

# Counterexamples to Kodaira's vanishing and Yau's inequality in positive characteristics

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*To the memory of Professor Masaki Maruyama*

**Abstract** We generalize Tango's theorem on the Frobenius map of the first cohomology groups to higher-dimensional algebraic varieties in characteristic  $p > 0$ . As an application we construct counterexamples to Kodaira vanishing in higher dimension and prove the Ramanujam-type vanishing on surfaces which are not of general type when  $p \geq 5$ .

Let  $X$  be a smooth complete algebraic variety over an algebraically closed field of positive characteristic  $p > 0$ , and let  $D$  be an effective divisor on  $X$ . In this article we study the kernel of the Frobenius map

$$(1) \quad F^* : H^1(X, \mathcal{O}_X(-D)) \rightarrow H^1(X, \mathcal{O}_X(-pD))$$

of the first cohomology groups of line bundles.

Tango [T1] described the kernel of  $F^*$  in terms of the exact differentials in the case of curves. First we generalize this result to varieties of arbitrary dimension, that is, we prove the following.

## THEOREM 1

*The kernel of the Frobenius map (1) is isomorphic to the vector space*

$$\{df \in \Omega_{Q(X)} \mid f \in Q(X), (df) \geq pD\},$$

*where  $Q(X)$  is the function field of  $X$  and  $(\omega) \geq pD$  means that a rational differential  $\omega \in \Omega_{Q(X)}$  belongs to  $\Gamma(X, \Omega_X(-pD))$ .*

Using this description and generalizing Raynaud's [Ra] method, we construct pathological varieties of higher dimension which are similar to his surfaces.

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## THEOREM 2

Let  $p$  be a prime number, and let  $n \geq 2$  be an integer. Then there exist an  $n$ -dimensional smooth projective variety  $X$  of characteristic  $p$  and an ample line bundle  $L$  such that

- (a)  $H^1(X, L^{-1}) \neq 0$ ;
- (b) the canonical divisor class  $K_X$  is ample, and the intersection number  $(c_i(X) \cdot K_X^{n-i})$  is negative for every  $i \geq 2$ ; and
- (c) there is a finite cover  $G$  of  $X$  and a sequence of morphisms

$$G = G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_2 \rightarrow G_1$$

such that  $G_{i+1} \rightarrow G_i$  is a  $\mathbb{P}^1$ -bundle for every  $i = 1, \dots, n-1$  and that  $G_1$  is a nonsingular curve. The Euler characteristic  $e(X)(:= \deg c_n(X))$  of  $X$  is equal to  $e(G) = 2^{n-1}e(G_1)$ .

Here  $c_i(X)$  is the  $i$ th Chern class of  $X$ .

When  $p = 2, 3$ , we obtain similar varieties  $X$  with quasi-elliptic fibrations  $X \rightarrow Y$ . In this case, the canonical classes  $K_X$  are the pullbacks of ample divisor classes on  $Y$ . By property (b) and Yau's [Y1] and [Y2] inequality or by (a) and [DI], we have the following.

## COROLLARY

The algebraic variety  $X$  in Theorem 2 is not liftable to characteristic zero.

Throughout this article R.V. (Ramanujam vanishing) on an algebraic surface  $X$  means the vanishing of  $H^1(X, L^{-1})$  for all nef and big line bundles on  $X$ . Conversely to the above counterexample, using Theorem 1 and [LM], we prove the following.

## THEOREM 3

In the case where  $X$  is of dimension two, we have the following.

- (a) Assume that  $X$  is not of general type and that the Iitaka fibration  $X \rightarrow C$  is not quasi-elliptic when the Kodaira dimension  $\kappa(X)$  is 1 and  $p = 2, 3$ . Then R.V. holds on  $X$ .
- (b) If R.V. does not hold on  $X$ , then there exist a birational morphism  $X' \rightarrow X$  and a morphism  $g: X' \rightarrow C$  onto a smooth algebraic curve  $C$  such that every fiber  $F$  of  $g$  is connected and singular. Furthermore, the cotangent sheaf  $\Omega_F$  has nonzero torsion.

Our counterexamples  $X$  in dimension two are sandwiched between two  $\mathbb{P}^1$ -bundles, and the general fibers  $F$  in Theorem 3(b) are rational for them. A curve of higher (geometric) genus appears as such a fiber  $F$  if we take a sufficiently general separable cover  $\pi: \tilde{X} \rightarrow X$  with  $(\deg \pi, p) = 1$ . Then R.V. does not hold on  $\tilde{X}$  either since  $L^{-1}$  is a direct summand of  $\pi_*\pi^*L^{-1}$ .

## CONVENTION

In the following we assume the characteristic  $p$  is positive, and by K.V. (Kodaira vanishing) we mean the vanishing of the first cohomology group  $H^1(X, L^{-1})$  for all ample line bundles  $L$  on  $X$ .

**1. Tango's theorem**

The feature of positive characteristic is the existence of the Frobenius morphisms  $F : X \rightarrow X$  and the Frobenius maps. Let  $L$  be a line bundle on  $X$ . The Frobenius morphism induces the Frobenius map

$$(2) \quad F^* : H^1(X, L^{-1}) \rightarrow H^1(X, L^{-p})$$

between the first cohomology groups. When  $X$  is normal and  $\dim X \geq 2$ , we have the following.

## LEMMA 1.1 (ENRIQUES–SEVERI–ZARISKI)

The vanishing  $H^1(X, L^{-m}) = 0$  holds if  $L$  is ample and  $m$  is sufficiently large.

Therefore, by the sequence

$$H^1(X, L^{-1}) \rightarrow H^1(X, L^{-p}) \rightarrow H^1(X, L^{-p^2}) \rightarrow \dots$$

of Frobenius maps, K.V. holds on  $X$  if and only if the following holds:

(\*)  $F^* : H^1(X, L^{-1}) \rightarrow H^1(X, L^{-p})$  is injective for every ample line bundle  $L$  on  $X$ .

**1.1. Tango–Raynaud curve**

The statement (\*) makes sense even when  $\dim X = 1$ . The following is fundamental for (\*) in this case.

## THEOREM 1.2 (TANGO [T1, LEMMA 12])

Let  $D$  be an effective divisor on a smooth algebraic curve  $X$ . Then the kernel of the Frobenius map (1) is isomorphic to the space of exact differentials  $df$  of rational functions  $f$  on  $X$  with  $(df) \geq pD$ .

The following example, which was found by Raynaud [Ra] in the case  $e = 1$ , shows that (\*) does not hold when  $\dim X = 1$ .

