The Daugavet equation in Banach spaces with alternatively convex-smooth duals

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Abstract This short paper gives a necessary and sufficient condition for the Daugavet equation $\|I+T\|=1+\|T\|$. A new characterization of the solution of the Daugavet equation in terms of invariant affine subspaces is given. We also study the notions of *alternatively convex or smooth* (*acs*) and *locally uniformly alternatively convex or smooth* (*luacs*).

1. Introduction

We remind the reader that the first person to study the equation $\|I+T\|=1+\|T\|$ was Daugavet. Indeed, in 1963 Daugavet [3] proved that every compact operator $T:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ satisfies the equation $\|I+T\|=1+\|T\|$. In 1965, Foias and Singer [4] extended Daugavet's result to arbitrary atomless $\mathcal{C}(K)$-spaces. This paper gives a necessary and sufficient condition for the Daugavet equation $\|I+T\|=1+\|T\|$ under the assumption that X^* is alternatively convex or smooth (acs). In particular, we show that it is not necessary to assume that T is a compact operator. Namely, we find a new technique for solving the Daugavet equation.

For a vector z in a Banach space X, consider the state space

$$\mathcal{U}_z := \{x^*\in X^* : x^*(z)=\|z\|, \|x^*\|=1\}.$$

By the Hahn–Banach theorem we get $\mathcal{U}_z\neq\emptyset$ for all $z\neq 0$. In this paper, for a normed space X, we denote by $S(X)$ the unit sphere in X and by $\operatorname{Ext} S(X)$ the set of all its extremal points. Given a normed space X and a Banach space Y, both over the same field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, we write $\mathcal{K}(X;Y)$ for the space of all compact operators going from X into Y, and $\mathcal{K}(X):=\mathcal{K}(X;X)$. The next theorem plays a crucial role in our investigations. The next result is well known.

THEOREM 1.1 (see [2])
For each $f\in\operatorname{Ext} S(\mathcal{K}(X;Y)^*)$ there exist $y^*\in\operatorname{Ext} S(Y^*)$ and $x^{**}\in\operatorname{Ext} S(X^{**})$ such that $f(K)=(x^{**}\otimes y^*)(K^*)$ for every $K\in\mathcal{K}(X;Y)$.

Kyoto Journal of Mathematics, Vol. 58, No. 4 (2018), 915–921
First published online June 20, 2018.
DOI 10.1215/21562261-2017-0039,
Received September 27, 2016. Revised December 22, 2016. Accepted December 28, 2016.
2010 Mathematics Subject Classification: Primary 46B20; Secondary 47A62.

We write $\mathcal{B}(X)$ for the space of all bounded operators going from X into X. $\mathcal{K}(X)$ is said to be an *M-ideal* in $\mathcal{B}(X)$ if $\mathcal{B}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^\perp$, where $\mathcal{K}(X)^\perp := \{f \in \mathcal{B}(X)^* : \mathcal{K}(X) \subset \ker f\}$, and if $f = f_1 + f_2$ is the unique decomposition of f in $\mathcal{B}(X)^*$, then $\|f\| = \|f_1\| + \|f_2\|$.

We remind the reader when $\mathcal{K}(X)$ is an M-ideal in $\mathcal{B}(X)$. Hennefeld [6] has proved that the $\mathcal{K}(l^p)$'s are M-ideals when $p \in (1,\infty)$. It is known that $\mathcal{K}(l^1)$ and $\mathcal{K}(l^\infty)$ are not M-ideals (see [8]).

An *affine subspace* in X is a set $\mathcal{A}$ such that, for every $x \in \mathcal{A}$, $\mathcal{A} - x$ is a linear subspace in X. We say that an affine subspace $\mathcal{A}$ is *nontrivial* if $\dim \mathcal{A} \geq 1$ and $0 \notin \mathcal{A}$; in particular, $\mathcal{A} \neq X$. It means that $\mathcal{A}$ is not a linear subspace.

2. Main result

In [7] the following notion was introduced. We say that a Banach space X is *alternatively convex or smooth* (acs) if for all $x, y \in S(X)$ and $x^* \in S(X^*)$ the implication

$$x^*(x) = 1, \qquad \|x + y\| = 2 \quad \Rightarrow \quad x^*(y) = 1 \tag{2.1}$$

holds. We remark that smooth spaces (or strictly convex spaces) are acs.

