1.1

(i) Theorem 1.1 is an extension of the result by Duyckaerts, Holmer and Roudenko [5]. See *Notes and Comments* below for the details.

(ii) Since Ω is clearly connected in the $H^1(\mathbb{R}^d)$-topology, we see from Theorem 1.1 that PW_+ is connected in the $H^1(\mathbb{R}^d)$-topology.

In contrast to the case of PW_+, the solutions with initial data from PW_- become singular.

THEOREM 1.2 (BLOWUP OR GROWUP)

Assume that $d\geq 1$, $2+(4/d)<p+1<2^*$ *and* $\psi_0\in\mathrm{PW}_-$. *Then, the corresponding solution* ψ *to the equation (NLS) satisfies the following:*

(i) ψ *stays in* PW_- *as long as it exists and satisfies*

$$\sup_{t\in I_{\max}}\mathcal{K}\big(\psi(t)\big)<0; \tag{1.38}$$

(ii) ψ *blows up in a finite time or grows up; that is,*

$$\sup_{t\in[0,T_{\max}^+)}\big\|\nabla\psi(t)\big\|_{L^2}=\sup_{t\in(-T_{\max}^-,0]}\big\|\nabla\psi(t)\big\|_{L^2}=\infty. \tag{1.39}$$

In particular, if $T_{\max}^\pm=\infty$*, then we have*

$$\limsup_{t\to\pm\infty}\int_{|x|>R}\big|\nabla\psi(x,t)\big|^2\,dx=\infty\quad\textit{for any } R>0. \tag{1.40}$$

REMARK 1.2

(i) We do not know whether a solution growing up at infinity exists.

(ii) We know (see [8]) that if $\psi_0\in H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d,|x|^2\,dx)$, then $T_{\max}^\pm<\infty$ and the corresponding solution ψ satisfies

$$\lim_{t\to\pm T_{\max}^\pm}\big\|\nabla\psi(t)\big\|_{L^2}=\infty.$$

For the case $\psi_0\notin L^2(\mathbb{R}^d,|x|^2\,dx)$, see Proposition 1.1 below (see also [22]).

Combining Theorems 1.1 and 1.2, we can show the instability of the ground states. Precisely, we have the following.

COROLLARY 1.3 (INSTABILITY OF GROUND STATE)

Let Q_ω be the ground state of the equation (1.16) for $\omega > 0$. Then, Q_ω has two unstable directions in the sense that $Q_\omega \in \overline{\mathrm{PW}_+} \cap \overline{\mathrm{PW}_-}$. In particular, for any $\varepsilon > 0$, there exist $f_+ \in \mathrm{PW}_+$ and $f_- \in \mathrm{PW}_-$ such that

$$\|Q_\omega - f_\pm\|_{H^1} \le \varepsilon.$$

REMARK 1.3

An example of $f_\pm$ is $(1 \mp (\varepsilon/(\|Q_\omega\|_{H^1})))Q_\omega$, where both upper or both lower signs should be chosen in the double signs.

Next, we consider singular solutions. The following theorem tells us that solutions with radially symmetric data from PW_- blow up in a finite time.

PROPOSITION 1.1 (EXISTENCE OF BLOWUP SOLUTION)

Assume that $d \ge 2$, $2 + (4/d) < p + 1 < 2^*$, and assume that $p \le 5$ if $d = 2$. Let ψ_0 be a radially symmetric function in PW_-, and let ψ be the corresponding solution to the equation (NLS). Then, we have

$$T^\pm_{\max} < \infty \qquad \text{and} \qquad \lim_{t \to \pm T^\pm_{\max}} \|\nabla \psi(t)\|_{L^2} = \infty. \tag{1.41}$$

Furthermore, for any $m > 0$, there exists a constant $R_m > 0$ such that

$$\int_{|x|>R} |\psi(x,t)|^2 \, dx < m \tag{1.42}$$

for any $R \ge R_m$ and $t \in I_{\max}$.

We do not know a lot of things about the asymptotic behavior of such singular solutions as found in Theorem 1.2. What we can say is the following. (For simplicity, we state the forward time case only.)

PROPOSITION 1.2

Assume that $d \ge 1$ and $2 + (4/d) < p + 1 < 2^*$. Let ψ be a solution to the equation (NLS) such that

$$\limsup_{t \to T^+_{\max}} \|\nabla \psi(t)\|_{L^2} = \limsup_{t \to T^+_{\max}} \|\psi(t)\|_{L^{p+1}} = \infty, \tag{1.43}$$

and let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence in $[0, T^+_{\max})$ such that

$$\lim_{n \to \infty} t_n = T^+_{\max}, \qquad \|\psi(t_n)\|_{L^{p+1}} = \sup_{t \in [0,t_n)} \|\psi(t)\|_{L^{p+1}}. \tag{1.44}$$

For this sequence $\{t_n\}$, we put

$$\lambda_n := \|\psi(t_n)\|_{L^{p+1}}^{-((p-1)(p+1))/(d+2-(d-2)p)} \tag{1.45}$$

and consider the scaled functions

$$\psi_n(x,t) := \lambda_n^{2/(p-1)} \overline{\psi(\lambda_n x, t_n - \lambda_n^2 t)}, \quad t \in \left(-\frac{T^+_{\max} - t_n}{\lambda_n^2}, \frac{t_n}{\lambda_n^2} \right]. \tag{1.46}$$

We define a "renormalized" functions Φ_n^{RN} *by*

$$\Phi_n^{RN}(x,t) = \psi_n(x,t) - e^{(i/2)t\Delta}\psi_n(x,0), \quad n \in \mathbb{N}. \tag{1.47}$$

Then, for any $T > 0$*, there exists a subsequence of* $\{\Phi_n^{RN}\}$ *(still denoted by the same symbol) with the following properties:*

$$\Phi_n^{RN} \in C\big([0,T]; H^1(\mathbb{R}^d)\big) \quad \textit{for any } n \in \mathbb{N}, \tag{1.48}$$

and there exists a nontrivial function $\Phi \in L^\infty([0,\infty); H^1(\mathbb{R}^d))$ *such that*

$$\lim_{n\to\infty} \Phi_n^{RN} = \Phi \quad \textit{in } C\big([0,T];\ \textit{weak-}H^1(\mathbb{R}^d)\big). \tag{1.49}$$

Here, Φ *solves the following equation:*

$$2i\frac{\partial \Phi}{\partial t} + \Delta\Phi = -F, \tag{1.50}$$

where F *is the nontrivial function in* $L^\infty([0,\infty); L^{(p+1)/p}(\mathbb{R}^d))$ *given by*

$$\lim_{n\to\infty} |\psi_n|^{p-1}\psi_n = F \quad \textit{weakly* in } L^\infty\big([0,T]; L^{(p+1)/p}(\mathbb{R}^d)\big). \tag{1.51}$$

Here, we discuss some relations between the previous works and our results.

NOTES AND COMMENTS

Our analysis in PW_+ is inspired by the previous work by Duyckaerts, Holmer, and Roudenko [5], [10] (see also Kenig and Merle [16]). They considered a typical nonlinear Schrödinger equation, the equation (NLS) with $d = p = 3$, and proved, in [5], that if $\psi_0 \in H^1(\mathbb{R}^3)$ satisfies

$$\mathcal{M}(\psi_0)\mathcal{H}(\psi_0) < \mathcal{M}(Q)\mathcal{H}(Q), \qquad \|\psi_0\|_{L^2}\|\nabla\psi_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}, \tag{1.52}$$

then the corresponding solution exists globally in time and has asymptotic states at $\pm\infty$, where Q denotes the ground state of the equation (1.16) with $\omega = 1$. We see from (1.25) and (1.33) that the condition (1.52) is equivalent to $\psi_0 \in \mathrm{PW}_+$. Hence, Theorem 1.1 is an extension of their result to all spatial dimensions $d \geq 1$ and L^2-supercritical and H^1-subcritical powers $2 + (4/d) < p + 1 < 2^*$. For the nonlinear Klein–Gordon equation, a result corresponding to Theorem 1.1 was obtained by Ibrahim, Masmoudi, and Nakanishi [12].

In [11], Holmer and Roudenko also considered the equation (NLS) and proved that if the initial datum ψ_0 satisfies $|x|\psi_0 \in L^2(\mathbb{R}^d)$ and either $\mathcal{H}(\psi_0) < 0$ or $\mathcal{H}(\psi_0) \geq 0$ and

$$\widetilde{\mathcal{N}}(Q) < \widetilde{\mathcal{N}}(\psi_0), \tag{1.53}$$

then the corresponding solution blows up in a finite time. In our terminology, their initial datum belongs to PW_-.

This paper is organized as follows. In Section 2, we discuss properties of the potential well PW. In Section 3, we introduce function spaces in which Strichartz-type estimates work well. We also give a small data theory and a long time perturbation theory. In Section 4, we give a proof of Theorem 1.1. Section 5 is devoted to proofs of Theorem 1.2 and Propositions 1.1 and 1.2. In the appendix, we introduce a generalized version of virial identity and give its fundamental properties.

NOTATION

We summarize the notation used in this paper.

We keep the letters d and p to denote the spatial dimension and the power of nonlinearity in the equation (NLS), respectively.

$\mathbb{N}$ denotes the set of natural numbers, that is, $\mathbb{N} = \{1, 2, 3, \dots\}$.

$I_{\max}$ denotes the maximal existence interval of the considering solution, which has the form

$$I_{\max} = (-T_{\max}^{-}, T_{\max}^{+}),$$

where $T_{\max}^{+} > 0$ is the maximal existence time for the future and $T_{\max}^{-} > 0$ is the one for the past.

The symbol $(\cdot, \cdot)$ denotes the inner product of $L^2(\mathbb{R}^d)$, that is,

$$(f, g) := \int_{\mathbb{R}^d} f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(\mathbb{R}^d).$$

$C_c^{\infty}(\mathbb{R}^d)$ denotes the set of infinitely differentiable functions from $\mathbb{R}^d$ to $\mathbb{C}$ with compact supports.

Using the Fourier transformation $\mathcal{F}$, for $s \in \mathbb{R}$ we define differential operators $|\nabla|^s$, $(-\Delta)^{(s/2)}$, and $(1-\Delta)^{(s/2)}$ by

$$|\nabla|^s f = (-\Delta)^{(s/2)} f := \mathcal{F}^{-1}\big[|\xi|^s \mathcal{F}[f]\big],$$
$$(1-\Delta)^{(s/2)} f := \mathcal{F}^{-1}\big[\big(1+|\xi|^2\big)^{(s/2)}\mathcal{F}[f]\big].$$

2. Potential well PW

In this section, we discuss fundamental properties of the sets PW, PW_-, and PW_+. In particular, we will prove that these sets are invariant under the flow defined by the equation (NLS). Moreover, we prove Corollary 1.3 here.

We begin with the following fact.

LEMMA 2.1

The set PW *does not contain any nontrivial function f with $\mathcal{K}(f) = 0$, that is,*

$$\big\{f \in H^1(\mathbb{R}^d) \setminus \{0\} \,\big|\, \mathcal{K}(f) = 0\big\} \cap \mathrm{PW} = \emptyset, \tag{2.1}$$

so that

$$\mathrm{PW} = \mathrm{PW}_+ \cup \mathrm{PW}_- \cup \{0\}. \tag{2.2}$$

Furthermore, PW *is invariant under the flow defined by (NLS); that is, letting $\psi_0 \in \mathrm{PW}$ and letting ψ be the corresponding solution, we have*

$$\psi(t) \in \mathrm{PW} \quad \textit{for any } t \in I_{\max}. \tag{2.3}$$

Proof of Lemma 2.1
Let f be a function in $H^1(\mathbb{R}^d) \setminus \{0\}$ with $\mathcal{K}(f) = 0$. Since $\mathcal{S}_\omega(f) = \omega^{1-s_p}\mathcal{S}(\omega^{-1/(p-1)}f(\omega^{-(1/2)}\cdot))$ and $\mathcal{K}(\omega^{--1/(p-1)}f(\omega^{-(1/2)}\cdot)) = 0$ for any $\omega > 0$, we see from (1.20) that $f \notin \mathrm{PW}$.

The invariance (2.3) follows from the mass and energy conservation laws (1.6) and (1.7). □

Now, we are in a position to prove Corollary 1.3.

Proof of Corollary 1.3
We consider a path $\Gamma_\omega\colon (0,\infty) \to H^1(\mathbb{R}^d)$ given by $\Gamma_\omega(\lambda) := \lambda^{d/2}Q_\omega(\lambda\cdot)$ for $\lambda > 0$. Then, one can verify that $\Gamma_\omega(\lambda) \in \mathrm{PW}_+$ for any $\lambda \in (0,1)$. This together with the continuity of Γ_ω shows that $Q_\omega \in \overline{\mathrm{PW}_+}$. Similarly, we can verify that $Q_\omega \in \overline{\mathrm{PW}_-}$. □

Next, we give the invariance results of the sets PW_+ and PW_- under the flow defined by the equation (NLS).

LEMMA 2.2
Let $\psi_0 \in \mathrm{PW}_+$, and let ψ be the corresponding solution to (NLS). Then, ψ exists globally in time and satisfies the following:

$$\psi(t) \in \mathrm{PW}_+ \quad \textit{for any } t \in \mathbb{R}, \tag{2.4}$$

$$\inf_{t\in\mathbb{R}} \mathcal{K}\big(\psi(t)\big) \geq \Big\{1 - \Big(\frac{\mathcal{N}(\psi_0)}{\mathcal{N}(Q)}\Big)^{(p-1)/2}\Big\}\mathcal{H}(\psi_0) > 0, \tag{2.5}$$

and

$$\sup_{t\in\mathbb{R}} \big\|\nabla\psi(t)\big\|_{L^2}^2 < \frac{d}{2s_p}\mathcal{H}(\psi_0). \tag{2.6}$$

On the other hand, let $\psi_0 \in \mathrm{PW}_-$, and let ψ be the corresponding solution to the equation (NLS). Then, we have

$$\psi(t) \in \mathrm{PW}- \quad \textit{for any } t \in I_{\max} \tag{2.7}$$

and

$$\sup_{t\in I_{\max}} \mathcal{K}\big(\psi(t)\big) \leq \mathcal{H}(\psi_0) - \mathcal{B}(\psi_0) < 0. \tag{2.8}$$

REMARK 2.1
We see from the proof below (see (2.10)) that

$$\Big\{1 - \Big(\frac{\widetilde{\mathcal{N}}(f)}{\widetilde{\mathcal{N}}(Q)}\Big)^{(p-1)/2}\Big\}\|\nabla f\|_{L^2}^2 \leq \mathcal{H}(f) \tag{2.9}$$

for any $f \in H^1(\mathbb{R}^d) \setminus \{0\}$ with $\widetilde{\mathcal{N}}(f) < \widetilde{\mathcal{N}}(Q)$.

Proof of Lemma 2.2

Let $f \in H^1(\mathbb{R}^d) \setminus \{0\}$ and $0 < \lambda < \sqrt{\frac{\widetilde{\mathcal{N}}(Q)}{\widetilde{\mathcal{N}}(f)}}$. Then, we have $\widetilde{\mathcal{N}}(\lambda f) = \lambda^2 \widetilde{\mathcal{N}}(f) < \widetilde{\mathcal{N}}(Q)$. This together with (1.25) shows that $\mathcal{K}(\lambda f) > 0$. Hence, we have

$$(2.10)\quad \mathcal{K}(f) = \lambda^{-(p+1)}\mathcal{K}(\lambda f) + (1-\lambda^{-(p-1)})\|\nabla f\|_{L^2}^2 > (1-\lambda^{-(p-1)})\|\nabla f\|_{L^2}^2.$$

Now, let $\psi_0 \in \mathrm{PW}_+$, and let ψ be the corresponding solution. Then, it follows from the conservation laws (1.6), (1.7), (1.25), (1.33), and $\mathcal{K}(\psi_0) > 0$ that

$$(2.11)\qquad 1 < \frac{\mathcal{N}(Q)}{\mathcal{N}(\psi_0)} = \frac{\mathcal{N}(Q)}{\mathcal{N}(\psi(t))} < \frac{\widetilde{\mathcal{N}}(Q)}{\widetilde{\mathcal{N}}(\psi_0)}.$$

Hence, taking $\lambda = \sqrt{\frac{\mathcal{N}(Q)}{\mathcal{N}(\psi_0)}}$ in (2.10), we obtain

$$(2.12)\qquad \begin{aligned} \mathcal{K}\big(\psi(t)\big) &> \left\{1 - \Big(\frac{\mathcal{N}(\psi_0)}{\mathcal{N}(Q)}\Big)^{(p-1)/2}\right\}\big\|\nabla\psi(t)\big\|_{L^2}^2 \\ &> \left\{1 - \Big(\frac{\mathcal{N}(\psi_0)}{\mathcal{N}(Q)}\Big)^{(p-1)/2}\right\}\mathcal{H}(\psi_0) \quad \text{for any } t \in I_{\max}. \end{aligned}$$

In particular, we have $\inf_{t\in I_{\max}} \mathcal{K}(\psi(t)) > 0$, which yields

$$(2.13)\quad \mathcal{H}(\psi_0) > \Big(1 - \frac{4}{d(p-1)}\Big)\big\|\nabla\psi(t)\big\|_{L^2}^2 = \frac{2s_p}{d}\big\|\nabla\psi(t)\big\|_{L^2}^2 \quad \text{for any } t \in I_{\max}.$$

Thus, we find $I_{\max} = \mathbb{R}$, and hence (2.12) and (2.13) give us the desired results (2.5) and (2.6).