## EXAMPLE 1.3

Let  $P(Y)$  be a polynomial of degree  $e$  in one variable  $Y$ , and let  $C \subset \mathbb{P}^2$  be the plane curve of degree  $pe$  defined by

$$(3) \quad P(Y^p) - Y = Z^{pe-1},$$

where  $(Y, Z)$  is a system of inhomogeneous coordinates of  $\mathbb{P}^2$ . It is easy to check that  $C$  is smooth and has exactly one point  $\infty$  on the line of infinity. By the

relation

$$-dY = -Z^{pe-2}dZ$$

between the differentials  $dY$  and  $dZ$ ,  $\Omega_C$  is generated by  $dZ$  over  $C \cap \mathbb{A}^2$ . In other words,  $dZ$  has no poles or zeros over  $C \cap \mathbb{A}^2$ . Since  $\deg \Omega_C = 2g(C) - 2 = pe(pe - 3)$ , we have  $(dZ) = pe(pe - 3)(\infty)$ . Therefore, by the above theorem of Tango (Theorem 1.2), the Frobenius map (1) is not injective for the divisor  $D = e(pe - 3)(\infty)$ .

A curve  $C$  of genus  $\geq 2$  is called a *Tango-Raynaud curve* if  $C$  satisfies the following mutually equivalent conditions:

- (a) there exists a line bundle  $L$  on  $C$  such that  $L^p \simeq \Omega_C$  and that the Frobenius map (2) is not injective, and
- (b) there exists a rational function  $f$  on  $C$  such that  $df \neq 0$  and that the divisor  $(df)$  is divisible by  $p$ .

The curve  $C$  in the Example 1.3 is a Tango-Raynaud curve.

## 1.2. Higher-dimensional generalization

Following [T1] we denote the cokernel of the natural ( $p$ th power) homomorphism  $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$  by  $\mathcal{B}_X$ . For a Cartier divisor  $D$  on  $X$  we have the exact sequence

$$(4) \quad 0 \rightarrow \mathcal{O}_X(-D) \rightarrow F_*(\mathcal{O}_X(-pD)) \rightarrow \mathcal{B}_X(-D) \rightarrow 0$$

and the associated long exact sequence

$$(5) \quad \begin{aligned} 0 \rightarrow H^0(\mathcal{O}_X(-D)) &\xrightarrow{F^*} H^0(\mathcal{O}_X(-pD)) \rightarrow H^0(\mathcal{B}_X(-D)) \\ &\xrightarrow{\delta} H^1(\mathcal{O}_X(-D)) \xrightarrow{F^*} H^1(\mathcal{O}_X(-pD)) \rightarrow \cdots \end{aligned}$$

If  $D$  is effective, then  $F^* : H^0(\mathcal{O}_X(-D)) \rightarrow H^0(\mathcal{O}_X(-pD))$  is surjective. Hence we have the following.

### LEMMA 1.4

If  $D$  is effective, then the coboundary map  $\delta$  of (5) induces the isomorphism

$$(6) \quad \text{Ker}[F^* : H^1(\mathcal{O}_X(-D)) \rightarrow H^1(\mathcal{O}_X(-pD))] \simeq H^0(\mathcal{B}_X(-D)).$$

Assume that  $X$  is normal, and consider the direct image of the derivation  $d : \mathcal{O}_X \rightarrow \Omega_X$  by  $F$ . By  $F_*d$ ,  $\mathcal{B}_X$  is regarded as a subsheaf of  $F_*\Omega_X$ . Let  $\Omega_{Q(X)}$  be the  $Q(X)$ -vector space of differentials. We denote the constant sheaf associated with  $Q(X)$  or  $\Omega_{Q(X)}$  on  $X$  by the same symbol and consider the intersection  $dQ(X) \cap \Omega_X$  in the constant sheaf  $\Omega_{Q(X)}$ . Then, more precisely,  $\mathcal{B}_X$  is contained in  $F_*(dQ(X) \cap \Omega_X)$ . We also have  $\mathcal{B}_X(-D) \hookrightarrow F_*(dQ(X) \cap \Omega_X(-pD))$ . Therefore, by the exact sequence (5), we have the following.

## PROPOSITION 1.5

If  $X$  is normal, then the kernel of the Frobenius map of  $H^1(\mathcal{O}_X(-D))$  is isomorphic to a subspace of the vector space

$$\{df \in \Omega_{Q(X)} \mid f \in Q(X), (df) \geq pD\}.$$

## COROLLARY

If  $X$  is normal and  $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(pD), \Omega_X) = 0$ , then the Frobenius map of  $H^1(\mathcal{O}_X(-D))$  is injective.

When  $X$  is smooth,  $\mathcal{B}_X = F_*(dQ(X) \cap \Omega_X)$  holds, by the existence of a  $p$ -basis. Hence  $\mathcal{B}_X(-D) = F_*(dQ(X) \cap \Omega_X(-pD))$  holds for a Cartier divisor  $D$ , and we have Theorem 1.

1.3. Purely inseparable covering in an  $\mathbb{A}^1$ -bundle

When a vector bundle  $E$  on  $X$  is given, we have the relative Frobenius morphism  $\mathbb{P}(E) \rightarrow \mathbb{P}(E^{(p)})$  over  $X$ . We denote this morphism by  $\varphi$ . We consider the special case where  $E$  is an extension of two line bundles

$$(**) \quad 0 \rightarrow \mathcal{O}_X(-D) \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0.$$

Then  $E^{(p)}$  is also an extension of line bundles

$$(***) \quad 0 \rightarrow \mathcal{O}_X(-pD) \rightarrow E^{(p)} \rightarrow \mathcal{O}_X \rightarrow 0.$$

Let  $F_\infty \subset \mathbb{P}(E)$  be the section corresponding to the exact sequence (\*\*). Then  $\mathbb{P}(E) \setminus F_\infty$  is an  $\mathbb{A}^1$ -bundle, and  $\mathbb{P}(E)$  is its compactification. Assume that the extension class  $\alpha$  of (\*\*) belongs to the kernel of the Frobenius map (1). Then (\*\*\*) have a splitting, which yields a section  $G'$  of  $\mathbb{P}(E^{(p)})$  disjoint from  $F'_\infty := \varphi(F_\infty)$ .

## DEFINITION 1.6

Let  $G = G(X, D, \alpha)$  be the (scheme-theoretic) inverse image of  $G'$  by the relative Frobenius morphism  $\varphi$ . We denote the restriction of the projection  $\bar{g}: \mathbb{P}(E) \rightarrow X$  to  $G$  by  $\tau$ .

By construction  $G$  is embedded in the  $\mathbb{A}^1$ -bundle  $\mathbb{P}(E) \setminus F_\infty$ . When  $\alpha$  corresponds to  $\eta = df \in H^0(\mathcal{B}_X(-D))$  in the way of Theorem 1, that is, when  $\alpha = \delta(\eta)$ , we denote  $G$  by  $G(X, D, \eta)$  also. The morphism  $\tau: G \rightarrow X$  is flat, finite of degree  $p$ , and ramifies everywhere. If  $X$  is normal, then the local equation of  $G$  in  $\mathbb{P}(E)$  is either irreducible or a  $p$ th power. Therefore, if  $X$  is normal and if  $\eta \neq 0$ , then  $G$  is a variety and its function field is a purely inseparable extension of  $Q(X)$ . By construction we have the following linear equivalence:

$$(7) \quad G - pF_\infty \sim -\bar{g}^*(pD).$$

Now we can state a criterion for  $G$  to be smooth.

