Moduli spaces of stable sheaves on Enriques surfaces

Kōta Yoshioka

Abstract We study the existence condition of μ-stable sheaves on Enriques surfaces. We also give a different proof of the irreducibility of the moduli spaces of rank 2 stable sheaves.

Contents

0. Introduction

Let X be an Enriques surface defined over an algebraically closed field k of characteristic not equal to 2. A number of authors have studied moduli spaces of stable sheaves on X, in particular in cases where the nonemptiness of the moduli spaces is completely determined (see [8], [14], [28]) and where the irreducibility of the moduli spaces is proved if X is unnodal and the associated Mukai vector is primitive (see [8], [14], [24]). In this article, we discuss the existence problem of μ-stable sheaves on Enriques surfaces. We also offer a remark on the irreducibility of the moduli spaces.

For a coherent sheaf E on X or an element E of $K(X)$, let $v(E) := \operatorname{ch}(E)\sqrt{\operatorname{td}_X} \in H^*(X,\mathbb{Q})$ be the Mukai vector of E. We introduce the Mukai pairing on $H^*(X,\mathbb{Q})$ by $\langle x, y\rangle := -\int_X x^\vee \wedge y$, where for $x = (x_0, x_1, x_2) \in H^*(X,\mathbb{Q})$, $x^\vee := (x_0, -x_1, x_2)$. Then

$$(v(K(X)), \langle\ ,\ \rangle) \tag{0.1}$$

is the Mukai lattice of X. For a Mukai vector v, we use the expressions

$$v = (r, \xi, a) = r + \xi + a\varrho_X, \quad r \in \mathbb{Z}, \xi \in \operatorname{NS}_{\mathrm{f}}(X), \frac{r}{2} + a \in \mathbb{Z},$$

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where $\varrho_X \in H^4(X,\mathbb{Z})$ is the fundamental class of X, and $\mathrm{NS}_{\mathrm{f}}(X)$ is the torsion-free quotient of $\mathrm{NS}(X)$; that is, $\mathrm{NS}_{\mathrm{f}}(X) = \mathrm{NS}(X)/\mathbb{Z}K_X$. The quotient $\mathrm{NS}_{\mathrm{f}}(X)$ is nothing but the numerical equivalence class group $\mathrm{Num}(X)$ of X.

For the Mukai vector $v \in H^*(X,\mathbb{Q})$ of a torsion-free sheaf, we assume that a polarization H is general with respect to v (see Definition 0.2). We point out that the problem of constructing μ-stable locally free sheaves was studied by Kim in the rank 2 case and by Nuer [14] in the rank 4 case.

For a Mukai vector v, $\mathcal{M}(v)$ denotes the moduli stack of coherent sheaves E with $v(E) = v$. Let H be an ample divisor on X. Let $\mathcal{M}_H(v)^{\mathrm{ss}}$ (resp., $\mathcal{M}_H(v)^{\mathrm{s}}$) denote the substack of $\mathcal{M}(v)$ consisting of (Gieseker) semistable sheaves (resp., stable sheaves). Let $\overline{M}_H(v)$ be the moduli scheme of S-equivalence classes of semistable sheaves, and let $M_H(v)$ be the open subscheme consisting of stable sheaves. If $v = (r,\xi,a)$ with $r > 0$, then $\mathcal{M}_H(v)^{\mu\mathrm{ss}}$ (resp., $\mathcal{M}_H(v)^{\mu\mathrm{s}}$) denotes the substack of $\mathcal{M}(v)$ consisting of μ-semistable sheaves (resp., μ-stable sheaves). As in [24], we also introduce $\mathcal{M}_H(v,L)^{\mathrm{ss}}$ (resp., $\mathcal{M}_H(v,L)^{\mathrm{s}}$, $\overline{M}_H(v,L), M_H(v,L)$) as the locus of $\mathcal{M}_H(v)^{\mathrm{ss}}$ (resp., $\mathcal{M}_H(v)^{\mathrm{s}}$, $\overline{M}_H(v), M_H(v)$) consisting of E with $c_1(E) = L$ in $\mathrm{NS}(X)$, where $[L \mod K_X] = \xi$.

For a $K3$ surface, the existence condition of μ-stable sheaves was completely described in [23]. For an Enriques surface, we get a similar result.

THEOREM 0.1 (Theorem 2.1)
Let $v = (lr, l\xi, \frac{s}{2})$ be a Mukai vector such that $\gcd(r,\xi) = 1$ and $\langle v^2\rangle \geq 0$. Let H be a general polarization with respect to v. Then $\mathcal{M}_H(v,L)^{\mathrm{ss}}$ contains a μ-stable sheaf if and only if

(i) *there is no stable sheaf E such that $v(E) = (r,\xi,b)$ and $\langle v(E)^2\rangle = -1, -2$, and $\langle v^2\rangle \geq 0$; or*

(ii) *there is a stable sheaf E such that $v(E) = (r,\xi,b)$ and $\langle v(E)^2\rangle = -1$, and $\langle v^2\rangle \geq l^2$; or*

(iii) *there is a stable sheaf E such that $v(E) = (r,\xi,b)$ and $\langle v(E)^2\rangle = -2$, and $\langle v^2\rangle \geq 2l^2$.*

Moreover, if $lr > 1$, then under the same condition, $\mathcal{M}_H(v,L)^{\mathrm{ss}}$ contains a μ-stable locally free sheaf.

In the second part of the article, we will study the irreducibility of the moduli spaces $\mathcal{M}_H(v,L)^{\mathrm{ss}}$. The irreducibility of these moduli spaces on an arbitrary surface was proved by Gieseker and Li [4] and O'Grady [15, Theorem D] when the expected dimension d is larger than a constant $N(r)$ that depends only on the rank r. We can expect a better estimate for $N(r)$ in the Enriques case, as occurs for $K3$ and Abelian surfaces, since an Enriques surface also has a numerically trivial canonical divisor. Let $v = (r,\xi,\frac{s}{2})$ be a primitive Mukai vector on an Enriques surface. If r is odd, then the irreducibility of $\mathcal{M}_H(v,L)^{\mathrm{ss}}$ was proved in [24]. If $r = 2$, then the irreducibility was investigated by Kim [8] and Nuer [14] if X is unnodal. Kim reduced the problem to the cases where $s = 1, 2$

and proved the irreducibility for $s = 1$. For the second case, Nuer reduced to the first case. Then by using Bridgeland stability, Nuer [14, Theorem 1.1] completed the proof of irreducibility for the even rank case. In this article, we will give a different proof of the irreducibility for $r = 2$. Combining a deformation argument with results in [24] and [28], we get the following result.

THEOREM 0.2

Let $v = (r, \xi, \frac{s}{2})$ be a primitive Mukai vector on an Enriques surface X, let L be a divisor on X with $[L \mod K_X] = \xi$, and let H be an ample divisor general for v. Then we have the following.