LEMMA 2.1

Let X be a Banach space such that X^* is acs. Assume that $\mathcal{K}(X)$ is an M-ideal in $\mathcal{B}(X)$. Let $T \in \mathcal{B}(X)$. Suppose that $\|T\| = 1$, $\operatorname{dist}(T, \mathcal{K}(X)) < 1$, and assume that T is weakly compact. The following conditions are equivalent:

- (a) $\|I + T\| = 2$;
- (b) *the number 1 is an eigenvalue of T^*;*
- (c) *there is a nontrivial closed invariant affine subspace for T.*

Moreover, the implications (c)⇒(b)⇒(a) *and* (b)⇒(c) *do not depend on the extra assumptions (i.e., M-ideal, acs, $\operatorname{dist}(T, \mathcal{K}(X)) < 1$).*

Proof

For the proof of (c)⇒(b) fix arbitrarily a nontrivial closed affine subspace $\mathcal{A}$ such that $T(\mathcal{A}) \subset \mathcal{A}$. Fix $a \in \mathcal{A}$. The Hahn–Banach theorem implies that there is a linear functional z^* such that $z^*(a) \neq 0$ and

$$\mathcal{A} - a \subset \ker z^*. \tag{2.2}$$

Fix a Banach limit $L\colon l^\infty \to \mathbb{K}$. Define the mapping $y^* : X \to \mathbb{K}$ by

$$y^*(x) := L\big(z^*(x), z^*(Tx), z^*(T^2x), z^*(T^3x), \ldots\big) \quad \text{for } x \in X.$$

Fix x in X. Note that $|z^*(T^n x)| \leq \|z^*\| \cdot \|T^n(x)\| \leq \|z^*\| \cdot \|x\|$ for all $n \in \mathbb{N}$. Thus, $(z^*(x), z^*(Tx), z^*(T^2x), z^*(T^3x), \ldots) \in l^\infty$. So, y^* is a well-defined function. It is easy to check that the above mapping is a continuous linear functional; that is, $y^* \in X^*$. Furthermore, for all $x \in X$, by a property of the Banach limits, we have

$$y^*(x) = L\big(z^*(x), z^*(Tx), z^*(T^2x), z^*(T^3x), \dots\big)$$
$$= L\big(z^*(Tx), z^*(T^2x), z^*(T^3x), \dots\big) = y^*(Tx).$$

It follows that $y^* = y^* \circ T$, which means $y^* = T^*y^*$. Since $T(\mathcal{A}) \subset \mathcal{A}$, $T^n a \in \mathcal{A}$ for all n. Then, by (2.2) we have $T^n(a) - a \in \ker z^*$. From this it is immediate to infer that $z^*(T^n a) = z^*(a)$ for all n. Therefore,

$$y^*(a) = L\big(z^*(a), z^*(Ta), z^*(T^2a), z^*(T^3a), \dots\big)$$
$$= L\big(z^*(a), z^*(a), z^*(a), z^*(a), \dots\big) = z^*(a) \neq 0,$$

and hence $y^* \neq 0$. Thus, we obtain $T^*y^* = y^*$ and $y^* \neq 0$. The proof of this implication is complete.

We will prove (b)⇒(a). From the assumption we have $T^*y^* = y^*$ for some $y^* \in X^* \setminus \{0\}$. Without loss of generality, we may assume that $\|y^*\| = 1$, and then $2 = \|y^* + y^*\| = \|y^* + T^*y^*\| = \|(I^* + T^*)(y^*)\| \leq \|I^* + T^*\| = \|I + T\|$. On the other hand, we have $\|I + T\| \leq \|I\| + \|T\| = 2$.