PROPOSITION 1.7

Assume that  $X$  is smooth. Then  $G = G(X, D, \eta)$  is smooth if and only if  $\eta \in H^0(\mathcal{B}_X(-D))$  is nowhere vanishing. If these equivalent conditions are satisfied, then the natural sequence

$$(8) \quad 0 \rightarrow \tau^* \mathcal{O}_X(pD) \xrightarrow{\times \eta} \tau^* \Omega_X \xrightarrow{\tau^*} \Omega_G \rightarrow \Omega_{G/X} \rightarrow 0$$

is exact and  $\Omega_{G/X}$  is isomorphic to  $\tau^* \mathcal{O}_X(D)$ . In particular the image of  $\tau^*$  is a vector bundle of rank  $n - 1$ .

*Proof*

Assume that  $D$  is given by a system  $\{g_i\}_{i \in I}$  of local equations for an open covering  $\{U_i\}_{i \in I}$  of  $X$ . We may assume that  $\eta$  is represented by a 0 cochain  $\{b_i\}_{i \in I}$  which satisfies

$$b_i = g_i^p c_i \in \Gamma(U_i, \mathcal{O}_X(-pD)), \quad b_j - b_i = a_{ij}^p \in \Gamma(U_i \cap U_j, \mathcal{O}_X(-pD)),$$

for some  $c_i \in \Gamma(U_i, \mathcal{O}_X)$  and  $a_{ij} \in \Gamma(U_i, \mathcal{O}_X(-D))$ . Then  $\{a_{ij}\}_{i,j \in I}$  is a 1-cocycle which represents  $\alpha = \delta(\eta)$ , and the vector bundle  $E$  in (\*\*\*) is defined by the 1-cocycle  $\left\{ \begin{pmatrix} g_i g_j^{-1} & 0 \\ a_{ij} g_j^{-1} & 1 \end{pmatrix} \right\}$  with coefficients in  $\text{GL}(2, \mathcal{O}_X)$ . Since

$$(c_i 1) \begin{pmatrix} g_i^p g_j^{-p} & 0 \\ a_{ij}^p g_j^{-p} & 1 \end{pmatrix} = (c_j 1)$$

holds, the 0 cocycle  $\{(c_i 1)\}_{i \in I}$  defines a splitting  $\mathcal{O}_X \rightarrow E^{(p)}$  of the extension (\*\*\*)

On each open set  $U_i$ ,  $G \subset \mathbb{P}(E) \setminus F_\infty$  is defined by the equation  $S_i^p = c_i$ , where  $S_i$  is a fiber coordinate of  $U_i \times \mathbb{A}^1$ . On their intersection,  $S_i^p = c_i$  (over  $U_i$ ) and  $S_j^p = c_j$  (over  $U_j$ ) are patched by the affine transformation  $g_j S_j = g_i S_i + a_{ij}$ . Let  $\mathcal{O}_X(pD) \xrightarrow{\times \eta} \Omega_X$  be the multiplication homomorphism by  $\eta$ . Since  $\tau^* dc_i = 0$ , we have the complex (8).

Let  $x$  be a point in  $U_i$ . If  $dc_i$  vanishes at  $x$ , then  $S_i^p = c_i$  is singular at  $x$ . Assume that  $dc_i$  is nonzero at  $x$ . Then  $G$  is smooth at  $\tau^{-1}(x)$ . Moreover, the cotangent space of  $X$  at  $x$  has a basis of the form  $\{\gamma_1, \dots, \gamma_{n-1}, dc_i\}$ , and  $\{\tau^* \gamma_1, \dots, \tau^* \gamma_{n-1}, dS_i\}$  is a basis of the cotangent space of  $G$  at  $\tau^{-1}(x)$ . Therefore, the kernel of  $\tau^*$  is spanned by  $dc_i$  and the cokernel by  $dS_i$ . Hence (8) is exact, and the image of  $\tau^*$  is a vector bundle of rank  $n - 1$ . Since  $g_j dS_j = g_i dS_i$  holds in  $\Omega_{G/X}$ ,  $\Omega_{G/X}$  is isomorphic to  $\tau^* \mathcal{O}_X(D)$ . □

COROLLARY 1.8

We have  $\tau^* K_X \sim K_G + (p - 1)\tau^* D$ .

We define the Euler number  $e(X)$  of  $X$  by the top Chern number  $\text{deg } c_{\text{top}}(X)$ .

COROLLARY 1.9

We have  $\tau^* c_n(X) \sim pc_n(G)$ , where  $n = \dim X$ . In particular, we have  $e(X) = e(G)$ .

*Proof*

Let  $B$  be the image of  $\tau^*$ . Then by the Proposition 1.7 we have  $c_n(G) \sim \tau^*(-D) \cdot c_{n-1}(B^\vee)$  and  $\tau^*c_n(X) \sim \tau^*(-pD) \cdot c_{n-1}(B^\vee)$ . Hence  $\tau^*c_n(X)$  is rationally equivalent to  $pc_n(G)$ . The second half of the corollary is obtained by taking the degree of these two 0 cycles.  $\square$

If  $X$  is a Tango–Raynaud curve, then  $\tau : G \rightarrow X$  is nothing but the Frobenius morphism of  $X$ .

REMARK 1.10

Purely inseparable coverings such as  $G(X, D, \alpha)$  in Definition 1.6 have been studied by many authors, and by now they are more or less well known. It is worthwhile to mention that Kollár [K, Chapter II.6] proved the vanishing of  $H^1$  in characteristic zero by this using this construction and mod  $p$  reduction.

The morphism  $G \rightarrow X$  in Proposition 1.7 is a special case of a quotient by 1-foliation, and the exact sequence (8) is described in Ekedahl [E1].

**2. Construction of counterexamples**

By a *Tango–Raynaud triple*, or a *TR triple* for short, we mean a triple  $(X, D, f)$  of a smooth variety  $X$ , a divisor  $D$  on  $X$ , and a rational function  $f \in Q(X)$  with  $(df) \geq pD$ . In this section, we shall construct a new TR triple  $(\tilde{X}, \tilde{D}, \tilde{f})$  from  $(X, D, f)$  under a certain divisibility assumption.

**2.1. New triple of higher dimension**

Let  $(X, D, f)$  be a TR triple. We assume that  $D = kD'$  for a divisor  $D'$  and an integer  $k \geq 2$  which is prime to  $p$ , and construct a new TR-triple  $(\tilde{X}, \tilde{D}, \tilde{f})$  with  $\dim \tilde{X} = \dim X + 1$ .