(1) *$\mathcal{M}_H(v, L)^{\mathrm{ss}}$ is connected.*
(2) *If X is unnodal or $\langle v^2 \rangle \geq 4$, then $\mathcal{M}_H(v, L)^{\mathrm{ss}}$ is irreducible.*

The strategy of our proof is the same as our proof for the similar problem on $K3$ surfaces (see [24, Theorem 3.18]). Thus we reduce the problem to the moduli of stable 1-dimensional sheaves by a relative Fourier–Mukai transform associated to an elliptic fibration. Then by a detailed estimate of the locus of stable sheaves whose supports are reducible or nonreduced, we show that the moduli space is birationally equivalent to an Abelian fiber space over an open subset of a projective space. Unlike in the case of $K3$ surfaces, we need the Mukai vector to be primitive. Indeed, if the Mukai vector is of the form $v = m(r, \xi, \frac{s}{2})$ ($m, r \in \mathbb{Z}_{>0}$, $\xi \in \mathrm{NS}(X)$, $2 \mid r - s$, and $2 \nmid r$), then we cannot reduce to the rank 0 case. Moreover, it is not so easy to study stable 1-dimensional sheaves on nonreduced curves. Hence we can only deal with 1-dimensional sheaves on nonreduced curves of multiplicity 2, which is sufficient to treat the primitive case. We give partial generalizations in Remarks 4.3 and 4.4. In the course of the proof, we also show that [17, Assumption 2.16] holds for $v = (0, \xi, \frac{s}{2})$ such that ξ is primitive (see Corollary 4.5). In particular, we get the following corollary by [17, Theorem 5.1] and [28].

COROLLARY 0.3 ([3, p. 83])

We have that $b_2(M_H(v, L)) = 11$ if X satisfies (1.3) and $v = (r, \xi, \frac{s}{2})$ is a primitive Mukai vector such that $2 \mid r$, $2 \nmid \xi$, and $\langle v^2 \rangle \geq 4$.

NOTATION

For an Enriques surface X, let $\varpi : \widetilde{X} \to X$ be the covering $K3$ surface, and let $\iota : \widetilde{X} \to \widetilde{X}$ be the covering involution.

0.1. Preliminaries

An Enriques surface X is nodal if there is a smooth rational curve, and X is unnodal if X is not nodal (see [3, p. 178]). For an unnodal Enriques surface, every effective divisor D is nef, and D is ample if and only if $(D^2) > 0$. Polarized Enriques surfaces form 10-dimensional moduli spaces, and general members are unnodal.

We collect some properties of the Mukai lattice of Enriques surfaces.

(i) We have an isomorphism of lattices:

$$(v(K(X)), \langle\ ,\ \rangle) \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8(-1). \tag{0.2}$$

(ii) For a Mukai vector $v = (r, \xi, \frac{s}{2})$, $\langle v^2 \rangle$ is even if and only if r is even.

(iii) A Mukai vector $v = (r, \xi, \frac{s}{2}) \in v(K(X))$ is primitive if and only if $r, s \in \mathbb{Z}$, $2 \mid (r+s)$, $\xi \in \mathrm{NS}_{\mathrm{f}}(X)$, and $\gcd(r, \xi, \frac{r+s}{2}) = 1$.

(iv) For a primitive Mukai vector $v = (r, c_1, \frac{s}{2})$, we set $\ell(v) := \gcd(r, c_1, s)$. Then $\ell(v) = 1, 2$ (see [5, Lemma 2.5]).

Thanks to the Auslander–Buchsbaum formula, the following facts are well known for a purely 1-dimensional sheaf on a smooth projective surface.

LEMMA 0.4

Let X be a smooth projective surface.

(1) *A 1-dimensional coherent sheaf E is purely 1-dimensional if and only if there is an injective homomorphism $\varphi : V_{-1} \to V_0$ of locally free sheaves V_{-1}, V_0 of the same rank and* $\operatorname{coker} \varphi = E$. *Thus we have a locally free resolution of E:*

$$0 \to V_{-1} \xrightarrow{\varphi} V_0 \to E \to 0. \tag{0.3}$$

(2) *A coherent sheaf E is purely 1-dimensional if and only if* $\mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(E, \mathcal{O}_X)[1] \in \mathrm{Coh}(X)$ *and is purely 1-dimensional.*

DEFINITION 0.1

For a coherent sheaf E on X, $\operatorname{rk} E$ denotes the rank of E. For a purely 1-dimensional sheaf E on X, $\operatorname{Div}(E)$ denotes the effective divisor C such that E is an $\mathcal{O}_C$-module and $c_1(E) = C$. By using a locally free resolution (0.3), $\operatorname{Div} E$ is given by the Cartier divisor $\det \varphi$.

0.2. Stabilities and their moduli stacks

Let X be a smooth projective surface, and let H be an ample divisor on X.

DEFINITION 0.2

(1) A torsion-free sheaf E is μ-semistable (resp., μ-stable) if

$$\frac{(c_1(F), H)}{\operatorname{rk} F} \underset{(<)}{\leq} \frac{(c_1(E), H)}{\operatorname{rk} E} \tag{0.4}$$

for any subsheaf F of E with $0 < \operatorname{rk} F < \operatorname{rk} E$.

(2) A polarization H is general with respect to v if for any μ-semistable sheaf E with $v(E) = v$ and any subsheaf F of E,

$$\frac{(c_1(F), H)}{\operatorname{rk} F} = \frac{(c_1(E), H)}{\operatorname{rk} E} \iff \frac{c_1(F)}{\operatorname{rk} F} = \frac{c_1(E)}{\operatorname{rk} E}. \tag{0.5}$$

DEFINITION 0.3 (see [10], [25])
Let G be an element of $K(X)$ with $\operatorname{rk} G > 0$.

(1) A torsion-free sheaf E is G-twisted semistable (resp., G-twisted stable) if

$$\frac{\chi(G, F(nH))}{\operatorname{rk} F} \underset{(<)}{\leq} \frac{\chi(G, E(nH))}{\operatorname{rk} E} \quad (n \gg 0) \tag{0.6}$$

for any subsheaf F of E with $0 < \operatorname{rk} F < \operatorname{rk} E$.

(2) A purely 1-dimensional sheaf E is G-twisted semistable (resp., G-twisted stable) if

$$\frac{\chi(G, F)}{(c_1(F), H)} \underset{(<)}{\leq} \frac{\chi(G, E)}{(c_1(E), H)} \tag{0.7}$$

for any proper subsheaf $F \neq 0$ of E.

(3) Since G-twisted semistability depends only on $v(G)$, we also define w-twisted semistability as a G-twisted semistability, where $v(G) = w$.

(4) $\mathcal{M}_H^G(v)^{\mathrm{ss}}$ (resp., $\mathcal{M}_H^G(v)^{\mathrm{s}}$) denotes the moduli stack of G-twisted semistable sheaves (resp., G-twisted stable sheaves).

REMARK 0.1

(1) G-twisted semistability depends only on $c_1(G)/\operatorname{rk} G$.

(2) If H is general with respect to v, then G-twisted semistability is independent of the choice of G.

