Equivalence of symplectic singularities

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Abstract After introducing an equivalence problem for symplectic singularities, we formulate an algebraic version of such a problem. Let X be an affine normal variety with a $\mathbf{C}^*$-action having only positive weights. Assume that the regular part X_{reg} of X admits an algebraic symplectic 2-form ω with weight l. Our main theorem asserts that any algebraic symplectic 2-form ω' on X_{reg} of weight l is equivalent to ω up to a $\mathbf{C}^*$-equivariant automorphism of X if $l \neq 0$. When $l = 0$ we have a counterexample to this statement. In the latter half of the article, we discuss the equivalence problem *up to constant*. We associate to X a projective variety $\mathbf{P}(X)$ and prove that $\mathbf{P}(X)$ has a contact orbifold structure. Moreover, when X has canonical singularities, the contact orbifold structure is rigid under a small deformation. The equivalence problem is then reduced to the uniqueness of the contact structures. In most examples the symplectic structures turn out to be unique up to constant with very few exceptions. In the final section we pose a splitting conjecture for symplectic singularities.

Introduction

Assume that X is a germ of a normal complex space whose regular locus X_{reg} admits a holomorphic symplectic 2-form ω. Two such pairs (X, ω) and (X', ω') are *equivalent* if there is an isomorphism $\phi: X \to X'$ such that $\omega = \phi^*(\omega')$. They are not, a priori, equivalent even if their underlying complex analytic structures are equivalent. The Darboux theorem asserts that any holomorphic symplectic structure on $(\mathbf{C}^{2n}, 0)$ is equivalent to the standard one $dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$. A general theme of this article is such an equivalence problem for a singular space.

The Darboux theorem is naturally extended to a symplectic quotient singularity (cf. Proposition 1.1). An essential idea for proving the Darboux theorem is due to Moser [Mo], and it seems rather difficult to develop this method for an arbitrary singular space.

In this article we formulate algebraic versions of the equivalence problem. Namely, we start with a normal affine variety X of dimension $2d$ with a $\mathbf{C}^*$-action. Assume that $0 \in X$ is a unique fixed point of the $\mathbf{C}^*$-action with positive weights. More precisely, the cotangent space $m_{X,0}/m_{X,0}^2$ of $0 \in X$ has only positive weights with respect to the $\mathbf{C}^*$-action, or equivalently, the affine ring R of X is positively graded: $\bigoplus_{i \geq 0} R_i$ with $R_0 = \mathbf{C}$. We call such a $\mathbf{C}^*$-action *good*. Let ω be an algebraic symplectic 2-form on X_{reg} with weight l. If we represent the $\mathbf{C}^*$-action

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by the family $\{\phi_t\}_{t\in\mathbf{C}^*}$ of automorphisms of X, then $\phi_t^*(\omega)=t^l\cdot\omega$. If we change the $\mathbf{C}^*$-action of X, l may possibly change. But the positivity of l reflects the properties of X itself. In fact, if X has canonical singularities, then l must be positive (see Lemma 2.2). Conversely, when $l>0$, one can show that X has canonical singularities under the assumption that X has an isolated singularity (see Lemma 2.4).

Let us consider two such pairs (X,ω) and (X',ω') with the same weight l. They are called equivalent if there is an $\mathbf{C}^*$-equivariant isomorphism $\phi : X\cong X'$ such that $\omega=\phi^*(\omega')$. In particular, if $X=X'$, then ω and ω' are called equivalent symplectic structures on X. Our main result is the following.

THEOREM 3.1

Let (X,ω) be a pair of a normal affine variety X with a good $\mathbf{C}^*$-action and an algebraic symplectic 2-form ω on X_{reg} with weight $l\neq 0$. Then ω is a unique symplectic structure with weight l up to equivalence.

If we drop the assumption $l\neq 0$, then the result does not hold. We have a counter-example when $l=0$ (see Remark 3.3).

Affine symplectic varieties are constructed in several different manners such as nilpotent orbit closures of a complex simple Lie algebra (cf. [CM]), Slodowy slices to such orbits (cf. [Sl]) and the symplectic (or hyper-Kähler) reductions. Note that these examples naturally come up with $\mathbf{C}^*$-actions. It often happens that the same $\mathbf{C}^*$-variety appears in different constructions. But Theorem 3.1 asserts that the symplectic structures on the same $\mathbf{C}^*$-variety are unique if they have the same weight. We explain below how Theorem 3.1 is applied to explicit examples.

Let $\mathbf{g}$ be a complex simple Lie algebra, and let $S\subset\mathbf{g}$ be a Slodowy slice to a nilpotent orbit O of $\mathbf{g}$. Let $\mathbf{h}$ be a Cartan subalgebra, and let W be the associated Weyl group of $\mathbf{g}$. We denote by $\chi : S\to\mathbf{h}/W$ the adjoint quotient map restricted to S. We write S_0 for the central fiber $\chi^{-1}(0)$ of χ. It is known that S_0 admits a (Kostant–Kirillov) symplectic structure ω together with a $\mathbf{C}^*$-action such that ω has weight 2.

EXAMPLE 1

(i) Let $\mathbf{g}$ be the simple Lie algebra of type B_n (resp., C_n, F_4, or G_2). Let $\mathbf{g}'$ be the simple Lie algebra of type A_{2n-1} (resp., D_{n+1}, E_6, or D_4). Consider the Slodowy slices S and S', respectively, for the subregular orbits of $\mathbf{g}$ and $\mathbf{g}'$. Then both S_0 and S'_0 have a du Val singularity of type A_{2n-1} (resp., D_{n+1}, E_6, or D_4). Moreover, they are isomorphic as $\mathbf{C}^*$-varieties (see [Sl, Section 7.4, Proposition 2, Section 8.3, Proposition 2]). According to Theorem 3.1[†] we see that (S_0,ω) and (S'_0,ω') are also equivalent as symplectic varieties. This fact has

[†]In this case one can check easily that they are symplectic equivalent without Theorem 3.1.

a nice application to the study of Poisson deformations of (S_0, ω). As is observed in [LNS], the Poisson deformation $S \to \mathbf{h}/W$ of (S_0, ω) is not the universal one. But, since $(S_0, \omega) \cong (S'_0, \omega')$, one can regard $S' \to \mathbf{h}'/W'$ as a Poisson deformation of (S_0, ω). Since $\mathbf{g}'$ is simply laced, this turns out to be universal.

(ii) Let $\mathbf{g}$ be the simple Lie algebra of type G_2, and let S be a Slodowy slice to the 8-dimensional nilpotent orbit of $\mathbf{g}$. Let $\mathbf{g}'$ be the simple Lie algebra of type C_3, and let S' be a Slodowy slice to the nilpotent orbit of $\mathbf{g}'$ of Jordan type $[4, 1^2]$. Then S_0 and S'_0 are isomorphic to the 4-dimensional quasi-homogeneous hypersurface

$$X := \{(a, b, x, y, z) \in \mathbf{C}^5; a^2x + 2aby + b^2z + (xz - y^2)^3 = 0\}$$

as $\mathbf{C}^*$-varieties (see [LNS, Section 10]). Then (S_0, ω) and (S'_0, ω') are equivalent as symplectic varieties by Theorem 3.1. As in (i), $S \to \mathbf{h}/W$ does not give the universal Poisson deformation of (S_0, ω). But $S' \to \mathbf{h}'/W'$ is the universal Poisson deformation of (S_0, ω).

EXAMPLE 2 ([LNSS, SECTION 3])

Quasi-homogeneous symplectic hypersurfaces

At this moment we know two kinds of quasi-homogeneous symplectic hypersurfaces. The first one is a series of examples X_n $(n \geq 2)$ of dimension 4:

$$X_n := \{(a, b, x, y, z) \in \mathbf{C}^5; a^2x + 2aby + b^2z + (xz - y^2)^n = 0\}.$$

The second one is a 6-dimensional example. For details on this example, see [LNS, Section 10].

One can put (homogeneous) symplectic structures on them in several different ways.

(a) Originally these examples were found as the central fibers S_0 of the Slodowy slices S to certain nilpotent orbits of $\mathbf{g}$. The X_n is the S_0 for the nilpotent orbit $O_{[2n-2,1^2]}$ of $\mathrm{sp}(2n)$ and the 6-dimensional example is the S_0 for the (unique) 6-dimensional nilpotent orbit of G_2. A Slodowy slice has a $\mathbf{C}^*$-action and admits a symplectic structure of weight 2.

(b) Let V be an even-dimensional representation of sl_2. One can put a Poisson structure on $A := \mathbf{C}[\mathrm{sl}_2 \oplus V]$ by using the Lie bracket of sl_2, the sl_2-representation V, and an sl_2-equivariant map $\varphi : \wedge^2 V \to \mathbf{C}[\mathrm{sl}_2]$. More precisely, for $x+v$, $y+w \in \mathrm{sl}_2 \oplus V$, we define

$$\{x + v, y + w\} := [x, y] + \varphi(v, w) + (x \cdot w + y \cdot v)$$

and extend this bracket to a Poisson structure on A by the Leibniz rule.

Take as V the standard 2-dimensional representation, and take as φ the $(n-1)$th power Δ^{n-1} of the Casimir element $\Delta \in \mathbf{C}[\mathrm{sl}_2]$. Then we have a Poisson structure on A. Notice that $\mathrm{Spec}\, A$ is a 5-dimensional affine space $\mathbf{A}^5$. The Poisson center $C_n := \{g \in A; \{g, A\} = 0\}$ is the polynomial ring $\mathbf{C}[f_n]$ generated by an element f_n of A. The ring homomorphism $\mathbf{C}[f_n] \to A$ induces a morphism of algebraic varieties $f_n : \mathbf{A}^5 \to \mathbf{A}^1$. One can prove that f_n coincides with the defining polynomial of X_n: $a^2x + 2aby + b^2z + (xz - y^2)^n$ after a suitable $\mathbf{C}^*$-equivariant coordinate change of $\mathbf{A}^5$. The Poisson structure on $\mathbf{A}^5$ induces a Poisson structure on the central fiber $X_n := \{f_n = 0\}$. This Poisson structure has weight -2, and it is generically nondegenerate; in other words, X_n admits a symplectic structure of weight 2.

Similarly, by using the symmetric product $S^3(\mathbf{C}^2)$ of the standard representation, we get a Poisson structure on A with a Poisson center f. Then f is equivalent to the equation of the 6-dimensional hypersurface.

(c) The series X_n of hypersurfaces can be also obtained as the symplectic reductions of Hanany and Mekareeya [HM] determined by unitrivalent graphs.

Thus we have three symplectic structures on X_n and two symplectic structures on the 6-dimensional example. They are all equivalent by Theorem 3.1.

In the latter half of the article we discuss the equivalence problem up to a constant. Let (X, ω) be the same one as in Theorem 3.1; namely, $l \neq 0$. A symplectic structure ω' on X is equivalent to ω up to constant when $\omega' = \lambda \cdot \omega$ with some $\lambda \in \mathbf{C}^*$. If the weight l of ω is nonzero, then the equivalence up to a constant implies the equivalence up to a $\mathbf{C}^*$-equivariant automorphism. Let R be the affine ring of X. By the assumption, R is positively graded: $R = \bigoplus_{i \geq 0} R_i$. We put $\mathbf{P}(X) := \mathrm{Proj}(\bigoplus_{i \geq 0} R_i)$. Roughly speaking, we reduce the equivalence problem for the symplectic structure on X to the uniqueness of the contact structure on $\mathbf{P}(X)$.

It is well known that a contact structure is an odd-dimensional counterpart of a symplectic structure in complex and differential geometry. The author thinks that this is a good occasion to give an appropriate formulation of the contact structure for singular varieties.

Recall that a contact structure on a complex manifold Z of dimension $2d+1$ is an exact sequence of vector bundles

$$0 \to D \to TZ \xrightarrow{\theta} M \to 0,$$

with $\mathrm{rank}(D) = 2d$ and $\mathrm{rank}(M) = 1$ so that $d\theta|_D$ induces a nondegenerate pairing on D. The line bundle M is called the contact line bundle. According to LeBrun [LeB], the contact structure is a unique one with the contact line bundle M if and only if $H^0(Z, O(D)) = 0$.

Let us consider the natural projection map $p : X - \{0\} \to \mathbf{P}(X)$. Then all fibers of p are isomorphic to $\mathbf{C}^*$, but some of them are multiple fibers. There exists an open dense subset $\mathbf{P}(X)^0$ of $\mathbf{P}(X)$ such that $\mathbf{P}(X)^0$ is smooth and p is a $\mathbf{C}^*$-bundle over $\mathbf{P}(X)^0$. Define $L := O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)^0}$. The symplectic form ω on X_{reg} of weight $l \neq 0$ determines a contact structure on $\mathbf{P}(X)^0$ with the contact line bundle $L^{\otimes l}$ (cf. Section 4.3). If $\mathrm{Codim}_{\mathbf{P}(X)}(\mathbf{P}(X) - \mathbf{P}(X)^0) \geq 2$, one can employ this contact structure on $\mathbf{P}(X)^0$ as a contact structure on $\mathbf{P}(X)$. But when $\mathrm{Codim}_{\mathbf{P}(X)}(\mathbf{P}(X) - \mathbf{P}(X)^0) = 1$, the contact structure on $\mathbf{P}(X)^0$ does not yet have enough information. This is the case, for example, when (X, ω) is a du Val singularity with a symplectic structure of weight 2. So, in a general case, we need to introduce the notion of a *contact orbifold structure* (see Section 4.4 for details). A contact orbifold structure on a normal variety Z consists of an orbifold structure Z^{orb} on Z, an orbifold line bundle $\mathcal{M}$ (i.e., contact line bundle) on Z^{orb}, and a global section θ of $\underline{\mathrm{Hom}}(\Theta_{Z^{\mathrm{orb}}}, \mathcal{M})$. Then one can prove the following.

THEOREM 4.4.1
The projectivized cone $\mathbf{P}(X)$ has a contact orbifold structure.