Under the same setting as in the proof of Proposition 1.7, we choose and fix a nonempty open subset  $U \subset X$  among  $U_i$ ’s,  $i \in I$ . We shrink  $U$  and replace  $f$  with  $f'$  satisfying  $df' = df$  if necessary, so that  $f$  is regular over  $U$ . We take a fiber coordinate  $S$  of  $\mathbb{P}(E) \rightarrow X$  over  $U$  such that the section of infinity  $F_\infty$  is defined by  $S = \infty$  and  $G = G(X, D, df)$  is defined by  $S^p - f = 0$ . Our new variety  $\tilde{X}$  is a model of the function field  $Q(X)(S, \sqrt[k]{S^p - f})$ . We construct it in two steps. Let  $m$  be a positive integer such that  $p + m$  is divisible by  $k$ . By the linear equivalence (7), we have

$$G + mF_\infty \sim k \left( \frac{p+m}{k} F_\infty - \tilde{g}^*(pD') \right),$$

that is,  $G + mF_\infty$  is the zero locus of a global section of  $M^{-k}$ , where  $M = \mathcal{O}_{\mathbb{P}}(-((p+m)/k)F_\infty + \tilde{g}^*(pD'))$ . First in the usual way we take the global  $k$ -fold cyclic covering

$$(9) \quad \text{Spec} \left( \bigoplus_{i=0}^{k-1} M^i \right) \rightarrow \mathbb{P}(E)$$

with algebra structure given by  $M^k \simeq \mathcal{O}_{\mathbb{P}(E)}(-G - mF_\infty) \hookrightarrow \mathcal{O}_{\mathbb{P}(E)}$ . Then we take the *relative normalization* of this covering over a neighborhood of  $F_\infty$ .

DEFINITION 2.1

We put

$$(10) \quad \tilde{X} = \text{Spec} \left( \bigoplus_{i=0}^{k-1} M^i([im/k]F_\infty) \right)$$

with a natural algebra structure induced by (9), where  $[ \ ]$  is the Gauss symbol. The composite of this  $k$ -fold cyclic covering  $\pi : \tilde{X} \rightarrow \mathbb{P}(E)$  and the structure morphism  $\mathbb{P}(E) \rightarrow X$  is denoted by  $g : \tilde{X} \rightarrow X$ . Furthermore, we set

$$\tilde{D} := (k - 1)F_\infty + g^*D' \quad \text{and} \quad \tilde{f} = \sqrt[k]{S^p - f} \in Q(\tilde{X}),$$

where the unique section of  $g$  lying over  $F_\infty$  is denoted by the same symbol.

The complete linear system  $|mF_\infty|$  defines an embedding outside  $G$  for sufficiently large  $m$ . Hence we have the following lemma.

LEMMA 2.2

*If  $D$  is ample, so is  $\tilde{D}$ .*

Now we assume further that  $\eta := df \in H^0(\mathcal{B}_X(-D))$  is nowhere vanishing. Then  $G$  is smooth by Proposition 1.7, and  $\tilde{X}$  is smooth since the branch locus  $F_\infty \sqcup G$  is smooth. Since  $\tilde{X}$  is defined by the equation  $T^k = S^p - f$  on  $g^{-1}(U)$ , taking the differential, we have  $kT^{k-1}dT = -df$ . Hence  $dT$  has no zero along  $G$ . The differential  $dT$  vanishes along the infinity section  $F_\infty$  with order  $p(k - 1)$ . Therefore,  $dT$  defines a nonzero global section of  $\Omega_{\tilde{X}}(-p(k - 1)F_\infty - ph^*D')$ . It is easily checked that  $d\tilde{f} \in H^0(\mathcal{B}_{\tilde{X}}(-\tilde{D}))$  is nowhere vanishing. Thus we have the following.

PROPOSITION 2.3

*If  $X$  is smooth and  $(X, D, f)$  is a TR triple with ample  $D$  and nowhere vanishing  $\eta = df$ , then  $\tilde{X}$  is smooth and  $(\tilde{X}, \tilde{D}, \tilde{f})$  is also a TR triple with ample  $\tilde{D}$  and nowhere vanishing  $\tilde{\eta} := d\tilde{f}$ .*

Every fiber of  $g$  is a rational curve with the unique singular point at the intersection with  $\pi^{-1}G$ . The singularity is the cusp of the form  $T^k = S^p$ .

Let  $\tilde{\tau} : \tilde{G} \rightarrow \tilde{X}$  be the everywhere ramified covering constructed from  $(\tilde{X}, \tilde{D}, \tilde{f})$  (see Definition 1.6). Since  $\sqrt[p]{\tilde{f}} = \sqrt[k]{S - \sqrt[p]{\tilde{f}}}$ , the composite  $g \circ \tilde{\tau}$  factors

through  $\tau$ , and we have the commutative diagram

$$\begin{array}{ccc} & h & \\ \tilde{G} & \longrightarrow & G \\ \tilde{\tau} \downarrow & & \downarrow \tau \\ \tilde{X} & \longrightarrow & X \\ & g & \end{array}$$

Moreover, this morphism  $h : \tilde{G} \rightarrow G$  is isomorphic to the  $\mathbb{P}^1$ -bundle  $\mathbb{P}(\mathcal{O}_G \oplus \mathcal{O}_G(\tau^*D'))$  over  $G$ . Let  $U$  and  $V$  be the infinity and zero sections, respectively, of the  $\mathbb{P}^1$ -bundle  $h$ . They are disjoint, and we have

$$(11) \quad U - V \sim h^*\tau^*D'.$$

The pullbacks  $\tilde{\tau}^*F_\infty$  and  $\tilde{\tau}^*G$  are  $U$  and  $pV$ , respectively. In particular, we have

$$(12) \quad \tilde{\tau}^*\tilde{D} \sim (k - 1)U + h^*\tau^*D' \sim kU - V.$$

**PROPOSITION 2.4**

Assume that  $X$  is smooth and  $(X, D, f)$  is a TR triple with  $(df) = pD$ . Then  $\tilde{G}$  is a  $\mathbb{P}^1$ -bundle over  $G$ , and the Euler number  $e(\tilde{X})$  of  $\tilde{X}$  is equal to  $2e(X)$ .

*Proof*

The first half is already shown above. This implies  $e(\tilde{G}) = e(\mathbb{P}^1)e(G) = 2e(G)$ . Hence the second half follows from Corollary 1.9.  $\square$

**REMARK 2.5**

The expression (10) of the cyclic covering  $\pi : \tilde{X} \rightarrow \mathbb{P}(E)$  was not given in the original [M1], though it is now standard (see, e.g., Hesnault and Viehweg [HV, Section 3]).