On the other hand, let $\psi_0 \in \mathrm{PW}_-$, and let ψ be the corresponding solution. It follows from (1.20) and the continuity of ψ in $H^1(\mathbb{R}^d)$ that $\psi(t) \in \mathrm{PW}_-$ for all $t \in I_{\max}$. Then, we see from (1.25) and (1.28) that

$$(2.14)\qquad 1 \ge \frac{\widetilde{\mathcal{N}}(Q)}{\widetilde{\mathcal{N}}(\psi(t))} = \frac{d}{2s_p}\frac{\mathcal{B}(\psi_0)}{\|\nabla\psi(t)\|_{L^2}^2},$$

which together with $\mathcal{K}(\psi(t)) < 0$ yields

$$(2.15)\quad \mathcal{B}(\psi_0) \le \frac{2s_p}{d}\big\|\nabla\psi(t)\big\|_{L^2}^2 \le \frac{s_p(p-1)}{p+1}\big\|\psi(t)\big\|_{L^{p+1}}^{p+1} = \mathcal{H}(\psi_0) - \mathcal{K}\big(\psi(t)\big).$$

Thus, we obtain (2.8). □

3. Strichartz-type estimate and scattering

In this section, we introduce Strichartz-type function spaces, which enables us to control the long-time behavior of solutions. Using these spaces, we prepare two important propositions: Proposition 3.5 in Section 3.2 (small data theory) and Proposition 3.6 in Section 3.3 (long-time perturbation theory). The former is used to avoid the vanishing and the latter to avoid the dichotomy in the "contradiction-compactness" argument due to Kenig and Merle [16, Section 4.2].

At the end of this section, we give an existence result of the wave operator in PW_+.

3.1. Auxiliary function space

To prove the scattering result ((1.37) in Theorem 1.1), we need to handle the inhomogeneous term of the integral equation associated with (NLS) in a suitable function space. Therefore, we will prepare a function space $X(I)$, $I \subset \mathbb{R}$, in which a Strichartz-type estimate works well.

Throughout this paper, we fix a number q_1 with $p+1 < q_1 < 2^*$. Then, we define indices r_0, r_1, and $\widetilde{r}_1$ by

$$\frac{1}{r_0} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{q_1}\Big), \tag{3.1}$$

$$\frac{1}{r_1} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{q_1} - \frac{s_p}{d}\Big), \tag{3.2}$$

$$\frac{1}{\widetilde{r}_1} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{q_1} + \frac{s_p}{d}\Big). \tag{3.3}$$

Here, the pair (q_1, r_0) is admissible. Besides these indices, we define a pair (q_2, r_2) by

$$\frac{p-1}{q_2} = 1 - \frac{2}{q_1}, \qquad \frac{1}{r_2} := \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{q_2} - \frac{s_p}{d}\Big). \tag{3.4}$$

For the relations among some pairs of these indices denoted by means of letters "q" and "r," see Figure 1. It is worthwhile to note that the Sobolev embedding and the Strichartz estimate lead us to the following estimate. For any pair (q, r) satisfying

$$\frac{d}{2}(p-1) \le q < 2^*, \qquad \frac{1}{r} = \frac{d}{2}\Big(\frac{1}{2} - \frac{1}{q} - \frac{s_p}{d}\Big), \tag{3.5}$$

we have

$$\|e^{(i/2)t\Delta} f\|_{L^r(I;L^q)} \lesssim \big\|(-\Delta)^{(s_p/2)} f\big\|_{L^2} \quad \text{for all } f \in \dot{H}^{s_p}(\mathbb{R}^d) \text{ and interval } I, \tag{3.6}$$

where the implicit constant depends only on d, p, and q. The pairs (q_1, r_1) and (q_2, r_2) satisfy the condition (3.5), so that the estimate (3.6) is valid for these pairs.

Now, for any interval I, we put

$$X(I) = L^{r_1}(I; L^{q_1}) \cap L^{r_2}(I; L^{q_2}), \tag{3.7}$$

$$S(I) = L^\infty(I; L^2) \cap L^{r_0}(I; L^{q_1}). \tag{3.8}$$

We find that Strichartz-type estimates work well in the space $X(I)$.

LEMMA 3.1

Assume that $d \ge 1$ and $2 + (4/d) < p+1 < 2^*$. Let $t_0 \in \mathbb{R}$, and let I be an interval whose closure contains t_0. Then, we have

$$\Big\|\int_{t_0}^{t} e^{i(t-t')\Delta} v(t')\, dt'\Big\|_{X(I)} \lesssim \|v\|_{L^{\widetilde{r}_1'}(I;L^{q_1'})}, \tag{3.9}$$

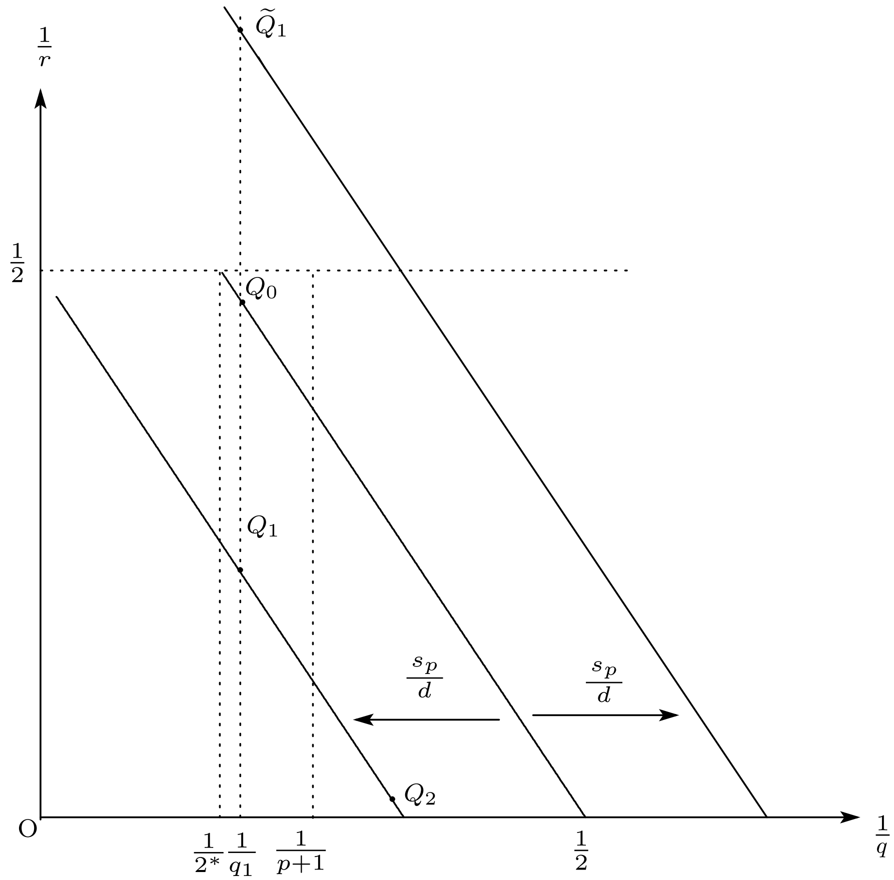

Figure 1. Strichartz-type estimates: $Q_0 : (\frac{1}{q_1}, \frac{1}{r_0})$, $Q_1 : (\frac{1}{q_1}, \frac{1}{r_1})$, $Q_2 : (\frac{1}{q_2}, \frac{1}{r_2})$, $\widetilde{Q}_1 : (\frac{1}{q_1}, \frac{1}{\widetilde{r}_1})$.

$$\Big\| \int_{t_0}^{t} e^{i(t-t')\Delta}(v_1 v_2)(t')\,dt' \Big\|_{X(I)} \lesssim \|v_1\|_{L^{r_1}(I;L^{q_1})} \|v_2\|_{L^{r_2/(p-1)}(I;L^{q_2/(p-1)})}, \tag{3.10}$$

where the implicit constants depend only on d, p, and q_1.

The estimate (3.9) in Lemma 3.1 is due to Foschi (see [6, Theorem 1.4]). The estimate (3.10) is an immediate consequence of (3.9) and the Hölder inequality.

The following lemma is frequently used in Section 4.

LEMMA 3.2

Assume that $d \ge 1$ and $2 + (4/d) < p+1 < 2^*$. Let $t_0 \in \mathbb{R}$, and let I be an interval whose closure contains t_0. Then, we have

$$\big\| |v|^{p-1} v \big\|_{L^{r_0'}(I;L^{q_1'})} \le \|v\|_{L^{r_0}(I;L^{q_1})} \|v\|_{L^{r_2}(I;L^{q_2})}^{p-1}, \tag{3.11}$$

$$\big\| \nabla\big(|v|^{p-1} v\big) \big\|_{L^{r_0'}(I;L^{q_1'})} \lesssim \|\nabla v\|_{L^{r_0}(I;L^{q_1})} \|v\|_{L^{r_2}(I;L^{q_2})}^{p-1}, \tag{3.12}$$

where the implicit constant depends only on d, p, and q_1.

Lemma 3.2 is easily obtained by the Hölder inequality and the chain rule.

We also need the interpolation estimate below in the next section (see Section 4.2).

LEMMA 3.3

For $j \in \{1,2\}$, there exist a constant $\theta_j \in (0,1)$ such that

$$\|e^{(i/2)t\Delta} f\|_{L^{r_j}(I;L^{q_j})} \lesssim \|e^{(i/2)t\Delta} f\|^{1-\theta_j}_{L^\infty(I;L^{(d/2)(p-1)})} \big\|(-\Delta)^{(s_p/2)} f\big\|^{\theta_j}_{L^2}$$

for all $f \in \dot{H}^{s_p}(\mathbb{R}^d)$,

where the implicit constant depends only on d, p, and q_1.

Proof of Lemma 3.3

Fix a pair (q,r) satisfying (3.5) and $q_1 < q < 2^*$. Applying the Hölder inequality first and (3.6) afterward, we obtain

$$\begin{aligned} \|e^{(i/2)t\Delta} f\|_{L^{r_j}(I;L^{q_j})} &\le \|e^{(i/2)t\Delta} f\|^{1-\theta_j}_{L^\infty(I;L^{\frac{d}{2}(p-1)})} \|e^{(i/2)t\Delta} f\|^{\theta_j}_{L^r(I;L^q)} \\ &\lesssim \|e^{(i/2)t\Delta} f\|^{1-\theta_j}_{L^\infty(I;L^{(d/2)(p-1)})} \big\|(-\Delta)^{s_p/2} f\big\|^{\theta_j}_{L^2}, \quad j = 1,2 \end{aligned} \tag{3.13}$$

where

$$\theta_j := \frac{q}{q_j} \frac{2q_j - d(p-1)}{2q - d(p-1)}.$$

Thus, we have proved the lemma. □

3.2. Sufficient conditions for scattering

We shall give two sufficient conditions for solutions to have asymptotic states in the energy space $H^1(\mathbb{R}^d)$. One of them is the small data theory (see Proposition 3.5).

We begin with the following proposition.

PROPOSITION 3.4 (SCATTERING IN THE ENERGY SPACE)

Assume that $d \ge 1$ and $2 + \frac{4}{d} < p+1 < 2^*$. Let ψ be a solution to the equation (NLS). Suppose that ψ exists on $[0,\infty)$ and satisfies

$$\|\psi\|_{X([0,\infty))} < \infty, \qquad \|\psi\|_{L^\infty([0,\infty);H^1)} < \infty. \tag{3.14}$$

Then, we have

$$\big\|(1-\Delta)^{1/2}\psi\big\|_{S([0,\infty))} < \infty \tag{3.15}$$

and there exists a unique $\phi_+ \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t\to\infty} \big\|\psi(t) - e^{(i/2)t\Delta}\phi_+\big\|_{H^1} = 0. \tag{3.16}$$

Since the proof of Proposition 3.4 is well known, we omit it.

The following proposition gives us a sufficient condition for the boundedness of X and S-norms.

PROPOSITION 3.5 (SMALL DATA THEORY)

Assume that $d \geq 1$ and $2+\frac{4}{d}<p+1<2^*$. Let $t_0 \in \mathbb{R}$, and let I be an interval whose closure contains t_0. Then, there exists a positive constant δ, depending only on d, p, and q_1, with the following property: for any $\psi_0 \in H^1(\mathbb{R}^d)$ satisfying

$$\|e^{\frac{i}{2}(t-t_0)\Delta}\psi_0\|_{X(I)} \leq \delta, \tag{3.17}$$

there exists a unique solution $\psi \in C(I;H^1(\mathbb{R}^d))$ to the equation (NLS) with $\psi(t_0)=\psi_0$ such that

$$\|\psi\|_{X(I)} < 2\|e^{\frac{i}{2}(t-t_0)\Delta}\psi_0\|_{X(I)}, \qquad \big\|(1-\Delta)^{\frac{1}{2}}\psi\big\|_{S(I)} \lesssim \|\psi_0\|_{H^1}, \tag{3.18}$$

where the implicit constant depends only on d, p, and q_1.

Proof of Proposition 3.5

We can prove this lemma by the standard contraction mapping principle. □

3.3. Long-time perturbation theory and wave operator

We will employ a concentration-compactness argument to prove that the solutions starting from PW_+ have asymptotic states (see Section 4). The following proposition plays a crucial role there.

PROPOSITION 3.6 (LONG-TIME PERTURBATION THEORY)

Assume that $d \geq 1$ and $2+(4/d)<p+1<2^*$. Then, for any $A>1$, there exists $\varepsilon>0$, depending only on A, d, p, and q_1, such that the following holds: Let I be an interval, and let u be a function in $C(I;H^1(\mathbb{R}^d))$ such that

$$\|u\|_{X(I)} \leq A, \tag{3.19}$$

$$\big\|2i\partial_t u+\Delta u+|u|^{p-1}u\big\|_{L^{\tilde{r}_1'}(I;L^{q_1'})} \leq \varepsilon. \tag{3.20}$$

If $\psi \in C(\mathbb{R};H^1)$ is a global solution to the equation (NLS) and satisfies

$$\big\|e^{(i/2)(t-t_1)\Delta}\big(\psi(t_1)-u(t_1)\big)\big\|_{X(I)} \leq \varepsilon \quad \text{for some } t_1 \in I, \tag{3.21}$$

then we have

$$\|\psi\|_{X(I)} \lesssim 1, \tag{3.22}$$

where the implicit constant depends only on d, p, q_1, and A.

Proof of Proposition 3.6

This proposition is essentially known (see [3]). Therefore, we omit the proof. □

The following proposition tells us that the wave operator is well defined on Ω.

PROPOSITION 3.7 (EXISTENCE OF WAVE OPERATOR)

Assume $d \geq 1$ and $2+(4/d) \leq p+1<2^*$. Then, for any $\phi_+ \in \Omega$, there exists a unique $\psi_0 \in \mathrm{PW}_+$ such that the corresponding solution ψ to the equation (NLS) *with $\psi(0)=\psi_0$ exists globally in time and satisfies the following:*

$$\psi \in X\big([0,\infty)\big), \tag{3.23}$$

$$\lim_{t\to+\infty}\big\|\psi(t)-e^{(i/2)t\Delta}\phi_+\big\|_{H^1}=0, \tag{3.24}$$

$$\mathcal{H}\big(\psi(t)\big)=\|\nabla\phi_+\|_{L^2}^2 \quad \textit{for any } t\in\mathbb{R}. \tag{3.25}$$

Here, the map $\phi_+\mapsto\psi_0$ is continuous from Ω into PW_+ in the $H^1(\mathbb{R}^d)$-topology.

Furthermore, if $\|\phi_+\|_{H^1}$ is sufficiently small, then we have

$$\|\psi\|_{X(\mathbb{R})}\lesssim\|\phi_+\|_{H^1}, \tag{3.26}$$

where the implicit constant depends only on d, p, and q_1.

Proof of Proposition 3.7
This proposition is essentially known (see, e.g., [10]). Therefore, we omit the proof. □

4. Analysis on PW_+

Our aim here is to prove Theorem 1.1. Obviously, Lemma 2.2 provides (1.35) and (1.36). Therefore, it remains to prove the asymptotic completeness in PW_+.