Let $\mathcal{L} \in \mathrm{Pic}(\mathrm{P}(X)^{\mathrm{orb}})$ be the tautologcal line bundle, and assume that $\mathcal{M} = \mathcal{L}^{\otimes l}$. Then one can completely recover the original symplectic structure (X, ω) from the data $(\mathbf{P}(X), \mathcal{M}, \theta)$.

For each du Val singularity (X, ω) of type ADE, a contact orbifold structure on $\mathbf{P}(X) \cong \mathbf{P}^1$ is determined. But these structures are all different even though the underlying space is the same $\mathbf{P}^1$. In other words, $\mathbf{P}^1$ has infinitely many different contact orbifold structures.

When X has canonical singularities, the projectivized cone $\mathbf{P}(X)$ is a singular Fano variety. But $\mathbf{P}(X)$ turns out to be a very special one. In fact, we prove that the contact orbifold structure $(\mathbf{P}(X), \mathcal{M}, \theta)$ is rigid under a small deformation if X has canonical singularities (see Proposition 5.2). When X is the closure of a minimal nilpotent orbit $O_{\min}$ of a simple Lie algebra, $\mathbf{P}(X)$ is a contact Fano homogeneous manifold. In this case the contact structure is known to be rigid under a small deformation (cf. [LeB]). Thus Proposition 5.2 generalizes this fact.

The equivalence problem for a symplectic structure on X is now reduced to the uniqueness of the contact orbifold structure on $\mathbf{P}(X)$. In most examples the symplectic structures turn out to be unique up to a constant with very few exceptions (see Section 6).

Section 7 is a speculation based on the analogy of the Bogomolov decomposition for compact Kähler manifolds with $c_1 = 0$. The contents of Section 6 are still fragmentary. However, the problems addressed in the final section would play a role as a working hypothesis in their future study.

1. Equivalence problem for complex analytic germs

Assume that X is a germ of a normal complex space whose regular locus X_{reg} admits a holomorphic symplectic 2-form ω. Two such pairs (X, ω) and (X', ω') are *equivalent* if there is an isomorphism $\phi : X \to X'$ such that $\omega = \phi^*(\omega')$. They are not, a priori, equivalent even if their underlying complex analytic structures are equivalent. The Darboux theorem asserts that any holomorphic symplectic structure on $(\mathbf{C}^{2n}, 0)$ is equivalent to the standard one $dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$.

One can generalize the Darboux theorem to a quotient singularity, which might be already known.

PROPOSITION 1.1

Let $(X,0)$ be a quotient symplectic singularity with a holomorphic symplectic form ω. Then any holomorphic symplectic form on $(X,0)$ is equivalent to ω.

Proof

Write $X = \mathbf{C}^{2n}/G$ with a finite group $G \subset \mathrm{Sp}(2n,\mathbf{C})$. Let $\pi : (\mathbf{C}^{2n},0) \to (X,0)$ be a natural projection. Let ω' be an arbitrary symplectic form on $(X,0)$. Let $\tilde{\omega}$ and $\tilde{\omega}'$ be, respectively, the pullbacks of ω and ω' by π. We shall prove that there is a G-equivariant isomorphism $\tilde{\varphi} : (\mathbf{C}^{2n},0) \to (\mathbf{C}^{2n},0)$ such that $\tilde{\varphi}^*(\tilde{\omega}') = \tilde{\omega}$. Then this $\tilde{\varphi}$ descends to an automorphism φ of $(X,0)$ such that $\varphi^*(\omega') = \omega$. We first prove a linear algebra version of this fact. □

LEMMA 1.2

Let V be a $2n$-dimensional complex representation of a finite group G. Assume that ω and ω' are G-invariant nondegenerate skew-symmetric 2-forms on V. Then there is a G-equivariant linear isomorphism ϕ such that $\phi^*(\omega) = \omega'$.

Proof

Denote by V^* the dual representation of V. We divide irreducible representations V of G into three types:

(I) $V \cong V^*$ and $\dim(\wedge^2 V^*)^G = 1$,
(II) $V \cong V^*$ and $(\wedge^2 V^*)^G = 0$,
(III) V is not isomorphic to V^* as a G-module.

Note that if V is irreducible and $V \cong V^*$, then $\mathbf{C} = \mathrm{Hom}_G(V,V) = (V \otimes V^*)^G = (V^* \otimes V^*)^G = (\wedge^2 V^*)^G \oplus (\mathrm{Sym}^2(V^*))^G$. In case (I) one has $\dim(\wedge^2 V^*)^G = 1$, and the isomorphism $V \cong V^*$ is given by a G-invariant nondegenerate skew-symmetric form which is unique up to a scalar. In case (II) one has $\dim(\mathrm{Sym}^2(V^*))^G = 1$, and $V \cong V^*$ is given by a G-invariant nondegenerate symmetric form which is unique up to a scalar. If V is of type (III), then $(\wedge^2 V^*)^G = 0$ because there is an injection $(\wedge^2 V^*)^G \to (V^* \otimes V^*)^G = \mathrm{Hom}_G(V,V^*) = 0$. Moreover, $\dim(V \otimes V^*)^G = 1$ because $(V \otimes V^*)^G = \mathrm{Hom}_G(V,V) = \mathbf{C}$. Finally, note that if V and V' are irreducible representations of different type, one has $(V \otimes V')^G = 0$ and $\mathrm{Hom}_G(V,V') = 0$.

Assume that V is of type I. An element

$$\varphi \in (\wedge^2 V^*)^G = \mathrm{Hom}_G(V,V^*)$$

is represented by a matrix X if we choose a basis of V and choose its dual basis of V^*. By changing the initial basis if necessarily, we may assume that $X = aJ$, where $a \in \mathbf{C}$ and where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Here notice that V is an even-dimensional $\mathbf{C}$-vector space. Similarly an element

$$\varphi \in \big(\wedge^2\big((V^*)^{\oplus n}\big)\big)^G \subset \mathrm{Hom}_G\big(V^{\oplus n}, (V^*)^{\oplus n}\big)$$

is represented by a matrix

$$X = \begin{pmatrix} a_{11}J & a_{12}J & \cdots & a_{1n}J \\ a_{21}J & a_{22}J & \cdots & a_{2n}J \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}J & a_{n2}J & \cdots & a_{nn}J \end{pmatrix},$$

where $A := (a_{ij})$ is a symmetric matrix. If φ is nondegenerate, then for a suitable matrix T of the form

$$T = \begin{pmatrix} t_{11}I & t_{12}I & \cdots & t_{1n}I \\ t_{21}I & t_{22}I & \cdots & t_{2n}I \\ \cdots & \cdots & \cdots & \cdots \\ t_{n1}I & t_{n2}I & \cdots & t_{nn}I \end{pmatrix},$$

we have

$${}^tTXA = \begin{pmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J \end{pmatrix}.$$

Assume that V is of type II. Then an element

$$\varphi \in (\mathrm{Sym}^2 V^*)^G = \mathrm{Hom}_G(V, V^*)$$

can be represented by a matrix of the form aI with $a \in \mathbf{C}^*$. Similarly an element

$$\varphi \in \big(\mathrm{Sym}^2\big((V^*)^{\oplus n}\big)\big)^G \subset \mathrm{Hom}_G\big(V^{\oplus n}, (V^*)^{\oplus n}\big)$$

is represented by a matrix

$$X = \begin{pmatrix} a_{11}I & a_{12}I & \cdots & a_{1n}I \\ a_{21}I & & \cdots & a_{2n}I \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}I & a_{n2}I & \cdots & a_{nn}I \end{pmatrix},$$

where $A := (a_{ij})$ is a skew-symmetric matrix. If φ is nondegenerate, then for a suitable matrix T of the form

$$T = \begin{pmatrix} t_{11}I & t_{12}I & \cdots & t_{1n}I \\ t_{21}I & t_{22}I & \cdots & t_{2n}I \\ \cdots & \cdots & \cdots & \cdots \\ t_{n1}I & t_{n2}I & \cdots & t_{nn}I \end{pmatrix},$$

we have

$${}^tTXA = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Finally, assume that V is of type III. Then $(\wedge^2(V \oplus V^*))^G \subset \mathrm{Hom}_G(V \oplus V^*, V^* \oplus V)$. An element $(\wedge^2(V \oplus V^*))^G$ is represented by a matrix

$$\begin{pmatrix} 0 & aI \\ -aI & 0 \end{pmatrix}.$$

Similarly an element

$$\varphi \in \left(\wedge^2\left(V^{\oplus n} \oplus (V^*)^{\oplus n}\right)\right)^G \subset \mathrm{Hom}_G\left(V^{\oplus n} \oplus (V^*)^{\oplus n}, (V^*)^{\oplus n} \oplus V^{\oplus n}\right)$$

is represented by a matrix

$$X = \begin{pmatrix} 0 & A \\ -{}^tA & 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} a_{11}I & a_{12}I & \cdots & a_{1n}I \\ a_{21}I & & \cdots & a_{2n}I \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}I & a_{n2}I & \cdots & a_{nn}I \end{pmatrix}.$$

If φ is nondegenerate, then for a suitable matrix T of the form

$$\begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix},$$

we have

$${}^tTXA = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Now let us consider the V in Lemma 1.2. Decompose V into the sum of irreducible representations

$$V = \bigoplus (V_i)^{\oplus l_i} \oplus \bigoplus (V_j')^{\oplus m_j} \oplus \bigoplus (W_k)^{\oplus n_k},$$

where V_i are of type (I), V_j' are of type (II), and W_k are of type (III). Since V admits a G-invariant nondegenerate 2-form φ, we see that in the third factor $\bigoplus (W_k)^{\oplus n_k}$ each irreducible representation and its dual one appear in a pairwise way. Thus the third factor can be written as $\bigoplus (W_k \oplus W_k^*)^{\oplus n_k}$. By the observations above we see that φ can be transformed to a standard G-equivariant 2-form after making a suitable G-equivariant base change of V.

Let us return to the proof of Proposition 1.1. Let $\tilde{\omega}(0) \in \wedge^2 T_0^*(\mathbf{C}^{2n})$ and $\tilde{\omega}'(0) \in \wedge^2 T_0^*(\mathbf{C}^{2n})$ be, respectively, the restriction of $\tilde{\omega}$ and $\tilde{\omega}'$ to the origin $0 \in \mathbf{C}^{2n}$. By the lemma above, we may assume from the first that $\tilde{\omega}(0) = \tilde{\omega}'(0)$. The rest of the argument is an equivariant version of Moser's standard argument. For $\tau \in \mathbf{R}$, define

$$\tilde{\omega}_\tau := (1-\tau)\tilde{\omega} + \tau\tilde{\omega}'.$$

We put

$$u := d\tilde{\omega}_\tau / d\tau.$$

Let us consider the complex $(\pi_*^G \Omega^{\cdot}_{\mathbf{C}^{2n}}, d)$, which is a resolution of the consant sheaf $\mathbf{C}_X$. Note that u is a section of $\pi_*^G \Omega^2_{\mathbf{C}^{2n}}$. Since u is d-closed, one can write $u = dv$ with a G-invariant 1-form v. Moreover, v can be chosen such that $v(\mathbf{0}) = 0$. Define a vector field X_τ on $(\mathbf{C}^{2n}, 0)$ by

$$i_{X_\tau} \tilde{\omega}_\tau = -v.$$

Since $\tilde{\omega}(\tau)$ is d-closed, we have

$$L_{X_\tau} \tilde{\omega}_\tau = -u,$$

where $L_{X_\tau} \tilde{\omega}_\tau$ is the Lie derivative of $\tilde{\omega}_\tau$ along X_τ. If we take a sufficiently small open set V of $\mathbf{0} \in \mathbf{C}^{2n}$, then the vector fields $\{X_\tau\}_{0 \le \tau \le 1}$ define a family of open immersions $\tilde{\varphi}_\tau : V \to \mathbf{C}^{2n}$ via

$$d\tilde{\varphi}_\tau / d\tau = X_\tau(\tilde{\varphi}_\tau), \qquad \tilde{\varphi}_0 = \mathrm{id}\,.$$

Since all $\tilde{\varphi}_\tau$ fix the origin and the X_τ are all G-invariant, the $\tilde{\varphi}_\tau$ induce G-invariant automorphisms of $(\mathbf{C}^{2n}, 0)$. We have

$$d(\tilde{\varphi}_\tau^* \tilde{\omega}_\tau)/d\tau = \tilde{\varphi}_\tau^* (d\tilde{\omega}_\tau / d\tau + L_{X_\tau} \tilde{\omega}_\tau) = 0.$$

In particular, $\tilde{\varphi}_0^* \tilde{\omega}_0 = \tilde{\varphi}_1^* \tilde{\omega}_1$. The left-hand side is $\tilde{\omega}$, and the right-hand side is $\tilde{\varphi}_1^* \tilde{\omega}'$. If we put $\tilde{\varphi} := \tilde{\varphi}_1$, then $\tilde{\varphi}$ is a desired G-equivariant automorphism of $(\mathbf{C}^{2n}, 0)$. □

2. Affine varieties with $\mathbf{C}^*$-actions and symplectic structures

Let X be a normal affine variety of dimension $2d$ with a $\mathbf{C}^*$-action. Assume that $0 \in X$ is a unique fixed point of the $\mathbf{C}^*$-action with positive weights. More precisely, the cotangent space $m_{X,0}/m_{X,0}^2$ of $0 \in X$ has only positive weights with respect to the $\mathbf{C}^*$-action, or equivalently, the affine ring R of X is positively graded: $\bigoplus_{i \ge 0} R_i$ with $R_0 = \mathbf{C}$. In the rest of the paper we call such a $\mathbf{C}^*$-action a *good* $\mathbf{C}^*$-action. Let ω be an algebraic symplectic 2-form on X_{reg} with weight l. If we represent the $\mathbf{C}^*$-action by the family $\{\phi_t\}_{t \in \mathbf{C}^*}$ of automorphisms of X, then $\phi_t^*(\omega) = t^l \cdot \omega$.

LEMMA 2.1

If ω' is another symplectic 2-form with weight l', then $l = l'$.