**2.2. The canonical classes of  $\tilde{G}$  and  $\tilde{X}$**

Let  $(X, D, f)$  be a TR triple with an ample divisor  $D$  and nowhere vanishing  $(df) \in H^0(\mathcal{B}_X(-D))$ . We compute the canonical classes of  $\tilde{G}$  and  $\tilde{X}$ . Since  $\tilde{G}$  is a  $\mathbb{P}^1$ -bundle over  $G$  with two disjoint sections  $U$  and  $V$ , the relative cotangent bundle  $\Omega_{\tilde{G}/G}$  is isomorphic to  $\mathcal{O}_{\tilde{G}}(-U - V)$ . Hence we have

$$(13) \quad K_{\tilde{G}} \sim -U - V + h^*K_G \sim -2U + h^*(K_G + \tau^*D')$$

by (11). By Corollary 1.8 and (12), we have

$$\begin{aligned} (14) \quad \tau^*K_X &\sim K_{\tilde{G}} + (p - 1)\tau^*\tilde{D} \\ &\sim -2U + h^*(K_G + \tau^*D') + (p - 1)\{(k - 1)U + h^*\tau^*D'\} \\ &\sim (pk - p - k - 1)U + h^*(K_G + p\tau^*D'). \end{aligned}$$

We note that  $pk - p - k - 1 \geq 0$  and the equality holds if and only if  $\{p, k\} = \{2, 3\}$ .

In the following we denote by  $\sim_{\mathbb{Q}}$  the  $\mathbb{Q}$ -linear (or  $\mathbb{Q}$ -rational) equivalence of  $\mathbb{Q}$ -divisors (or  $\mathbb{Q}$ -cycles). For the later use we put

$$(15) \quad J := K_G + \frac{1}{k-1} \tau^* D \quad \text{and} \quad \tilde{J} := K_{\tilde{G}} + \frac{1}{\tilde{k}-1} \tilde{\tau}^* \tilde{D}$$

for an integer  $\tilde{k}$ . Since  $D' \sim_{\mathbb{Q}} D/k$ , we have

$$(16) \quad \begin{aligned} \tilde{J} &\sim_{\mathbb{Q}} -U - V + h^* K_G + \frac{1}{\tilde{k}-1} \left\{ (k-1)U + \frac{1}{k} h^* \tau^* D \right\} \\ &\sim_{\mathbb{Q}} \left( \frac{k-1}{\tilde{k}-1} - 2 \right) U + h^* \left\{ K_G + \left( \frac{1}{\tilde{k}} + \frac{1}{k(\tilde{k}-1)} \right) \tau^* D \right\} \\ &\sim_{\mathbb{Q}} \left( \frac{k-1}{\tilde{k}-1} - 2 \right) U + h^* \left\{ J + \frac{1}{\tilde{k}} \left( \frac{1}{\tilde{k}-1} - \frac{1}{k-1} \right) \tau^* D \right\} \end{aligned}$$

by (13).

### 2.3. Chern numbers of $\tilde{X}$

For the same reason as (13), we have

$$(17) \quad \begin{aligned} c_1(\tilde{G}) &\sim (1 + U + V) \cdot h^* c(G), \\ c_i(\tilde{G}) &\sim h^* c_i(G) + (U + V) \cdot h^* c_{i-1}(G). \end{aligned}$$

Since  $U \cap V = \emptyset$ , we have

$$(18) \quad \begin{aligned} U \cdot V &\sim 0, \\ U^2 &\sim ((U - V) + V) \cdot U \sim k^{-1} h^* \tau^* D \cdot U, \\ V^2 &\sim (U - (U - V)) \cdot V \sim -k^{-1} h^* \tau^* D \cdot V \end{aligned}$$

by (11). More generally, we have

$$(19) \quad \begin{aligned} U^m &\sim k^{-m+1} h^* \tau^* D^{m-1} \cdot U \quad \text{and} \\ V^m &\sim (-k)^{-m+1} h^* \tau^* D^{m-1} \cdot V \end{aligned}$$

for every integer  $m \geq 1$ .

#### PROPOSITION 2.6

Let  $\lambda$  and  $\mu$  be nonnegative integers such that  $\lambda + i + \mu = \dim \tilde{G}$ . Then we have

$$\begin{aligned} &c_1(\tilde{G})^\lambda \cdot c_i(\tilde{G}) \cdot \tilde{\tau}^* \tilde{D}^\mu \\ &= \sum_{\alpha=0}^{\lambda} \binom{\lambda}{\alpha} (c_1(G)^{\lambda-\alpha} \cdot c_i(G) \cdot \tau^* D^{\mu+\alpha-1}) (k^{1-\alpha} + (-1)^\mu k^{1-\alpha-\mu}) \\ &\quad + \sum_{\alpha=0}^{\lambda} \binom{\lambda}{\alpha} (c_1(G)^{\lambda-\alpha} \cdot c_{i-1}(G) \cdot \tau^* D^{\mu+\alpha}) (k^{-\alpha} + (-1)^\mu k^{-\alpha-\mu}). \end{aligned}$$

*Proof*

By (12) and (17), we have

$$\begin{aligned}
& (c_1(\tilde{G})^\lambda \cdot c_i(\tilde{G}) \cdot \tilde{\tau}^* \tilde{D}^\mu) \\
&= (c_1(\tilde{G})^\lambda \cdot h^* c_i(G) \cdot \tilde{\tau}^* \tilde{D}^\mu) + (c_1(\tilde{G})^\lambda \cdot h^* c_{i-1}(\tilde{G}) \cdot (U + V) \cdot \tilde{\tau}^* \tilde{D}^\mu) \\
&= \sum_{\alpha=0}^{\lambda} \binom{\lambda}{\alpha} (h^* c_1(G)^{\lambda-\alpha} h^* c_i(G) \cdot (U + V)^\alpha \cdot (kU - V)^\mu) \\
&\quad + \sum_{\alpha=0}^{\lambda} \binom{\lambda}{\alpha} (h^* c_1(G)^{\lambda-\alpha} h^* c_{i-1}(G) \cdot (U + V)^{\alpha+1} \cdot (kU - V)^\mu) \\
&= \sum_{\alpha=0}^{\lambda} \binom{\lambda}{\alpha} (h^* c_1(G)^{\lambda-\alpha} \cdot h^* c_i(G) \cdot (k^\mu U^{\alpha+\mu} + (-1)^\mu V^{\alpha+\mu})) \\
&\quad + \sum_{\alpha=0}^{\lambda} \binom{\lambda}{\alpha} (h^* c_1(G)^{\lambda-\alpha} \cdot h^* c_{i-1}(G) \cdot (k^\mu U^{\alpha+\mu+1} + (-1)^\mu V^{\alpha+\mu+1})) \\
&= \sum_{\alpha=0}^{\lambda} \binom{\lambda}{\alpha} (h^* c_1(G)^{\lambda-\alpha} \cdot h^* c_i(G) \cdot h^* \tau^* D^{\alpha+\mu-1} \cdot (k^{1-\alpha} U + (-1)^\mu k^{1-\alpha-\mu} V)) \\
&\quad + \sum_{\alpha=0}^{\lambda} \binom{\lambda}{\alpha} (h^* c_1(G)^{\lambda-\alpha} \cdot h^* c_{i-1}(G) \cdot h^* \tau^* D^{\alpha+\mu} \cdot (k^{-\alpha} U + (-1)^\mu k^{-\alpha-\mu} V)).
\end{aligned}$$