Let us recall a quotient stack description of $\mathcal{M}_H(v)^{\mu\mathrm{ss}}$ and its open substacks. For an ample divisor H' on X, let $Q(mH', v)$ be the open subscheme of the quot-scheme $\operatorname{Quot}_{\mathcal{O}_X(-mH')^{\oplus N}/X}$ consisting of points

$$\lambda : \mathcal{O}_X(-mH')^{\oplus N} \to E \tag{0.8}$$

such that

(i) $v(E) = v$,
(ii) λ induces an isomorphism $H^0(X, \mathcal{O}_X^{\oplus N}) \cong H^0(X, E(mH'))$, and
(iii) $H^i(X, E(mH')) = 0$, $i > 0$.

Let $\mathcal{O}_{Q(mH',v) \times X}(-mH')^{\oplus N} \to \mathcal{Q}_v$ be the universal quotient. We set $V_v := \mathcal{O}_X(-mH')^{\oplus N}$. For our purpose, the choice of mH' is not so important. Hence we simply denote $Q(mH', v)$ by $Q(v)$. Let $\mathcal{M}(v)$ be the moduli stack of coherent sheaves E with $v(E) = v$, and let $q_v : Q(v) \to \mathcal{M}(v)$ be the natural map. We denote the pullbacks $q_v^{-1}(\mathcal{M}_H(v)^{\mu\mathrm{ss}}), q_v^{-1}(\mathcal{M}_H(v)^{\mathrm{ss}}), \dots$ by $Q(v)^{\mu\mathrm{ss}}, Q(v)^{\mathrm{ss}}, \dots$, respectively. If we choose a suitable $Q(v)$, then $q_v : Q(v)^{\mu\mathrm{ss}} \to \mathcal{M}_H(v)^{\mu\mathrm{ss}}$ is surjective and $\mathcal{M}_H(v)^{\mu\mathrm{ss}}$ is a quotient stack of $Q(v)^{\mu\mathrm{ss}}$ by the natural action of $\mathrm{GL}(N)$:

$$\mathcal{M}_H(v)^{\mu\mathrm{ss}} \cong \left[Q(v)^{\mu\mathrm{ss}} / \mathrm{GL}(N)\right]. \tag{0.9}$$

From now on, we assume that $q_v : Q(v)^{\mu ss} \to \mathcal{M}_H(v)^{\mu ss}$ is surjective. We have

$$\dim \mathcal{M}_H(v)^{\mu ss} = \dim Q(v)^{\mu ss} - \dim \mathrm{GL}(N). \tag{0.10}$$

REMARK 0.2
Since $\mathrm{PGL}(N)$ acts freely on $Q_H(v)^{\mathrm{s}}$, we have $\dim \mathcal{M}_H(v)^{\mathrm{s}} = \dim M_H(v) - 1$.

LEMMA 0.5
Let $\mathcal{M}$ be an irreducible component of $\mathcal{M}_H(v)^{\mu ss}$. Then $\dim \mathcal{M} \geq \langle v^2 \rangle$.

Proof
We take a quotient (0.8). Then we see that $\mathrm{Ext}^2(\ker \lambda, E) = 0$. By the deformation theory of the quot-scheme, the Zariski tangent space of the quot-scheme at (0.8) is $\mathrm{Hom}(\ker \lambda, E)$ and the obstruction space is $\mathrm{Ext}^1(\ker \lambda, E) \cong \mathrm{Ext}^2(E, E)$. Hence the dimension of an irreducible component of $Q(v)^{\mu ss}$ containing the point (0.8) is at least

$$\dim \mathrm{Hom}(\ker \lambda, E) - \dim \mathrm{Ext}^1(\ker \lambda, E) = N^2 - \chi(E, E) = \langle v^2 \rangle + \dim \mathrm{GL}(N).$$

Hence we get the claim. □

The following formula is used frequently in this article.

LEMMA 0.6 ([9, Lemma 5.2])
Let $\mathcal{F}(v_1, v_2)$ be the stack of filtrations $0 \subset E_1 \subset E$ such that E_1 is a coherent sheaf with $v(E_1) = v_1$ and $E_2 := E/E_1$ is a coherent sheaf with $v(E_2) = v_2$. We have a morphism $p_{v_1,v_2} : \mathcal{F}(v_1, v_2) \to \mathcal{M}(v_1) \times \mathcal{M}(v_2)$ by sending $E_1 \subset E$ to $(E_1, E/E_1)$. We set

$$\begin{aligned} &\mathcal{N}^n(v_1, v_2) \\ &\quad := \{(E_1, E_2) \in \mathcal{M}(v_1) \times \mathcal{M}(v_2) \mid \dim \mathrm{Hom}(E_1, E_2(K_X)) = n\}, \\ &\mathcal{F}^n(v_1, v_2) \\ &\quad := p_{v_1,v_2}^{-1}(\mathcal{N}^n(v_1, v_2)) \\ &\quad = \{(F_1 \subset E) \in \mathcal{F}(v_1, v_2) \mid \dim \mathrm{Hom}(F_1, (E/F_1)(K_X)) = n\}. \end{aligned} \tag{0.11}$$

Then

$$\dim \mathcal{F}^n(v_1, v_2) = \dim \mathcal{N}^n(v_1, v_2) + \langle v_1, v_2 \rangle + n. \tag{0.12}$$

Proof
Since $\dim \mathrm{Ext}^2(E_2, E_1) = \dim \mathrm{Hom}(E_1, E_2(K_X)) = n$, the same proof of [9, Lemma 5.2] works. □

1. The dimension of moduli stacks

In this section, we assume that X is an Enriques surface, and we estimate the dimension of various substacks of $\mathcal{M}_H(v)$. We also show that $\mathcal{M}_H(v)^{\mathrm{ss}}$ is a reduced stack if $\langle v^2\rangle > 0$ or v is a primitive and isotropic Mukai vector. Before giving estimates, we first recall the nonemptiness of the moduli stacks.

THEOREM 1.1 ([14, Theorem 1.1], [28, Theorem 3.1])

Let X be an Enriques surface over k. We take $r,s\in\mathbb{Z}$ $(r>0)$ and $L\in \mathrm{NS}(X)$ such that $r+s$ is even. We set $\xi := [L \mod K_X]$. Assume that $\gcd(r,\xi,\frac{r+s}{2})=1$*; that is, the Mukai vector $v:=(r,\xi,\frac{s}{2})$ is primitive. Then $\mathcal{M}_H(v,L)^{\mathrm{s}}\neq\emptyset$ for a general H if and only if*

(i) $\ell(v)=1$ *and* $\langle v^2\rangle \geq -1$, *or*
(ii) $\ell(v)=2$ *and* $\langle v^2\rangle \geq 2$, *or*
(iii) $\ell(v)=2$, $\langle v^2\rangle = 0$, *and* $L\equiv \frac{r}{2}K_X \mod 2$, *or*
(iv) $\langle v^2\rangle = -2$, $L\equiv D+\frac{r}{2}K_X \mod 2$, *where D is a nodal cycle, that is,* $(D^2)=-2$ *and* $H^1(\mathcal{O}_D)=0$.

By taking a direct sum, we get the following corollary of Theorem 1.1.

COROLLARY 1.2

Let $v=(r,\xi,\frac{s}{2})$ be a primitive Mukai vector with $\langle v^2\rangle>0$. Then $\mathcal{M}_H(lv,L)^{\mathrm{ss}}\neq\emptyset$ for a general H, where $[L \mod K_X]=l\xi$.