To prove the existence of asymptotic states for a solution ψ, it suffices to show that $\|\psi\|_{X(\mathbb{R})}<\infty$ by virtue of Proposition 3.4 and Lemma 2.2. To this end, we introduce a set $\mathrm{PW}_+(\delta)$ for $\delta>0$:

$$\mathrm{PW}_+(\delta):=\big\{f\in H^1(\mathbb{R}^d)\;\big|\;\mathcal{K}(f)>0,\mathcal{N}(f)<\delta\big\}. \tag{4.1}$$

It follows from (1.33) that $\mathrm{PW}_+=\mathrm{PW}_+(\mathcal{N}(Q))$.

We also define a number N_c by

$$\begin{aligned} N_c &:= \sup\big\{\delta>0\;\big|\;\|\psi\|_{X(\mathbb{R})}<\infty \text{ for any } \psi_0\in\mathrm{PW}_+(\delta)\big\}\\ &= \inf\big\{\delta>0\;\big|\;\|\psi\|_{X(\mathbb{R})}=\infty \text{ for some } \psi_0\in\mathrm{PW}_+(\delta)\big\}, \end{aligned} \tag{4.2}$$

where ψ denotes the solution to (NLS) with $\psi(0)=\psi_0$.

The small data theory (Proposition 3.5) shows $N_c>0$, and the existence of the ground state Q shows $N_c\le\mathcal{N}(Q)$. Thus, our task is to prove $N_c=\mathcal{N}(Q)$.

4.1. Critical element versus virial identity

In this section, we give an outline of the proof of $N_c=\mathcal{N}(Q)$. We suppose the contrary that $N_c<\mathcal{N}(Q)$. In this undesired situation, we can find a "critical element" in PW_+ which is a solution to (NLS) and whose orbit is precompact modulo the invariant transformation group (see Proposition 4.1 below). Then, its behavior contradicts the one described by the generalized virial identity (A.9), so that we conclude that $N_c=\mathcal{N}(Q)$. At the end of this Section 4.1, we actually show this, provided that the critical element exists.

The construction of the critical element is rather long. We divide it into two parts; in Section 4.2, one finds its candidate, and in Section 4.3, one sees that the candidate is actually the critical element.

We here briefly explain how to find a candidate for the critical element. If $N_c < \mathcal{N}(Q)$, then we can take a sequence $\{\psi_n\}$ of global solutions to (NLS) such that

$$\begin{aligned} &\psi_n(t) \in \mathrm{PW}_+ \quad \text{for any } t \in \mathbb{R}, \\ &\|\psi_n\|_{X(\mathbb{R})} = \infty, \qquad \lim_{n\to\infty} \mathcal{N}\big(\psi_n(0)\big) = N_c. \end{aligned} \tag{4.3}$$

We consider the integral equation for ψ_n:

$$\psi_n(t) = e^{(i/2)t\Delta}\psi_{0,n} + \frac{i}{2}\int_0^t e^{(i/2)(t-t')\Delta}\big\{\big|\psi_n(t')\big|^{p-1}\psi_n(t')\big\}\,dt', \tag{4.4}$$

where we put $\psi_{0,n} = \psi_n(0)$. We first observe that the linear part of this integral equation possibly behaves as follows*:

$$e^{(i/2)t\Delta}\psi_{0,n}(x) \sim \sum_{l\ge 1} e^{(i/2)(t-\tau_n^l)\Delta} e^{-\eta_n^l\cdot\nabla} f^l(x) \tag{4.5}$$

for some nontrivial functions $f^l \in \mathrm{PW}_+$, $\tau_n^l \in \mathbb{R}$, and $\eta_n^l \in \mathbb{R}^d$. Of course, this is not a good approximation to ψ_n. So, putting $\tau_\infty^l = \lim_{n\to\infty}\tau_n^l$ (possibly $\tau_\infty^l = \pm\infty$), we solve our equation (NLS) with the initial datum $e^{(i/2)\tau_\infty^l\Delta} f^l$ at $t = -\tau_\infty^l$:

$$\begin{aligned} \psi^l(t) = {} & e^{(i/2)(t+\tau_\infty^l)\Delta} e^{-(i/2)\tau_\infty^l\Delta} f^l \\ & + \frac{i}{2}\int_{-\tau_\infty^l}^t e^{(i/2)(t-t')\Delta}\big\{\big|\psi^l(t')\big|^{p-1}\psi^l(t')\big\}\,dt'. \end{aligned} \tag{4.6}$$

Here, in case of $\tau_\infty^l = \pm\infty$, we are regarding this as the final value problem:

$$e^{-(i/2)t\Delta}\psi^l(t) = f^l + \frac{i}{2}\int_{\mp\infty}^t e^{-(i/2)t'\Delta}\big\{\big|\psi^l(t')\big|^{p-1}\psi^l(t')\big\}\,dt'. \tag{4.7}$$

Then, instead of (4.5), we consider the superposition of these solutions with the space-time translations:

$$\begin{aligned} \psi_n^{\mathrm{app}}(x,t) &:= \sum_{l\ge 1}\big(e^{-\tau_n^l \frac{\partial}{\partial t} - \eta_n^l\cdot\nabla}\psi^l\big)(x,t) \\ &= \sum_{l\ge 1}\psi^l(x-\eta_n^l, t-\tau_n^l). \end{aligned} \tag{4.8}$$

We will see that this formal object ψ_n^{app} is an "almost" solution to our equation (NLS) with the initial datum $\sum_{l\ge 1} e^{-(i/2)\tau_n^l\Delta}e^{-\eta_n^l\cdot\nabla} f^l$ and is supposed to be a good approximation to ψ_n. In other words, a kind of superposition principle holds valid in an asymptotic sense as $n \to \infty$. By virtue of the long-time perturbation theory (Proposition 3.6), the sum in ψ_n^{app} consists of a finite number of solutions. Actually, as a consequence of the minimizing property of the sequence $\{\psi_n\}$ (see (4.3)), the summand is just one: put $\Psi := \psi^1$. Then, it turns out that Ψ is a critical element which we are looking for. In fact, we can prove the following.

*$e^{-\eta_n^l\cdot\nabla}$ denotes the space translation by $-\eta_n$. We may expect that the number of summands f^l is finite.

PROPOSITION 4.1 (CRITICAL ELEMENT IN PW_+)

Suppose that $N_c < \mathcal{N}(Q)$. Then, there exists a global solution $\Psi \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ to the equation (NLS) with the following properties:

(i) Ψ *is a critical element for N_c in (4.2) in the sense that*

$$\|\Psi\|_{X(\mathbb{R})} = \infty, \qquad \mathcal{N}\big(\Psi(t)\big) = N_c, \qquad \Psi(t) \in \mathrm{PW}_+ \quad \text{for any } t \in \mathbb{R}, \tag{4.9}$$

(ii) Ψ *satisfies*

$$\big\|\Psi(t)\big\|_{L^2} = \big\|\Psi(0)\big\|_{L^2} = 1 \quad \text{for any } t \in \mathbb{R}, \tag{4.10}$$

and

$$\sup_{t\in\mathbb{R}} \big\|\nabla\Psi(t)\big\|_{L^2} \le N_c^{1/(p-1)s_p}, \tag{4.11}$$

(iii) Ψ *has zero momentum*

$$\Im \int_{\mathbb{R}^d} \overline{\Psi}(x,t) \nabla\Psi(x,t)\, dx = 0 \quad \text{for any } t \in \mathbb{R}. \tag{4.12}$$

(iv) $\{\Psi(t)\}_{t\ge 0}$ *is tight in $H^1(\mathbb{R}^d)$ in the following sense. For any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ and a continuous path $\gamma_\varepsilon \in C([0,\infty); \mathbb{R}^d)$ with $\gamma_\varepsilon(0) = 0$ such that*

$$\int_{|x-\gamma_\varepsilon(t)|<R_\varepsilon} \big|\Psi(x,t)\big|^2\, dx > 1 - \varepsilon \quad \text{for any } t \in [0,\infty), \tag{4.13}$$

and

$$\int_{|x-\gamma_\varepsilon(t)|<R_\varepsilon} \big|\nabla\Psi(x,t)\big|^2\, dx > \big\|\nabla\Psi(t)\big\|_{L^2}^2 - \varepsilon \quad \text{for any } t \in [0,\infty). \tag{4.14}$$

We will give the proof of Proposition 4.1 in Sections 4.2 and 4.3.

To prove $N_c = \mathcal{N}(Q)$, however, we need to know more subtle behavior of the path. For a sufficiently long time, we can take γ_ε as the almost center of mass, say $\gamma_\varepsilon^{\mathrm{ac}}$.

LEMMA 4.2 (ALMOST CENTER OF MASS)

Let Ψ be a global solution to the equation (NLS) satisfying the properties (4.9)–(4.14). Let R_ε be a radius found in of Proposition 4.1(iv) for $\varepsilon > 0$. We define an "almost center of mass" by

$$\gamma_{\varepsilon,R}^{\mathrm{ac}}(t) := \big(\vec{w}_{20R}, \big|\Psi(t)\big|^2\big) \quad \text{for any } \varepsilon \in (0,(1/100)) \text{ and } R > R_\varepsilon, \tag{4.15}$$

where $\vec{w}_R$ is the function defined by (A.3). Then, we have

$$\gamma_{\varepsilon,R}^{\mathrm{ac}} \in C^1\big([0,\infty); \mathbb{R}^d\big), \tag{4.16}$$

and there exists a constant $\alpha > 0$, depending only on d and p, such that

$$\big|\gamma_{\varepsilon,R}^{\mathrm{ac}}(t)\big| \le 20R \quad \text{for any } t \in \Big[0, \alpha\frac{R}{\sqrt{\varepsilon}}\Big], \tag{4.17}$$

$$\int_{|x-\gamma_{\varepsilon,R}^{\mathrm{ac}}(t)|\le 4R} \big|\Psi(x,t)\big|^2 + \big|\nabla\Psi(x,t)\big|^2\,dx \ge \big\|\Psi(t)\big\|_{H^1}^2 - \varepsilon \tag{4.18}$$
$$\textit{for any } t\in\Big[0,\alpha\frac{R}{\sqrt{\varepsilon}}\Big].$$

REMARK 4.1

In the proof below, we find that the following estimate holds (see (4.30)):

$$\Big|\frac{d\gamma_{\varepsilon,R}^{\mathrm{ac}}}{dt}(t)\Big| \lesssim \sqrt{\varepsilon} \quad \text{for any } t\in\Big[0,\alpha\frac{R}{\sqrt{\varepsilon}}\Big],$$

where the implicit constant depends only on d and q.

Proof of Lemma 4.2

We easily verify (see, e.g., [21, Proposition B.1]) that

$$\begin{aligned}\gamma_{\varepsilon,R}^{\mathrm{ac}}(t) &= \big(\vec{w}_{20R},\big|\Psi(0)\big|^2\big)\\ &\quad + \Big(2\Im\int_0^t\int_{\mathbb{R}^d}\nabla\vec{w}_{20R}^{j}(x)\cdot\nabla\Psi(x,s)\overline{\Psi(x,s)}\,dx\,ds\Big)_{j=1,\dots,d}.\end{aligned}\tag{4.19}$$

This formula, with the help of (4.10), (4.11), and $\|\nabla\vec{w}_{20R}\|_{L^\infty}\lesssim 1$, shows (4.16): $\gamma_{\varepsilon,R}^{\mathrm{ac}}\in C^1([0,\infty);\mathbb{R}^d)$.

Next, we prove the properties (4.17) and (4.18). Let γ_ε be a path found in Proposition 4.1, and let t_ε be the first time such that the size of γ_ε reaches $10R$, that is,

$$t_\varepsilon := \inf\big\{t\ge 0 \,\big|\, \big|\gamma_\varepsilon(t)\big| = 10R\big\}. \tag{4.20}$$

Since $\gamma_\varepsilon\in C([0,\infty);\mathbb{R}^d)$ with $\gamma_\varepsilon(0)=0$, we have $t_\varepsilon>0$ and

$$\big|\gamma_\varepsilon(t)\big| \le 10R \quad \text{for any } t\in[0,t_\varepsilon]. \tag{4.21}$$

We claim that

$$\big|\gamma_{\varepsilon,R}^{\mathrm{ac}}(t)-\gamma_\varepsilon(t)\big| < 2R \quad \text{for any } t\in[0,t_\varepsilon]. \tag{4.22}$$

It follows from property (4.10) that

$$\begin{aligned}\big|\gamma_{\varepsilon,R}^{\mathrm{ac}}(t)-\gamma_\varepsilon(t)\big| &= \big|\big(\vec{w}_{20R},\big|\Psi(t)\big|^2\big)-\gamma_\varepsilon(t)\big\|\Psi(t)\big\|_{L^2}^2\big|\\ &\le \int_{|x-\gamma_\varepsilon(t)|\le R}\big|x-\gamma_\varepsilon(t)\big|\big|\Psi(x,t)\big|^2\,dx\\ &\quad + \int_{|x-\gamma_\varepsilon(t)|\ge R}\big|\vec{w}_{20R}-\gamma_\varepsilon(t)\big|\big|\Psi(x,t)\big|^2\,dx.\end{aligned}\tag{4.23}$$

Moreover, applying (4.21) and the estimate $\|\vec{w}_{20R}\|_{L^\infty}\le 10R$ to the second term on the right-hand side above, we obtain

$$\big|\gamma_{\varepsilon,R}^{\mathrm{ac}}(t)-\gamma_\varepsilon(t)\big| \le R\big\|\Psi(t)\big\|_{L^2}^2 + 50R\int_{|x-\gamma_\varepsilon(t)|\ge R}\big|\Psi(x,t)\big|^2\,dx \tag{4.24}$$
$$\text{for any } t\in[0,t_\varepsilon].$$

Hence, this inequality (4.24) together with (4.10) and the tightness (4.13) yields

$$\big|\gamma^{\mathrm{ac}}_{\varepsilon,R}(t)-\gamma_\varepsilon(t)\big| \le R+50R\varepsilon<2R \quad \text{for any } \varepsilon<\frac{1}{100} \text{ and } t\in[0,t_\varepsilon]. \tag{4.25}$$

Now, we have by (4.21) and (4.22) that

$$\big|\gamma^{\mathrm{ac}}_{\varepsilon,R}(t)\big| \le 12R \quad \text{for any } t\in[0,t_\varepsilon]. \tag{4.26}$$

Moreover, (4.22) also gives us

$$B_R\big(\gamma_\varepsilon(t)\big)\subset B_{4R}\big(\gamma^{\mathrm{ac}}_{\varepsilon,R}(t)\big) \quad \text{for any } t\in[0,t_\varepsilon], \tag{4.27}$$

so that the tightness of $\{\Psi(t)\}_{t\ge 0}$ in $H^1(\mathbb{R}^d)$ (see (4.13) and (4.14)) gives us

$$\int_{|x-\gamma^{\mathrm{ac}}_{\varepsilon,R}(t)|\le 4R}\big|\Psi(x,t)\big|^2+\big|\nabla\Psi(t)\big|^2\,dx \ge \big\|\Psi(t)\big\|_{H^1}^2-\varepsilon \quad \text{for any } t\in[0,t_\varepsilon]. \tag{4.28}$$

Therefore, for the desired results (4.17) and (4.18), it suffices to show that there exists a constant $\alpha>0$, depending only on d and p, such that

$$\alpha\frac{R}{\sqrt{\varepsilon}}\le t_\varepsilon. \tag{4.29}$$

To this end, we prove that

$$\Big|\frac{d\gamma^{\mathrm{ac}}_{\varepsilon,R}}{dt}(t)\Big| \lesssim \sqrt{\varepsilon} \quad \text{for any } t\in[0,t_\varepsilon]. \tag{4.30}$$

Before proving (4.30), we describe how it yields (4.29). It follows from (4.30) that

$$\big|\gamma^{\mathrm{ac}}_{\varepsilon,R}(t_\varepsilon)\big|-\big|\gamma^{\mathrm{ac}}_{\varepsilon,R}(0)\big| \le \int_0^{t_\varepsilon}\Big|\frac{d\gamma^{\mathrm{ac}}_{\varepsilon,R}}{dt}(t)\Big|\,dt \lesssim \sqrt{\varepsilon}t_\varepsilon. \tag{4.31}$$

Hence, we see from (4.22) and $\gamma_\varepsilon(0)=0$ that

$$\sqrt{\varepsilon}t_\varepsilon \gtrsim \big|\gamma^{\mathrm{ac}}_{\varepsilon,R}(t_\varepsilon)\big|-2R \ge \big|\gamma_\varepsilon(t_\varepsilon)\big|-\big|\gamma^{\mathrm{ac}}_{\varepsilon,R}(t_\varepsilon)-\gamma_\varepsilon(t_\varepsilon)\big|-2R\ge 6R, \tag{4.32}$$

which gives (4.29).