Since both  $U$  and  $V$  are sections of  $h: \tilde{G} \rightarrow G$ , we have  $(h^* Z \cdot U) = \deg Z$  for every 0 cycle  $Z$  on  $G$ . Therefore the proposition follows from the last expression.  $\square$

#### COROLLARY

The intersection number  $(c_1(\tilde{G})^\lambda \cdot c_i(\tilde{G}) \cdot \tilde{\tau}^* \tilde{D}^\mu)$  is of degree  $\leq 1$  as a Laurent polynomial in the variable  $k$ . Moreover, the coefficient of  $k$  is equal to  $(c_1(G)^\lambda \cdot c_i(G) \cdot \tau^* D^{\mu-1})$  if  $\mu \geq 1$  and 0 otherwise.

#### 2.4. Proof of Theorem 2

Now we are ready to construct an  $n$ -dimensional TR triple  $(X_n, D_n, df_n)$ . We define two sequences  $\{k_i\}_{1 \leq i \leq n-1}$  and  $\{e_i\}_{1 \leq i \leq n-1}$  of positive integers inductively by the rule

$$k_i = 1 + c_i e_{i-1} \quad \text{and} \quad e_i = e_{i-1} k_i$$

for  $2 \leq i \leq n-1$ , where  $\{c_i\}_i$  is a nondecreasing sequence of integers  $c_i \geq 2$  such that  $k_i$ 's are not divisible by  $p$ . (The simplest choice is  $c_i := p$  for every  $i$ .) We start with an arbitrary positive integer  $k_1 \geq 2$  prime to  $p$  and  $e_1 := k_1$ .

The first TR triple  $(X_1, D_1, df_1)$  consists of a Tango–Raynaud curve  $X_1$ , a divisor  $D_1$ , and an exact differential  $df_1$  with  $(df_1) = pD_1$  such that  $D_1$  is divisible by  $e_{n-1}$ . Then we apply the construction in Section 2.1 by taking a  $k_{n-1}$ -fold covering of the  $\mathbb{P}^1$ -bundle  $\mathbb{P}(E_1)$  over  $X_1$  and putting  $(X_2, D_2, df_2) =$

$(\tilde{X}_1, \tilde{D}_1, d\tilde{f}_1)$ . This is a TR triple of dimension 2. We repeat this process  $n - 1$  times. We note that the divisor  $D_2 = (k_{n-1} - 1)F_\infty + D_1/k_{n-1}$  is divisible by  $e_{n-2}$ . In particular,  $D_2$  is divisible by  $k_{n-2}$ . Hence, taking a  $k_{n-2}$ -fold covering of  $\mathbb{P}(E_2)$  over  $X_2$ , we obtain  $(X_3, D_3, df_3) = (\tilde{X}_2, \tilde{D}_2, d\tilde{f}_2)$ , which is a TR triple of dimension 3 such that  $D_3$  is divisible by  $e_{n-3}$ , and so on. In the final  $(n - 1)$ st step we take the  $k_1$ -fold covering of  $\mathbb{P}(E_{n-1})$  since  $D_{n-1}$  is divisible by  $e_1 = k_1$ . We obtain a new TR triple  $(X_n, D_n, df_n)$ , which is an  $n$ -dimensional counterexample to Kodaira's vanishing by Proposition 2.3.

The first half of Theorem 2(b) is a consequence of the following.

**PROPOSITION 2.7**

*The canonical class  $K_{X_n}$  is ample if  $\{p, k_1\} \neq \{2, 3\}$  and is the pullback of an ample divisor on  $X_{n-1}$  if  $\{p, k_1\} = \{2, 3\}$ .*

*Proof*

Since  $\tau_n : G_n \rightarrow X_n$  is finite, it suffices to show that  $K_{G_{n-1}} + (p/k_1)\tau_{n-1}^*D_{n-1}$  is ample by (14). We put

$$J_i := K_{G_i} + \frac{1}{k_{n-i} - 1} \tau_i^* D_i$$

for every  $1 \leq i \leq n - 1$  after (15). Since  $p/k_1 \geq 1/(k_1 - 1)$ , it suffices to show the following.

**CLAIM 1**

*$J_i$  is ample.*

We prove it by induction on  $i$ . In the case  $i = 1$ , both  $K_{G_1}$  and  $D_1$  are ample. Hence  $J_1$  is ample. Assume that  $i \geq 2$ . We have

$$\frac{k_{n-i+1} - 1}{k_{n-i} - 1} = \frac{c_{n-i+1}e_{n-i}}{c_{n-i}e_{n-i-1}} \geq k_{n-i} \geq 2$$

if  $n - i \geq 2$ , and  $(k_2 - 1)/(k_1 - 1) = c_2k_1/(k_1 - 1) > 2$ . By the formula (16),  $J_i$  is ample since so is  $J_{i-1}$  and since  $k_{n-i+1} > k_{n-i}$ . This finishes the proof of Proposition 2.7. □

Now we consider the sequence of the morphisms

$$G_n \xrightarrow{h_{n-1}} G_{n-1} \xrightarrow{h_{n-2}} \dots \xrightarrow{h_2} G_2 \xrightarrow{h_1} G_1,$$

in order to investigate the asymptotic behavior of certain Chern numbers of  $X_n$  as  $k_1, \dots, k_{n-1}$  go to  $\infty$ , where  $G_j := \tilde{G}_{j-1}$  for  $j = 2, \dots, n$ . Since  $G_1$  is a curve, we have  $-\deg c_1(G_1) = \deg \tau_1^* D_1 = 2g - 2$ , where  $g$  is the genus of the Tango-Raynaud curve  $G_1 \simeq X_1$ . Applying Proposition 2.6 (or its corollary) successively to the above morphisms  $h_i$ , we have the following.

**PROPOSITION 2.8**

The intersection number  $(c_1(G_n)^\lambda \cdot c_i(G_n) \cdot \tau_n^* D_n^\mu)$  is a Laurent polynomial in the variables  $k_1, \dots, k_{n-1}$  whose coefficients are integers independent of  $X_1$  and  $D_1$ . The degree of the Laurent polynomial is at most 1 with respect to every variable. Moreover, the coefficient of  $k_1 \cdots k_{n-1}$  is equal to

$$\begin{cases} 2g - 2 & \text{if } (\lambda, i, \mu) = (0, 0, n), \\ -(2g - 2) & \text{if } (\lambda, i, \mu) = (1, 0, n - 1), (0, 1, n - 1), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore we have the following proposition.