LEMMA 1.3

Let $v=(r,\xi,\frac{s}{2})$ be a Mukai vector with $\langle v^2\rangle>0$. Then

(1) $\mathcal{M}_H(v,L)^{\mathrm{s}}$ *is reduced and* $\dim \mathcal{M}_H(v,L)^{\mathrm{s}}=\langle v^2\rangle$,
(2) $\mathcal{M}_H(v,L)^{\mathrm{s}}$ *is normal, unless*
  (i) $v=2v_0$ *with* $\langle v_0^2\rangle=1$ *and* $L\equiv\frac{r}{2}K_X \mod 2$, *or*
  (ii) $\langle v^2\rangle = 2$.

Proof

(1) We introduce the substack

$$\mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}} := \{E\in\mathcal{M}_H(v,L)^{\mathrm{s}} \mid \mathrm{Ext}^2(E,E)\neq 0\} = \{E\in\mathcal{M}_H(v,L)^{\mathrm{s}} \mid E\cong E(K_X)\}, \tag{1.1}$$

which is expected to be the singular locus of $\mathcal{M}_H(v,L)^{\mathrm{s}}$. Indeed, the singular locus of $\mathcal{M}_H(v,L)^{\mathrm{s}}$ is contained in $\mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}}$, and they coincide if $\dim\mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}}<\langle v^2\rangle$ by the deformation theory of coherent sheaves. So we will estimate $\dim\mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}}$. We set $v=(r,c_1,\frac{s}{2})$. If r is odd, then $\mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}}=\emptyset$ since $\det(E(K_X))\cong(\det E)(K_X)$. Hence we assume that r is even. By [7] (see also Remark 1.1) or [18], $\dim\mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}}$ is odd and $\dim\mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}}\leq\frac{\langle v^2\rangle}{2}+1$. Moreover, if the equality holds, then $2\mid c_1$ and $L\equiv\frac{r}{2}K_X \mod 2$, and if v is primitive, then $\langle v^2\rangle\equiv 0 \mod 8$ (see Lemma 1.4).

In particular, we have

$$\langle v^2\rangle - \dim \mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}} \geq \begin{cases} \frac{\langle v^2\rangle}{2} - 1, & \langle v^2\rangle \equiv 0 \mod 4, \\ \frac{\langle v^2\rangle}{2}, & \langle v^2\rangle \equiv 2 \mod 4. \end{cases}$$

Since $\mathcal{M}_H(v,L)^{\mathrm{s}} \setminus \mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}}$ is smooth of dimension $\langle v^2\rangle$, by the proof of Lemma 0.5, we see that $\mathcal{M}_H(v,L)^{\mathrm{s}}$ is a locally complete intersection stack of $\dim \mathcal{M}_H(v,L)^{\mathrm{s}} = \langle v^2\rangle$. In particular, $\mathcal{M}_H(v,L)^{\mathrm{s}}$ is reduced.

(2) In order to prove the normality of $\mathcal{M}_H(v,L)^{\mathrm{s}}$, it is sufficient to prove

$$\langle v^2\rangle - \dim \mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}} \geq 2. \tag{1.2}$$

If $\langle v^2\rangle \geq 6$, then obviously (1.2) holds. If $\langle v^2\rangle = 4$ and v does not satisfy (i), then $\dim \mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}} < \frac{\langle v^2\rangle}{2} + 1 = 3$, which implies that $\dim \mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}} \leq 1$. In particular, $\langle v^2\rangle - \dim \mathcal{M}_H(v,L)^{\mathrm{s}}_{\mathrm{sing}} \geq 3$. Therefore (2) holds. □

LEMMA 1.4 (Nuer [14], Saccà [17, Theorem 2.9])

Assume that

$$\varpi^*\big(\mathrm{Pic}(X)\big) = \mathrm{Pic}(\widetilde{X}); \tag{1.3}$$

thus ι acts on $\mathrm{Pic}(\widetilde{X})$ trivially.

Let $v := (r, \xi, \frac{s}{2})$ be a primitive Mukai vector. Then $\mathcal{M}_H(v,L)^{\mathrm{s}}$ is smooth of $\dim \mathcal{M}_H(v,L)^{\mathrm{s}} = \langle v^2\rangle$, unless $\ell((r,\xi,\frac{s}{2})) = 2$ and $L \equiv \frac{r}{2}K_X \mod 2$.

Proof

By using (-1)-reflection (see Remark 1.1), we may assume that $\mathcal{M}_H(v,L)^{\mathrm{s}}$ consists of μ-stable locally free sheaves. Assume that $E \cong E(K_X)$. Then r is even and there is a locally free sheaf F such that $\varpi_*(F) = E$. Then $\varpi^*(E) \cong F \oplus \iota^*(F)$. By our assumption on X, $\iota^*(c_1(F)) = c_1(F)$ and $c_1(F) = c_1(\varpi^*(L))$, where $L \in \mathrm{Pic}(X)$. Hence $\det F = \varpi^*(L)$ and $\varpi^*(c_1(E)) = c_1(F) + \iota^*(c_1(F)) = 2\varpi^*(c_1(L))$. Then

$$c_1(E) = c_1\big(\varpi_*(F)\big) = c_1\big(\varpi_*(\det(F))\big) + \Big(\frac{r}{2} - 1\Big)K_X = 2c_1(L) + \frac{r}{2}K_X \tag{1.4}$$

by [28, Lemma 3.5]. Hence $c_1(E) \equiv \frac{r}{2}K_X \mod 2$. □

REMARK 1.1

In [7], it is assumed that $E \in \mathcal{M}_H(v)^{\mathrm{s}}_{\mathrm{sing}}$ is locally free. Indeed $\mathcal{M}_H(v)^{\mathrm{s}}$ is isomorphic to a moduli stack of μ-stable locally free sheaves by using an autoequivalence Φ of $\mathbf{D}(X)$ called (-1)-*reflection* in [24, Section 4]. A similar claim to [26, Theorem 1.7] holds for Φ (see [28, Remark 2.19]) and for a sufficiently large n (depending only on v),

$$\Phi(E) = \ker\big(H^0\big(E(nH)\big) \otimes \mathcal{O}_X \oplus H^0\big(E(K_X + nH)\big) \otimes \mathcal{O}_X(K_X) \to E(nH)\big) \tag{1.5}$$

is a μ-stable locally free sheaf. Thus we can reduce the general case to the case of μ-stable locally free sheaves.

LEMMA 1.5

Let v be a Mukai vector with $\langle v^2\rangle > 0$. We set

$$\mathcal{M}_H(v)^{\mathrm{pss}} := \{E \in \mathcal{M}_H(v)^{\mathrm{ss}} \mid E \text{ is properly semistable}\}. \tag{1.6}$$

Assume that H is general with respect to v. Then

(1) $\dim \mathcal{M}_H(v)^{\mathrm{pss}} \le \langle v^2\rangle - 1$*; moreover,* $\dim \mathcal{M}_H(v)^{\mathrm{pss}} \le \langle v^2\rangle - 2$ *unless* $v = 2v_0$ *with* $\langle v_0^2\rangle = 1$*;*

(2) $\mathcal{M}_H(v)^{\mathrm{s}} \ne \emptyset$ *and* $\dim \mathcal{M}_H(v)^{\mathrm{ss}} = \langle v^2\rangle$.