Proof

Assume that $l < l'$. Since ${\omega'}^d$ is a generator of the canonical line bundle K_X, one can write $\omega^d = g \cdot {\omega'}^d$ with a homogeneous regular function g on X with negative weight $l - l'$. But this contradicts the assumption that X is positively weighted. □

REMARK

The lemma shows that if we fix a $\mathbf{C}^*$-action on X, then l is uniquely determined. But if we replace the $\mathbf{C}^*$-action on X by a different one, l may possibly change.

For example, let X be a 2-dimensional quotient singularity $\mathbf{C}^2/G$ where G is a cyclic group of order m acting on $\mathbf{C}^2$ as $x \to \zeta \cdot x$ and $y \to \zeta^{-1} \cdot y$ with a primitive mth root ζ of unity. Introduce a $\mathbf{C}^*$-action on $\mathbf{C}^2$ by $x \to t^p \cdot x$ and $y \to t^q \cdot y$ with positive integers p and q which are coprime to each other. Put $u := x^m$, $v := y^m$, and $w := xy$. Then X is an affine subvariety of $\mathbf{C}^3(u,v,w)$ defined by the equation $uv - w^m = 0$. The $\mathbf{C}^*$-action on $\mathbf{C}^2(x,y)$ descends to a $\mathbf{C}^*$-action on X. With respect to this $\mathbf{C}^*$-action, we have

$$\big(wt(u), wt(v), wt(w)\big) = (mp, mq, p+q).$$

If we choose p, q in such a way that $p+q$ and m are coprime, then $\mathrm{GCD}(mp, mq, p+q) = 1$. By definition X has a symplectic 2-form

$$\omega := du \wedge dv / w^{m-1},$$

which has weight $p+q$.

Before going to the next lemma, we recall the notions of a symplectic singularity and a canonical singularity. Let (X, ω) be a normal affine variety with a $\mathbf{C}^*$-action and an algebraic symplectic 2-form ω with weight l. Since $\omega^d := \omega \wedge \cdots \wedge \omega$ is a generator of the dualizing sheaf ω_X, the canonical divisor K_X is a Cartier divisor. Let $\pi : Y \to X$ be a resolution, and let E_i $(1 \le i \le n)$ be the π-exceptional divisors. One can write $K_Y = \pi^* K_X + \Sigma a_i E_i$ with some integers a_i. If $a_i \ge 0$ for all i, then we say that X has *canonical singularities*. On the other hand, if ω is pulled back to a regular 2-form on Y, we say that X has *symplectic singularities* (see [Be]). By [Na2], X has canonical singularities if and only if X has symplectic singularities.

In order to check that X does not have canonical singularities, we only have to find a *partial* resolution $f : Z \to X$ such that $\mathrm{Exc}(f)$ contains a divisor E such that $f^*(\omega^d)$ has a pole along E.

LEMMA 2.2

If X has only canonical singularities, then l is positive.

Proof

We prove that if $l \le 0$, then X does not have canonical singularities. Let R be the affine ring of X. By the $\mathbf{C}^*$-action of X, R has a grading $R = \bigoplus_{k \ge 0} R_k$ with $R_0 = \mathbf{C}$. Let $x_0, \ldots, x_n$ be homogeneous minimal generators of the $\mathbf{C}$-algebra R, and put $a_i := \mathrm{wt}(x_i)$. We assume that $\mathrm{GCD}(a_0, \ldots, a_n) = 1$. The affine variety X is embedded in $\mathbf{C}^{n+1}$ by x_i's. Let $\pi : V \to \mathbf{C}^{n+1}$ be the weighted blowing up of $\mathbf{C}^{n+1}$ with weight $(a_0, \ldots, a_n)$. By the definition, V is covered by open sets V_i $(0 \le i \le n)$ and there is a $\mathbf{Z}/a_i\mathbf{Z}$-Galois cover

$$p_i : \mathbf{C}^{n+1} \to V_i$$

such that

$$(\pi \circ p_i)^* x_i = (x_i')^{a_i},$$

$$(\pi \circ p_i)^* x_j = (x_i')^{a_j} x_j' \quad (j \neq i),$$

and p_i is the quotient map of the $\mathbf{Z}/a_i\mathbf{Z}$-action on $\mathbf{C}^{n+1}$,

$$x_i' \to \zeta \cdot x_i',$$
$$x_j' \to \zeta^{-a_j} \cdot x_j'$$

with an a_ith primitive root ζ of unity. The exceptional divisor $E := \pi^{-1}(0)$ is isomorphic to the weighted projective space $\mathbf{P}(a_0, \dots, a_n)$. Let us observe the restriction of p_i to $p_i^{-1}(E \cap V_i)$. Note that $p_i^{-1}(E \cap V_i)$ is a divisor of $\mathbf{C}^{n+1}$ defined by the equation $x_i' = 0$ and that the $\mathbf{Z}/a_i\mathbf{Z}$-action on $p_i^{-1}(E \cap V_i)$ is given by

$$x_j' \to \zeta^{-a_j} \cdot x_j'.$$

By the assumption $\mathrm{GCD}(a_0, \dots, a_n) = 1$, we see that $\mathbf{Z}/a_i\mathbf{Z}$ acts effectively on $p_i^{-1}(E \cap V_i)$. Therefore,

$$p_i^{-1}(E \cap V_i) \to E \cap V_i$$

is a $\mathbf{Z}/a_i\mathbf{Z}$-Galois covering. Let $p \in E$ be a general point. Then V is smooth at p. Let $\tilde{X}$ be the proper transform of $X \subset \mathbf{C}^{n+1}$ by the weighted blowing up $\pi : V \to \mathbf{C}^{n+1}$, and let

$$\pi_X : \tilde{X} \to X$$

be the induced birational morphism. Note that

$$E \cap \tilde{X} = \mathrm{Proj}\Big(\bigoplus_{k \geq 0} R_k\Big).$$

Since $E \cap \tilde{X}$ is generically smooth and E is a Cartier divisor at a general point $p \in E \cap \tilde{X}$, we can see that $\tilde{X}$ is also smooth at such a point p.

Now let us consider the $2d$-form ω^d and regard it as a section of the canonical line bundle K_X. We shall prove that $(\pi_X)^*\omega^d$ has a pole along $E \cap \tilde{X}$ if $l \leq 0$. Take a general point $p \in E \cap \tilde{X}$, and assume that $p \in V_i$. We put $\tilde{X}_i := (p_i)^{-1}(\tilde{X} \cap V_i)$ and $E_i := (p_i)^{-1}(E \cap V_i)$. Recall that $p_i^{-1}(E \cap V_i) \to E \cap V_i$ is a $\mathbf{Z}/a_i\mathbf{Z}$-Galois covering whose branch locus is contained in the divisor $\prod_{j \neq i} x_j = 0$ of $E = \mathbf{P}(a_0, \dots, a_n)$. Since $\mathrm{Proj}(\bigoplus_{k \geq 0} R_k)$ is not contained in the divisor $\prod x_j = 0$ of $\mathbf{P}(a_0, \dots, a_n)$, we see that

$$E_i \cap \tilde{X}_i \to E \cap \tilde{X} \cap V_i$$

is a $(\mathbf{Z}/a_i\mathbf{Z})$-Galois cover. This implies that the order of the zeros (or the poles) of $(\pi_X)^*\omega^d$ along $E \cap \tilde{X}$ coincides with the order of the zeros (or the poles) of $(\pi_X \circ p_i|_{\tilde{X}_i})^*\omega^d$ along $E_i \cap \tilde{X}_i$. Let $q \in \tilde{X}_i$ be a point such that $p_i(q) = p$. One can choose the local coordinates of $q \in \tilde{X}_i$ from $x_j' - x_j'(q)$ $(0 \leq j \leq n)$. Since E_i is smooth at q, we can include x_i' among the local coordinates. (Note that $x_i'(q) = 0$.) Assume that x_i', $x_{j_1}' - x_{j_1}'(q)$, $\dots$, $x_{j_{2d-1}}' - x_{j_{2d-1}}'(q)$ are local coordinates. Recall that V has a natural $\mathbf{C}^*$-action and this $\mathbf{C}^*$-action extends to the $\mathbf{C}^*$-action on

$(x'_0, \dots, x'_n) \in \mathbf{C}^{n+1}$ by

$$x'_i \to t \cdot x'_i$$

and

$$x'_j \to x'_j \quad (j \neq i).$$

Since ω has weight l, the weight of $(\pi_X \circ p_i|_{\tilde{X}_i})^* \omega^d$ is $d \cdot l$. Around $q \in \tilde{X}_i$, one can write

$$(\pi_X \circ p_i|_{\tilde{X}_i})^* \omega^d = h \cdot dx'_i \wedge dx'_{j_1} \wedge \cdots \wedge dx'_{j_{2d-1}}$$

with a meromorphic function h of degree $d \cdot l - 1$. This means that $(\pi_X \circ p_i|_{\tilde{X}_i})^* \omega^l$ has poles of order $1 - d \cdot l$ along $E_i \cap \tilde{X}_i$ if $l \leq 0$. □

COROLLARY 2.3
If $\mathrm{Codim}_X \mathrm{Sing}(X) \geq 4$, *then* $l > 0$.

Proof
If $\mathrm{Sing}(X)$ has at least codimension 4 in X, then the symplectic 2-form ω extends to a regular 2-form on an arbitrary resolution $\tilde{X}$ of X by Flenner [Fl]. This implies that X has only canonical singularities. □

LEMMA 2.4
If X has an isolated singularity and $l > 0$, then X has only canonical singularities.

Proof
Let $\pi : Y \to X$ be a $\mathbf{C}^*$-equivariant resolution. Let Y_c be a relatively compact open subset of Y such that $\pi^{-1}(0) \subset Y_c$. Write $K_Y = \pi^* K_X + \Sigma a_i E_i$ where E_i are π-exceptional divisors. Since K_X is Cartier (because of the existence of ω), all coefficients a_i are integers. In order to prove that $a_i \geq 0$, we only need to prove that $a_i > -1$. This condition is equivalent to the L^2-condition (cf. the proof of [Ko, Proposition 3.20]):

$$\int_{Y_c} \pi^* \omega^d \wedge \pi^* \bar{\omega}^d < \infty.$$

Since $\mathbf{R}_{>0}$ is naturally contained in $\mathbf{C}^*$, each element $t \in \mathbf{R}_{>0}$ acts on X as an automorphism ϕ_t of X. Let U be an open neighborhood of $0 \in X$ such that $\phi_t(U) \subset U$ for all $t \in (0, 1]$. Put $V := \pi^{-1}(U)$. Fix $\epsilon_0 \in (0, 1)$, and put $U_n := \phi_{\epsilon_0^n}(U) - \phi_{\epsilon_0^{n+1}}(U)$. Define $V_n := \pi^{-1}(U_n)$. Since $\phi_t^* \omega = t^l \cdot \omega$, we have

$$\int_{V_n} \pi^* \omega^d \wedge \pi^* \bar{\omega}^d = \epsilon_0^{2dnl} \cdot \int_{V_0} \pi^* \omega^d \wedge \pi^* \bar{\omega}^d.$$

By the definition we have

$$\int_V \pi^* \omega^d \wedge \pi^* \bar{\omega}^d = \sum_{n=0}^{\infty} \int_{V_n} \pi^* \omega^d \wedge \pi^* \bar{\omega}^d.$$

But the right-hand side equals

$$\Big(\sum_{n=0}^{\infty}\epsilon_0^{2dnl}\Big)\int_{V_0}\pi^*\omega^d\wedge\pi^*\bar{\omega}^d<\infty.$$

The desired L^2-condition has now been proved.

Note that this proof is not valid for a nonisolated case because $\int_{V_n}\pi^*\omega^d\wedge\pi^*\bar{\omega}^d$ might be infinite. □

3. Algebraic version of equivalence problems

In this section (X,ω) is a pair of a normal affine variety X of dimension $2d$ with a good $\mathbf{C}^*$-action and an algebraic symplectic 2-form ω on X_{reg} with weight l. We shall consider the equivalence problem for a pair (X,ω). Let (X',ω') be another pair. Then (X,ω) and (X',ω') are equivalent if there is a $\mathbf{C}^*$-equivariant isomorphism $\phi: X\cong X'$ such that $\omega=\phi^*(\omega')$. In particular, if $X=X'$, then ω and ω' are called equivalent symplectic structures on X. A purpose of this section is to prove the following theorem.

THEOREM 3.1

Assume that $l\neq 0$. Then ω is a unique symplectic structure with weight l on X up to equivalence.

We shall briefly recall some basic results on Poisson structures and their deformations (for details see [Na1]). Note that the symplectic 2-form ω gives a natural Poisson structure $\{\ ,\ \}$ on X_{reg}. By the normality of X, this Poisson structure extends to a Poisson structure X. We denote this bracket also by $\{\ ,\ \}$. The bracket $\{\ ,\ \}$ has weight $-l$ with respect to the $\mathbf{C}^*$-action because ω has weight l. Namely, if f and g are homogeneous element of O_X of degree a and b, then $\{f,g\}$ is a homogeneous element of degree $a+b-l$.