In order to prove (a)⇒(c), assume that $\|I + T\| = 2$. First, we want to show that $\mathcal{U}_{I+T} \subset \mathcal{U}_I \cap \mathcal{U}_T$. Let $f \in \mathcal{U}_{I+T}$. Then $f(I + T) = 2$ and $\|f\| = 1$. So it suffices to show that $f(I) = 1$ and $f(T) = 1$. Note that $2 = f(I + T) = f(I) + f(T)$, $|f(I)| \leq 1$, and $|f(T)| \leq 1$. Hence, $f(I) = 1$, $f(T) = 1$, and we may consider $\mathcal{U}_{I+T} \subset \mathcal{U}_I \cap \mathcal{U}_T$ as shown. Note that $\mathcal{U}_{I+T}$ is a nonempty weak*-closed subset of the weak*-compact unit ball of $\mathcal{B}(X)^*$. In particular, $\mathcal{U}_{I+T}$ is weak*-compact and convex. The Krein–Milman theorem implies that there is an extremal point e of the set $\mathcal{U}_{I+T}$. The set $\mathcal{U}_{I+T}$ is an extremal subset of clball$(\mathcal{B}(\mathrm{X})^*)$, so every extreme point of $\mathcal{U}_{I+T}$ is an extreme point of clball$(\mathcal{B}(\mathrm{X})^*)$. Thus, we obtain $e \in \operatorname{Ext} S(\mathcal{B}(X)^*)$.

We want to show that $e \in \operatorname{Ext} S(\mathcal{K}(X)^*)$. From the assumption, we have that $\mathcal{B}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^\perp$. Let $e = e_1 + e_2$ be the associated decomposition of e; that is, $e_1 \in \mathcal{K}(X)^*$ and $e_2 \in \mathcal{K}(X)^\perp$. Then $1 = \|e_1\| + \|e_2\|$. From this it is very easy to prove that $e \in \operatorname{Ext} S(\mathcal{K}(X)^*)$ or $e \in \operatorname{Ext} S(\mathcal{K}(X)^\perp)$. So it suffices to prove that $e_2 = 0$. Suppose this is not so. Thus, $e_2 \neq 0$. By the assumption, $\operatorname{dist}(T, \mathcal{K}(X)) < 1$. From this it is immediate to infer that there exists a compact operator $W \in \mathcal{K}(X)$ such that $\|T - W\| < 1$. Hence, $1 = e_2(T) = e_2(T - W) \leq \|T - W\| < 1$, a contradiction. Thus, $e = e_1 \in \operatorname{Ext} S(\mathcal{K}(X)^*)$. By Theorem 1.1, $e = b^{**} \otimes a^*$ for some $b^{**} \in \operatorname{Ext} S(X^{**})$ and $a^* \in \operatorname{Ext} S(X^*)$.

To summarize, it has been shown that $b^{**} \otimes a^* \in \mathcal{U}_{I+T}$, $b^{**} \otimes a^* \in \mathcal{U}_I$, and $b^{**} \otimes a^* \in \mathcal{U}_T$, which yields $b^{**}(a^* + T^*a^*) = \|I + T\|$, $b^{**}(a^*) = \|I\|$, and $b^{**}(T^*a^*) = \|T\|$. It follows easily that $\|a^* + T^*a^*\| = 2$, $\|a^*\| = 1$, and $\|T^*a^*\| = 1$.

Consider a functional $x_o^{**} \in S_{X^{**}}$ such that $x_o^{**}(a^*) = 1$. By the acs property of X^* one has $x_o^{**}(T^*a^*) = 1$. Therefore, $x_1^{**} := x_o^{**} \circ T^*$ attains the value 1 at a^* and hence belongs to $S_{X^{**}}$. Again, using (2.1), we obtain $x_1^{**}(T^*a^*) = 1$. Applying the same argument inductively shows that $x_o^{**}((T^*)^n a^*) = 1$ for all

$n = 0,1,2,\dots$. This implies that

$$\big((T^{**})^n x_o^{**}\big)(a^*) = 1 \tag{2.3}$$

for all $n = 0,1,2,\dots$. Recall that the function $Q\colon X^* \to X^{***}$ defined by $Qy^*(x^{**}) := x^{**}(y^*)$ for all x^{**} in X^{**} is a linear isometry of X^* into X^{***}.

Define a functional $h\colon X^{**} \to \mathbb{K}$ by the formula $h := Qa^*$. It follows that $h \in X^{***}$ and $h((T^{**})^n(x_o^{**})) = 1$ for all n. It follows from (2.3) that

$$h\big((T^{**})^n x_o^{**}\big) = 1 \tag{2.4}$$

for all n. Fix a Banach limit $L\colon l^\infty \to \mathbb{K}$. Define the mapping $y^{***} : X^{**} \to \mathbb{K}$ by

$$y^{***}(x^{**}) := L\big(h(x^{**}), h\big((T^{**})x^{**}\big), h\big((T^{**})^2 x^{**}\big), h\big((T^{**})^3 x^{**}\big), \dots\big)$$

for $x^{**} \in X^{**}$. Therefore, by (2.4) and by the properties of the Banach limits, we have $y^{***}(x_o^{**}) = L(1,1,1,\dots) = 1$, so $y^{***} \neq 0$.