Proof

We set $v = lv_0$, where v_0 is primitive and $l \in \mathbb{Z}_{>0}$. We first note that the first claim of (1) implies (2) by Lemma 0.5, Lemma 1.3, and Corollary 1.2. The proof of (1) is almost the same as that of [9, Lemma 3.2]. So we only remark that [9, Lemma 5.1] is replaced by Lemma 0.5, and [9, (3.4)] is replaced by

$$\dim J(v_1, v_2) \le \langle v^2\rangle - \big(\langle v_1, v_2\rangle - \max\{l_2/l_1 - 1, 0\}\big),$$

where $J(v_1, v_2)$ is the substack whose member E fits in an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{1.7}$$

such that E_1 is a stable sheaf with $v_1 := l_1 v_0$ and E_2 is a semistable sheaf with $v_2 := l_2 v_0$. We note that -1 on the right-hand side comes from the vanishing $\mathrm{Hom}(E_1, E_2(K_X)) = 0$ for a general stable sheaf E_1, since E_1 is not rigid. We first assume that $\langle v_0^2\rangle \ge 2$. Then

$$\langle v_1, v_2\rangle - \max\{l_2/l_1 - 1, 0\} = l_1 l_2 \langle v_0^2\rangle - \max\{l_2/l_1 - 1, 0\} \ge 2.$$

Hence $\dim J(v_1, v_2) \le \langle v^2\rangle - 2$, so the second claim of (1) holds if $\langle v_0^2\rangle > 1$, and a fortiori the first claim. So we may assume that $\langle v_0^2\rangle = 1$. Then the first claim of (1) clearly holds from the dimension estimate on $J(v_1, v_2)$, so let us prove the second claim. We set

$$H_k := \{(E_1, E_2) \mid E_1 \in \mathcal{M}_H(v_1)^{\mathrm{s}}, E_2 \in \mathcal{M}_H(v_2)^{\mathrm{ss}}, \dim \mathrm{Hom}\big(E_1, E_2(K_X)\big) = k\}.$$

For $(E_1, E_2) \in H_k$, $E_2/(E_1(K_X)^{\oplus k})$ is a semistable sheaf with the Mukai vector $v_2 - kv_1$ by [9, Lemma 3.1]. Hence E_1 is determined by E_2 as a factor of a Jordan–Hölder filtration of E_2. Moreover, if $k \ge 2$, then E_2 is properly semistable. Therefore $\dim H_1 \le \langle v_2^2\rangle$ and $\dim H_k \le \langle v_2^2\rangle - 1$ for $k > 1$. If $k \ge 2$, then

$$\begin{aligned} \langle v_1, v_2\rangle + k + \dim H_k &\le \langle v^2\rangle - \big(\langle v_1, v_2\rangle + \langle v_1^2\rangle + 1 - k\big) \\ &\le \langle v^2\rangle - 2. \end{aligned} \tag{1.8}$$

If $k = 1$ and $l_1 l_2 \ge 2$, then

$$\begin{aligned} \langle v_1, v_2\rangle + k + \dim H_k &\le \langle v^2\rangle - \big(\langle v_1, v_2\rangle + \langle v_1^2\rangle - 1\big) \\ &\le \langle v^2\rangle - 2. \end{aligned} \tag{1.9}$$

Therefore $\dim J(v_1, v_2) \le \max_k(\dim H_k + \langle v_1, v_2\rangle + k) \le \langle v^2\rangle - 2$ if $l_1 l_2 \ge 2$. The remaining case is $v = 2v_0$, which is excluded. Therefore (1) holds. □

COROLLARY 1.6

Let $v=(r,\xi,\frac{s}{2})$ be a Mukai vector with $\langle v^2\rangle>0$. Then for a general polarization H, we have the following.

(1) $\mathcal{M}_H(v,L)^{\mathrm{ss}}$ *is reduced and* $\dim\mathcal{M}_H(v,L)^{\mathrm{ss}}=\langle v^2\rangle$.
(2) $\mathcal{M}_H(v,L)^{\mathrm{ss}}$ *is normal, unless*
  (i) $v=2v_0$ *with* $\langle v_0^2\rangle=1$ *and* $L\equiv\frac{r}{2}K_X \mod 2$, *or*
  (ii) $\langle v^2\rangle=2$.

Proof

By Lemmas 1.3 and 1.5, (1) holds. Moreover, (2) also holds unless (i) $v=2v_0$ with $\langle v_0^2\rangle=1$ or (ii) $\langle v^2\rangle=2$. Therefore we will treat the moduli stack $\mathcal{M}_H(2v_0,L)^{\mathrm{ss}}$ with $\langle v_0^2\rangle=1$ and $L\not\equiv\frac{r}{2}K_X \mod 2$. By Lemma 1.3(2), $\mathcal{M}_H(2v_0,L)^{\mathrm{s}}$ is normal.

We will prove that $\mathcal{M}_H(2v_0,L)^{\mathrm{ss}}$ is smooth in a neighborhood of the boundary. Since $\langle v_0^2\rangle=1$, $\operatorname{rk}v_0$ is odd, which implies that $\frac{r}{2}K_X\equiv K_X \mod 2$. Since $2\mid\xi$ in $\mathrm{NS}_{\mathrm{f}}(X)$, we have $L=2D,2D+K_X$ $(D\in\mathrm{NS}(X))$. Therefore $L=2D$ by $L\not\equiv\frac{r}{2}K_X\equiv K_X \mod 2$. Assume that $E\in\mathcal{M}_H(2v_0,L)^{\mathrm{ss}}$ is S-equivalent to $E_1\oplus E_2$. By $\det E_1=(\det E_2^{\vee})(L)$ and the fact that $\operatorname{rk}v_0$ is odd, $\operatorname{Hom}(E_i,E_j(K_X))=0$ for all $1\le i,j\le 2$. Thus $\operatorname{Ext}^2(E,E)=\operatorname{Hom}(E,E(K_X))^{\vee}=0$, which implies that $\mathcal{M}_H(2v_0,L)^{\mathrm{ss}}$ is smooth at E. Therefore $\mathcal{M}_H(2v_0,L)^{\mathrm{ss}}$ is a normal stack. □

1.1. Isotropic case

Let $v=(r,\xi,\frac{s}{2})$ be a primitive and isotropic Mukai vector, and take a general polarization H with respect to v.

LEMMA 1.7

(1) *If $\ell(v)=1$, then $\mathcal{M}_H(v)^{\mathrm{s}}=\mathcal{M}_H(v)^{\mathrm{ss}}$ is a reduced stack of* $\dim\mathcal{M}_H(v)^{\mathrm{s}}=\langle v^2\rangle=0$.