By using the Poisson bracket we define the Lichnerowicz–Poisson complex

$$0\to\Theta_{X_{\mathrm{reg}}}\stackrel{\delta_1}{\to}\wedge^2\Theta_{X_{\mathrm{reg}}}\stackrel{\delta_2}{\to}\cdots$$

by

$$\begin{aligned}
&\delta_p f(da_1\wedge\cdots\wedge da_{p+1})\\
&\quad:=\sum_{i=1}^{p+1}(-1)^{i+1}\{a_i,f(da_1\wedge\hat{da_i}\wedge\cdots\wedge da_{p+1})\}\\
&\qquad+\sum_{j<k}(-1)^{j+k}f\big(d\{a_j,a_k\}\wedge da_1\wedge\cdots\wedge\hat{da_j}\wedge\cdots\wedge\hat{da_k}\wedge\cdots\wedge da_{p+1}\big).
\end{aligned}$$

In the Lichnerowicz–Poisson complex, $\wedge^p\Theta_{X_{\mathrm{reg}}}$ is placed in degree p. By the symplectic form ω, each term $\wedge^p\Theta_{X_{\mathrm{reg}}}$ can be identified with the sheaf $\Omega^p_{X_{\mathrm{reg}}}$ of p-forms. Moreover, the Lichnerowicz–Poisson complex is identified with the truncated de Rham complex

$$0 \to \Omega^1_{X_{\mathrm{reg}}} \xrightarrow{d} \Omega^2_{X_{\mathrm{reg}}} \xrightarrow{d} \cdots.$$

Put $S_1 := \operatorname{Spec} \mathbf{C}[\epsilon]$. Then the second cohomology $\mathbf{H}^2(\Gamma(X_{\mathrm{reg}}, \wedge^{\geq 1}\Theta_{X_{\mathrm{reg}}}))$ describes the equivalence classes of the O_{S_1}-bilinear Poisson structures $\{\ ,\ \}_\epsilon$ on $X_{\mathrm{reg}} \times S_1$ which are extensions of the original Poisson structure $\{\ ,\ \}$ on $X_{\mathrm{reg}} \times \{0\}$. In fact, for $\varphi \in \Gamma(X_{\mathrm{reg}}, \wedge^2\Theta_{X_{\mathrm{reg}}})$, we define a bracket $\{\ ,\ \}_\epsilon$ on $O_{X_{\mathrm{reg}}} \oplus \epsilon O_{X_{\mathrm{reg}}}$ by

$$\{f + \epsilon f', g + \epsilon g'\}_\epsilon := \{f, g\} + \epsilon\big(\varphi(df \wedge dg) + \{f, g'\} + \{f', g\}\big).$$

Then this bracket is a Poisson bracket if and only if $\delta(\varphi) = 0$. On the other hand, an element $\theta \in \Gamma(X_{\mathrm{reg}}, \Theta_{X_{\mathrm{reg}}})$ corresponds to an automorphism ϕ_θ of $X_{\mathrm{reg}} \times S_1$ over S_1 which restricts to give the identity map of $X_{\mathrm{reg}} \times \{0\}$. Let $\{\ ,\ \}_{\epsilon,1}$ and $\{\ ,\ \}_{\epsilon,2}$ be the Poisson structures determined, respectively, by elements φ_1 and φ_2 of $\Gamma(X_{\mathrm{reg}}, \wedge^2\Theta_{X_{\mathrm{reg}}})$. Then the two Poisson structures are equivalent under ϕ_θ if $\varphi_1 - \varphi_2 = \delta(\theta)$.

Note that a Poisson structure $\{\ ,\ \}_\epsilon$ on $X_{\mathrm{reg}} \times S_1$ uniquely extends to a Poisson structure on $X \times S_1$. This means that $\mathbf{H}^2(\Gamma(X_{\mathrm{reg}}, \wedge^{\geq 1}\Theta_{X_{\mathrm{reg}}}))$ also describes equivalence classes of the O_{S_1}-bilinear Poisson structures $\{\ ,\ \}_\epsilon$ on $X \times S_1$ which are extensions of the original Poisson structure $\{\ ,\ \}$ on $X \times \{0\}$.

Let us introduce a $\mathbf{C}^*$-action on $X \times S_1$ in such a way that it acts on the first factor by the original action and acts trivially on the second factor. The following proposition is a $\mathbf{C}^*$-equivariant version of the above observation.

PROPOSITION 3.2 (RIGIDITY PROPOSITION)

Let $\{\ ,\ \}_{\epsilon,1}$ and $\{\ ,\ \}_{\epsilon,2}$ be two Poisson structures on $X \times S_1$ relative to S_1, both of which have weight $-l \neq 0$ and induce the original Poisson structure on $X \times \{0\}$. Then there is a $\mathbf{C}^*$-equivariant automorphism of $X \times S_1$ over S_1 such that it induces the identity map of $X \times \{0\}$ and it sends $\{\ ,\ \}_{\epsilon,1}$ to $\{\ ,\ \}_{\epsilon,2}$.

Proof

Let $(\wedge^{\geq 1}\Theta_{X_{\mathrm{reg}}}, \delta)$ be the Lichnerowicz–Poisson complex for a Poisson manifold X_{reg}. The algebraic torus $\mathbf{C}^*$ acts on $\Gamma(X_{\mathrm{reg}}, \wedge^p\Theta_{X_{\mathrm{reg}}})$, and there is an associated grading

$$\Gamma(X_{\mathrm{reg}}, \wedge^p\Theta_{X_{\mathrm{reg}}}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X_{\mathrm{reg}}, \wedge^p\Theta_{X_{\mathrm{reg}}})(n).$$

The coboundary map δ has degree $-l$; thus we have a complex

$$\Gamma(X_{\mathrm{reg}}, \wedge^1\Theta_{X_{\mathrm{reg}}})(0) \xrightarrow{\delta_1} \Gamma(X_{\mathrm{reg}}, \wedge^2\Theta_{X_{\mathrm{reg}}})(-l) \xrightarrow{\delta_2} \Gamma(X_{\mathrm{reg}}, \wedge^3\Theta_{X_{\mathrm{reg}}})(-2l).$$

The middle cohomology $\operatorname{Ker}(\delta_2)/\operatorname{Im}(\delta_1)$ of this complex describes the equivalence classes of the extension of the Poisson structure $\{\ ,\ \}$ on X_{reg} to that on $X_{\mathrm{reg}} \times S_1$ with weight $-l$ up to $\mathbf{C}^*$-equivariant automorphism of $X_{\mathrm{reg}} \times S_1$ over S_1 that induces the identity map of $X_{\mathrm{reg}} \times \{0\}$. Since each Poisson structure $X_{\mathrm{reg}} \times S_1$ uniquely extends to that on $X \times S_1$, $\operatorname{Ker}(\delta_2)/\operatorname{Im}(\delta_1)$ also describes the equivalence classes of the extension of the Poisson structure $\{\ ,\ \}$ on X to that on $X \times S_1$ with weight $-l$ up to $\mathbf{C}^*$-equivariant automorphism of $X \times S_1$ over S_1 that induces the identity map of $X \times \{0\}$.

The Lichnerowicz–Poisson complex $(\wedge^{\geq 1}\Theta_{X_{\mathrm{reg}}}, \delta)$ is identified with the truncated de Rham complex $(\Omega^{\geq 1}_{X_{\mathrm{reg}}}, d)$ by the symplectic form ω. The algebraic torus $\mathbf{C}^*$ acts on $\Gamma(X_{\mathrm{reg}}, \Omega^p_{X_{\mathrm{reg}}})$ and there is an associated grading

$$\Gamma(X_{\mathrm{reg}}, \Omega^p_{X_{\mathrm{reg}}}) = \bigoplus_{n \in \mathbf{Z}} \Gamma(X_{\mathrm{reg}}, \Omega^p_{X_{\mathrm{reg}}})(n).$$

The coboundary map d has degree zero; thus we have a complex

$$\Gamma(X_{\mathrm{reg}}, \Omega^1_{X_{\mathrm{reg}}})(l) \xrightarrow{d_1} \Gamma(X_{\mathrm{reg}}, \Omega^2_{X_{\mathrm{reg}}})(l) \xrightarrow{d_2} \Gamma(X_{\mathrm{reg}}, \Omega^3_{X_{\mathrm{reg}}})(l).$$

Since ω has weight l, this complex is identified with the three-term complex above.

We shall prove that $\mathrm{Ker}(d_2)/\mathrm{Im}(d_1) = 0$. The $\mathbf{C}^*$-action on X defines a vector field ζ on X_{reg}. According to Naruki [Nar, Lemma 2.1.1] we define

$$\Delta : \Gamma(X_{\mathrm{reg}}, \Omega^2_{X_{\mathrm{reg}}}) \to \Gamma(X_{\mathrm{reg}}, \Omega^1_{X_{\mathrm{reg}}})$$

by $\Delta(v) := i_\zeta v$. Since ζ is a $\mathbf{C}^*$-equivariant vector field, Δ induces a map

$$\Delta : \Gamma(X_{\mathrm{reg}}, \Omega^2_{X_{\mathrm{reg}}})(l) \to \Gamma(X_{\mathrm{reg}}, \Omega^1_{X_{\mathrm{reg}}})(l).$$

For $v \in \Gamma(X_{\mathrm{reg}}, \Omega^2_{X_{\mathrm{reg}}})(l)$, the Lie derivative $L_\zeta v$ of v along ζ equals $l \cdot v$. If moreover v is d-closed, then one has $l \cdot v = d(i_\zeta v)$ by the Cartan relation

$$L_\zeta v = d(i_\zeta v) + i_\zeta(dv).$$

This means that v is d-exact. □

Proof of Theorem 3.1
Denote by R the affine ring of X. By definition, R has a natural grading $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = \mathbf{C}$. Let $j : X_{\mathrm{reg}} \to X$ be the inclusion map. Since $j_*\Omega^2_{X_{\mathrm{reg}}}$ is a coherent O_X-module, $M := \Gamma(X_{\mathrm{reg}}, \Omega^2_{X_{\mathrm{reg}}})$ is a finitely generated, graded R-module: $M = \bigoplus M_i$. Each M_i is a finite-dimensional $\mathbf{C}$-vector space because $R_i = 0$ for $i < 0$ and $R_0 = \mathbf{C}$. Our ω is an element of M_l by the definition. Let $M_{l,\mathrm{closed}}$ be the subspace of M_l which consists of d-closed 2-forms. Let $\mathrm{Aut}^{\mathbf{C}^*}(X)$ be the algebraic group of $\mathbf{C}^*$-equivariant automorphisms of X. Then $\mathrm{Aut}^{\mathbf{C}^*}(X)$ acts on $M_{l,\mathrm{closed}}$. Let $M^0_{l,\mathrm{closed}}$ be the Zariski open subset of $M_{l,\mathrm{closed}}$ which consists of nondegenerate 2-forms. In particular, $M^0_{l,\mathrm{closed}}$ is connected. Since $\mathrm{Aut}^{\mathbf{C}^*}(X)$ preserves $M^0_{l,\mathrm{closed}}$ as a set, we see that $M^0_{l,\mathrm{closed}}$ is a single orbit of $\mathrm{Aut}^{\mathbf{C}^*}(X)$ by Proposition 3.2. □

REMARK 3.3
Let X be an affine variety defined by $f := x^3 + y^3 + z^3 = 0$ in $\mathbf{C}^3$. Then X has a natural $\mathbf{C}^*$-action with a fixed point $0 \in X$ and with $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$. Then regular part X_{reg} admits a symplectic form $\omega := \mathrm{Res}(dx \wedge dy \wedge dz/f)$. The

weight l of ω is zero. The blowing up of X at zero gives us a resolution $\pi : \tilde{X} \to X$ with an exceptional curve E, which is an elliptic curve. The pullback $\pi^*(\omega)$ is a meromorphic 2-form which has a pole along E. Thus (X, ω) is *not* a symplectic variety in the sense of [Be]. In this case, rigidity does not hold. In fact, $(X, t \cdot \omega)$ $(t \in \mathbf{C}^*)$ is a nontrivial Poisson deformation of (X, ω) (cf. [EG]). We shall give here a short proof of this fact. By the argument of the proof of the rigidity proposition, it suffices to prove that $\omega \in \Gamma(X_{\text{reg}}, \Omega^2_{X_{\text{reg}}})$ is not in the image of $d : \Gamma(X_{\text{reg}}, \Omega^1_{X_{\text{reg}}}) \to \Gamma(X_{\text{reg}}, \Omega^2_{X_{\text{reg}}})$. Note that ω is a meromorphic 2-form on $\tilde{X}$ having a pole along E at order 1. Thus one has $\omega \in \Gamma(\tilde{X}, \Omega^2_{\tilde{X}}(\log E))$. It can be checked that $\Gamma(\tilde{X}, \Omega^1_{\tilde{X}}(\log E)) \cong \Gamma(X_{\text{reg}}, \Omega^1_{X_{\text{reg}}})$. Let us consider the commutative diagram

$$
\begin{array}{ccc}
\Gamma\big(\tilde{X}, \Omega^1_{\tilde{X}}(\log E)\big) & \xrightarrow{\text{Res}} & \Gamma(E, O_E) \\
{\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
\Gamma\big(\tilde{X}, \Omega^2_{\tilde{X}}(\log E)\big) & \xrightarrow{\text{Res}} & \Gamma(E, \Omega^1_E)
\end{array}
\tag{1}
$$

Suppose that $\omega = d\eta$ for $\eta \in \Gamma(\tilde{X}, \Omega^1_{\tilde{X}}(\log E))$. Then one can write

$$\text{Res}(\omega) = d\,\text{Res}(\eta)$$

by the commutative diagram. For any 1-cycle γ on E, one has

$$\int_\gamma \text{Res}(\omega) = \int_\gamma d\,\text{Res}(\eta) = 0.$$

On the other hand, since $\text{Res}(\omega)$ is a nowhere vanishing 1-form on E, we should have

$$\int_\gamma \text{Res}(\omega) \neq 0$$

for some 1-cycle γ on E. This is a contradiction.

REMARK 3.4

Assume that X has canonical singularities. Then the complex

$$\Gamma(X_{\text{reg}}, \Omega^1_{X_{\text{reg}}}) \xrightarrow{d} \Gamma(X_{\text{reg}}, \Omega^2_{X_{\text{reg}}}) \xrightarrow{d} \Gamma(X_{\text{reg}}, \Omega^3_{X_{\text{reg}}})$$

is exact. In particular, the complex

$$\Gamma(X_{\text{reg}}, \Omega^1_{X_{\text{reg}}})(0) \xrightarrow{d} \Gamma(X_{\text{reg}}, \Omega^2_{X_{\text{reg}}})(0) \xrightarrow{d} \Gamma(X_{\text{reg}}, \Omega^3_{X_{\text{reg}}})(0)$$

is also exact.