Finally, we prove (4.30). Using (4.19) and the property (4.12), we obtain

$$\Big|\frac{d\gamma^{\mathrm{ac}}_{\varepsilon,R}}{dt}(t)\Big|^2 \le 4\sum_{j=1}^{d}\|\nabla\vec{w}^{j}_{20R}\|_{L^\infty}^2\big\|\Psi(t)\big\|_{L^2}^2\int_{|x|\ge 20R}\big|\nabla\Psi(x,t)\big|^2\,dx \quad \text{for any } t\ge 0. \tag{4.33}$$

Applying (4.10) and the estimate $\|\nabla\vec{w}_{20R}\|_{L^\infty}\lesssim 1$ to the right-hand side above, we further obtain

$$\Big|\frac{d\gamma^{\mathrm{ac}}_{\varepsilon,R}}{dt}(t)\Big|^2 \lesssim \int_{|x|\ge 20R}\big|\nabla\Psi(x,t)\big|^2\,dx \quad \text{for any } t\ge 0. \tag{4.34}$$

Since the estimate (4.21) shows that

$$B_R\big(\gamma_\varepsilon(t)\big)\subset B_{20R}(0) \quad \text{for any } t\in[0,t_\varepsilon], \tag{4.35}$$

the estimate (4.34) together with the tightness (4.14) leads to (4.30). □

Lemma 4.2 implies that Ψ found in Proposition 4.1 is in a bound motion, rather, a standing wave. On the other hand, the generalized virial identity (A.9) suggests that Ψ is in a scattering motion. As we already mentioned at the beginning of this Section 4.1, these two facts contradict each other; Thus, we see that $N_c = \mathcal{N}(Q)$. Here, we show this point precisely.

The generalized virial identity (A.9) together with (A.16) and (A.17) yields

$$\begin{aligned}
&\big(W_R, |\Psi(t)|^2\big) \\
&\quad \ge \big(W_R, |\Psi(0)|^2\big) + 2t\Im\big(\vec{w}_R \cdot \nabla\Psi(0), \Psi(0)\big) \\
&\quad + 2\int_0^t \int_0^{t'} \mathcal{K}\big(\Psi(t'')\big)\, dt''\, dt' \\
&\quad - 2\int_0^t \int_0^{t'} \int_{|x|\ge R} \rho_1(x)\big|\nabla\Psi(x,t'')\big|^2 + \rho_2(x)\Big|\frac{x}{|x|}\cdot\nabla\Psi(x,t'')\Big|^2\, dx\, dt''\, dt' \\
&\quad - \frac{1}{2}\int_0^t \int_0^{t'} \big\|\Delta(\operatorname{div}\vec{w}_R)\big\|_{L^\infty}\big\|\Psi(t'')\big\|_{L^2}^2\, dt''\, dt' \quad \text{for any } R>0.
\end{aligned} \tag{4.36}$$

Applying the estimates (4.10), $\|\Delta(\operatorname{div}\vec{w}_R)\|_{L^\infty} \lesssim 1/R^2$, and (A.20) to the right-hand side above, we obtain

$$\begin{aligned}
&\big(W_R, |\Psi(t)|^2\big) \\
&\quad \ge \big(W_R, |\Psi(0)|^2\big) + 2t\Im\big(\vec{w}_R \cdot \nabla\Psi(0), \Psi(0)\big) \\
&\quad + 2\int_0^t \int_0^{t'} \mathcal{K}\big(\Psi(t'')\big)\, dt''\, dt' \\
&\quad - C_1\int_0^t \int_0^{t'} \int_{|x|\ge R} \big|\nabla\Psi(x,t'')\big|^2\, dx\, dt''\, dt' - \frac{C_2}{R^2}t^2 \quad \text{for any } R>0,
\end{aligned} \tag{4.37}$$

where C_1 and C_2 are some positive constants independent of R. Moreover, it follows from (2.5) in Lemma 2.2 that

$$\begin{aligned}
&\big(W_R, |\Psi(t)|^2\big) \\
&\quad \ge \big(W_R, |\Psi(0)|^2\big) + 2t\Im\big(\vec{w}_R \cdot \nabla\Psi(0), \Psi(0)\big) + t^2\omega_0\mathcal{H}\big(\Psi(0)\big) \\
&\quad - C_1\int_0^t \int_0^{t'} \int_{|x|\ge R} \big|\nabla\Psi(x,t'')\big|^2\, dx\, dt''\, dt' - \frac{C_2}{R^2}t^2 \quad \text{for any } R>0,
\end{aligned} \tag{4.38}$$

where $\omega_0 := 1 - ((\mathcal{N}(\Psi(0)))/(\mathcal{N}(Q)))^{(p-1)/2}$. Here, we have by Lemma 4.2 that for any $\varepsilon \in (0,(1/100))$, there exists $R_\varepsilon > 0$ with the following property. For any $R \ge R_\varepsilon$, there exists $\gamma^{\mathrm{ac}}_{\varepsilon,R} \in C^1([0,\infty);\mathbb{R}^d)$ such that

$$\big|\gamma^{\mathrm{ac}}_{\varepsilon,R}(t)\big| \le 20R \quad \text{for any } t \in \Big[0, \alpha\frac{R}{\sqrt{\varepsilon}}\Big], \tag{4.39}$$

$$\int_{|x-\gamma^{\mathrm{ac}}_{\varepsilon,R}(t)|\ge 4R} \big|\nabla\Psi(x,t)\big|^2\, dx < \varepsilon \quad \text{for any } t \in \Big[0, \alpha\frac{R}{\sqrt{\varepsilon}}\Big], \tag{4.40}$$

where α is some constant depending only on d and p.

We see from (4.39) that

$$\begin{aligned}&\left|x-\gamma^{\mathrm{ac}}_{\varepsilon,R}(t)\right|\ge 4R\\ &\qquad\text{for any } R\ge R_\varepsilon, t\in\Big[0,\alpha\frac{R}{\sqrt{\varepsilon}}\Big], \text{ and } x\in\mathbb{R}^d \text{ with } |x|\ge 24R.\end{aligned}\tag{4.41}$$

Hence, (4.38) together with the tightness (4.40) leads to

$$\begin{aligned}&\big(W_{50R},\left|\Psi(t)\right|^2\big)\\ &\ge\big(W_{50R},\left|\Psi(0)\right|^2\big)+2t\Im\big(\vec{w}_{50R}\cdot\nabla\Psi(0),\Psi(0)\big)+t^2\omega_0\mathcal{H}\big(\Psi(0)\big)\\ &\qquad -C_1t^2\varepsilon-\frac{C_2}{(50R)^2}t^2 \quad\text{for any } R\ge R_\varepsilon \text{ and } t\in\Big[0,\alpha\frac{R}{\sqrt{\varepsilon}}\Big].\end{aligned}\tag{4.42}$$

We choose ε so small that

$$0<\varepsilon<\min\Big\{\frac{1}{100},\frac{\omega_0}{4C_2}\mathcal{H}\big(\Psi(0)\big)\Big\},\tag{4.43}$$

and we choose R so large that

$$R\ge\max\Big\{R_\varepsilon,\frac{\sqrt{C_2}}{\sqrt{\omega_0\mathcal{H}(\Psi(0))}}\Big\}.\tag{4.44}$$

Then, it follows from (4.42) that

$$\begin{aligned}&\big(W_{50R},\left|\Psi(t)\right|^2\big)\\ &\ge\big(W_{50R},\left|\Psi(0)\right|^2\big)+2t\Im\big(\vec{w}_{50R}\cdot\nabla\Psi(0),\Psi(0)\big)+\frac{t^2}{2}\omega_0\mathcal{H}\big(\Psi(0)\big)\\ &\qquad\text{for any } t\in\Big[0,\alpha\frac{R}{\sqrt{\varepsilon}}\Big].\end{aligned}\tag{4.45}$$

Dividing the both sides of (4.45) by t^2 and applying the estimates (4.10) and $\|W_R\|_{L^\infty}\le 8R^2$, we obtain

$$\begin{aligned}\frac{8(50R)^2}{t^2}&\ge\frac{1}{t^2}\big(W_{50R},\left|\Psi(0)\right|^2\big)+\frac{2}{t}\Im\big(\vec{w}_{50R}\cdot\nabla\Psi(0),\Psi(0)\big)\\ &\qquad+\frac{\omega_0}{2}\mathcal{H}\big(\Psi(0)\big)\quad\text{for any } t\in\Big[0,\alpha\frac{R}{\sqrt{\varepsilon}}\Big].\end{aligned}\tag{4.46}$$

In particular, when $t=\alpha\frac{R}{\sqrt{\varepsilon}}$, we have by (4.10), $\|W_R\|_{L^\infty}\le 8R^2$, and $\|\vec{w}_R\|_{L^\infty}\le 2R$ that

$$\begin{aligned}\frac{8(50)^2\varepsilon}{\alpha^2}&\ge\frac{\varepsilon}{\alpha^2R^2}\big(W_{50R},\left|\Psi(0)\right|^2\big)+\frac{2\sqrt{\varepsilon}}{\alpha R}\Im\big(\vec{w}_{50R}\cdot\nabla\Psi(0),\Psi(0)\big)\\ &\qquad+\frac{\omega_0}{2}\mathcal{H}\big(\Psi(0)\big)\\ &\ge-\frac{8(50)^2\varepsilon}{\alpha^2}-\frac{200\sqrt{\varepsilon}}{\alpha}\big\|\nabla\Psi(0)\big\|_{L^2}+\frac{\omega_0}{2}\mathcal{H}\big(\Psi(0)\big),\end{aligned}\tag{4.47}$$

so that

$$\frac{8(50)^2\varepsilon}{\alpha^2}+\frac{8(50)^2\varepsilon}{\alpha^2}+\frac{200\sqrt{\varepsilon}}{\alpha}\big\|\nabla\Psi(0)\big\|_{L^2}\ge\frac{\omega_0}{2}\mathcal{H}\big(\Psi(0)\big).\tag{4.48}$$

However, taking $\varepsilon \to 0$ in (4.48), we obtain a contradiction. This absurd conclusion comes from the existence of critical element Ψ (see Proposition 4.1). Thus, it must hold that $N_c = \mathcal{N}(Q)$, provided that Proposition 4.1 is valid.

4.2. Solving the variational problem for $\widetilde{N}_c$

In this section, we construct a candidate for the critical element, considering the variational problem for N_c.

Suppose to the contrary that $N_c < \mathcal{N}(Q)$. Then, we can take a minimizing sequence $\{\delta_n\}_{n\in\mathbb{N}}$ such that

$$N_c < \delta_n < \mathcal{N}(Q) \quad \text{for any } n \in \mathbb{N}, \qquad \lim_{n\to\infty} \delta_n = N_c. \tag{4.49}$$

Moreover, using the scale transformation (1.22), we can take a sequence $\{\psi_{0,n}\}_{n\in\mathbb{N}}$ in PW_+ such that

$$N_c < \mathcal{N}(\psi_{0,n}) < \delta_n \quad \text{for any } n \in \mathbb{N}, \tag{4.50}$$

$$\|\psi_{0,n}\|_{L^2} = 1 \quad \text{for any } n \in \mathbb{N}. \tag{4.51}$$

Note that (4.50) together with (4.49) leads to

$$\lim_{n\to\infty} \mathcal{N}(\psi_{0,n}) = N_c. \tag{4.52}$$

We also find that

$$\limsup_{n\to\infty} \|e^{(i/2)t\Delta}\psi_{0,n}\|_{L^\infty(\mathbb{R};L^{(d/2)(p-1)})} > 0. \tag{4.53}$$

Let ψ_n be the solution to (NLS) with $\psi_n(0) = \psi_{0,n}$. Then, (4.50) together with the definition of N_c (see (4.2)) implies that

$$\|\psi_n\|_{X(\mathbb{R})} = \infty \quad \text{for any } n \in \mathbb{N}. \tag{4.54}$$

If (4.53) failed, then the small data theory (Proposition 3.5) concludes that

$$\|\psi_n\|_{X(\mathbb{R})} < \infty \quad \text{for any sufficiently large } n \in \mathbb{N}, \tag{4.55}$$

which contradicts (4.54). Hence (4.53) holds.

Now, we apply the profile decomposition given in [5] (see also [1], [17]) to the sequence $\{\psi_{0,n}\}$, so that we have the following.

LEMMA 4.3

We can extract some subsequence of $\{\psi_{0,n}\}$ (still denoted by the same symbol) with the following properties. There exist

(i) *a family of nontrivial functions $\{f^1, f^2, f^3, \dots\}$ in $H^1(\mathbb{R}^d)$ and*

(ii) *a family of sequences $\{\{(\eta_n^1, \tau_n^1)\}, \{(\eta_n^2, \tau_n^2)\}, \{(\eta_n^3, \tau_n^3)\}, \dots\}$ in $\mathbb{R}^d \times \mathbb{R}$ with*

$$\lim_{n\to\infty} \tau_n^l = \tau_\infty^l \in \mathbb{R} \cup \{\pm\infty\} \quad \textit{for any } l \geq 1 \tag{4.56}$$

and

$$\lim_{n\to\infty} |\tau_n^l - \tau_n^k| + |\eta_n^l - \eta_n^k| = \infty \quad \textit{for any } 1 \leq k < l, \tag{4.57}$$

such that, putting

$$f_n^0 := \psi_{0,n}, \qquad f^0 := 0, \qquad \tau_n^0 := 0, \qquad \eta_n^0 := 0,$$

$$f_n^l := e^{(i/2)(\tau_n^l - \tau_n^{l-1})\Delta} e^{(\eta_n^l - \eta_n^{l-1})\cdot\nabla}(f_n^{l-1} - f^{l-1}) \quad \textit{for } l \geq 1,$$

we have that, for any $l \geq 1$ *and* $q \in [2, 2^*)$*,*

$$\lim_{n\to\infty} f_n^l = f^l \quad \textit{weakly in } H^1(\mathbb{R}^d) \textit{ and strongly in } L^q_{\mathrm{loc}}(\mathbb{R}^d), \tag{4.58}$$

$$\lim_{n\to\infty}\left\{\left\||\nabla|^s\psi_{0,n}\right\|_{L^2}^2 - \left\||\nabla|^s(f_n^l - f^l)\right\|_{L^2}^2\right\} = \sum_{k=1}^{l}\left\||\nabla|^s f^k\right\|_{L^2}^2 \quad \textit{for any } s \in [0,1], \tag{4.59}$$

$$\lim_{n\to\infty}\left\{\|\psi_{0,n}\|_{L^q}^q - \left\|e^{-(i/2)\tau_n^l\Delta}(f_n^l - f^l)\right\|_{L^q}^q - \sum_{k=1}^{l}\|e^{-(i/2)\tau_n^k\Delta}f^k\|_{L^q}^q\right\} = 0, \tag{4.60}$$

$$\lim_{n\to\infty}\left\{\mathcal{H}(\psi_{0,n}) - \mathcal{H}\big(e^{-(i/2)\tau_n^l\Delta}(f_n^l - f^l)\big) - \sum_{k=1}^{l}\mathcal{H}(e^{-(i/2)\tau_n^k\Delta}f^k)\right\} = 0. \tag{4.61}$$

Furthermore, putting $N := \#\{f^1, f^2, f^3, \dots\}$*, we have the alternatives: if* N *is finite, then*

$$\lim_{n\to\infty}\left\|e^{(i/2)t\Delta}(f_n^N - f^N)\right\|_{L^\infty(\mathbb{R};L^{(d/2)(p-1)})\cap X(\mathbb{R})} = 0; \tag{4.62}$$

if $N = \infty$*, then*

$$\lim_{l\to\infty}\lim_{n\to\infty}\left\|e^{(i/2)t\Delta}(f_n^l - f^l)\right\|_{L^\infty(\mathbb{R};L^{(d/2)(p-1)})\cap X(\mathbb{R})} = 0. \tag{4.63}$$

Besides the above properties, we have

$$\mathcal{K}\big(e^{-\frac{i}{2}\tau_n^l\Delta}(f_n^l - f^l)\big) > 0 \quad \text{for any sufficiently large } n \in \mathbb{N} \tag{4.64}$$

and

$$\begin{cases} e^{-(i/2)\tau_\infty^l\Delta}f^l \in \mathrm{PW}_+ & \text{if } \tau_\infty^l \in \mathbb{R}, \\ f^l \in \Omega & \text{if } \tau_\infty^l = \pm\infty \end{cases} \tag{4.65}$$

for any $l \geq 1$. We shall prove these properties. It follows from (2.6) in Lemma 2.2, (4.52), and $N_c < \mathcal{N}(Q)$ that

$$\limsup_{n\to\infty}\widetilde{\mathcal{N}}(\psi_{0,n}) \leq \limsup_{n\to\infty}\mathcal{N}(\psi_{0,n}) = N_c < \mathcal{N}(Q). \tag{4.66}$$

This together with (4.59) shows that

$$\widetilde{\mathcal{N}}\big(e^{-(i/2)\tau_n\Delta}(f_n^l - f^l)\big) = \widetilde{\mathcal{N}}(f_n^l - f^l) < \mathcal{N}(Q) \quad \text{for any sufficiently large } n \in \mathbb{N}. \tag{4.67}$$

Hence, we see from (1.25) that (4.64) holds.