(2) *If $\ell(v)=2$ and $L\equiv\frac{r}{2}K_X \mod 2$, then $\mathcal{M}_H(v,L)^{\mathrm{s}}=\mathcal{M}_H(v,L)^{\mathrm{ss}}$ is smooth of* $\dim\mathcal{M}_H(v,L)^{\mathrm{s}}=1$ *and* $\mathcal{M}_H(v,L+K_X)^{\mathrm{ss}}=\emptyset$.

Proof

(1) Assume that $\ell(v)=1$. Since $2\nmid\xi$, by the proof of Lemma 1.3, we see that $\dim\mathcal{M}_H(v)^{\mathrm{s}}_{\mathrm{sing}}=-1$. Thus $\mathcal{M}_H(v)^{\mathrm{s}}$ is reduced and $\dim\mathcal{M}_H(v)^{\mathrm{s}}=0$.

(2) If $\ell(v)=2$ and $L\equiv\frac{r}{2}K_X \mod 2$, then there is an irreducible component M of the coarse moduli scheme $\overline{M}_H(v,L)=M_H(v,L)$ with $\dim M\ge 2$ (see [14, Theorem 5.2], [28, Section 3]). Since the Zariski tangent space $\operatorname{Ext}^1(E,E)$ of $M_H(v,L)$ at $E\in M_H(v,L)$ satisfies

$$\dim\operatorname{Ext}^1(E,E)=\dim\operatorname{Hom}\big(E,E(K_X)\big)+1\le 2,$$

M is smooth of dimension 2 and $E(K_X)\cong E$ for $E\in M$. Since the Mukai lattice is unimodular, $M_H(v,L)$ is a fine moduli space. Then by using the Fourier–Mukai transform, we see that $M_H(v,L)$ itself is smooth of $\dim M_H(v,L)=2$ and $\overline{M}_H(v,L+K_X)=\emptyset$ (see also the proof of [27, Lemma 3.1.6]). For the claim on $\dim\mathcal{M}_H(v,L)^{\mathrm{ss}}$, we recall Remark 0.2. □

We next study the nonprimitive case. We assume that H is a general polarization with respect to lv. Then H is also a general polarization with respect to $l'v$ for $1 \le l' \le l$. For $E_0 \in \mathcal{M}_H(l_0 v)^{\mathrm{s}}$, we set

$$\mathcal{J}(l, E_0) := \{E \in \mathcal{M}_H(lv)^{\mathrm{ss}} \mid E \text{ is generated by } E_0(pK_X),\ p \in \mathbb{Z}\}, \tag{1.10}$$

where $l_0 \mid l$.

REMARK 1.2
If $\ell(v) = 2$, then $E_0(K_X) \cong E_0$ for all $E_0 \in \mathcal{M}_H(v)^{\mathrm{s}}$, and if $\ell(v) = 1$, then $E_0(K_X) \not\cong E_0$ for a general $E_0 \in \mathcal{M}_H(v)^{\mathrm{s}}$.

LEMMA 1.8
We have $\dim \mathcal{J}(l, E_0) \le -1$.

Proof
For $F \in \{E_0(pK_X) \mid p \in \mathbb{Z}\}$, we set

$$\mathcal{J}(l, E_0, F^{\oplus n}) := \{E \in \mathcal{J}(l, E_0) \mid \dim \operatorname{Hom}(F, E) = n\}. \tag{1.11}$$

For $E \in \mathcal{J}(l, E_0, F^{\oplus n})$, we have an exact sequence

$$0 \to \operatorname{Hom}(F, E) \otimes F \to E \to E' \to 0 \tag{1.12}$$

and $E' \in \mathcal{J}(l - nl_0, E_0, F(K_X)^{\oplus n'})$ $(n' \ge 0)$. Let $p : \mathcal{J}(l, E_0, F^{\oplus n}) \to \mathcal{J}(l - nl_0, E_0)$ be the morphism sending $E \in \mathcal{J}(l, E_0, F^{\oplus n})$ to E' in (1.12). Then the proof of Lemma 0.6 implies that

$$\dim p^{-1}(E') \le nn' - n^2, \quad E' \in \mathcal{J}\big(l - nl_0, E_0, F(K_X)^{\oplus n'}\big). \tag{1.13}$$

Then the same proof of [9, (3.8)] works here. □

PROPOSITION 1.9
Assume that X is an Enriques surface. Let v be an isotropic and primitive Mukai vector.

(1) *Assume that $\mathcal{M}_H(lv)^{\mathrm{s}}$ is nonempty. Then $l = 1, 2$.*
(2) *$\mathcal{M}_H(2v, L)^{\mathrm{s}} \ne \emptyset$ if and only if $\ell(v) = 1$ and $L \equiv 0 \mod 2$. Moreover,*

$$\mathcal{M}_H(2v)^{\mathrm{s}} = \{\varpi_*(F) \mid F \in \mathcal{M}^w_{\varpi^*(H)}(w)^{\mathrm{s}}, \iota^*(F) \not\cong F\},$$

where $w = \varpi^*(v)$. In particular, $\mathcal{M}_H(2v)^{\mathrm{s}}$ is smooth of dimension 1.
(3) *We have* $\dim \mathcal{M}_H(lv)^{\mathrm{ss}} \le l$. *If $\ell(v) = 1$, then* $\dim \mathcal{M}_H(lv)^{\mathrm{ss}} \le [\frac{l}{2}]$.

Proof
(1) Since H is general, $\mathcal{M}_H(lv)^{\mathrm{s}}$ is the same as the moduli stack of v-twisted stable sheaves. Let w be a primitive and isotropic Mukai vector of $\widetilde{X}$ with $\varpi^*(v) = mw$ $(m \in \mathbb{Z}_{>0})$. For $E \in \mathcal{M}_H(lv)^{\mathrm{s}}$, $\varpi^*(E)$ is w-twisted semistable with respect to $\varpi^*(H)$. Indeed, by the uniqueness of the Harder–Narasimhan filtration of $\varpi^*(E)$, it is ι-invariant. Since ϖ is étale, it comes from a filtration on X. In order to show

$l=1,2$, we first treat the case where $\varpi^*(E)$ is properly w-twisted semistable in (a), and then we treat the other case in (b).

(a) Assume that $\varpi^*(E)$ is not w-twisted stable, and let F be a w-twisted stable proper subsheaf of $\varpi^*(E)$ with

$$\frac{\chi(\varpi^*(E),F(n\varpi^*(H)))}{\operatorname{rk} F}=\frac{\chi(\varpi^*(E),\varpi^*(E)(n\varpi^*(H)))}{\operatorname{rk}\varpi^*(E)} \quad (n\in\mathbb{Z}). \tag{1.14}$$

Then $\iota^*(F)$ is also a w-twisted stable subsheaf of $\varpi^*(E)$ with

$$\frac{\chi(\varpi^*(E),\iota^*(F)(n\varpi^*(H)))}{\operatorname{rk}\iota^*(F)}=\frac{\chi(\varpi^*(E),\varpi^*(E)(n\varpi^*(H)))}{\operatorname{rk}\varpi^*(E)} \quad (n\in\mathbb{Z}). \tag{1.15}$$