The proof goes as follows. Let $f : \tilde{X} \to X$ be a $\mathbf{C}^*$-equivariant resolution. Let α be a d-closed holomorphic 2-form on X_{reg}. By [Na2, Theorem 4] one has $\Gamma(\tilde{X}, \Omega^2_{\tilde{X}}) = \Gamma(X_{\text{reg}}, \Omega^2_{X_{\text{reg}}})$. Thus $f^*\alpha$ is a holomorphic 2-form on $\tilde{X}$. The following argument is based on [Na3, Section 1]. One can also find a similar argument in [Ka].

We first show that $[f^*\alpha] \in H^2(\tilde{X}, \mathbf{C})$ is zero. It is sufficient to prove that, for a small open neighborhood U of $0 \in X$ (in the classical topology), $[f^*\alpha|_{f^{-1}(U)}] \in H^2(f^{-1}(U), \mathbf{C})$ is zero. In fact, the restriction map $H^2(\tilde{X}, \mathbf{C}) \to H^2(f^{-1}(U), \mathbf{C})$ is an isomorphism by the $\mathbf{C}^*$-action. One can blow up $\tilde{X}$ further to have a resolution $g : Z \to \tilde{X}$ so that $E := (f \circ g)^{-1}(0)$ is a simple normal crossing divisor of Z. Since $R^1 g_* \mathbf{C} = 0$, we have an injection $H^2(f^{-1}(U), \mathbf{C}) \to H^2((f \circ g)^{-1}(U), \mathbf{C})$. If we take U sufficiently small, then $H^2((f \circ g)^{-1}(U), \mathbf{C}) \cong H^2(E, \mathbf{C})$. To prove that $[f^*\alpha|_{f^{-1}(U)}] = 0$ in $H^2(f^{-1}(U), \mathbf{C})$, we only have to check that $[(f \circ g)^*\alpha|_E] = 0$ in $H^2(E, \mathbf{C})$. Here we note that $H^2(E, \mathbf{C})$ has a mixed Hodge structure and $F^2(H^2(E, \mathbf{C})) = H^0(E, \hat{\Omega}^2_E)$. The sheaf $\hat{\Omega}^2_E$ is the quotient sheaf of Ω^2_E by the torsion subsheaf supported on $\mathrm{Sing}(E)$. Since $(f \circ g)^*\alpha|_E$ is a holomorphic 2-form on E, we have $[(f \circ g)^*\alpha|_E] \in H^0(E, \hat{\Omega}^2_E)$. But $H^0(E, \hat{\Omega}^2_E) = 0$ by [Na3, Lemma 1.2]. As a consequence, we have proved that $[f^*\alpha] \in H^2(\tilde{X}, \mathbf{C})$ is zero.

Now look at the Hodge spectral sequence

$$E_1^{p,q} = H^q(\tilde{X}, \Omega^p_{\tilde{X}}) \Rightarrow H^{p+q}(\tilde{X}, \mathbf{C}).$$

Then $f^*\alpha = d\eta$ mod. $E_2^{0,1}$ with some holomorphic 1-form η on $\tilde{X}$. Since X has rational singularities, we have $E_1^{0,1} = 0$; hence $E_2^{0,1} = 0$. Thus $f^*\alpha = d\eta$. This clearly shows that the original complex is exact.

PROPOSITION 3.5 (STRONG RIGIDITY)

Assume, in addition, that X has canonical singularities. Let $(X_1, \{\ ,\ \}_\epsilon)$ be a $\mathbf{C}^*$-equivariant Poisson deformation of $(X, \{\ ,\ \})$ over S_1 in such a way that $\mathbf{C}^*$ acts on S_1 trivially. Then this Poisson deformation is a trivial one.

Proof

The difference from Proposition 3.2 is that we do not assume that $X_1 = X \times S_1$. Let $f : \mathcal{X} \to \mathbf{A}^d$ be a $\mathbf{C}^*$-equivariant universal Poisson deformation of X over an affine space $\mathbf{A}^d$ constructed in [Na1]. Note that there is a Poisson isomorphism $\iota : X \cong f^{-1}(0)$. The $\mathbf{C}^*$-action on X induces a $\mathbf{C}^*$-action on the base space $\mathbf{A}^d$ of the universal Poisson deformation. By the construction of f (see [Na1, Section 5.2]), this action has only positive weights.

The infinitesimal Poisson deformation $X_1 \to S_1$ determines a map $S_1 \to \mathbf{A}^d$ which sends the closed point of S_1 to the origin of $\mathbf{A}^d$. Assume that this is a closed immersion; namely, $S_1 \subset \mathbf{A}^d$. By the assumption, the $\mathbf{C}^*$-action on $\mathbf{A}^d$ restricts to the trivial action on S_1. This contradicts the fact that the $\mathbf{C}^*$-action on $\mathbf{A}^d$ has only positive weights. Thus the map $S_1 \to \mathbf{A}^d$ is a constant map. □

4. Projectivized cone and contact structures

In this section (X, ω) is a pair of a normal affine variety X of dimension $2d$ with a good $\mathbf{C}^*$-action and an algebraic symplectic 2-form ω on X_{reg} with *positive* weight l.

4.1. Projectivized cone

Let R be the affine ring of X. By definition, R has a natural grading $R = \bigoplus_{i\geq 0} R_i$ with $R_0 = \mathbf{C}$. We put

$$\mathbf{P}(X) := \mathrm{Proj}\Big(\bigoplus_{i\geq 0} R_i\Big).$$

Let x_0, $x_1, \ldots, x_n$ be homogeneous minimal generators of the $\mathbf{C}$-algebra R, and put $a_i = \mathrm{wt}(x_i)$. Then $\mathbf{P}(X)$ is naturally embedded in the weighted projective space $\mathbf{P}(a_0, a_1, \ldots, a_n)$. Let $V \to \mathbf{C}^{n+1}$ be the weighted blowing up of $\mathbf{C}^{n+1}$ with weight $(a_0, \ldots, a_n)$. Then the fiber over the origin $0 \in \mathbf{C}^{n+1}$ is isomorphic to $\mathbf{P}(a_0, \ldots, a_n)$. The singular locus $\mathrm{Sing}(V)$ of V is contained in the fiber over the origin; hence one can regard $\mathrm{Sing}(V)$ as a subset of $\mathbf{P}(a_0, \ldots, a_n)$. In this identification $\mathrm{Sing}(V)$ is the locus where the projection map $\mathbf{C}^{n+1} - \{0\} \to \mathbf{P}(a_0, \ldots, a_n)$ is not a $\mathbf{C}^*$-bundle. As the subsets of $\mathbf{P}(a_0, \ldots, a_n)$, we may take the intersection of $\mathbf{P}(X)$ and $\mathrm{Sing}(V)$. We assume that $\mathbf{P}(X) \cap \mathrm{Sing}(V)$ has codimension at least 2 in $\mathbf{P}(X)$. Notice that this condition is not necessarily satisfied. For example, A_m-surface singularity $x_0^2 + x_1^2 + x_2^{m+1} = 0$ for $m \geq 2$ does not satisfy this condition.

Let $\mathbf{P}(X)^0$ be the open subset obtained by excluding this subset and $\mathrm{Sing}(\mathbf{P}(X))$ from $\mathbf{P}(X)$. Note that $\mathrm{Codim}_{\mathbf{P}(X)}(\mathbf{P}(X) - \mathbf{P}(X)^0) \geq 2$. There is a natural projection

$$p : X - \{0\} \to \mathbf{P}(X),$$

which is a $\mathbf{C}^*$-fibration and is actually a $\mathbf{C}^*$-fiber bundle over $\mathbf{P}(X)^0$. We put $X^0 := p^{-1}(\mathbf{P}(X)^0)$. Let $O(1)$ be the tautological sheaf on $\mathbf{P}(a_0, \ldots, a_n)$, and put $O_{\mathbf{P}(X)}(1) := O(1) \otimes_{O_{\mathbf{P}(a_0,\ldots,a_n)}} O_{\mathbf{P}(X)}$. Then $O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)^0}$ is an invertible sheaf on $\mathbf{P}(X)^0$. Let $L \in \mathrm{Pic}(\mathbf{P}(X)^0)$ be the corresponding line bundle to this sheaf. More exactly, $O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)^0}$ is the sheaf of sections of L. Denote by L^{-1} the dual line bundle of L, and denote by $(L^{-1})^\times$ the $\mathbf{C}^*$-bundle which is obtained from L^{-1} by removing the zero section. Then X^0 coincides with $(L^{-1})^\times$, and the natural projection

$$\pi : (L^{-1})^\times \to \mathbf{P}(X)^0$$

coincides with $p|_{X^0}$. Note that there is a canonical trivialization

$$\pi^* L \cong O_{(L^{-1})^\times}.$$

Recall that l is the weight of ω. Later we will use the trivialization

$$\pi^*(L^{\otimes l}) \to O_{(L^{-1})^\times}$$

induced by this canonical trivialization.

4.2. Contact structure on a complex manifold

We shall briefly review a contact complex manifold according to LeBrun [LeB]. Let Z be a complex manifold of dimension $2d + 1$. A contact structure on Z is an exact sequence of vector bundles

$$0 \to D \to TZ \xrightarrow{\theta} M \to 0,$$

with $\mathrm{rank}(D) = 2d$ and $\mathrm{rank}(M) = 1$, so that $d\theta|_D$ induces a nondegenerate pairing on D. By using the formula for exterior derivation

$$d\theta(x,y) = x\big(\theta(y)\big) - y\big(\theta(x)\big) - \theta\big([x,y]\big),$$

one can check that this is equivalent to saying that $[\ ,\ ] : D \times D \to TZ/D(= M)$ is nondegenerate. We call M the contact line bundle. As is well known, infinitesimal automorphisms of Z are controlled by the cohomology group $H^0(Z, \Theta_Z)$. An infinitesimal automorphism of Z is said to be contact if it preserves the contact structure.

PROPOSITION 4.2.1 ([LeB, PROPOSITION 2.1])
Let

$$0 \to O(D) \to \Theta_Z \xrightarrow{\theta} O(M) \to 0$$

be the exact sequence of sheaves determined by the contact structure. Then there is a map $s : O(M) \to \Theta_Z$ *of* $\mathbf{C}$*-modules (not of* O_Z*-modules) that splits the sequence above, and the group of infinitesimal contact automorphisms coincides with* $s(H^0(Z, O(M)))$.

COROLLARY 4.2.2 (CF. [LeB, PROPOSITION 2.2])
Fix a line bundle M *on* Z. *Assume that* $TZ \xrightarrow{\theta} M$ *is a contact structure on* Z *such that* $H^0(Z, O(D)) = 0$. *Then* θ *is a unique contact structure with contact line bundle* M.

4.3. Quasi-contact structure on $\mathbf{P}(X)$

One can generalize the notion of contact structures to a singular variety. Let Z be a normal variety. Here a quasi-contact structure[†] on Z is just a contact structure on an open set $Z^0 \subset Z_{\mathrm{reg}}$ with $\mathrm{codim}_Z(Z - Z^0) \geq 2$. By the definition, there are a line bundle M on Z^0 and a vector bundle D on Z^0 of rank $2d$ which fit into an exact sequence

$$0 \to O(D) \to \Theta_{Z^0} \to O(M) \to 0.$$

Since the degeneracy locus of a contact form has codimension one, a contact structure on Z^0 uniquely extends to that on Z_{reg}. Thus we may say that a qusi-contact structure on Z is a contact structure on Z_{reg}. Let $j : Z^0 \to Z$ be the natural inclusion map. Then we have an exact sequence

$$0 \to j_*O(D) \to \Theta_Z \to j_*O(M) \to 0.$$

Note that the last map is surjective by Proposition 4.2.1.

[†]We do not assume that j_*M is a line bundle on Z. As we will define in Section 4.4, if j_*M is a line bundle on Z, we call it a contact structure on Z.

Let us return to the original situation. The complement of $\mathbf{P}(X)^0$ in $\mathbf{P}(X)$ has at least codimension 2. Let us introduce a quasi-contact structure on $\mathbf{P}(X)$. This is a slight modification of the argument in [LeB, p. 425], where the case $l=1$ is treated. Recall that we have a $\mathbf{C}^*$-bundle $p|_{X^0}: X^0 \to \mathbf{P}(X)^0$ and it is identified with $\pi: (L^{-1})^\times \to \mathbf{P}(X)^0$.

For $\theta \in H^0(\mathbf{P}(X)^0, \Omega^1_{\mathbf{P}(X)^0}(L^{\otimes l}))$, the pullback $\pi^*(\theta)$ is regarded as an element of $H^0((L^{-1})^\times, \Omega^1_{(L^{-1})^\times})$ by the trivialization $\pi^*(L^{\otimes l}) \to O_{(L^{-1})^\times}$.

By the assumption we have a symplectic 2-form ω on $(L^{-1})^\times$ with weight l. As a $\mathbf{C}^*$-bundle, there is a natural $\mathbf{C}^*$-action on $(L^{-1})^\times$. Let ζ be the vector field which generates the $\mathbf{C}^*$-action. Then one can write $\omega(\zeta, \cdot) = \pi^*\theta$ with an element $\theta \in H^0(\mathbf{P}(X)^0, \Omega^1_{\mathbf{P}(X)^0}(L^{\otimes l}))$. This θ gives a contact structure on $\mathbf{P}(X)^0$ with contact line bundle $L^{\otimes l}$. Conversely, if a contact structure $\theta \in H^0(\mathbf{P}(X)^0, \Omega^1_{\mathbf{P}(X)^0}(L^{\otimes l}))$ is given to $\mathbf{P}(X)^0$, then $d\pi^*(\theta)$ becomes a holomorphic symplectic 2-form on $(L^{-1})^\times$ with weight l. Note that we need the assumption $l \neq 0$ to get the correspondence between symplectic structures of weight l and contact structures.

4.4. Contact orbifold structure and Jacobi orbifold structure

In Section 4.1 we imposed a rather technical assumption; namely, $\mathbf{P}(X) \cap \mathrm{Sing}(V)$ has at least codimension 2 in $\mathbf{P}(X)$.