In a similar way as in the proof of the implication (c)⇒(b) we obtain the equality $y^{***} = y^{***} \circ T^{**}$. Therefore, if we define a closed affine subspace $\mathcal{A}'' \subset X^{**}$ by $\mathcal{A}'' := (y^{***})^{-1}(\{1\})$, then we have $T^{**}(\mathcal{A}'') \subset \mathcal{A}''$ and $0 \notin \mathcal{A}''$.

It is standard that $X \subset X^{**}$ and $T^{**}|_X = T$. Define a closed affine subspace $\mathcal{A} \subset X$ by the formula $\mathcal{A} := X \cap \mathcal{A}''$. Since T is weakly compact, it follows that $T^{**}(X^{**}) \subset X$. Since $\operatorname{codim}\mathcal{A}'' = 1$ and $0 \notin \mathcal{A}''$, we obtain $\emptyset \neq \mathcal{A} \subset X$ and $0 \notin \mathcal{A} \neq X$. It is a straightforward verification to show that $T(\mathcal{A}) \subset \mathcal{A}$.

The equivalence of (a), (b), and (c) is proved, but we show that the implication (b)⇒(c) does not depend on the extra assumptions. Indeed, suppose $T^*y^* = y^* \neq 0$; that is, $y^* \circ T = y^*$. If $\mathcal{A} := (y^*)^{-1}(\{1\})$, then $T(\mathcal{A}) \subset \mathcal{A}$ and $0 \notin \mathcal{A}$. □

Two vectors x and y in a normed space satisfy $\|x+y\| = \|x\| + \|y\|$ if and only if $\|\alpha x + \beta y\| = \|\alpha x\| + \|\beta y\|$ holds for $\alpha, \beta \geq 0$. In particular, a continuous operator $T \in \mathcal{B}(X)$ satisfies the Daugavet equation if and only if the operator $\frac{T}{\|T\|}$ satisfies the Daugavet equation. From here we get the following consequence.

THEOREM 2.2

Assume that X is a Banach space such that X^* is acs. Suppose that $\mathcal{K}(X)$ is an M-ideal. Let $T \in \mathcal{B}(X)$, $\operatorname{dist}(T, \mathcal{K}(X)) < \|T\|$. Then a continuous operator T satisfies the Daugavet equation if and only if there exists a nontrivial closed invariant affine subspace for $\frac{T}{\|T\|}$.

Which Banach spaces X have the property that there is a bounded operator on X with no nontrivial closed invariant linear subspaces? The question is unanswered even if X is a Hilbert space. However, for certain specific classes of operators we can prove that the set of invariant subspaces is not trivial.

THEOREM 2.3

Let X, $\mathcal{B}(X)$, $\mathcal{K}(X)$, and T be as in Lemma 2.1. If T satisfies the Daugavet equation, then T has a nontrivial closed invariant linear subspace.

Proof

Define $W := \frac{T}{\|T\|}$. By Lemma 2.1, there is a $y^* \neq 0$ with $W^* y^* = y^*$. This implies that $N := \mathrm{cl}(I - W)(X) \neq X$. Note that $W(N) \subset N$, and so $T(N) \subset N$. □

REMARK 2.4

Let $T \in \mathcal{B}(l^2)$ be the operator defined by $T(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$. It is easy to see that T has a nontrivial closed invariant linear subspace. On the other hand, there is no nontrivial closed invariant affine subspace for T (see the implications (b)⇔(c) in Lemma 2.1).

3. The anti-Daugavet property

For terminology and notation, we follow [7]. We say that a Banach space X is *locally uniformly alternatively convex or smooth* (*luacs*) if for all $x_n, y \in S(X)$ and $x^* \in S(X^*)$ the implication

$$x^*(x_n) \to 1, \qquad \|x_n + y\| \to 2 \quad \Rightarrow \quad x^*(y) = 1$$

holds. Clearly, luacs implies acs.