Next, we prove (4.65). Suppose first that $\tau_\infty^l \in \mathbb{R}$. Then, (4.59) and (4.66) give us

$$\widetilde{\mathcal{N}}(e^{-(i/2)\tau_\infty^l \Delta} f^l) = \widetilde{\mathcal{N}}(f^l) < \mathcal{N}(Q). \tag{4.68}$$

Hence, we find from (1.14) and (1.25) that

$$0 < \mathcal{K}(e^{-(i/2)\tau_\infty^l \Delta} f^l) < \mathcal{H}(e^{-(i/2)\tau_\infty^l \Delta} f^l). \tag{4.69}$$

Moreover, we have by (4.59) and (4.61) that

$$\mathcal{N}(e^{-(i/2)\tau_\infty^l \Delta} f^l) = \lim_{n\to\infty} \mathcal{N}(e^{-(i/2)\tau_n^l \Delta} f^l) \le \lim_{n\to\infty} \mathcal{N}(\psi_{0,n}). \tag{4.70}$$

Combining (4.52) with (4.70), we obtain

$$\mathcal{N}(e^{-\frac{i}{2}\tau_\infty^l \Delta} f^l) \le N_c < \mathcal{N}(Q). \tag{4.71}$$

Hence, (1.33) together with (4.69) and (4.71) shows that $e^{-(i/2)\tau_\infty^l \Delta} f^l \in \mathrm{PW}_+$.

We next suppose that $\tau_\infty^l \in \{\pm\infty\}$. Then, (4.61) together with (1.14) and (4.64) yields

$$\begin{aligned} \|\nabla f^l\|_{L^2}^2 &= \mathcal{H}(e^{-(i/2)\tau_n^l \Delta} f^l) + \frac{2}{p+1} \|e^{-(i/2)\tau_n^l \Delta} f^l\|_{L^{p+1}}^{p+1} \\ &\le \mathcal{H}(\psi_{0,n}) + o_n(1) + \frac{2}{p+1} \|e^{-(i/2)t_n^l \Delta} f^l\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{4.72}$$

Here, it follows from $\tau_\infty^l \in \{\pm\infty\}$ that

$$\lim_{n\to\infty} \|e^{-(i/2)t_n^l \Delta} f^l\|_{L^{p+1}} = 0. \tag{4.73}$$

Hence, we have

$$\|\nabla f^l\|_{L^2}^2 \le \lim_{n\to\infty} \mathcal{H}(\psi_{0,n}). \tag{4.74}$$

Combining (4.74), (4.59), and (4.52), we obtain

$$\widetilde{\mathcal{N}}(f^l) \le \Big(\frac{2s_p}{d}\Big)^{s_p} \lim_{n\to\infty} \mathcal{N}(\psi_{0,n}) = \Big(\frac{2s_p}{d}\Big)^{s_p} N_c < \Big(\frac{2s_p}{d}\Big)^{s_p} \mathcal{N}(Q), \tag{4.75}$$

so that $f^l \in \Omega$ (see (1.34) for the definition of Ω).

Now, let ψ_n be the solution to (NLS) with $\psi_n(0) = \psi_{0,n}$, so that ψ_n exists on the whole interval $\mathbb{R}$ and

$$\|\psi_n\|_{X(\mathbb{R})} = \infty. \tag{4.76}$$

Then, we find the following fact, which gives us a candidate for the critical element in Proposition 4.1.

LEMMA 4.4

We can extract a subsequence of $\{\psi_n\}$ (still denoted by the same symbol) satisfying the following properties. There exist

(i) *a nontrivial global solution $\Psi \in C(\mathbb{R}; H^1(\mathbb{R}^d))$ to the equation (NLS) with*

$$\|\Psi\|_{X(\mathbb{R})} = \infty, \tag{4.77}$$

$$\Psi(t) \in \mathrm{PW}_+ \quad \textit{for any } t \in \mathbb{R}, \tag{4.78}$$

$$\big\|\Psi(t)\big\|_{L^2} = 1 \quad \textit{for any } t \in \mathbb{R}, \tag{4.79}$$

$$\mathcal{N}\big(\Psi(t)\big) = N_c \quad \textit{for any } t \in \mathbb{R}, \tag{4.80}$$

and

(ii) *a nontrivial function* $f \in \mathrm{PW}_+$*, a sequence* $\{\tau_n\}$ *in* $\mathbb{R}$ *with* $\lim_{n\to\infty} \tau_n = \tau_\infty$ *for some* $\tau_\infty \in \mathbb{R}$*, and a sequence* $\{\eta_n\}$ *in* $\mathbb{R}^d$ *such that*

$$\lim_{n\to\infty} e^{(i/2)\tau_n\Delta} e^{\eta_n\cdot\nabla}\psi_n(0) = f \quad \textit{strongly in } H^1(\mathbb{R}^d), \tag{4.81}$$

$$\lim_{n\to\infty} \big\|\Psi(-\tau_n) - e^{-(i/2)\tau_n\Delta} f\big\|_{H^1} = 0. \tag{4.82}$$

Especially, we have

$$\|f\|_{L^2} = \big\|\Psi(t)\big\|_{L^2} \quad \textit{for any } t \in \mathbb{R}, \qquad \|\nabla f\|_{L^2} = \lim_{n\to\infty} \big\|\nabla\Psi(-\tau_n)\big\|_{L^2}. \tag{4.83}$$

Proof of Lemma 4.4
Note that $\{\psi_n\}$ satisfies

$$\psi_n(t) \in \mathrm{PW}_+ \quad \text{for any } t \in \mathbb{R} \text{ and } n \in \mathbb{N}, \tag{4.84}$$

$$\big\|\psi_n(t)\big\|_{L^2} = 1 \quad \text{for any } t \in \mathbb{R} \text{ and } n \in \mathbb{N}, \tag{4.85}$$

$$\sup_{n\in\mathbb{N}} \big\|\psi_n(0)\big\|_{H^1} < \infty, \tag{4.86}$$

$$\lim_{n\to\infty} \mathcal{N}\big(\psi_n(t)\big) = N_c \quad \text{for any } t \in \mathbb{R}, \tag{4.87}$$

$$\|\psi_n\|_{X(\mathbb{R})} = \infty \quad \text{for any } n \in \mathbb{N}. \tag{4.88}$$

We begin with proving $N = 1$ in Lemma 4.3. To this end, we introduce an approximate solution ψ_n^{app} of ψ_n: Let $L = N$ if $N < \infty$, and let L be a sufficiently large number specified later if $N = \infty$. Then, we define

$$\psi_n^{\mathrm{app}}(x,t) := \sum_{l=1}^{L} \psi^l(x - \eta_n^l, t - \tau_n^l). \tag{4.89}$$

Here, each ψ^l is the solution to (4.6) (or (4.7)) with f^l found in Lemma 4.3, and each $\{(\eta_n^l, \tau_n^l)\}$ is the sequence found in Lemma 4.3, so that we find that

$$\lim_{n\to\infty} \big\|\psi^l(-\tau_n^l) - e^{-(i/2)\tau_n^l\Delta} f^l\big\|_{H^1} = 0 \quad \text{for any } 1 \le l \le L, \tag{4.90}$$

$$\psi^l(t) \in \mathrm{PW}_+ \quad \text{for any } 1 \le l \le L \text{ and } t \in \mathbb{R}. \tag{4.91}$$

We note again that if $\tau_\infty^l = \pm\infty$, then ψ^l is the solution to the final value problem (4.7). Indeed, since $f^l \in \Omega$ if $\tau_\infty^l = \pm\infty$ (see (4.65)), we actually obtain the desired solution ψ^l by Proposition 3.7.

We shall show that

$$\|\psi^l\|_{X(\mathbb{R})} < \infty \quad \text{for any } 1 \le l \le L, \text{if } N \ge 2. \tag{4.92}$$

To prove this, it suffices to show that $\mathcal{N}(\psi^l(0)) < N_c$ (see (4.2)). Suppose $N \ge 2$, so that $L \ge 2$. Then, we see from (4.59), (4.61), and (4.64) that

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{N}\big(\psi_n(0)\big)\\ &\ge \limsup_{n\to\infty}\Big(\frac{1}{1-s_p}\sum_{k=1}^{L}\mathcal{M}(f^k)\Big)^{1-s_p}\Big(\frac{1}{s_p}\sum_{k=1}^{L}\mathcal{H}(e^{-(i/2)\tau_n^k\Delta}f^k)\Big)^{s_p}.\end{aligned} \tag{4.93}$$

Here, it follows from (4.65) and $\lim_{t\to\pm\infty}\|e^{(i/2)t\Delta}f^k\|_{L^{p+1}} = 0$ that

$$\mathcal{H}(e^{-(i/2)\tau_n^k\Delta}f^k) > 0 \quad \text{for any } k \ge 1 \text{ and sufficiently large } n. \tag{4.94}$$

Hence, we find from (4.87), (4.93), and (4.90) that

$$\begin{aligned}N_c = \lim_{n\to\infty}\mathcal{N}\big(\psi_n(0)\big) &> \lim_{n\to\infty}\Big(\frac{\mathcal{M}(f^l)}{1-s_p}\Big)^{1-s_p}\Big(\frac{\mathcal{H}(e^{-(i/2)\tau_n^l\Delta}f^l)}{s_p}\Big)^{s_p}\\ &= \lim_{n\to\infty}\Big(\frac{\mathcal{M}(\psi^l(-\tau_n^l))}{1-s_p}\Big)^{1-s_p}\Big(\frac{\mathcal{H}(\psi^l(-\tau_n^l))}{s_p}\Big)^{s_p}\\ &= \mathcal{N}\big(\psi^l(0)\big) \quad \text{for any } 1 \le l \le L.\end{aligned} \tag{4.95}$$

Thus, (4.92) holds.

We know by (4.92) that

$$\sup_{n\in\mathbb{N}}\|\psi_n^{\mathrm{app}}\|_{X(\mathbb{R})} < \infty. \tag{4.96}$$

Furthermore, when $N = \infty$, there exists $A > 0$ with the following property. For any $L \in \mathbb{N}$ (the number of components of ψ_n^{app}; see (4.89)), there exists $n_L \in \mathbb{N}$ such that

$$\sup_{n\ge n_L}\|\psi_n^{\mathrm{app}}\|_{X(\mathbb{R})} \le A. \tag{4.97}$$

We shall prove this fact. Recall that $X(\mathbb{R}) = L^{r_1}(\mathbb{R};L^{q_1}) \cap L^{r_2}(I;L^{q_2})$. One can see that

$$\begin{aligned}&\|\psi_n^{\mathrm{app}}\|_{L^{r_j}(\mathbb{R};L^{q_j})}^{q_j}\\ &\le \sum_{l=1}^{L}\|\psi^l\|_{L^{r_j}(\mathbb{R};L^{q_j})}^{q_j}\\ &\quad + C_L\sum_{l=1}^{L}\sum_{\substack{1\le k\le L;\\ k\ne l}}\Big\||\psi^k(\cdot-\eta_n^k,\cdot-\tau_n^k)||\psi^l(\cdot-\eta_n^l,\cdot-\tau_n^l)|^{q_j-1}\Big\|_{L^{r_j/q_j}(\mathbb{R};L^1)}\\ &=: I_j + II_j \quad \text{for } j = 1,2,\end{aligned} \tag{4.98}$$

where $C_L > 0$ is some constant depending only on d, p, q_1, and L. We first consider the term I_j. It follows from (4.59) and (4.86) that

$$\sum_{l=1}^{\infty}\|f^l\|_{H^1}^2 < \infty, \tag{4.99}$$

so that

$$\lim_{l\to\infty} \|f^l\|_{H^1} = 0. \tag{4.100}$$

Hence, we see from Proposition 3.5 (when $\tau^l_\infty \in \mathbb{R}$) and Proposition 3.7 (when $\tau^l_n = \pm\infty$) that there exists $l_0 \in \mathbb{N}$, independent of L (the number of components of ψ_n^{app}), such that

$$\|\psi^l\|_{X(\mathbb{R})} \lesssim \|f^l\|_{H^1} \le 1 \quad \text{for any } l \ge l_0, \tag{4.101}$$

where the implicit constant depends only on d, p, and q_1. Since $q_1, q_2 > 2$, we have by (4.92), (4.99), and (4.101) that

$$I_j \lesssim \sum_{l=1}^{l_0} \|\psi^l\|^{q_j}_{X(\mathbb{R})} + \sum_{l=l_0+1}^{\infty} \|f^l\|^2_{H^1} < \infty. \tag{4.102}$$

Next, we consider the term II_j. Using the condition (4.57), we can take $n_L \in \mathbb{N}$ such that for any $n \ge n_L$ and $k, l \le L$ with $k \ne l$,

$$\Big\| \big|\psi^k(\cdot - \eta^k_n, \cdot - \tau^k_n)\big| \big|\psi^l(\cdot - \eta^l_n, \cdot - \tau^l_n)\big|^{q_j - 1} \Big\|_{L^{r_j/q_j}(\mathbb{R}; L^1)} \le \frac{1}{C_L L^2}, \tag{4.103}$$

where C_L is the constant given in (4.98). Hence, we find that

$$II_j \le 1 \quad \text{for any } n \ge n_L. \tag{4.104}$$

Combining (4.98) with (4.102) and (4.103), we obtain (4.97).

We show that the case $N \ge 2$ cannot occur. Note that ψ_n^{app} solves the following equation:

$$2i\frac{\partial}{\partial t}\psi_n^{\mathrm{app}} + \Delta\psi_n^{\mathrm{app}} + |\psi_n^{\mathrm{app}}|^{p-1}\psi_n^{\mathrm{app}} = e_n, \tag{4.105}$$

where

$$\begin{aligned} e_n(x,t) &:= \big|\psi_n^{\mathrm{app}}(x,t)\big|^{p-1}\psi_n^{\mathrm{app}}(x,t) \\ &\quad - \sum_{l=1}^{L} \big|\psi^l(x - \eta^l_n, t - \tau^l_n)\big|^{p-1}\psi^l(x - \eta^l_n, t - \tau^l_n). \end{aligned} \tag{4.106}$$

Proposition 3.6 (the long-time perturbation theory), with the help of (4.97), tells us that there exists $\varepsilon_1 > 0$, independent of L when $N = \infty$, with the following property: if there exists $n \ge n_L$ (n_L is the number found in (4.97) if $N = \infty$, $n_L = 1$ if $N < \infty$) such that

$$\big\| e^{(i/2)t\Delta}\big(\psi_n(0) - \psi_n^{\mathrm{app}}(0)\big)\big\|_{X(\mathbb{R})} \le \varepsilon_1 \tag{4.107}$$

and

$$\|e_n\|_{L^{\tilde{r}'}(\mathbb{R}; L^{q_1'})} \le \varepsilon_1, \tag{4.108}$$

then

$$\|\psi_n\|_{X(\mathbb{R})} < \infty. \tag{4.109}$$

In the sequel, we show that if $N \geq 2$, then (4.107) and (4.108) hold valid for some L, which shows $N = 1$, since (4.109) contradicts (4.88). It is worthwhile noting here that

$$\psi_n(0) = \psi_{0,n} = \sum_{l=1}^{L} e^{-(i/2)\tau_n^l \Delta} e^{-\eta_n^l \cdot \nabla} f^l + e^{-(i/2)\tau_n^L \Delta} e^{-\eta_n^L \cdot \nabla}(f_n^L - f^L). \tag{4.110}$$

Using the condition (4.57), we find that (4.108) holds for any sufficiently large n. On the other hand, the formula (4.110), with the help of (3.6), shows that

$$\begin{aligned} &\left\| e^{(i/2)t\Delta}\big(\psi_n(0) - \psi_n^{\mathrm{app}}(0)\big)\right\|_{X(\mathbb{R})} \\ &= \left\| e^{(i/2)t\Delta}\Big(\psi_{0,n} - \sum_{l=1}^{L} e^{-\eta_n^l \cdot \nabla}\psi^l(-\tau_n^l)\Big)\right\|_{X(\mathbb{R})} \\ &\leq \left\| e^{(i/2)(t-\tau_n^L)\Delta} e^{-\eta_n^L \cdot \nabla}(f_n^L - f^L)\right\|_{X(\mathbb{R})} \\ &\quad + \sum_{l=1}^{L} \left\| e^{(i/2)t\Delta}\big(e^{-(i/2)\tau_n^l \Delta} e^{-\eta_n^l \cdot \nabla} f^l - e^{-\eta_n^l \cdot \nabla}\psi^l(-\tau_n^l)\big)\right\|_{X(\mathbb{R})} \\ &\leq \left\| e^{(i/2)t\Delta}(f_n^L - f^L)\right\|_{X(\mathbb{R})} + C\sum_{l=1}^{L}\left\| e^{-(i/2)\tau_n^l \Delta} f^l - \psi^l(-\tau_n^l)\right\|_{H^1}, \end{aligned} \tag{4.111}$$

where C is some constant depending only on d, p, and q_1. Here, we have, by (4.62) or (4.63), that

$$\lim_{n\to\infty}\left\| e^{(i/2)t\Delta}(f_n^L - f^L)\right\|_{X(\mathbb{R})} \leq \frac{\varepsilon_1}{4} \tag{4.112}$$

for any sufficiently large L ($L = N$ if $N < \infty$). Hence, for any $L \in \mathbb{N}$ satisfying (4.112), there exists $n_{L,1} \in \mathbb{N}$ such that

$$\left\| e^{(i/2)t\Delta}(f_n^L - f^L)\right\|_{X(\mathbb{R})} \leq \frac{\varepsilon_1}{2} \quad \text{for any } n \geq n_{L,1}. \tag{4.113}$$

Moreover, (4.90) shows that for any $L \leq N$, there exists $n_{L,2} \in \mathbb{N}$ such that

$$\left\| e^{-(i/2)\tau_n^l \Delta} f^l - \psi^l(-\tau_n^l)\right\|_{H^1} \leq \frac{\varepsilon_1}{2CL} \quad \text{for any } n \geq n_{L,2} \text{ and } 1 \leq l \leq L, \tag{4.114}$$

where C is the constant found in (4.111). Combining (4.111) with (4.113) and (4.114), we find that for any $L \in \mathbb{N}$ satisfying (4.112), there exists $n_{L,3} \in \mathbb{N}$ such that

$$\left\| e^{(i/2)t\Delta}\big(\psi_n(0) - \psi_n^{\mathrm{app}}(0)\big)\right\|_{X(\mathbb{R})} \leq \varepsilon_1 \quad \text{for any } n \geq n_{L,3}, \tag{4.115}$$

which gives (4.107).