If $\iota^*(F)\cong F$, then we can introduce an action of ι on F, and hence there is a coherent sheaf E_1 on X such that $F\cong\varpi^*(E_1)$. Since

$$\begin{aligned}\operatorname{Hom}\big(F,\varpi^*(E)\big)&=\operatorname{Hom}\big(E_1,\varpi_*\big(\varpi^*(E)\big)\big)\\&=\operatorname{Hom}\big(E_1,E\oplus E(K_X)\big),\end{aligned}$$

we see that E_1 or $E_1(K_X)$ is a subsheaf of E, which shows that E is properly v-twisted semistable. Hence $\iota^*(F)\not\cong F$. We note that $\phi: F\oplus\iota^*(F)\to\varpi^*(E)$ is injective. Indeed, let G be a w-twisted stable subsheaf of $\ker\phi$ with

$$\frac{\chi(\varpi^*(E),G(n\varpi^*(H)))}{\operatorname{rk} G}=\frac{\chi(\varpi^*(E),\varpi^*(E)(n\varpi^*(H)))}{\operatorname{rk}\varpi^*(E)} \quad (n\in\mathbb{Z}). \tag{1.16}$$

Then $G\to F$ and $G\to\iota^*(F)$ are isomorphic or zero. Since F and $\iota^*(F)$ are subsheaves of $\varpi^*(E)$, we get $G\cong F$ and $G\cong\iota^*(F)$, which is a contradiction. Therefore ϕ is injective. By the v-twisted stability of E, ϕ is also surjective. Thus $F\oplus\iota^*(F)\cong\varpi^*(E)$. Then $\varpi_*(F)^{\oplus 2}\cong E\oplus E(K_X)$ implies that $\varpi_*(F)\cong E\cong E(K_X)$. By [27, Lemma 2.3.6], F is a factor of a Jordan–Hölder filtration of a w-twisted semistable sheaf of Mukai vector w, and hence $\operatorname{rk} F\le\operatorname{rk} w$ and the equality holds if and only if $v(F)=w$. Hence $lm\operatorname{rk} w=\operatorname{rk} E\le 2\operatorname{rk} w$, which shows that $lm\le 2$. Moreover, $lm=2$ implies that $v(F)=w$.

(b) If $\varpi^*(E)$ is w-twisted stable, then by using [27, Lemma 2.3.6], we have $\operatorname{rk} E\le\operatorname{rk} w$, which shows that $l=m=1$.

(2) In the proof of (1), for $E\in\mathcal{M}_H(2v)^{\mathrm{s}}$, we have $\ell(v)=1$ and $E=\varpi_*(F)$ with $\iota^*(v(F))=v(F)$. In particular, $v(F)=w$ and $\mathcal{M}_H(2v)^{\mathrm{s}}$ is smooth of dimension 1. By (1.3) and $4\mid\operatorname{rk} E$, we have $2\mid c_1(E)$ in $\operatorname{NS}(X)$.

We note that for a primitive and isotropic Mukai vector v with $\ell(v)=1$, $w:=\varpi^*(v)$ is a primitive and isotropic Mukai vector on $\widetilde{X}$ such that $\iota^*(w)=w$. For such a vector w, we have $M^w_{\varpi^*(H)}(w)^{\mathrm{s}}\neq\emptyset$ (see [27, Corollary 1.3.3]), and the fixed point set of the ι^*-action on $M^w_{\varpi^*(H)}(w)^{\mathrm{s}}$ is 1-dimensional by Lemma 1.7. For $F\in M^w_{\varpi^*(H)}(w)^{\mathrm{s}}$ with $\iota^*(F)\neq F$, $\iota^*(F)\oplus F\cong\varpi^*(\varpi_*(F))$ does not contain an ι-invariant proper subsheaf G satisfying (1.16) for $E=\varpi_*(F)$, where $\varpi_*(F)$ is a stable sheaf with respect to H. Therefore (2) holds.

(3) We have a decomposition

$$\overline{M}_H(lv)=\bigcup_{(n_1l_1,\dots,n_tl_t)\in S_l}\prod_i S^{n_i}M_H(l_iv),$$

where

$$S_l := \Big\{(n_1 l_1, \dots, n_t l_t) \,\Big|\, l_1 < l_2 < \cdots < l_t, n_1, \dots, n_t \in \mathbb{Z}_{>0}, \sum_i n_i l_i = l\Big\}.$$

We have a morphism $\phi : \mathcal{M}_H(lv)^{\mathrm{ss}} \to \overline{M}_H(lv)$. Let x be a point of $\overline{M}_H(lv)$. Then there are stable sheaves $E_i \in \mathcal{M}_H(k_i v)^{\mathrm{s}}$ such that $E_i \not\cong E_j(pK_X)$ for $i \neq j$ and $x = \bigoplus_{i=1}^t \bigoplus_p E_i(pK_X)^{\oplus n_{ip}}$. Since

$$\mathrm{Hom}(E_i, E_j) = \mathrm{Ext}^2(E_i, E_j) = 0$$

for $i \neq j$ and $\chi(E_i, E_j) = 0$, for $E \in \phi^{-1}(x)$, there are $G_i \in \mathcal{J}(\sum_p n_{ip} k_i, E_i)$ and $E = \bigoplus_i G_i$. By Lemma 1.8, we see that $\dim \phi^{-1}(x) \leq -t$. We note that $\dim M_H(v(E_i))^{\mathrm{s}} = 1, 2$ by (1), (2), and Lemma 1.7. We set

$$t_1 := \big\{i \,\big|\, \dim M_H\big(v(E_i)\big) = 1\big\}, \qquad t_2 := \big\{i \,\big|\, \dim M_H\big(v(E_i)\big) = 2\big\}. \tag{1.17}$$

Then we have

$$\dim \mathcal{M}_H(lv)^{\mathrm{ss}} \leq \max_{x \in \overline{M}_H(lv)} \{-t + t_1 + 2t_2\} = \max_{x \in \overline{M}_H(lv)} t_2 \leq l.$$

Moreover, if $\ell(v) = 1$, then $\dim M_H(v(E_i)) = 2$ implies that $v(E_i) = 2v$, which implies that $t_2 \leq l/2$. Hence the second claim also holds. □

REMARK 1.3

Assume that $\ell(v) = 1$. If l is even, then $\bigoplus_{i=1}^{l/2} E_i$ $(E_i \in \mathcal{M}_H(2v)^{\mathrm{s}})$ forms a component of dimension $l/2$. If l is odd, then $F \oplus \bigoplus_{i=1}^{(l-1)/2} E_i$ $(F \in \mathcal{M}_H(v)^{\mathrm{s}}, E_i \in \mathcal{M}_H(2v)^{\mathrm{s}})$ forms a component of dimension $(l-1)/2$. Thus the equality holds in (3).

REMARK 1.4

Assume that $\ell(v) = 1$. If E is a singular point of $\mathcal{M}_H(v)^{\mathrm{s}}$, then $E(K_X) \cong E$, and hence $E = \varpi_*(F)$ $(F \in \mathrm{Coh}(\widetilde{X}))$. In this case, $\varpi^*(E)$ is properly w-semistable and $l = m = 1$.