In the remainder of this section we do *not* assume this.

In a general case a possible structure would be a *contact orbifold structure*. Let us consider a normal variety Z and a line bundle M on Z. A contact structure on Z (with contact line bundle M) is a contact structure on the Zariski open set Z_{reg} (as a complex manifold) with contact line bundle $M|_{Z_{\mathrm{reg}}}$. A contact form θ is regarded as a section of $\underline{\mathrm{Hom}}(\Theta_Z, M)$. A contact orbifold Y is a normal variety with the following data: $Y = \bigcup U_\alpha$ is an open covering of Y and, for each α, there is a finite Galois covering $\varphi_\alpha: \tilde{U}_\alpha \to U_\alpha$ such that the (possibly singular but normal) variety $\tilde{U}_\alpha$ admits a line bundle M_α and a contact form θ_α with contact line bundle M_α. These data should satisfy a compatibility condition. If $U_\alpha \cap U_\beta \neq \emptyset$, then we form a diagram

$$\tilde{U}_\alpha \stackrel{p_\alpha}{\leftarrow} \tilde{U}_\alpha \times_Y \tilde{U}_\beta \stackrel{p_\beta}{\to} \tilde{U}_\beta.$$

Let $(\tilde{U}_\alpha \times_Y \tilde{U}_\beta)^n$ be the normalization of $\tilde{U}_\alpha \times_Y \tilde{U}_\beta$. Denote by p^n_α the composite of the normalization map and p_α. We then assume that p^n_α and p^n_β are both étale. Moreover, as the compatibility condition we assume that there is an isomorphism of line bundles

$$g_{\beta,\alpha}: (p^n_\alpha)^* M_\alpha \to (p^n_\beta)^* M_\beta$$

and that

$$(p^n_\alpha)^*(\theta_\alpha) = (p^n_\beta)^*(\theta_\beta).$$

Finally, for any α, β, and γ with $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, we should have

$$g_{\alpha,\beta} \circ g_{\beta,\gamma} \circ g_{\gamma,\alpha} = \mathrm{id}$$

on

$$(\tilde{U}_\alpha \times_Y \tilde{U}_\beta \times_Y \tilde{U}_\gamma)^n.$$

In other words, $\{M_\alpha\}$ is an orbifold line bundle $\mathcal{M}$ on Y^{orb}, and $\{\theta_\alpha\}$ is a global section of $\underline{\mathrm{Hom}}(\Theta_{Y^{\mathrm{orb}}}, \mathcal{M})$.

The most natural structure would be actually a *Jacobi structure* (see [Li]). This is very similar to the fact that a Poisson structure would be more natural than a symplectic structure in the singular case. If a normal variety has a contact structure in the sense above, then we have a pairing map

$$O(M)|_{Z_{\mathrm{reg}}} \times O(M)|_{Z_{\mathrm{reg}}} \to O(M)|_{Z_{\mathrm{reg}}}$$

defined by $(u,v) \to \theta([s(u), s(v)])$. Here s is the map defined in Proposition 4.2.1 in Section 6. By the normality this pairing uniquely extends to

$$\{\ ,\ \} : O(M) \times O(M) \to O(M).$$

The bracket satisfies the Jacobi identity, but it is no more a biderivation. We call it a Jacobi structure on Z. The Jacobi structure is generalized to the orbifold version in a similar way as the contact orbifold structure is defined. A contact orbifold structure determines a Jacobi orbifold structure.

THEOREM 4.4.1

The projectivized cone $\mathbf{P}(X)$ *has a contact orbifold structure.*

Proof

First note that $\mathbf{P}(a_0, \dots, a_n)$ has a natural orbifold structure. In fact, let $\mathbf{C}^{n+1} - \{0\} \to \mathbf{P}(a_0, \dots, a_n)$ be the quotient map of the $\mathbf{C}^*$-action $(x_0, \dots, x_n) \to (t^{a_0}x_0, \dots, t^{a_n}x_n)$. Restrict this map to $W_i := \{x_i = 1\} \subset \mathbf{C}^{n+1}$. Then one has a map $W_i \to \mathbf{P}(a_0, \dots, a_n)$ for each i, and these maps give an orbifold structure of $\mathbf{P}(a_0, \dots, a_n)$. We show that $\mathbf{P}(a_0, \dots, a_n)$ admits an orbifold line bundle $O_{\mathbf{P}(a_0,\dots,a_n)}(1)$. There is a finite Galois cover

$$\mathbf{P}(a_0, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) \to \mathbf{P}(a_0, \dots, a_n)$$

defined by

$$(x_0, \dots, x_n) \to (x_0, \dots, x_i^{a_i}, \dots, x_n)$$

for each i. One can identify W_i with the open set of $\mathbf{P}(a_0, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n)$ defined by $x_i \neq 0$. Let

$$\tilde{L}_i := O_{\mathbf{P}(a_0,\dots,a_{i-1},1,a_{i+1},\dots,a_n)}(1)|_{W_i}.$$

Then $\{\tilde{L}_i\}_{0 \le i \le n}$ gives an orbifold line bundle on $\mathbf{P}(a_0, \dots, a_n)$. In fact, the $(\mathbf{Z}/a_0\mathbf{Z} \times \cdots \times \mathbf{Z}/a_n\mathbf{Z})$-Galois cover

$$\mathbf{P}(1, \dots, 1) \to \mathbf{P}(a_0, \dots, a_n)$$

is a smooth global cover (cf. [Mu, Section 2]) in the sense that it is factorized as

$$\mathbf{P}(1, \dots, 1) \to \mathbf{P}(a_0, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) \to \mathbf{P}(a_0, \dots, a_n)$$

for any i. The tautological line bundle $O_{\mathbf{P}(1,\ldots,1)}(1)$ has a $G := (\mathbf{Z}/a_0\mathbf{Z} \times \cdots \times \mathbf{Z}/a_n\mathbf{Z})$ linearization defined by $x_i \to \zeta_i^{m_i} x_i$ $(0 \le i \le n)$ for a primitive a_ith root ζ_i of unity and $m_i \in \mathbf{Z}/a_i\mathbf{Z}$. Then $O_{\mathbf{P}(1,\ldots,1)}(1)|_{x_i \neq 0}$ with the action of $G_i := \mathbf{Z}/a_i\mathbf{Z}$ is the pullback of $\tilde{L}_i$. This is equivalent to giving an orbifold line bundle of $\mathbf{P}(a_0, \ldots, a_n)$ (cf. [Mu, Section 2]). The merit of introducing the orbifold structure is the following. Let $\Sigma \subset \mathbf{P}(a_0, \ldots, a_n)$ be the union† of the ramification loci of the coverings $W_i \to \mathbf{P}(a_0, \ldots, a_n)$. Each fiber of the projection map $\mathbf{C}^{n+1} - \{0\} \to \mathbf{P}(a_0, \ldots, a_n)$ is isomorphic to $\mathbf{C}^*$, but the fibers over the points contained in Σ are multiple fibers. However, if we take the normalization $(W_i \times_{\mathbf{P}(a_0,\ldots,a_n)} (\mathbf{C}^{n+1} - \{0\}))^n$ of the fiber product of W_i and $\mathbf{C}^{n+1} - \{0\}$ over $\mathbf{P}(a_0, \ldots, a_n)$, then the first projection

$$\big(W_i \times_{\mathbf{P}(a_0,\ldots,a_n)} \big(\mathbf{C}^{n+1} - \{0\}\big)\big)^n \to W_i$$

is a $\mathbf{C}^*$-bundle and the second projection

$$\big(W_i \times_{\mathbf{P}(a_0,\ldots,a_n)} \big(\mathbf{C}^{n+1} - \{0\}\big)\big)^n \to \mathbf{C}^{n+1} - \{0\}$$

is an étale map.

Put $U_i := W_i \cap X$ and $L_i := \tilde{L}_i|_{U_i}$. Then an orbifold structure of $\mathbf{P}(X)$ is given by $\{U_i \to \mathbf{P}(X)\}$. Moreover, $\{L_i\}$ gives an orbifold line bundle $\mathcal{L}$ on $\mathbf{P}(X)$. The orbifold line bundle $\mathcal{L}$ is called the tautological line bundle. Let $\mathcal{M}$ be the orbifold line bundle on $\mathbf{P}(X)$ defined by $\mathcal{L}^{\otimes l}$.

Let X_i be the normalization of the fiber product $U_i \times_{\mathbf{P}(X)} (X - \{0\})$. Then the first projection $X_i \to U_i$ is a $\mathbf{C}^*$-bundle, and the second projection $X_i \to X - \{0\}$ is an étale map. Let ω_i be the pullback of ω by the map $(X_i)_{\mathrm{reg}} \to X_{\mathrm{reg}}$. As in Section 4.3, ω_i defines a contact structure on $(U_i)_{\mathrm{reg}}$ with contact line bundle $L_i^{\otimes l}|_{(U_i)_{\mathrm{reg}}}$. These contact structures are glued together to give a contact orbifold structure on $\mathbf{P}(X)$ with contact line bundle $\mathcal{M}$. □

We shall briefly recall the cohomology for an orbifold. Let $\pi_i : U_i \to \mathbf{P}(X)$ be the orbifold charts with $G_i = \mathrm{Gal}(\pi_i)$, and let $p_{i,j}^n : (U_i \times_{\mathbf{P}(X)} U_j)^n \to U_i$ be the projection maps from the normalization of $U_i \times_{\mathbf{P}(X)} U_j$ to U_i. An orbifold $O_{\mathbf{P}(X)}$-module F is a collection $\{F_i\}$ of O_{U_i}-modules glued together by

$$g_{i,j} : (p_{i,j}^n)^* F_i \cong (p_{j,i}^n)^* F_j$$

compatible on the triple overlaps. If we put, in particular, $j = i$, then this means that F_i has a G_i-linearization. The sheaves $(\pi_i)_*^{G_i} F_i$ of G_i-invariant sections of F_i are glued together to give a sheaf $\bar{F}$ on $\mathbf{P}(X)$. The space $\Gamma(\mathbf{P}(X), F)$ of global sections of F is nothing but $\Gamma(\mathbf{P}(X), \bar{F})$. We define $H^p(\mathbf{P}(X), F) := H^p(\mathbf{P}(X), \bar{F})$. When F_i are all invertible sheaves, F is called an orbifold line bundle. Even if F is an orbifold line bundle, $\bar{F}$ is not necessarily a line bundle. Let $\mathrm{Pic}(\mathbf{P}(X)^{\mathrm{orb}})$ be the group of isomorphism classes of orbifold line bundles on

†If we identify $\mathbf{P}(a_0, \ldots, a_n)$ with the central fiber of the weighted blowup $V \to \mathbf{C}^{n+1}$, then Σ coincides with $\mathrm{Sing}(V)$.

$\mathbf{P}(X)$. In general $\mathrm{Pic}(\mathbf{P}(X)^{\mathrm{orb}})$ is not isomorphic to $H^1(\mathbf{P}(X), O^*_{\mathbf{P}(X)})$. In order to capture the orbifold line bundles we consider the Čech complex

$$\prod_i \Gamma(U_i, O^*_{U_i}) \to \prod_{i,j} \Gamma(U_{i,j}, O^*_{U_{i,j}}) \to \prod_{i,j,k} \Gamma(U_{i,j,k}, O^*_{U_{i,j,k}}) \to \cdots .$$

We denote by $H^p_{\mathrm{orb}}(\mathcal{U}, O^*_{\mathbf{P}(X)})$ the pth cohomology of this complex. The inductive limit $\lim H^p_{\mathrm{orb}}(\mathcal{U}, O^*_{\mathbf{P}(X)})$ for the admissible orbifold charts is the pth Čech orbifold cohomology $H^p_{\mathrm{orb}}(\mathbf{P}(X), O^*_{\mathbf{P}(X)})$ of $O^*_{\mathbf{P}(X)}$. Then

$$\mathrm{Pic}\big(\mathbf{P}(X)^{\mathrm{orb}}\big) \cong H^1_{\mathrm{orb}}\big(\mathbf{P}(X), O^*_{\mathbf{P}(X)}\big).$$

As in the proof of Theorem 4.4.1, let $\mathcal{L}$ be the tautological line bundle on $\mathbf{P}(X)$.

We put

$$R := \bigoplus_{n\geq 0} H^0\big(\mathbf{P}(X), \mathcal{L}^{\otimes n}\big).$$

Then $X = \mathrm{Spec}(R)$. If we pull back the projection map $p : X - \{0\} \to \mathbf{P}(X)$ by $U_i \to \mathbf{P}(X)$ and take the normalization, then we have a $\mathbf{C}^*$-bundle $X_i \to U_i$ and the induced map $X_i \to X - \{0\}$ is étale. The contact structure θ_i induces a symplectic structure ω_i of weight l on X_i. These symplectic structures $\{\omega_i\}$ descend to a symplectic structure ω of weight l on X. In this way, (X, ω) is recovered from the contact structure $(\mathcal{M}, \theta)$.

EXAMPLE 4.4.2

(i) Let us consider a du Val singularity X of type A_n, D_n $(n \geq 4)$, or E_n $(n = 6, 7, 8)$ with a symplectic structure ω of weight 2. By Theorem 4.4.1, $\mathbf{P}(X)$ has a contact orbifold structure. Different types of du Val singularities determine mutually different contact orbifold structures. But the underlying variety $\mathbf{P}(X)$ are all $\mathbf{P}^1$. In other words, $\mathbf{P}^1$ has infinitely many different contact orbifold structures.

(ii) The odd-dimensional projective space $\mathbf{P}^{2n+1}$ has a contact structure with a contact line bundle $M = O(2)$. We have two choices of the tautological line bundle L: $L = O(1)$ or $L = O(2)$. The weights l are then, respectively, 2 and 1. The corresponding symplectic variety (X, ω) is isomorphic to $\mathbf{C}^{2n+2}$ with the standard symplectic structure in the first case. In the second case (X, ω) is isomorphic to $\mathbf{C}^{2n+2}/\{+1, -1\}$ with a symplectic 2-form ω_0. Here -1 acts on $\mathbf{C}^{2n+2}$ by $x_i \to -x_i$ $(1 \leq i \leq 2n+2)$, and ω_0 is the symplectic structure induced from the standard symplectic form $dx_1 \wedge dx_2 + \cdots + dx_{2n+1} \wedge dx_{2n+2}$.