We have just proved $N = 1$, and therefore L should be one:

$$\psi_n^{\mathrm{app}}(x,t) = \psi^1(x - \eta_n^1, t - \tau_n^1) = (e^{-\tau_n^1 \frac{\partial}{\partial t} - \eta_n^1 \cdot \nabla}\psi^1)(x,t).$$

Put $\Psi = \psi^1$, $f = f^1$, $(\gamma_n, \tau_n) = (\gamma_n^1, \tau_n^1)$, and $\tau_\infty = \tau_\infty^1$. Then, these are what we want. Indeed, we have already shown that these satisfy the properties (4.78) and

(4.82) (see (4.91) for (4.78), and see (4.90) for (4.82)). Moreover, the property (4.83) immediately follows from (4.82).

It remains to prove (4.77), (4.79), (4.80), (4.81), and $\tau_\infty \in \mathbb{R}$.

We first prove that Ψ satisfies the property (4.77): $\|\Psi\|_{X(\mathbb{R})} = \infty$. Suppose to the contrary that $\|\Psi\|_{X(\mathbb{R})} < \infty$. Then, a quite similar argument above works well, so that we obtain an absurd conclusion

$$\|\psi_n\|_{X(\mathbb{R})} < \infty \quad \text{for any sufficiently large } n \in \mathbb{N}. \tag{4.116}$$

Thus, (4.77) holds.

Before proving (4.79), we shall prove (4.80) and (4.81). To this end, we show that there exists a subsequence of $\{\psi_n\}$ (still denoted by the same symbol) such that

$$\|f\|_{L^2} = \lim_{n\to\infty} \big\|\psi_n(0)\big\|_{L^2}, \tag{4.117}$$

$$\lim_{n\to\infty} \mathcal{H}(e^{-(i/2)\tau_n\Delta} f) = \lim_{n\to\infty} \mathcal{H}\big(\psi_n(0)\big). \tag{4.118}$$

Extracting some subsequence, we have by (4.59), (4.61), (4.64), and (4.90) (or (4.82)) that

$$\big\|\Psi(0)\big\|_{L^2} = \lim_{n\to\infty} \big\|\Psi(-\tau_n)\big\|_{L^2} = \|f\|_{L^2} \le \lim_{n\to\infty} \big\|\psi_n(0)\big\|_{L^2} \tag{4.119}$$

and

$$\mathcal{H}\big(\Psi(0)\big) = \lim_{n\to\infty} \mathcal{H}\big(\Psi(-\tau_n)\big) = \lim_{n\to\infty} \mathcal{H}(e^{-(i/2)\tau_n\Delta} f) \le \lim_{n\to\infty} \mathcal{H}\big(\psi_n(0)\big). \tag{4.120}$$

Hence, if (4.117) or (4.118) failed, then we have by (4.87) that

$$\begin{aligned} \mathcal{N}\big(\Psi(0)\big) &= \Big(\frac{\mathcal{M}(f)}{1-s_p}\Big)^{1-s_p} \lim_{n\to\infty} \Big(\frac{\mathcal{H}(e^{-(i/2)\tau_n\Delta} f)}{s_p}\Big)^{s_p} \\ &< \lim_{n\to\infty} \Big(\frac{\mathcal{M}(\psi_n(0))}{1-s_p}\Big)^{1-s_p} \lim_{n\to\infty} \Big(\frac{\mathcal{H}(\psi_n(0))}{s_p}\Big)^{s_p} \\ &= \lim_{n\to\infty} \mathcal{N}\big(\psi_n(0)\big) = N_c. \end{aligned} \tag{4.121}$$

This estimate together with the definition of N_c (see (4.2)) leads to the conclusion that $\|\Psi\|_{X(\mathbb{R})} < \infty$, which contradicts $\|\Psi\|_{X(\mathbb{R})} = \infty$. Thus, (4.117) and (4.118) hold. Then, the inequality in (4.121) becomes the equality, so that (4.80) holds. Moreover, we see from (4.61) together with (4.118) that

$$\lim_{n\to\infty} \mathcal{H}\big(e^{-(i/2)\tau_n\Delta}(f_n^1 - f)\big) = 0, \tag{4.122}$$

and from (4.59) and (4.117) that

$$\lim_{n\to\infty} \widetilde{\mathcal{N}}\big(e^{-(i/2)\tau_n\Delta}(f_n^1 - f)\big) = 0. \tag{4.123}$$

Hence, it follows from Remark 2.1 that

$$\lim_{n\to\infty} \big\|\nabla(f_n^1 - f)\big\|_{L^2} = 0. \tag{4.124}$$

Since $f_n^1 = e^{(i/2)\tau_n\Delta} e^{\eta_n} \psi_n(0)$, we find from (4.117) and (4.124) that (4.81) holds.

We prove (4.79). The properties (4.81) and (4.82) (or (4.83)) together with (4.85) and the mass conservation law (1.6) yield (4.79).

Finally, we show that $\tau_\infty \in \mathbb{R}$. Since $\|\Psi\|_{X(\mathbb{R})} = \infty$, we have $\|\Psi\|_{X([0,\infty))} = \infty$ or $\|\Psi\|_{X((-\infty,0])} = \infty$. The time reversibility of (NLS) allows us to assume that $\|\Psi\|_{X([0,\infty))} = \infty$; if not, we consider $\overline{\Psi(x,-t)}$ instead of $\Psi(x,t)$.

Put

$$\Psi_n(t) := \Psi(t - \tau_n),$$

and suppose to the contrary that $\tau_\infty = -\infty$ or $\tau_\infty = +\infty$. If $\tau_\infty = -\infty$, then (3.6) and (4.82) show that

$$\begin{aligned}\left\|e^{(i/2)t\Delta}\Psi_n(0)\right\|_{X((-\infty,0])} &= \left\|e^{(i/2)(t+\tau_n)\Delta}\Psi_n(0)\right\|_{X((-\infty,-\tau_n])} \\ &\le \|e^{(i/2)t\Delta} f\|_{X((-\infty,-\tau_n])} \\ &\quad + \left\|e^{(i/2)t\Delta}\big(e^{(i/2)\tau_n\Delta}\Psi_n(0) - f\big)\right\|_{X((-\infty,-\tau_n])} \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{4.125}$$

Then, the small data theory (Proposition 3.5) leads us to

$$\|\Psi\|_{X((-\infty,-\tau_n])} = \|\Psi_n\|_{X((-\infty,0])} \lesssim 1 \quad \text{for any sufficiently large } n \in \mathbb{N}. \tag{4.126}$$

Hence, taking $n \to \infty$ in (4.126), we obtain

$$\|\Psi\|_{X(\mathbb{R})} < \infty, \tag{4.127}$$

which contradicts $\|\Psi\|_{X(\mathbb{R})} = \infty$. Thus, the case $\tau_\infty = -\infty$ never happens. Similarly, when $\tau_\infty = +\infty$, we have by (3.6) and (4.82) that

$$\begin{aligned}\left\|e^{(i/2)t\Delta}\Psi_n(0)\right\|_{X([0,+\infty))} &= \left\|e^{(i/2)(t+\tau_n)\Delta}\Psi_n(0)\right\|_{X([-\tau_n,+\infty))} \\ &\le \|e^{(i/2)t\Delta} f\|_{X([-\tau_n,+\infty))} \\ &\quad + \left\|e^{(i/2)t\Delta}\big(e^{(i/2)\tau_n\Delta}\Psi_n(0) - f\big)\right\|_{X([-\tau_n,+\infty))} \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{4.128}$$

Hence, the small data theory (Proposition 3.5) shows that

$$\|\Psi\|_{X([\tau_n,+\infty))} = \|\Psi_n\|_{X([0,+\infty)} < \infty \quad \text{for any sufficiently large } n, \tag{4.129}$$

so that

$$\|\Psi\|_{X([0,+\infty))} < \infty. \tag{4.130}$$

However, (4.130) contradicts our working hypothesis $\|\Psi\|_{X([0,+\infty))} = \infty$. Thus, we find that $\tau_\infty \in \mathbb{R}$. □

4.3. Proof of Proposition 4.1

We shall prove Proposition 4.1, showing that the candidate Ψ found in Lemma 4.4 is actually critical element.

Proof of Proposition 4.1
The properties (4.9) and (4.10) have been obtained in Lemma 4.4. Moreover, (2.6) together with (4.9) and (4.10) yields (4.11).

We shall prove (4.12); the momentum of Ψ is zero. We apply the Galilei transformation to Ψ:

$$\Psi_\xi(x,t) := e^{(i/2)(x\xi - t|\xi|^2)}\Psi(x-\xi t, t), \quad \xi \in \mathbb{R}^d. \tag{4.131}$$

It is easy to verify that

$$\big\|\Psi_\xi(t)\big\|_{L^q} = \big\|\Psi(t)\big\|_{L^q} \quad \text{for any } \xi \in \mathbb{R}^d,\ q \in [2, 2^*) \text{ and } t \in \mathbb{R}, \tag{4.132}$$

$$\|\Psi_\xi\|_{X(\mathbb{R})} = \|\Psi\|_{X(\mathbb{R})} = \infty \quad \text{for any } \xi \in \mathbb{R}^d. \tag{4.133}$$

Moreover, a simple calculation together with the mass and momentum conservation laws (1.6) and (1.8) shows that

$$\begin{aligned}\big\|\nabla\Psi_\xi(t)\big\|_{L^2}^2 &= |\xi|^2\big\|\Psi(0)\big\|_{L^2}^2 + \big\|\nabla\Psi(t)\big\|_{L^2}^2 \\ &\quad + 2\xi\cdot\Im\int_{\mathbb{R}^d}\overline{\Psi}(x,0)\nabla\Psi(x,0)\,dx \quad \text{for any } \xi \in \mathbb{R}^d.\end{aligned} \tag{4.134}$$

This together with the energy conservation law (1.7) yields

$$\begin{aligned}\mathcal{H}\big(\Psi_\xi(t)\big) &= \mathcal{H}\big(\Psi(0)\big) + |\xi|^2\big\|\Psi(0)\big\|_{L^2}^2 \\ &\quad + 2\xi\cdot\Im\int_{\mathbb{R}^d}\overline{\Psi}(x,0)\nabla\Psi(x,0)\,dx \quad \text{for any } \xi \in \mathbb{R}^d.\end{aligned} \tag{4.135}$$

Put

$$\xi_0 := -\frac{\Im\int_{\mathbb{R}^d}\overline{\Psi}(x,0)\nabla\Psi(x,0)\,dx}{\|\Psi(0)\|_{L^2}}. \tag{4.136}$$

Then, we have by (4.134) and (4.135) that

$$\begin{aligned}\big\|\nabla\Psi_{\xi_0}(t)\big\|_{L^2}^2 &= \big\|\nabla\Psi(t)\big\|_{L^2}^2 - \Big(\Im\int_{\mathbb{R}^d}\overline{\Psi}(x,0)\nabla\Psi(x,0)\,dx\Big)^2 \\ &\qquad \text{for any } t \in \mathbb{R},\end{aligned} \tag{4.137}$$

$$\begin{aligned}\mathcal{H}\big(\Psi_{\xi_0}(t)\big) &= \mathcal{H}\big(\Psi(0)\big) - \Big(\Im\int_{\mathbb{R}^d}\overline{\Psi}(x,0)\nabla\Psi(x,0)\,dx\Big)^2 \\ &\qquad \text{for any } t \in \mathbb{R}.\end{aligned} \tag{4.138}$$

Now, we suppose that

$$\Im\int_{\mathbb{R}^d}\overline{\Psi}(x,t)\nabla\Psi(x,t)\,dx = \Im\int_{\mathbb{R}^d}\overline{\Psi}(x,0)\nabla\Psi(x,0)\,dx \neq 0. \tag{4.139}$$

Then, it follows from (4.132), (4.137), (2.6), and (4.9) that

$$\widetilde{\mathcal{N}}\big(\Psi_{\xi_0}(t)\big) < \widetilde{\mathcal{N}}\big(\Psi(t)\big) < \mathcal{N}\big(\Psi(t)\big) = N_c < \mathcal{N}(Q) \quad \text{for any } t \in \mathbb{R}. \tag{4.140}$$

This inequality together with (1.25) implies that

$$\mathcal{K}\big(\Psi_{\xi_0}(t)\big) > 0 \quad \text{for any } t \in \mathbb{R}. \tag{4.141}$$

Moreover, using (4.132), (4.138), and (4.9), we obtain

$$\mathcal{N}\big(\Psi_{\xi_0}(t)\big) < N_c, \tag{4.142}$$

so that $\|\Psi_{\xi_0}\|_{X(\mathbb{R})} < \infty$. However, this contradicts (4.133). Thus, the momentum of Ψ must be zero.

Next, we prove the claim (4.13)–(4.14). Since $\|\Psi\|_{X(\mathbb{R})} = \infty$, we have $\|\Psi\|_{X([0,\infty))} = \infty$ or $\|\Psi\|_{X((-\infty,0])} = \infty$. The time reversibility of (NLS) allows us to assume that $\|\Psi\|_{X([0,\infty))} = \infty$; if not, we consider $\overline{\Psi(x,-t)}$ instead of $\Psi(x,t)$.

In order to prove the claim (4.13)–(4.14), it suffices to show that $\{\Psi(t) \mid t \in [0,\infty)\}$ is precompact in $H^1(\mathbb{R}^d)$ modulo the space translation (see [5, Appendix A]). Take any sequence $\{t_n\}$ in $\mathbb{R}$, and put

$$\Psi_n(x,t) := \Psi(x, t+t_n). \tag{4.143}$$

Then, we can apply the argument in Section 4.2 to the sequence $\{\Psi_n(0)\}$, so that there exists a subsequence of $\{\Psi_n\}$ (still denoted by the same symbol) with the following properties. There exist a nontrivial function $\Phi \in \mathrm{PW}_+$, $\tau_\infty \in \mathbb{R}$ and a sequence $\{\gamma_n\}$ in $\mathbb{R}^d$, such that

$$\lim_{n\to\infty} \Psi_n(\cdot+\gamma_n, 0) = \Psi(\cdot+\gamma_n, t_n) = e^{-(i/2)\tau_\infty \Delta}\Phi \quad \text{strongly in } H^1(\mathbb{R}^d), \tag{4.144}$$

which shows that $\{\Psi(t) \mid t \in [0,\infty)\}$ is precompact in $H^1(\mathbb{R}^d)$ modulo the space translation. Thus, we have completed the proof. □

4.4. Proof of Theorem 1.1

We shall prove Theorem 1.1.