**PROPOSITION 2.9**

The intersection number  $(K_{X_n}^{n-i} \cdot c_i(X_n))$  is a Laurent polynomial in the variables  $k_1, \dots, k_{n-1}$ , and the degree is at most 1 with respect to each variable. If  $i \geq 2$ , then the coefficient of the highest monomial  $k_1 \cdots k_{n-1}$  in the Laurent expression of  $(K_{X_n}^{n-i} \cdot c_i(X_n))$  is equal to  $-p^{-n}(p - 1)^{n-i}(n - i)(2g - 2)$ .

*Proof*

By (17),  $\tau_n^* c_i(X_n)$  is rationally equivalent to

$$\begin{aligned} & c_i(G_n) + (1 - p) \sum_{j=1}^i c_{i-j}(G_n) \cdot \tau_n^* D_n^j \\ & \sim (1 - p) \tau_n^* D_n^i + (1 - p) c_1(G_n) \tau_n^* D_n^{i-1} + (\text{lower terms in } D_n). \end{aligned}$$

Since  $\tau_n^* c_1(X_n) \sim (1 - p) \tau_n^* D_n + c_1(G_n)$ , we have

$$\begin{aligned} & p^n (c_1(X_n)^{n-i} \cdot c_i(X_n)) \\ & = (\tau_n^* c_1(X_n)^{n-i} \cdot \tau_n^* c_i(X_n)) \\ & = (1 - p)^{n-i+1} (\tau_n^* D_n^n) + (1 - p)^{n-i} (n - i) (c_1(G_n) \cdot \tau_n^* D_n^{n-1}) \\ & \quad + (1 - p)^{n-i+1} (c_1(G_n) \cdot \tau_n^* D_n^{n-1}) + (\text{lower terms in } D_n). \end{aligned}$$

Hence our assertion follows from Proposition 2.8. □

By the proposition,  $(K_X^{n-i} \cdot c_i(X_n))$  is negative for sufficiently large choices of  $k_1, \dots, k_{n-1}$  for  $i \geq 2$ . This shows Theorem 2(b). Theorem 2(c) is a direct consequence of Proposition 2.4.

**2.5. Properties of  $(X_2, D_2, df_2)$**

Here we remark a few properties of 2-dimensional counterexample  $(X, D, df) := (X_2, D_2, df_2)$ , which is a  $k$ -fold covering of a  $\mathbb{P}^1$ -bundle over a Tango–Raynaud curve  $C$ . By Proposition 1.7, the cokernel of the multiplication map by  $df$  is locally free. In our case, the cokernel is a line bundle. Hence we have the exact

sequence

$$(20) \quad 0 \rightarrow \mathcal{O}_X(pD) \xrightarrow{\times df} \Omega_X \rightarrow \mathcal{O}_X(K_X - pD) \rightarrow 0.$$

**PROPOSITION 2.10**

- (a) *The complete linear system  $|p(pD - K_X)|$  is nonempty.*
- (b) *If  $k \equiv -1(p)$ , then  $X$  has a nonzero vector field, that is,  $H^0(T_X) \neq 0$ .*
- (c) *When  $\{p, k\} \neq \{2, 3\}$ , the canonical class  $K_X$  is ample and *K.V.* holds for  $K_X$ , that is,  $H^1(\mathcal{O}_X(-K_X)) = 0$ .*

*Proof*

First we compute the canonical class  $K_X$  more rigorously than in Section 2.2. Since  $K_{\mathbb{P}(E)/C} = -2F_\infty + D_1$  and since the  $k$ -fold cyclic covering  $\pi : X \rightarrow \mathbb{P}(E)$  has branch locus  $G \sqcup F_\infty$ , we have

$$K_{X/C} = \pi^* K_{\mathbb{P}(E)/C} + (k-1)G + (k-1)F_\infty \sim -(k+1)F_\infty + (k-1)G + g^*D_1.$$

The rational function  $S^p - f$  gives the linear equivalence  $G \sim p(F_\infty - D_1)$  on  $\mathbb{P}(E)$ , which is (7). Hence its  $k$ th root  $\sqrt[k]{S^p - f} \in Q(X)$  gives the equivalence  $G \sim p(F_\infty - D_1/k)$  on  $X$ . Therefore, we have

$$(21) \quad K_X \sim K_{X/C} + pD \sim (pk - p - k - 1)F_\infty + (p+k)D_1/k$$

and  $pD - K_X \sim (k+1)F_\infty - D_1$ . Now we are ready to prove our assertions.

(a) The linear system  $|p(pD - K_X)|$  is nonempty since  $p(pD - K_X)$  is linearly equivalent to  $(k+1)G + pD_1/k$ .

(b) Put  $k = ap - 1$  for a nonnegative integer  $a$ . Then we have  $pD - K_X \sim apF_\infty - D_1 \sim aG + D_1/k$ . Since  $T_X$  contains  $\mathcal{O}_X(pD - K_X)$  as a line subbundle, we have  $H^0(T_X) \neq 0$ .

(c)  $K_X$  is ample by Proposition 2.7. Since  $p(pk - p - k - 1) > k - 1$ , we have  $\text{Hom}(\mathcal{O}_X(p^m K_X), \mathcal{O}_X(D)) = 0$  for every  $m \geq 1$ . Hence we have  $\text{Hom}(\mathcal{O}_X(p^m K_X), \Omega_X) = 0$  by (20) and (a). Therefore, we have  $H^1(\mathcal{O}_X(-K_X)) = 0$  by the corollary of Proposition 1.5 and Lemma 1.1. □

By (a) of the proposition the cotangent bundle  $\Omega_X$  is not slope stable with respect to any ample line bundle. Since any positive-dimensional algebraic group does not act on a surface of general type, the group scheme  $\text{Aut } X$  is not reduced by (b). See [Ru] and [La] for alternative treatment of (generalized) Raynaud’s surface from this viewpoint. We refer to [E2] and [SB] for the pluricanonical maps of surfaces of general type in positive characteristic.

**3. Surfaces on which R.V. does not hold**

In this section we prove Theorem 3. By virtue of the following result, R.V. on a (smooth complete) surface  $X$  is equivalent to the injectivity of the Frobenius map (2) for all nef and big line bundle  $L$ .

PROPOSITION 3.1 (SZPIRO [Sz, PROPOSITION 2.1], LEWIN-MÉNÉGAUX [LM, PROPOSITION 2])

$H^1(X, L^{-m}) = 0$  holds for  $m \gg 0$  if  $L$  is nef and big.

The following is inspired by a similar statement [T2, Corollary 8]. This is not absolutely necessary for our proof but makes it more transparent.

PROPOSITION 3.2

Let  $X'$  be the blowup of a surface  $X$  at a point. The R.V. holds on  $X'$  if and only if so does on  $X$ .