We say that a Banach space X has the *anti-Daugavet property* for a class $\mathcal{M}$ of operators if, for $T \in \mathcal{M}$, the equivalence

$$\|I + T\| = 1 + \|T\| \quad \Leftrightarrow \quad \|T\| \in \sigma(T)$$

holds. We set $\mathcal{M}_X := \{T \in \mathcal{B}(X) : \mathrm{dist}(T, \mathcal{K}(X)) < \|T\|\}$. Clearly, $\mathcal{K}(X) \subset \mathcal{M}_X$.

THEOREM 3.1

Let X be a Banach space such that X^* is acs. Suppose that $\mathcal{K}(X)$ is an M-ideal. Then the space X has the anti-Daugavet property for the class $\mathcal{M}_X$.

Proof

It is easy to see that $\|T\| \in \sigma(T) \Leftrightarrow \|T^*\| \in \sigma(T^*)$. It follows from Lemma 2.1 that X has the anti-Daugavet property for $\mathcal{M}_X$. □

The acs and luacs spaces were originally introduced in [7] to obtain geometric characterizations of the anti-Daugavet property, which was introduced in the same paper. Clearly, rotundity and smoothness both imply acs. Note that, by compactness, in the case $\dim X < \infty$ the notions of acs and luacs spaces coincide. Clearly, luacs implies acs. Hardtke [5] proved the following theorem.

THEOREM 3.2 ([5, Proposition 2.15])

If X^* is acs, then X is acs.

In some sense, the above result can be extended.

THEOREM 3.3

Let $\mathcal{K}(X)$ be an M-ideal. If X^* is acs, then X is luacs.

Proof

In [7] it was proved that X has the anti-Daugavet property for compact operators if and only if X is luacs (see [7, Theorem 4.3]). It is easy to check that $\|T\| \in \sigma(T) \Leftrightarrow \|T^*\| \in \sigma(T^*)$. Therefore, from Lemma 2.1 (i.e., (a)⇔(b)) it follows that X has the anti-Daugavet property for compact operators. So, X is luacs. □

Combining Theorems 3.2 and 3.3, we immediately get the following result.

THEOREM 3.4

Let X be a reflexive Banach space. Let $\mathcal{K}(X)$ be an M-ideal in $\mathcal{B}(X)$. The Banach space X is acs if and only if X is luacs.

Now we demonstrate how the Daugavet equation, a purely isometric property, can be used to obtain some geometrical conclusion regarding operator spaces. Namely, we are able to prove the following criterion for checking when $\mathcal{K}(X)$ is not an M-ideal in $\mathcal{B}(X)$. From Theorem 3.3, we obtain the next result.

THEOREM 3.5

If X^* is acs and X is not luacs, then $\mathcal{K}(X)$ is not an M-ideal.

4. Daugavet equation and eigenvalues

Abramovich, Aliprantis, and Burkinshaw proved the following theorem (see [1, Corollary 2.4]). In the present section, we will generalize Theorem 4.1. The method of proof presented here is different from that of [1].

THEOREM 4.1

Let $1 < p < \infty$. A compact operator $T \in \mathcal{B}(l^p)$ satisfies the Daugavet equation if and only if its norm $\|T\|$ is an eigenvalue of T.

The authors of [1] proved a far more general theorem in that they assumed only uniform convexity; here we prove the special case of ℓ^p and do not generalize their original version. We want to show that it is not necessary to assume that T is a compact operator. So, our result also generalizes and complements Theorem 4.1.

PROPOSITION 4.2

Let $1 < p < \infty$, $T \in \mathcal{B}(l^p)$. Suppose that $\operatorname{dist}(T, \mathcal{K}(l^p)) < \|T\|$. The operator T satisfies the Daugavet equation if and only if its norm $\|T\|$ is an eigenvalue of T.

Proof
It is helpful to recall that $\mathcal{K}(l^p)$ and $\mathcal{K}((l^p)^*)$ are M-ideals (see [6]). The spaces l^p and $(l^p)^*$ are strictly convex. Therefore, the spaces l^p and $(l^p)^*$ are acs. Note that Lemma 2.1(a) is obviously self-dual. So, all one has to do is apply Lemma 2.1 to the dual of l^p and the adjoint of T. □

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