Proof of Theorem 1.1

The claims (i) and (ii) are direct consequences of Lemma 2.2.

We shall prove (iii). We consider the forward time only. Proposition 3.7 shows that the wave operator W_+ exists on Ω and is continuous. It remains to prove the bijectivity of W_+ from Ω to PW_+ and the continuity of W_+^{-1}. These proofs are standard (see, e.g., [4]), and therefore we prove the surjectivity only. Take any $\psi_0 \in \mathrm{PW}_+$, and let ψ be the solution to (NLS) with $\psi(0) = \psi_0$. Since $N_c = \mathcal{N}(Q)$, we have $\|\psi\|_{X(\mathbb{R})} < \infty$. Hence, it follows from Proposition 3.4 that ψ has an asymptotic state ϕ_+ in $H^1(\mathbb{R}^d)$. It remains to show that $\phi_+ \in \Omega$. Since

$$\lim_{t\to+\infty} \big\|\psi(t)\big\|_{L^{p+1}} = \lim_{t\to+\infty} \|e^{(i/2)t\Delta}\phi_+\|_{L^{p+1}} = 0, \tag{4.145}$$

we see from the energy conservation law (1.7) that

$$\begin{aligned} \mathcal{H}(\psi_0) &= \lim_{t\to+\infty}\Big(\big\|\nabla\psi(t)\big\|_{L^2}^2 - \frac{2}{p+1}\big\|\psi(t)\big\|_{L^{p+1}}^{p+1}\Big) \\ &= \|\nabla\phi_\pm\|_{L^2}^2. \end{aligned} \tag{4.146}$$

Moreover, the mass conservation law (1.6) gives us

$$\|\psi_0\|_{L^2}^2 = \lim_{t\to+\infty}\big\|\psi(t)\big\|_{L^2}^2 = \|\phi_+\|_{L^2}^2. \tag{4.147}$$

Combining (4.146), (4.147), and (1.33), we find that

$$\widetilde{\mathcal{N}}(\phi_+) = \Big(\frac{2s_p}{d}\Big)^{s_p}\mathcal{N}(\psi_0) < \Big(\frac{2s_p}{d}\Big)^{s_p}\mathcal{N}(Q). \tag{4.148}$$

Thus, we have $\phi_+ \in \Omega$. □

5. Analysis on PW_-

We shall give proofs of Theorem 1.2, Proposition 1.1, and Proposition 1.2.

Proof of Theorem 1.2
We have already proved the claim (i) in Lemma 2.2. Proofs of (1.39) and (1.40) remain. For simplicity, we consider the forward time only. The problem for the backward time can be proved in a similar way.

Take any $\psi_0 \in \mathrm{PW}_-$, and let ψ be the corresponding solution to the equation (NLS) with $\psi(0) = \psi_0$. When the maximal existence time $T^+_{\max}$ is finite, we have (1.39) as mentioned in (1.5). Therefore, it suffices to prove (1.40). We prove this by employing the idea of Nawa [21].

We suppose to the contrary that (1.40) fails when $T^+_{\max} = \infty$, so that there exists $R_0 > 0$ such that

$$M_0 := \sup_{t\in[0,\infty)} \int_{|x|\ge R_0} \big|\nabla\psi(x,t)\big|^2\,dx < \infty. \tag{5.1}$$

Then, we shall derive a contradiction in three steps.

In what follows, we put $\varepsilon_0 := \mathcal{B}(\psi_0) - \mathcal{H}(\psi_0)$; we see from (1.27) that $\varepsilon_0 > 0$.

Step 1. We claim that there exists a constant $m_0 > 0$ such that for any $R > 0$,

$$\begin{aligned} m_0 < \inf\Big\{ &\int_{|x|\ge R} \big|v(x)\big|^2\,dx \;\Big| \\ &v \in H^1(\mathbb{R}^d), \mathcal{K}^R(v) \le -\frac{1}{4}\varepsilon_0, \|\nabla v\|^2_{L^2(|x|\ge R)} \le M_0, \|v\|_{L^2} \le \|\psi_0\|_{L^2}\Big\}, \end{aligned} \tag{5.2}$$

where $\mathcal{K}^R$ is the functional given by (A.10). Let us prove this. Take any $v \in H^1(\mathbb{R}^d)$ with the following properties:

$$\mathcal{K}^R(v) \le -\frac{1}{4}\varepsilon_0, \qquad \|\nabla v\|^2_{L^2(|x|\ge R)} \le M_0, \qquad \|v\|_{L^2} \le \|\psi_0\|_{L^2}. \tag{5.3}$$

Then, we see from the first property in (5.3) and (A.17) that

$$\frac{1}{4}\varepsilon_0 \le -\mathcal{K}^R(v) \le \int_{|x|\ge R} \rho_3(x)\big|v(x)\big|^{p+1}\,dx. \tag{5.4}$$

Moreover, using the Hölder inequality, the Sobolev embedding, and the second property in (5.3), we obtain

$$\int_{|x|\ge R} \rho_3(x)\big|v(x)\big|^{p+1}\,dx \lesssim \|\rho_3\|_{L^\infty}\|v\|^{p+1-(d(p-1)/2)}_{L^2(|x|\ge R)} M_0^{d(p-1)/4}, \tag{5.5}$$

where the implicit constant depends only on d and p. This estimate together with (5.4) yields

$$\frac{\varepsilon_0}{\|\rho_3\|_{L^\infty} M_0^{d(p-1)/4}} \lesssim \|v\|_{L^2(|x|\ge R)}^{p+1-(d(p-1)/2)}, \tag{5.6}$$

where the implicit constant depends only on d and p. Since $\|\rho_3\|_{L^\infty} \lesssim 1$ (see (A.20)), the estimate (5.6) gives us the desired result (5.2).

Step 2. Let m_0 be a constant found in (5.2). Then, we prove that

$$\sup_{t\in[0,\infty)} \int_{|x|\ge R} \left|\psi(x,t)\right|^2 dx \le m_0 \tag{5.7}$$

for any R satisfying the following properties:

$$R \ge R_0, \tag{5.8}$$

$$\frac{1}{R^2}\|\psi_0\|_{L^2}^2 \ll \varepsilon_0, \tag{5.9}$$

$$\int_{|x|\ge R} \left|\psi_0(x)\right|^2 dx < m_0, \tag{5.10}$$

$$\frac{1}{R^2}\Big(1+\frac{2}{\varepsilon_0}\|\nabla\psi_0\|_{L^2}^2\Big)\big(W_R, |\psi_0|^2\big) < m_0. \tag{5.11}$$

We remark that Lemma A.1 shows that (5.11) holds for any sufficiently large R.

Now, for $R>0$ satisfying (5.8)–(5.11), we put

$$T_R := \sup\Big\{T>0 \,\Big|\, \sup_{t\in[0,T)} \int_{|x|\ge R} \left|\psi(x,t)\right|^2 dx \le m_0\Big\}. \tag{5.12}$$

Note here that since $\psi \in C(\mathbb{R}; L^2(\mathbb{R}^d))$ and $\psi(0) = \psi_0$, we have by (5.10) that $T_R > 0$. It is clear that $T_R = \infty$ shows (5.7).

We suppose the contrary that $T_R < \infty$. Then, it follows from $\psi \in C(\mathbb{R}; L^2(\mathbb{R}^d))$ that

$$\int_{|x|\ge R} \left|\psi(x,T_R)\right|^2 dx = m_0. \tag{5.13}$$

Hence, the definition of m_0 (see (5.2)) together with (5.1), (5.8), and the mass conservation law (1.6) leads us to

$$-\frac{1}{4}\varepsilon_0 < \mathcal{K}^R\big(\psi(T_R)\big). \tag{5.14}$$

Moreover, the generalized virial identity (A.9) together with (5.14) and (2.8) shows that

$$\begin{aligned}\big(W_R, \left|\psi(T_R)\right|^2\big) &< \big(W_R, |\psi_0|^2\big) + 2T_R \Im(\vec{w}_R\cdot\nabla\psi_0, \psi_0) - \varepsilon_0^2 T_R^2 \\ &\quad + \frac{1}{4}\varepsilon_0 T_R^2 - \frac{1}{2}\int_0^{T_R}\int_0^{t'} \big(\Delta(\operatorname{div}\vec{w}_R), \left|\psi(t'')\right|^2\big)\, dt''\, dt'.\end{aligned} \tag{5.15}$$

Here, using the estimate $\|\Delta(\operatorname{div}\vec{w}_R)\|_{L^\infty} \lesssim \frac{1}{R^2}$, the mass conservation law (1.6), and (5.9), we estimate the last term on the right-hand side above as follows:

$$-\frac{1}{2}\int_0^{T_R}\int_0^{t'}\big(\Delta(\operatorname{div}\vec{w}_R),\big|\psi(t'')\big|^2\big)\,dt''\,dt' \lesssim \int_0^{T_R}\int_0^{t'}\frac{1}{R^2}\big\|\psi(t'')\big\|_{L^2}^2\,dt''\,dt' \le \frac{1}{4}\varepsilon_0 T_R^2. \tag{5.16}$$

Combining (5.15) with (5.16) and the estimate $|\vec{w}_R(x)|^2 \le W_R(x)$, we obtain

$$\begin{aligned}\big(W_R,\big|\psi(T_R)\big|^2\big) &< \big(W_R,|\psi_0|^2\big) + 2T_R\Im(\vec{w}_R\cdot\nabla\psi_0,\psi_0) - \frac{1}{2}T_R^2\varepsilon_0\\ &= \big(W_R,|\psi_0|^2\big) - \frac{1}{2}\varepsilon_0\Big\{T_R - \frac{2}{\varepsilon_0}\Im(\vec{w}_R\cdot\nabla\psi_0,\psi_0)\Big\}^2\\ &\quad + \frac{2}{\varepsilon_0}\big|(\vec{w}_R\cdot\nabla\psi_0,\psi_0)\big|^2\\ &\le \big(W_R,|\psi_0|^2\big) + \frac{2}{\varepsilon_0}\|\nabla\psi_0\|_{L^2}^2\int_{\mathbb{R}^d}W_R(x)|\psi_0(x)|^2\,dx.\end{aligned} \tag{5.17}$$

This together with (5.11) concludes that

$$\big(W_R,\big|\psi(T_R)\big|^2\big) \le \Big(1+\frac{2}{\varepsilon_0}\|\nabla\psi_0\|_{L^2}^2\Big)\big(W_R,|\psi_0|^2\big) < R^2 m_0. \tag{5.18}$$

On the other hand, since $W_R(x) \ge R^2$ for $|x| \ge R$, we have

$$\begin{aligned}\int_{|x|\ge R}\big|\psi(x,T_R)\big|^2\,dx &= \frac{1}{R^2}\int_{|x|\ge R}R^2\big|\psi(x,T_R)\big|^2\,dx\\ &\le \frac{1}{R^2}\big(W_R,\big|\psi(T_R)\big|^2\big).\end{aligned} \tag{5.19}$$

Thus, we see from (5.18) and (5.19) that

$$\int_{|x|\ge R}\big|\psi(x,T_R)\big|^2\,dx < m_0, \tag{5.20}$$

which contradicts (5.13), so that $T_R = \infty$ and (5.7) holds.

Step 3. We complete the proof of Theorem 1.2. The definition of m_0 together with the mass conservation law (1.6), (5.1), and (5.7) shows that

$$-\frac{1}{4}\varepsilon_0 \le \mathcal{K}^R\big(\psi(t)\big) \tag{5.21}$$

for any $R > 0$ satisfying (5.8)–(5.11) and any $t \ge 0$. Combining the generalized virial identity (A.9) with (5.21), we obtain the following estimate as well as step 2 (see (5.15) and (5.16)):

$$\big(W_R,\big|\psi(t)\big|^2\big) \le \big(W_R,|\psi_0|^2\big) + 2t\Im(\vec{w}_R\cdot\nabla\psi_0,\psi_0) - \frac{1}{2}t^2\varepsilon_0 \tag{5.22}$$

for any $t \ge 0$. This inequality means that $(W_R,|\psi(t)|^2)$ becomes negative in a finite time, so that $T_{\max}^+$ must be finite. However, this contradicts $T_{\max}^+ = \infty$. Hence, (5.1) derives an absurd conclusion. Thus, (1.40) holds. □

Next, we shall give a proof of Proposition 1.1.

Proof of Proposition 1.1
We can prove (1.41) in a way similar to [22] (see also [10]). Hence, we omit it.

It remains to prove (1.42). We consider the term $\mathcal{K}^R$ in the generalized virial identity (A.9) (see (A.10) and Remark A.1). Integrating by parts, we have

$$(5.23)\qquad \begin{aligned} -\int_0^t \int_0^{t'} \mathcal{K}^R\bigl(\psi(t'')\bigr)\,dt''\,dt' &= -\int_0^t (t-t')\mathcal{K}^R\bigl(\psi(t')\bigr)\,dt' \\ &\lesssim \int_0^{T_{\max}} (T_{\max}-t)\bigl\|\psi(t)\bigr\|_{L^{p+1}(|x|\geq R)}^{p+1}\,dt, \end{aligned}$$

where we have used (A.16), (A.17), and (A.20). Here, we recall the following estimate (see [4], [19]):

$$(5.24)\qquad \int_0^{T_{\max}} (T_{\max}-t)\bigl\|\nabla\psi(t)\bigr\|_{L^2}^2\,dt < \infty.$$

This estimate together with the Hamiltonian conservation law (1.7) yields

$$(5.25)\qquad \int_0^{T_{\max}} (T_{\max}-t)\bigl\|\psi(t)\bigr\|_{L^{p+1}}^{p+1}\,dt < \infty.$$

Combining (5.23) with the Lebesgue dominated convergence theorem and (5.25), we find that

$$(5.26)\qquad -\int_0^t \int_0^{t'} \mathcal{K}^R\bigl(\psi(t'')\bigr)\,dt''\,dt' \leq 0, \quad \text{when } R\to+\infty.$$

Now, take any $m>0$. Then, for any sufficiently large $R>0$ depending on m, the generalized virial identity (A.9) together with the estimates $W_R(x)\geq R^2$ for $|x|\geq R$, $\|\vec{w}_R\|_{L^\infty}\lesssim R$, (2.8), $\|\Delta(\operatorname{div}\vec{w}_R)\|_{L^\infty}\lesssim \frac{1}{R^2}$, and (5.26) shows that

$$(5.27)\qquad \begin{aligned} \int_{|x|\geq R} \bigl|\psi(x,t)\bigr|^2\,dx &\leq \frac{1}{R^2}\bigl(W_R, \bigl|\psi(t)\bigr|^2\bigr) \\ &\leq \frac{1}{R^2}\bigl(W_R, |\psi_0|^2\bigr) + 2\frac{T_{\max}^+}{R}\|\psi_0\|_{L^2}\|\nabla\psi_0\|_{L^2} - \frac{\varepsilon_0}{R^2}(T_{\max}^+)^2 \\ &\quad + \frac{m}{2} + \frac{(T_{\max}^+)^2}{R^4}\|\psi_0\|_{L^2}^2. \end{aligned}$$

This together with Lemma A.1 yields the desired estimate (1.42). □

Finally, we shall give the proof of Proposition 1.2.

Proof of Proposition 1.2
Let ψ, $\{t_n\}$, $\{\lambda_n\}$, $\{\psi_n\}$, and $\{\Psi_n^{RN}\}$ be as in Proposition 1.2.

We can easily verify that

$$(5.28)\qquad \lim_{n\to\infty} \lambda_n = 0,$$

$$(5.29)\qquad 2i\frac{\partial\psi_n}{\partial t} + \Delta\psi_n + |\psi_n|^{p-1}\psi_n = 0 \quad \text{in } \mathbb{R}^d \times \Bigl(-\frac{T_{\max}^+ - t_n}{\lambda_n^2}, \frac{t_n}{\lambda_n^2}\Bigr],$$

(5.30) $$\|\psi_n(t)\|_{L^2} = \lambda_n^{-s_p}\|\psi_0\|_{L^2} \quad \text{for any } t \in \Big(-\frac{T^+_{\max} - t_n}{\lambda_n^2}, \frac{t_n}{\lambda_n^2}\Big],$$

(5.31) $$\mathcal{H}\big(\psi_n(t)\big) = \lambda_n^{2-2s_p}\mathcal{H}(\psi_0) \quad \text{for any } t \in \Big(-\frac{T^+_{\max} - t_n}{\lambda_n^2}, \frac{t_n}{\lambda_n^2}\Big],$$

(5.32) $$\sup_{t\in[0,t_n/\lambda_n^2]} \|\psi_n(t)\|_{L^{p+1}} = 1,$$

where we put $\psi_0 = \psi(0)$. Besides, we have

(5.33) $$\sup_{t\in[0,t_n/\lambda_n^2]} \|\nabla\psi_n(t)\|^2_{L^2} \le 1 \quad \text{for any sufficiently large } n \in \mathbb{N}.$$

Indeed, (5.31) and (5.32) lead us to

(5.34) $$\begin{aligned}\|\nabla\psi_n(t)\|^2_{L^2} &= \mathcal{H}\big(\psi_n(t)\big) + \frac{2}{p+1}\|\psi_n(t)\|^{p+1}_{L^{p+1}} \\ &\le \lambda_n^{2-2s_p}\mathcal{H}(\psi_0) + \frac{2}{p+1} \quad \text{for any } t \in \Big[0, \frac{t_n}{\lambda_n^2}\Big],\end{aligned}$$

which together with (5.28) immediately yields (5.33).