*Proof*

Let  $x \in X$  be the center of the blowup  $\pi : X' \rightarrow X$ . If  $L$  is a nef and big line bundle on  $X$ , then so is the pullback  $\pi^*L$ . If R.V. holds on  $X'$ , then  $H^1(X', \pi^*L^{-1})$  vanishes. Since  $H^1(X, L^{-1})$  is isomorphic to  $H^1(X', \pi^*L^{-1})$ , R.V. holds also on  $X$ .

Conversely assume that R.V. holds on  $X$ , and let  $D'$  be a nef and big divisor on  $X'$ . Then  $D := \pi_*D'$  is also nef and big. By Theorem 1, the vector space  $\{df \in \Omega_{Q(X)} \mid f \in Q(X), (df) \geq pD\}$  is zero. The space  $\{df \in \Omega_{Q(X')} \mid f \in Q(X'), (df) \geq pD'\}$  is also zero since  $(df) \geq pD$  is a divisorial condition. Therefore, R.V. holds on  $X'$ .  $\square$

We first prove (b). Let  $X$  be a surface on which R.V. does not hold. By Proposition 1.5, there exist a rational function  $f$  and a nef and big divisor  $D$  with  $(df) \geq pD$ . The function  $f$  gives a rational map from  $X$  to the projective line  $\mathbb{P}^1$ . By taking suitable blowups  $X' \rightarrow X$  and the Stein factorization, we have the morphism  $g : X' \rightarrow C$  with  $g_*\mathcal{O}_{X'} = \mathcal{O}_C$ .  $C$  is smooth since so is  $X$ . Every fiber of  $g$  is connected. Let  $L$  be the image of the multiplication homomorphism  $\mathcal{O}_{X'}(pD) \rightarrow \Omega_{X'}$  by  $df$ . The relative cotangent sheaf  $\Omega_{X'/C} = \Omega_{X'}/g^*\Omega_C$  contains  $T := L/[L \cap g^*\Omega_C]$  as a subsheaf. On a nonempty subset of  $C$ ,  $\Omega_C$  contains  $df$  as its global section. Hence,  $L \cap g^*\Omega_C \neq 0$  and  $T$  is a torsion sheaf. There exists an effective divisor  $A$  with  $\text{Supp } A = \text{Supp } T$  which is linearly equivalent to  $c_1(L) - c_1(L \cap g^*\Omega_C)$ ;  $c_1(L) = pD$  is a nef and big divisor on  $X'$ , and  $c_1(L \cap g^*\Omega_C) \leq g^*K_C$  holds. Hence  $A$  contains a component  $G$  different from fibers of  $g$ . Then for every fibers  $B$  of  $g$ ,  $\Omega_B$  has nonzero torsion at the intersection  $B \cap G$ . In particular,  $B$  is singular at  $B \cap G$ .

Now we prove (a). By Proposition 3.2, we may assume that  $X$  is a (relatively) minimal model.

PROPOSITION 3.3

If  $X$  is a ruled surface or an elliptic surface, then R.V. holds on  $X$ .

*Proof*

Let  $h : X \rightarrow C$  be a  $\mathbb{P}^1$ -bundle or an elliptic fibration of  $X$ . Then there exists an exact sequence

$$0 \rightarrow h^*\Omega_C \rightarrow \Omega_X \rightarrow \Omega_{X/C} \rightarrow 0$$

of torsion-free sheaves on  $X$ . Let  $L$  be a nef and big line bundle on  $X$ . The degree of  $L$ ,  $h^*\Omega_C$ , and  $\Omega_{X/C}$  restricted to general fibers of  $h$  are positive, zero and nonpositive, respectively. Therefore, we have  $\mathrm{Hom}_{\mathcal{O}_X}(L, h^*\Omega_C) = \mathrm{Hom}_{\mathcal{O}_X}(L, \Omega_{X/C}) = 0$ . By the exact sequence, we have  $\mathrm{Hom}_{\mathcal{O}_X}(L, \Omega_X) = 0$ . Hence R.V. holds on  $X$ . (This argument is taken from [T2, Corollary 6].)  $\square$

#### CASE 1

If  $\kappa(X) = -\infty$ , then we can take a  $\mathbb{P}^1$ -bundle as a relatively minimal model. Hence R.V. holds by the proposition.

#### CASE 2

If  $\kappa(X) = 1$ , then the minimal model  $X$  is an elliptic surface by our assumption. Hence R.V. holds by the proposition.

#### CASE 3

Assume that  $\kappa(X) = 0$ . By the classification of Bombieri and Mumford [BM2],  $X$  and the second Betti number  $B_2(X)$  satisfy one of the following:

- (a)  $B_2(X) = 6$  and  $X$  is an abelian surface;
- (b)  $B_2(X) = 22$  and  $X$  is a K3 surface;
- (c)  $B_2(X) = 10$  and  $X$  is either a classical, singular or supersingular Enriques surface; the last two types occur only when  $p = 2$ ;
- (d)  $B_2(X) = 6$  and  $X$  is either hyperelliptic or quasi-hyperelliptic. The latter appears only when  $p = 2, 3$ .

In the case (a), R.V. holds by the corollary of Proposition 1.5 since  $\Omega_X \simeq \mathcal{O}_X^{\oplus 2}$ . In the case (d), R.V. holds by Proposition 3.3 since  $X$  has an elliptic fibration also (over  $\mathbb{P}^1$ ) by [BM2, Theorem 3]. Our proof of Theorem 3 is completed by the following proposition.

#### PROPOSITION 3.4

*R.V. holds on a K3 and an Enriques surfaces.*

#### *Proof*

It suffices to show injectivity of (1) for all nef and big divisor  $D$  on  $X$ . Assume the contrary. Then, by Lemma 1.4,  $H^0(\mathcal{B}_X(-D))$  is nonzero. By the multiplication map

$$(22) \quad H^0(\mathcal{O}_X(D)) \times H^0(\mathcal{B}_X(-D)) \longrightarrow H^0(\mathcal{B}_X),$$

and by the Riemann–Roch inequality

$$(23) \quad \dim H^0(\mathcal{O}_X(D)) \geq \frac{1}{2}(D^2) + \chi(\mathcal{O}_X) \geq 2,$$

we have

$$\dim \operatorname{Ker} [F^* : H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_X)] = \dim H^0(\mathcal{B}_X) \geq 2.$$

This is a contradiction since  $H^1(\mathcal{O}_X)$  is at most 1-dimensional by [BM2] and [BM1, Lemma 1].  $\square$

*Acknowledgments.* All results of this article are contained in either [M1] or [M2] except for Proposition 3.2. The report [M1] is an outcome of the author's seminar around 1977 on [Mum] and a preprint of [Ra] with Professor Masaki Maruyama, to whose advice and encouragement the author expresses his sincere gratitude on this occasion. The author is also grateful to the referees for their careful reading and suggestion of useful references.

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