5. Rigidity of contact orbifold structures

5.1.

Let (X, ω) be a pair of a normal affine variety X of dimension $2d$ with a good $\mathbf{C}^*$-action and an algebraic symplectic 2-form ω on X_{reg} with positive weight l.

In Section 4 we have attached a contact orbifold structure $(\mathcal{M}, \theta)$ to the projectivized cone $\mathbf{P}(X)$. In this section we consider the deformation of such a contact orbifold.

Let $(Y, \{U_\alpha\}, \{M_\alpha\}, \{\theta_\alpha\})$ be a contact orbifold, and let S be a punctured space with $0 \in S$. A flat deformation of $(Y, \{U_\alpha\}, \{M_\alpha\}, \{\theta_\alpha\})$ is a flat surjective map $\mathcal{Y} \to S$ together with the covering charts $\Phi_\alpha : \tilde{\mathcal{U}}_\alpha \to \mathcal{Y}$ such that

(i) $\mathcal{Y}$ is a flat deformation of Y;

(ii) for each α, Φ_α is a Galois covering with the Galois group G_α, which is a flat deformation of $\varphi_\alpha : \tilde{U}_\alpha \to Y$ over S so that the maps

$$(\tilde{\mathcal{U}}_\alpha \times_{\mathcal{Y}} \tilde{\mathcal{U}}_\beta)^n \to \tilde{\mathcal{U}}_\alpha$$

are étale;

(iii) there are line bundles $\mathcal{M}_\alpha$ on $\tilde{\mathcal{U}}_\alpha$ that restrict to give M_α on $\tilde{U}_\alpha$, and $\{\mathcal{M}_\alpha\}$ are glued together to give an orbifold line bundle of $\mathcal{Y}$; and finally,

(iv) all contact structures θ_α on $\tilde{U}_\alpha$ extend compatibly to the contact structures Θ_α on $\tilde{\mathcal{U}}_\alpha$ with the contact line bundles $\mathcal{M}_\alpha$.

Of course, one can start from a different admissible orbifold charts $\{U'_\alpha\}$ of Y and consider its flat deformation. A flat deformation of the contact orbifold structure $(Y, \mathcal{M}, \theta)$ is exactly an equivalence classe of that of $(Y, \{U_i\}, \{M_\alpha\}, \{\theta_\alpha\})$.

Now let us return to the contact orbifold $(\mathbf{P}(X), \mathcal{M}, \theta)$.

PROPOSITION 5.2

Assume that X has only canonical singularities. Then the contact orbifold structure $(\mathbf{P}(X), \mathcal{M}, \theta)$ is rigid under a small flat deformation.

This is a counterpart of Proposition 3.5 in contact geometry.

Proof

Recall the construction of the contact orbifold structure (cf. proof of Theorem 4.4.1). With the same notation as in the proof of Theorem 4.4.1, the map $X_i \to X - \{0\}$ is an étale map. Since X has canonical singularities, X_i also has canonical singularities. Since X_i is a $\mathbf{C}^*$-bundle over U_i, we see that U_i has canonical singularities; hence all orbifold charts of $\mathbf{P}(X)$ have canonical singularities.

Let $(Y, \mathcal{M}_1, \theta_1)$ be an infinitesimal deformation of $(\mathbf{P}(X), \mathcal{M}, \theta)$ over $S_1 := \operatorname{Spec} \mathbf{C}[\epsilon]$. The orbifold line bundle $\mathcal{L}$ defines an element $[\mathcal{L}]$ of $H^1_{\text{orb}}(\mathbf{P}(X), O^*_{\mathbf{P}(X)})$. Note that there is an exact sequence

$$0 \to O_{\mathbf{P}(X)} \to O^*_Y \to O^*_{\mathbf{P}(X)} \to 1,$$

where the map $O_{\mathbf{P}(X)} \to O^*_Y$ is defined by $f \to 1 + \epsilon \cdot f$. This exact sequence yields the exact sequence

$$H^1_{\text{orb}}\big(\mathbf{P}(X), O_{\mathbf{P}(X)}\big) \to H^1_{\text{orb}}(Y, O^*_Y) \to H^1_{\text{orb}}\big(\mathbf{P}(X), O^*_{\mathbf{P}(X)}\big) \xrightarrow{\delta} H^2_{\text{orb}}\big(\mathbf{P}(X), O_{\mathbf{P}(X)}\big).$$

Note that $\mathcal{M} = \mathcal{L}^{\otimes l}$. Since $\mathcal{M}$ extends to $\mathcal{M}_1$, one has $\delta([\mathcal{M}]) = 0$. This means that $l \cdot \delta([\mathcal{L}]) = 0$. As $H^2_{\mathrm{orb}}(\mathbf{P}(X), O_{\mathbf{P}(X)})$ is a $\mathbf{C}$-vector space, one has $\delta([\mathcal{L}]) = 0$. Thus one can find an orbifold line bundle $\mathcal{L}_1$ on Y that is an extension of $\mathcal{L}$. Then $l \cdot [\mathcal{L}_1] - [\mathcal{M}_1] \in H^1_{\mathrm{orb}}(Y, O_Y^*)$ is the image of an element $\eta \in H^1_{\mathrm{orb}}(\mathbf{P}(X), O_{\mathbf{P}(X)})$ by the map $H^1_{\mathrm{orb}}(\mathbf{P}(X), O_{\mathbf{P}(X)}) \to H^1_{\mathrm{orb}}(Y, O_Y^*)$. If we replace $\mathcal{L}_1$ by the orbifold line bundle corresponding to $[\mathcal{L}_1] - 1/l \cdot \eta$, then we may assume that $\mathcal{L}_1^{\otimes l} = \mathcal{M}_1$. The exact sequences

$$0 \to \epsilon \cdot \mathcal{L}^{\otimes n} \to \mathcal{L}_1^{\otimes n} \to \mathcal{L}^{\otimes n} \to 0$$

yield the exact sequences

$$H^0(Y, \mathcal{L}_1^{\otimes n}) \to H^0\big(\mathbf{P}(X), \mathcal{L}^{\otimes n}\big) \to H^1\big(\mathbf{P}(X), \mathcal{L}^{\otimes n}\big).$$

By the next lemma, we see that the maps $H^0(Y, \mathcal{L}_1^{\otimes n}) \to H^0(\mathbf{P}(X), \mathcal{L}^{\otimes n})$ are all surjective for $n \geq 0$. We put

$$\mathcal{R} := \bigoplus_{n \geq 0} H^0(Y, \mathcal{L}_1^{\otimes n}).$$

and define $\mathcal{X} := \mathrm{Spec}(\mathcal{R})$. Then $\mathcal{X}$ is an infinitesimal deformation of X over S_1. Moreover, the contact structure $(\mathcal{M}_1, \theta_1)$ of Y defines a symplectic structure ω_1 on $\mathcal{X}$. As a consequence, we have obtained an infinitesimal deformation $(\mathcal{X}, \omega_1)$ of (X, ω). By the construction $(\mathcal{X}, \omega_1)$ has a $\mathbf{C}^*$-action. If one regards S_1 as a $\mathbf{C}^*$-space with trivial action, then the map $\mathcal{X} \to S_1$ is $\mathbf{C}^*$-equivariant. By Proposition 3.5, $(\mathcal{X}, \omega_1)$ is a trivial deformation of (X, ω). In particular, $(Y, \mathcal{M}_1, \theta_1)$ is also a trivial deformation of $(\mathbf{P}(X), \mathcal{M}, \theta)$. □

LEMMA 5.2.1
We have

$$H^1\big(\mathbf{P}(X), \mathcal{L}^{\otimes n}\big) = 0$$

for all $n \geq 0$.

Proof
As remarked above, each orbifold of $\mathbf{P}(X)$ has rational Gorenstein singularities. Since $\mathbf{P}(X)$ is locally the quotient variety of an orbifold chart by a finite group, the log variety $(\mathbf{P}(X), 0)$ (with the zero boundary divisor) has log terminal singularities by [Kaw, Proposition 1.7].† Moreover, $\mathbf{P}(X)$ has a contact orbifold structure; thus $K^{\mathrm{orb}}_{\mathbf{P}(X)} = \mathcal{M}^{-d-1}$ if $\dim \mathbf{P}(X) = 2d+1$. Since $\mathcal{M} = \mathcal{L}^{\otimes l}$ is ample, $K^{\mathrm{orb}}_{\mathbf{P}(X)}$ is negative.

Let $\pi_i : U_i \to \mathbf{P}(X)$ be an orbifold chart. Then $U_i \to \pi_i(U_i)$ is a $(\mathbf{Z}/a_i\mathbf{Z})$-Galois cover. One can write $K_{U_i} = \pi_i^* K_{\mathbf{P}(X)} + B_i$ with an effective divisor B_i on U_i whose support coincides with the ramification divisor of π_i. We have in this way a collection $\{B_i\}$ of $(\mathbf{Z}/a_i\mathbf{Z})$-stable Cartier divisors B_i on U_i such that

†The map $\pi : V \to X$ in Proposition 1.7 is assumed to be étale in codimension 1. But this condition is not necessary to prove the “if” part.

$(p^n)^*_{i,j}(B_i) = (p^n)^*_{j,i}(B_j)$. Then $O(B) := \{O(B_i)\}$ becomes an orbifold line bundle, and by the definition of B, we have $O(B) \cong \mathcal{L}^{\otimes m}$ for some $m \geq 0$. Since $-K^{\mathrm{orb}}_{\mathbf{P}(X)}$ and $O(B)$ are ample and nef, respectively, the $\mathbf{Q}$-Cartier divisor $-K_{\mathbf{P}(X)}$ is ample. Since $\bar{\mathcal{L}}^{\otimes n} - K_{\mathbf{P}(X)}$ is ample and $(\mathbf{P}(X), 0)$ has log terminal singularities, the lemma is a direct consequence of the Kawamata–Viehweg vanishing (see [KMM, Theorem 1-2-5]). $\square$

6. Equivalence up to a constant

Let (X, ω) be a pair of a normal affine variety X of dimension $2d$ with a good $\mathbf{C}^*$-action and an algebraic symplectic 2-form ω on X_{reg} with *positive* weight l. An algebraic symplectic 2-form ω' on X_{reg} is said to be equivalent to ω up to constant when $\omega' = \lambda \cdot \omega$ with some $\lambda \in \mathbf{C}^*$.

Let us consider the hypersurfaces

$$X_n := \{(a, b, x, y, z) \in \mathbf{C}^5; a^2x + 2aby + b^2z + (xz - y^2)^n = 0\},$$

where $n \geq 2$. These are central fibers of Slodowy slices to nilpotent orbits of $\mathrm{sp}(2n)$ with Jordan type $[2n-2, 1^2]$ (see [LNS]); hence they admit natural symplectic 2-forms ω_n of weight 2. One can also construct symplectic 2-forms ω'_n of weight 2 on X_n by using representations of sl_2 (cf. Section 0, Example 2(b)). Moreover, X_3 coincides with the central fiber of the Slodowy slice to the subsubregular nilpotent orbit of the Lie algebra of type G_2 (see [LNS, Section 10]). Thus X_3 admits a symplectic 2-form σ_3 induced from the Kostant–Kirillov form on $\mathbf{g}_2$. By Theorem 3.1 we already know that they are equivalent up to a $\mathbf{C}^*$-equivariant automorphism. But we can say more.

PROPOSITION 6.1

Each hypersurface X_n admits a unique holomorphic symplectic 2-form of weight 2 up to a constant.

Proof

We put $X := X_n$. In this case, as explained below, $\mathrm{Codim}_{\mathbf{P}(X)}(\mathbf{P}(X) - \mathbf{P}(X)^0) = 2$ and $\mathbf{P}(X)^0 = \mathbf{P}(X)_{\mathrm{reg}}$. As in Section 4.3, ω_n defines a contact form $\theta \in H^0(\mathbf{P}(X)_{\mathrm{reg}}, \Omega^1_{\mathbf{P}(X)_{\mathrm{reg}}} \otimes L^{\otimes 2})$. It is enough to check that θ is a unique contact structure with contact line bundle $L^{\otimes 2}$.