Moreover, we see from (5.28), (5.29), (5.32), and (5.33) that for any $T > 0$, there exists a subsequence of $\{\psi_n\}$ in $C([0,T]; H^1(\mathbb{R}^d))$ (still denoted by the same symbol) with the following properties:

(5.35) $$\sup_{t\in[0,T]} \|\psi_n(t)\|_{L^{p+1}} = 1 \quad \text{for any } n \in \mathbb{N},$$

(5.36) $$\sup_{t\in[0,T]} \|\nabla\psi_n(t)\|_{L^2} \le 1 \quad \text{for any } n \in \mathbb{N},$$

(5.37) $$2i\frac{\partial\psi_n}{\partial t} + \Delta\psi_n + |\psi_n|^{p-1}\psi_n = 0 \quad \text{in } \mathbb{R}^d \times [0,T].$$

For such a subsequence $\{\psi_n\}$, we define Φ_n^{RN} by

(5.38) $$\Phi_n^{RN}(x,t) = \psi_n(x,t) - e^{(i/2)t\Delta}\psi_n(x,0).$$

Here, it is worthwhile noting that

(5.39) $$\Phi_n^{RN} \in C\big([0,T]; H^1(\mathbb{R}^d)\big) \quad \text{for any } n \in \mathbb{N},$$

(5.40) $$\Phi_n^{RN}(t) = \frac{i}{2}\int_0^t e^{(i/2)(t-t')\Delta}|\psi_n(t')|^{p-1}\psi_n(t')\,dt' \quad \text{for any } t \in [0,T] \text{ and } n \in \mathbb{N}.$$

We shall show that

(5.41) $$\sup_{n\in\mathbb{N}} \|\Phi_n^{RN}\|_{L^\infty([0,T];H^1)} \le C_T$$

for some constant $C_T > 0$ depending only on d, p, and T. Applying the Strichartz estimate to the formula (5.40), and using (5.35), we obtain the following two estimates:

$$
(5.42)\quad
\begin{aligned}
&\|\Phi_n^{RN}\|_{L^\infty([0,T];L^2)}\\
&\quad\lesssim \big\||\psi_n|^{p-1}\psi_n\big\|_{L^{\frac{4(p+1)}{4(p+1)-d(p-1)}}([0,T];L^{(p+1)/p})}\\
&\quad\le T^{1-(d(p-1))/(4(p+1))}\|\psi_n\|^p_{L^\infty([0,T];L^{p+1})}\\
&\quad\le T^{1-(d(p-1))/(4(p+1))},
\end{aligned}
$$

$$
(5.43)\quad
\begin{aligned}
&\|\nabla\Phi_n^{RN}\|_{L^\infty([0,T];L^2)}\\
&\quad\lesssim \big\|\nabla\big(|\psi_n|^{p-1}\psi_n\big)\big\|_{L^{4(p+1)/(4(p+1)-d(p-1))}([0,T];L^{(p+1)/p})}\\
&\quad\le T^{1-(d(p-1))/(2(p+1))}\|\psi_n\|^{p-1}_{L^\infty([0,T];L^{p+1})}\|\nabla\psi_n\|_{L^{(4(p+1))/(d(p-1))}([0,T];L^{p+1})}\\
&\quad\le T^{1-(d(p-1))/(2(p+1))}\|\nabla\psi_n\|_{L^{(4(p+1))/(d(p-1))}([0,T];L^{p+1})},
\end{aligned}
$$

where the implicit constants depend only on d and p. Therefore, for the desired estimate (5.41), it suffices to show that

$$
(5.44)\qquad \sup_{n\in\mathbb{N}}\|\nabla\psi_n\|_{L^{(4(p+1))/(d(p-1))}([0,T];L^{p+1})}\le D_T
$$

for some constant $D_T>0$ depending only on d, p, and T. Here, note that the pair $(p+1,4(p+1)/d(p-1))$ is admissible. In order to prove (5.44), we introduce an admissible pair (q,r) with $q=p+2$ if $d=1,2$ and $q=(1/2)(p+1+2^*)$ if $d\ge 3$, so that $p+1<q<2^*$. Then, it follows from the integral equation for ψ_n and the Strichartz estimate that

$$
(5.45)\quad
\begin{aligned}
&\|\nabla\psi_n\|_{L^r([0,T];L^q)}\\
&\quad\lesssim \big\|\nabla\psi_n(0)\big\|_{L^2}+\|\psi_n\|^{p-1}_{L^{(2(p+1)(p-1))/(d+2-(d-2)p}([0,T];L^{p+1})}\\
&\qquad\times\|\nabla\psi_n\|_{L^{(4(p+1))/(d(p-1))}([0,T];L^{p+1})}\\
&\quad\le \big\|\nabla\psi_n(0)\big\|_{L^2}+T^{\frac{d+2-(d-2)p}{2(p+1)}}\|\psi_n\|^{p-1}_{L^\infty([0,T];L^{p+1})}\\
&\qquad\times\|\nabla\psi_n\|^{1-\frac{q(p-1)}{(q-2)(p+1)}}_{L^\infty([0,T];L^2)}\|\nabla\psi_n\|^{\frac{q(p-1)}{(q-2)(p+1)}}_{L^r([0,T];L^q)}.
\end{aligned}
$$

Combining (5.45) with (5.35) and (5.36), we obtain

$$
(5.46)\qquad \|\nabla\psi_n\|_{L^r([0,T];L^q)}\lesssim 1+T^{\frac{d+2-(d-2)p}{2(p+1)}}\|\nabla\psi_n\|^{\frac{q(p-1)}{(q-2)(p+1)}}_{L^r([0,T];L^q)}.
$$

Since $0<q(p-1)/((q-2)(p+1))<1$, this estimate together with the Young inequality yields

$$
(5.47)\qquad \|\nabla\psi_n\|_{L^r([0,T];L^q)}\lesssim 1+T^{\frac{(q-2)\{d+2-(d-2)p\}}{4\{q-(p+1)\}}}.
$$

Hence, interpolating (5.36) and (5.47), we obtain (5.44), so that (5.41) holds.

Next, we shall show that $\{\Phi_n^{RN}\}$ is an equicontinuous sequence in $C([0,T];L^q(\mathbb{R}^d))$ for any $q\in[2,2^*)$. Differentiating the both sides of (5.40), we obtain

$$
(5.48)\qquad \partial_t\Phi_n^{RN}(t)=\frac{i}{2}\big|\psi_n(t)\big|^{p-1}\psi_n(t)+\frac{i}{2}\Delta\Phi_n^{RN}(t)\quad\text{in } H^{-1}(\mathbb{R}^d).
$$

This formula (5.48) and the Hölder inequality show that

$$
\begin{aligned}
&\left\|\Phi_n^{RN}(t)-\Phi_n^{RN}(s)\right\|_{L^2}^2 \\
&=\int_s^t \frac{d}{dt'}\left\|\Phi_n^{RN}(t')-\Phi_n^{RN}(s)\right\|_{L^2}^2\,dt' \\
&=2\Re\int_s^t\int_{\mathbb{R}^d}\overline{\partial_t\Phi_n^{RN}(x,t')}\left\{\Phi_n^{RN}(x,t')-\Phi_n^{RN}(x,s)\right\}dx\,dt' \\
&\lesssim |t-s|\|\psi_n\|^p_{L^\infty([0,T];L^{p+1})}\|\Phi_n^{RN}\|_{L^\infty([0,T];L^{p+1})} \\
&\quad+|t-s|\|\nabla\Phi_n^{RN}\|^2_{L^\infty([0,T];L^2)}.
\end{aligned}
\tag{5.49}
$$

Combining this estimate with (5.35) and (5.41), we obtain

$$
\left\|\Phi_n^{RN}(t)-\Phi_n^{RN}(s)\right\|_{L^2}^2\lesssim|t-s|\quad\text{for any } s,t\in[0,T],
\tag{5.50}
$$

where the implicit constant depends only on d, q, and T. Moreover, the Gagliardo–Nirenberg inequality together with (5.41) and (5.50) shows that $\{\Phi_n^{RN}\}$ is an equicontinuous sequence in $C([0,T];L^q(\mathbb{R}^d))$ for any $q\in[2,2^*)$.

We see from the Ascoli–Arzelá theorem together with (5.41) and the equicontinuity that there exist a subsequence of $\{\Phi_n^{RN}\}$ (still denoted by the same symbol) and a nontrivial function $\Phi\in L^\infty([0,\infty);H^1(\mathbb{R}^d))$ such that

$$
\lim_{n\to\infty}\Phi_n^{RN}=\Phi\quad\text{in } C\big([0,T];\text{weak-}H^1(\mathbb{R}^d)\big),
\tag{5.51}
$$

$$
\lim_{n\to\infty}\Phi_n^{RN}=\Phi\quad\text{strongly in } C\big([0,T];L^q_{\mathrm{loc}}(\mathbb{R}^d)\big)\text{ for any } q\in[2,2^*).
\tag{5.52}
$$

It remains to prove (1.50). We see from (5.35) that there exists $F\in L^\infty([0,\infty);L^{(p+1)/p}(\mathbb{R}^d))$ such that

$$
\lim_{n\to\infty}|\psi_n|^{p-1}\psi_n=F\quad\text{weakly* in } L^\infty\big([0,T];L^{\frac{p+1}{p}}(\mathbb{R}^d)\big).
\tag{5.53}
$$

Then, it follows from (5.48) and (5.51) that

$$
2i\frac{\partial\Phi}{\partial t}+\Delta\Phi+F=0.
\tag{5.54}
$$

Here, if F were trivial, then Φ is so, since $\Phi(0)=\lim_{n\to\infty}\Phi_n^{RN}(0)=0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Therefore, F is nontrivial. $\square$

Appendix: Generalized virial identity

The proofs of Theorem 1.1, Theorem 1.2, and Proposition 1.1 are based on a generalization of the virial identity. To state it, we first introduce a positive function w in $W^{3,\infty}([0,\infty))$, which is a variant of the function in [21]–[22]:

$$
w(r)=\begin{cases}
r & \text{if } 0\le r<1,\\
r-(r-1)^{(d/2)(p-1)+1} & \text{if } 1\le r\le 1+\left(\frac{2}{d(p-1)+2}\right)^{2/(d(p-1))},\\
\text{smooth and } w'\le 0 & \text{if } 1+\left(\frac{2}{d(p-1)+2}\right)^{2/(d(p-1))}<r<2,\\
0 & \text{if } 2\le r.
\end{cases}
\tag{A.1}
$$

Since w is determined by d and p only, we may assume that

$$\|w\|_{W^{3,\infty}} \lesssim 1. \tag{A.2}$$

Using this w, we define

$$\vec{w}_R(x) = \big(\vec{w}_R^1(x), \ldots, \vec{w}_R^d(x)\big) := \frac{x}{|x|} R w\Big(\frac{|x|}{R}\Big) \tag{A.3}$$

and

$$W_R(x) := 2R \int_0^{|x|} w\Big(\frac{r}{R}\Big)\, dr \tag{A.4}$$

for $R > 0$ and $x \in \mathbb{R}^d$.

LEMMA A.1

Assume $d \geq 1$ and $2 + (4/d) \leq p + 1 < 2^*$. Then, for any $m > 0$, $C > 0$ and $f \in L^2(\mathbb{R}^d)$, there exists $R_1 > 0$ such that

$$\frac{C}{R^2}\big(W_R, |f|^2\big) < m \quad \textit{for any } R \geq R_1. \tag{A.5}$$

Proof of Lemma A.1

For any $m > 0$, $C > 0$, and $f \in L^2(\mathbb{R}^d)$, we can take $R_1' > 0$ such that

$$\int_{|x| \geq R_1'} \big|f(x)\big|^2\, dx < \frac{m}{16C}. \tag{A.6}$$

Since $\|W_R\|_{L^\infty} \leq 8R^2$, we see from (A.6) that

$$\begin{aligned} \frac{C}{R^2} \int_{|x| \geq R_1'} W_R(x) \big|f(x)\big|^2\, dx &\leq 8C \int_{|x| \geq R_1'} \big|f(x)\big|^2\, dx \\ &< \frac{m}{2} \quad \text{for any } R > 0. \end{aligned} \tag{A.7}$$

On the other hand, we have by the definition of W_R (see (A.4)) that

$$\begin{aligned} \frac{C}{R^2} \int_{|x| < R_1'} W_R(x) \big|f(x)\big|^2\, dx &\lesssim C \frac{2R_1'}{R} \|f\|_{L^2}^2 \\ &\ll \frac{m}{2} \quad \text{for any } R \gg \frac{4CR_1'\|f\|_{L^2}^2}{m}. \end{aligned} \tag{A.8}$$

Combining (A.7) and (A.8), we obtain the desired result. □

We introduce a generalized virial identity (cf. [21]–[22]):

$$\begin{aligned} \big(W_R, \big|\psi(t)\big|^2\big) &= \big(W_R, |\psi_0|^2\big) + 2t\Im(\vec{w}_R \cdot \nabla\psi_0, \psi_0) + 2\int_0^t \int_0^{t'} \mathcal{K}\big(\psi(t'')\big)\, dt''\, dt' \\ &\quad - 2\int_0^t \int_0^{t'} \mathcal{K}^R\big(\psi(t'')\big)\, dt''\, dt' \\ &\quad - \frac{1}{2}\int_0^t \int_0^{t'} \big(\Delta(\operatorname{div} \vec{w}_R), \big|\psi(t'')\big|^2\big)\, dt''\, dt' \quad \text{for } R > 0. \end{aligned} \tag{A.9}$$

Here, $\mathcal{K}^R$ is defined by

$$\mathcal{K}^R(f) = \int_{\mathbb{R}^d} \rho_1(x)\big|\nabla f(x)\big|^2 + \rho_2(x)\Big|\frac{x}{|x|}\cdot\nabla f(x)\Big|^2 - \rho_3(x)\big|f(x)\big|^{p+1}\,dx, \tag{A.10}$$

where

$$\rho_1(x) := 1 - \frac{R}{|x|}w\Big(\frac{|x|}{R}\Big), \tag{A.11}$$

$$\rho_2(x) := \frac{R}{|x|}w\Big(\frac{|x|}{R}\Big) - w'\Big(\frac{|x|}{R}\Big), \tag{A.12}$$

$$\rho_3(x) := \frac{p-1}{2(p+1)}\Big\{d - w'\Big(\frac{|x|}{R}\Big) - \frac{d-1}{|x|}Rw\Big(\frac{|x|}{R}\Big)\Big\}. \tag{A.13}$$

REMARK A.1

If $d=1$ or ψ is radially symmetric, then we have

$$\mathcal{K}^R(\psi) = \int_{\mathbb{R}^d} \rho_0(x)\big|\nabla\psi(x)\big|^2 - \rho_3(x)\big|\psi(x)\big|^{p+1}\,dx, \tag{A.14}$$

where

$$\rho_0(x) := 1 - w'\Big(\frac{|x|}{R}\Big) = \rho_1(x) + \rho_2(x). \tag{A.15}$$

In the next lemma, we give several properties of the weight functions ρ_1, ρ_2, ρ_3, and ρ_0.

LEMMA A.2

Assume $d \ge 1$ and $2 + (4/d) \le p+1 < 2^*$. Then, for any $R > 0$, we have the following:

$$\operatorname{supp}\rho_j = \{x \in \mathbb{R}^d \mid |x| \ge R\} \quad \textit{for any } R>0 \textit{ and } j = 0,1,2,3, \tag{A.16}$$

$$\inf_{x\in\mathbb{R}^d}\rho_j(x) \ge 0 \quad \textit{for any } R>0 \textit{ and } j=0,1,2,3, \tag{A.17}$$

$$\rho_0(x) = 1 \quad \textit{if } |x| \ge 2R, \tag{A.18}$$

$$\rho_3(x) = \frac{d(p-1)}{2(p+1)} \quad \textit{if } |x| \ge 2R, \tag{A.19}$$

$$\|\rho_j\|_{L^\infty} \lesssim 1 \quad \textit{for any } j = 0,1,2,3. \tag{A.20}$$

Proof of Lemma A.2

We easily verify this lemma from the definitions of ρ_1, ρ_2, ρ_3, and ρ_0. □

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