First note that $\mathbf{P}(X)$ is not quasi-smooth; X has a du Val singularity of type D_{n+1} along $\{a = b = xz - y^2 = 0\}$. When $n = 2$, we understand that $D_3 = A_3$. The singular locus of $\mathbf{P}(X)$ is the disjoint union of two smooth rational curves

$$\{a = b = xz - y^2 = 0\} \cup \{x = y = z = 0\}$$

in $\mathbf{P}(2n-1, 2n-1, 2, 2, 2)$. Along the first component, $\mathbf{P}(X)$ has a D_{2n} surface singularity, and along the second component, it has quotient singularity of type $(1/(2n-1))(1,1)$. Take points p_1 and p_2, respectively, from the first and second components, and consider the complex analytic germs $(\mathbf{P}(X), p_i)$. Then

$$(\mathbf{P}(X), p_1) \cong (\mathbf{C}^1, 0) \times D_{2n},$$

$$\big(\mathbf{P}(X), p_2\big) \cong (\mathbf{C}^1, 0) \times \frac{1}{2n-1}(1,1).$$

Let $\mathrm{Cl}(\mathbf{P}(X))$ (resp., $\mathrm{Cl}(\mathbf{P}(X), p_i)$) be the divisor class group of $\mathbf{P}(X)$ (resp., $(\mathbf{P}(X), p_i)$). One has an exact sequence

$$0 \to \mathrm{Pic}\big(\mathbf{P}(X)\big) \to \mathrm{Cl}\big(\mathbf{P}(X)\big) \to \bigoplus_{1\le i\le 2} \mathrm{Cl}\big(\mathbf{P}(X), p_i\big).$$

By the same argument as in [Do, 3.2.5, 3.2.6], we see that $\mathrm{Pic}(\mathbf{P}(X)) = \mathbf{Z}\cdot[O_{\mathbf{P}(X)}(4n-2)]$. Since $\mathrm{Cl}(\mathbf{P}(X), p_i)$ are finite abelian groups, we see that $\mathrm{Cl}(\mathbf{P}(X))$ is a finitely generated abelian group; in particular it is discrete. Let ϕ be an automorphism of $\mathbf{P}(X)$ contained in the neutral component $\mathrm{Aut}^0(\mathbf{P}(X))$ of the automorphism group $\mathrm{Aut}(\mathbf{P}(X))$. Then $\phi_*([O_{\mathbf{P}(X)}(i)]) = [O_{\mathbf{P}(X)}(i)]$ for all i. Note that there is an exact sequence

$$0 \to O_{\mathbf{P}((2n-1)^2,2^3)}(i-4n) \to O_{\mathbf{P}((2n-1)^2,2^3)}(i) \to O_{\mathbf{P}(X)}(i) \to 0.$$

Applying these exact sequences, we have

$$H^0\big(\mathbf{P}\big((2n-1)^2, 2^3\big), O_{\mathbf{P}((2n-1)^2,2^3)}(i)\big) \cong H^0\big(\mathbf{P}(X), O_{\mathbf{P}(X)}(i)\big)$$

for $i = 2, 2n-1$. Note that

$$H^0\big(\mathbf{P}\big((2n-1)^2, 2^3\big), O_{\mathbf{P}((2n-1)^2,2^3)}(2)\big) = \mathbf{C}x \oplus \mathbf{C}y \oplus \mathbf{C}z$$

and

$$H^0\big(\mathbf{P}\big((2n-1)^2, 2^3\big), O_{\mathbf{P}((2n-1)^2,2^3)}(2n-1)\big) = \mathbf{C}a \oplus \mathbf{C}b.$$

The automorphism ϕ induces linear automorphisms of $H^0(\mathbf{P}(X), O_{\mathbf{P}(X)}(i))$ ($i = 2, 2n-1$) and hence those of $\mathbf{C}x \oplus \mathbf{C}y \oplus \mathbf{C}z$ and $\mathbf{C}a \oplus \mathbf{C}b$. Such linear automorphisms induce an automorphism of $\mathbf{P}(2n-1, 2n-1, 2, 2, 2)$. Thus ϕ extends to an automorphism of the ambient space $\mathbf{P}(2n-1, 2n-1, 2, 2, 2)$.

We shall use Corollary 4.2.2 to prove the uniqueness of θ. Let $j : \mathbf{P}(X)_{\mathrm{reg}} \to \mathbf{P}(X)$ be the inclusion map. As we noted in Section 4.3, the contact structure θ induces an exact sequence

$$0 \to j_*O(D) \to \Theta_{\mathbf{P}(X)} \to j_*(L^{\otimes 2}) \to 0.$$

Since $j_*(L^{\otimes 2}) = O_{\mathbf{P}(X)}(2)$, we see that $h^0(\mathbf{P}(X), j_*(L^{\otimes 2})) = 3$.

On the other hand, $h^0(\mathbf{P}(X), \Theta_{\mathbf{P}(X)}) = 3$. A geometric explanation of this fact is the following. As we have seen above, all infinitesimal automorphisms of $\mathbf{P}(X)$ come from those of the ambient space $\mathbf{P}(2n-1, 2n-1, 2, 2, 2)$. The set of linear transformations of (x, y, z) preserving the quadratic form $xz - y^2$ becomes a 3-dimensional algebraic subgroup of $\mathrm{GL}(3, \mathbf{C})$. Fix such a linear transformation φ. Then there is a unique linear transformation of (a, b) (up to sign) which sends the cubic form $a^2\varphi(x) + 2ab\varphi(y) + b^2\varphi(z)$ to $a^2x + 2aby + b^2z$. Since the exact sequence attached to the contact structure always splits (as $\mathbf{C}$-modules), we conclude that $h^0(j_*O(D)) = 0$. □

Let $O \subset \mathbf{g}$ be a nilpotent adjoint orbit of a complex simple Lie algebra. Let $\tilde{O}$ be the normalization of the closure $\bar{O}$. Since O admits a Kostant–Kirillov 2-form, $\tilde{O}$ has a holomorphic symplectic structure of weight 1.

PROPOSITION 6.2

Assume that $\tilde{O}$ is a Richardson orbit with a Springer map $\pi : T^*(G/P) \to \tilde{O}$ for some parabolic subgroup P of G. Then $\tilde{O}$ has a unique symplectic structure of weight 1 up to constant.

Proof

Let $\mathbf{P} := \mathbf{P}(T^*(G/P))$ be the projectivized tangent bundle of G/P. Then π induces a generically finite proper map $\bar{\pi} : \mathbf{P} \to \mathbf{P}(\bar{O})$, and the contact 1-form $\theta \in H^0(\mathbf{P}(O), \Omega^1_{\mathbf{P}(O)} \otimes O_{\mathbf{P}(O)}(1))$ is pulled back (and is extended) to a contact 1-form

$$\bar{\pi}^*\theta \in H^0\big(\mathbf{P}, \Omega^1_{\mathbf{P}} \otimes O_{\mathbf{P}}(1)\big).$$

We prove that this is a unique contact structure on $\mathbf{P}$ with contact line bundle $O_{\mathbf{P}}(1)$. Let

$$0 \to O(D) \to \Theta_{\mathbf{P}} \overset{\bar{\pi}^*\theta}{\to} O_{\mathbf{P}}(1) \to 0$$

be the corresponding exact sequence. Let $p : \mathbf{P} \to G/P$ be the projection map of the projective space bundle. Since $p_* O_{\mathbf{P}}(1) = \Theta_{G/P}$, we have

$$h^0\big(\mathbf{P}, O_{\mathbf{P}}(1)\big) = h^0(G/P, \Theta_{G/P}).$$

On the other hand, by the exact sequences

$$0 \to O_{\mathbf{P}} \to p^*\Omega^1_{G/P} \otimes O_{\mathbf{P}}(1) \to \Theta_{\mathbf{P}/(G/P)} \to 0,$$

one has an exact sequence

$$0 \to H^0(O_{\mathbf{P}}) \to H^0\big(\underline{\mathrm{Hom}}(\Theta_{G/P}, \Theta_{G/P})\big) \to H^0(\Theta_{\mathbf{P}/(G/P)}) \to H^1(O_{\mathbf{P}}).$$

Since $\Theta_{G/P}$ is a simple vector bundle (see [A-B]), we have $H^0(\underline{\mathrm{Hom}}(\Theta_{G/P}, \Theta_{G/P})) \cong \mathbf{C}$. As $H^1(O_{\mathbf{P}}) = 0$, we see that $H^0(\Theta_{\mathbf{P}/(G/P)}) = 0$. By the exact sequence

$$0 \to H^0(\Theta_{\mathbf{P}/(G/P)}) \to H^0(\Theta_{\mathbf{P}}) \to H^0(p^*\Theta_{G/P}),$$

it is clear that $h^0(\Theta_{\mathbf{P}}) = h^0(G/P, \Theta_{G/P})$. This implies that $H^0(\mathbf{P}, O(D)) = 0$. □

REMARK

Let O be a nilpotent orbit (where O is not necessarily a Richardson orbit). Consider the contact structure on $\mathbf{P}(O)$:

$$0 \to O(D) \to \Theta_{\mathbf{P}(O)} \overset{\theta}{\to} O_{\mathbf{P}(O)}(1) \to 0.$$

Since O is a homogeneous space acted on by G, there is a natural map $\mathbf{g} \to H^0(\Theta_{\mathbf{P}(O)})$. Then the composition map

$$\theta|_{\mathbf{g}} : \mathbf{g} \to H^0\big(O_{\mathbf{P}(O)}(1)\big)$$

is injective. The following is a proof. Let ω be the Kostant–Kirillov 2-form on O. As in Section 4.3, let ζ be the vector field on O which generates the $\mathbf{C}^*$-action. Let $\pi : O \to \mathbf{P}(O)$ be the projection map. By definition, $\pi^*\theta = \omega(\zeta, \cdot)$. For $x \in O$, we denote by $\bar{x} \in \mathbf{P}(O)$ the corresponding point. Let us consider T_xO as a linear subspace of $\mathbf{g}$. Then $\zeta_x = x$ by the definition. For $v \in \mathbf{g}$, we have $[x, v] \in T_xO$; hence

$$\big(\theta|_{\mathbf{g}}(v)\big)_{\bar{x}} = \omega_x\big(x, [v, x]\big).$$

One can write $x = [a_x, x]$ with some $a_x \in \mathbf{g}$. Let κ be the Killing form on $\mathbf{g}$. By the definition of the Kostant–Kirillov 2-form we have

$$\omega_x\big(x, [v, x]\big) = \kappa\big(x, [a_x, v]\big) = \kappa\big([x, a_x], v\big) = -\kappa(x, v).$$

If $v \in \ker(\theta_{\mathbf{g}})$, then $\kappa(x, v) = 0$ for all $x \in O$. Note that x is contained in the cone $\bar{O} \subset \mathbf{g}$. Since $T_0\bar{O}$ is invariant under the adjoint G-action and the adjoint representation is irreducible, $T_0\bar{O} = \mathbf{g}$. This means that, if x runs inside O, they span $\mathbf{g}$ as a $\mathbf{C}$-vector space. Since κ is nondegenerate, we conclude that $v = 0$. Now we have the following.

PROBLEM

When does $\mathbf{g}$ coincide with $H^0(\mathbf{P}(O), \Theta_{\mathbf{P}(O)})$?

When $O_{\min}$ is the minimal nilpotent orbit of $\mathbf{g}$, $\mathbf{P}(O_{\min})$ is a flag variety G/P with a parabolic subgroup P. Let $M := G/P$ be a flag variety, where G is a connected simple complex Lie group acting effectively on M. Then, by Onishchik (cf. [GO, Theorem 4.10]), the neutral component $\mathrm{Aut}^0(G/P)$ is isomorphic to G except in the following three cases:

(i) $G = P\,\mathrm{Sp}(2n)$ and P is the stabilizer subgroup of an isotropic flag of type $(1, 2n-2, 1)$ in the vector space $\mathbf{C}^{2n}$ acted by G;

(ii) $G = G_2 \subset \mathrm{SO}(7)$ and M is a quadric 5-fold in $\mathbf{P}^6$.

(iii) $G = \mathrm{SO}(2n+1)$ and P is the stabilizer subgroup of an isotropic flag of type $(n, 1, n)$ in $\mathbf{C}^{2n+1}$.

In (ii) and (iii), $M = G/P$ is not realized as the projectivized cone $\mathbf{P}(O_{\min})$ of the minimal nilpotent orbit $O_{\min}$. But in the case (i), $G/P = \mathbf{P}(O_{\min})$ with $O_{\min} \subset \mathrm{sp}(2n)$. Thus we have proved the following.

PROPOSITION 6.3

Assume that $O_{\min}$ is the minimal nilpotent orbit of $\mathbf{g}$. Then $\tilde{O}_{\min}$ has a unique symplectic structure of weight 1 up to constant except when $\mathbf{g} = \mathrm{sp}(2n)$.

Note that the exceptional case corresponds to the quotient singularity $\mathbf{C}^{2n}/\mathbf{Z}_2$ by the action $(z_1, \ldots, z_{2n}) \to (-z_1, \ldots, -z_{2n})$.

7. Problems

Let (X, ω) be a pair of a normal affine variety X of dimension $2d$ with a good $\mathbf{C}^*$-action and an algebraic symplectic 2-form ω with *positive* weight l. Let us call (X, ω) *irreducible* of weight l when ω is a unique symplectic structure of weight l up to a constant.

PROBLEM 7.1

Does (X, ω) have symplectic singularities, or equivalently, canonical singularities?

PROBLEM 7.2

Is the fundamental group $\pi_1(X_{\mathrm{reg}})$ of the regular part of X finite?

When $G := \pi_1(X_{\mathrm{reg}})$ is finite, one can take a finite G-Galois covering $\pi : Y \to X$ in such a way that the induced map $\pi^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is the universal covering of X_{reg}. Let m be the order of G. Let $\mathbf{C}^* \times X \to X$ $(t, x) \to \phi_t(x)$ be the given $\mathbf{C}^*$-action on X. We consider the $\mathbf{C}^*$-action on X defined as its mth power:

$$\mathbf{C}^* \times X \to X(t, x) \to \phi_{t^m}(x).$$

Then Y has a $\mathbf{C}^*$-action so that π is $\mathbf{C}^*$-equivariant.

Recall here the Bogomolov splitting theorem for a compact Kähler manifold X with $c_1 = 0$. It states, in particular, that if X is a holomorphic symplectic manifold with a finite fundamental group, then its universal cover $\tilde{X}$ splits into the product of irreducible symplectic manifolds X_i $(i = 1, \ldots, r)$ such that $h^0(X_i, \Omega^2_{X_i}) = 1$.

The following is an analogue of the splitting theorem in affine symplectic varieties with good $\mathbf{C}^*$-actions.

PROBLEM 7.3

Is there a $\mathbf{C}^*$-equivariant isomorphism of symplectic varieties

$$(Y, \pi^*\omega) \cong \prod_{1 \le i \le k} (Y_i, \omega_i)$$

where each (Y_i, ω_i) is irreducible of weight $m \cdot l$?

For example, as an (X, ω), take the quotient singularity $\mathbf{C}^{2n}/\mathbf{Z}_2$ defined at the end of Section 6, and take the symplectic form induced from $\tilde{\omega} := dz_1 \wedge dz_2 + \cdots + dz_{2n-1} \wedge dz_{2n}$. Then (X, ω) is not irreducible, but

$$(\mathbf{C}^{2n}, \tilde{\omega}) \cong \prod_{1 \le i \le n} (\mathbf{C}^2, dz_{2i-1} \wedge dz_{2i}).$$

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