REMARK 1.5

Let $\pi : X \to C$ be an elliptic surface, and let mD be a tame multiple fiber. Let $v := (0, rD, d)$ be a primitive Mukai vector; that is, $\gcd(r, d) = 1$. For a semistable sheaf E with $v(E) = lv$ and $\mathrm{Div}(E) = lrD$, we will show in Lemma 3.9 that E is S-equivalent to $\bigoplus_i E_i$, where $E_i \in \mathcal{M}_H(v)^{\mathrm{s}}$. Assume that $m \nmid r$. Then $E_i \otimes K_X \not\cong E_i$, which implies that $\dim \mathcal{M}_H(v)^{\mathrm{s}} = 0$. Hence we see that $\dim \mathcal{M}_H(lv)^{\mathrm{ss}} \leq [\frac{lm_0}{m}]$, where $m_0 = \gcd(r, m)$.

2. μ-stability

In this section, we continue to assume that X is an Enriques surface, and we study the existence condition of μ-stable locally free sheaves. For a Mukai vector v of $\mathrm{rk}\, v > 0$, we have a decomposition $v = (lr, l\xi, \frac{s}{2})$, where $\gcd(r, \xi) = 1$, $l \in \mathbb{Z}_{>0}$, $s \in \mathbb{Z}$, $lr - s \in 2\mathbb{Z}$. We divide this into three cases.

(A) There is no stable sheaf E such that $v(E) = (r, \xi, b)$ and $\langle v(E)^2 \rangle = -1, -2$.

(B) There is a stable sheaf E such that $v(E) = (r, \xi, b)$ and $\langle v(E)^2 \rangle = -1$.

(C) There is a stable sheaf E such that $v(E) = (r, \xi, b)$ and $\langle v(E)^2 \rangle = -2$.

REMARK 2.1

We note that r is odd for case B and even for case C.

We use a case-by-case approach to prove the following result.

THEOREM 2.1

Let $v = (lr, l\xi, \frac{s}{2})$ be a Mukai vector such that $\gcd(r, \xi) = 1$ and $\langle v^2 \rangle \geq 0$. Let H be a general polarization with respect to v. Then for $L \in \mathrm{NS}(X)$ with $[L \mod K_X] = l\xi$, $\mathcal{M}_H(v, L)^{\mathrm{ss}}$ contains a μ-stable sheaf if and only if

(A) *there is no stable sheaf E such that $v(E) = (r, \xi, b)$, $\langle v(E)^2 \rangle = -1, -2$, and $\langle v^2 \rangle \geq 0$; or*

(B) *there is a stable sheaf E such that $v(E) = (r, \xi, b)$, $\langle v(E)^2 \rangle = -1$, and $\langle v^2 \rangle \geq l^2$; or*

(C) *there is a stable sheaf E such that $v(E) = (r, \xi, b)$, $\langle v(E)^2 \rangle = -2$, and $\langle v^2 \rangle \geq 2l^2$.*

Moreover, if $lr > 1$, then under the same condition, $\mathcal{M}_H(v, L)^{\mathrm{ss}}$ contains a μ-stable locally free sheaf.

By Corollary 1.2, $\mathcal{M}_H(v, L)^{\mu\mathrm{ss}} \neq \emptyset$ if $\langle v^2 \rangle > 0$. Hence it is sufficient to compute the codimension of $\mathcal{M}_H(v, L)^{\mu\mathrm{ss}} \setminus \mathcal{M}_H(v, L)^{\mu\mathrm{s}}$. Although the arguments in this section are similar to those of [23], we repeat them here since several estimates are slightly different. Throughout this section, H is a general polarization with respect to v.

2.1. Case A

In this section, we treat case A. Let $v := l(r + \xi) + a\varrho_X \in H^*(X, \mathbb{Q})$ be a Mukai vector. We first estimate the dimension of various locally closed substacks of $\mathcal{M}(v)$.

LEMMA 2.2

If $\mathcal{M}_H(v)^{\mu\mathrm{ss}} \neq \emptyset$, then $\langle v^2 \rangle \geq 0$. If the equality holds, then $\mathcal{M}_H(v)^{\mu\mathrm{ss}} = \mathcal{M}_H(v)^{\mathrm{ss}}$ and $E \in \mathcal{M}_H(v)^{\mathrm{ss}}$ is S-equivalent to $\bigoplus_i E_i$, where the E_i's are μ-stable locally free sheaves with $v(E_i) \in \mathbb{Q}v$.

Proof

Let E be a μ-semistable sheaf of $v(E) = v$, and choose a Jordan–Hölder filtration of E with respect to μ-stability whose factors are μ-stable sheaves E_i $(1 \leq i \leq s)$.

We set

$$v(E_i) := l_i(r+\xi) + a_i \varrho_X, \quad 1 \le i \le s. \tag{2.1}$$

By our assumption, $\langle v(E_i)^2\rangle = l_i(l_i(\xi^2) - 2ra_i) \neq -1, -2$. Thus $\langle v(E_i)^2\rangle \geq 0$ for all i. Since

$$\frac{\langle v^2\rangle}{l} = \sum_{i=1}^{s} \frac{\langle v(E_i)^2\rangle}{l_i}, \tag{2.2}$$

we get $\langle v^2\rangle \geq 0$. Assume that $\langle v^2\rangle = 0$. Then $\langle v(E_i)^2\rangle = 0$ for all i. Since

$$\begin{aligned} \frac{\langle v(E_i)^2\rangle}{\operatorname{rk}(E_i)^2} &= (\xi^2) - 2\frac{a_i}{rl_i}, \\ \frac{\chi(E_i)}{\operatorname{rk}(E_i)} &= \frac{1}{2} + \frac{a_i}{rl_i}, \end{aligned} \tag{2.3}$$

we see that $\chi(E_i)/\operatorname{rk}(E_i) = \chi(E)/\operatorname{rk}(E)$ for all i. Thus E is semistable. Since $E_i^{\vee\vee}$ are μ-stable locally free sheaves with $\langle v(E_i^{\vee\vee})^2\rangle \geq 0$ and $0 = \langle v(E_i)^2\rangle = \langle v(E_i^{\vee\vee})^2\rangle + 2\operatorname{rk} E_i \chi(E_i^{\vee\vee}/E_i)$, we see that $\chi(E_i^{\vee\vee}/E_i) = 0$. Thus all E_i's are locally free, which shows that E is also locally free. □

COROLLARY 2.3

If $\langle v^2\rangle = 0$, then $\mathcal{M}_H(v)^{\mu ss}$ consists of locally free sheaves.

DEFINITION 2.1

Let $w = l_0(r+\xi) + a_0\varrho_X$ ($l_0 > 0$) be a primitive Mukai vector such that $\langle w^2\rangle = 0$. Since $a_0/l_0 = (\xi^2)/(2r)$, w is uniquely determined.

By Lemma 2.2 and Corollary 2.3, $\mathcal{M}_H(w)^{\mu ss}$ consists of μ-stable locally free sheaves. Since $l_0(\xi^2) - 2a_0 r = 0$, $l_0 r$ is even. In particular, $a_0 \in \mathbb{Z}$.

LEMMA 2.4

For a Mukai vector $u = (lr, l\xi, a)$, $r \mid \langle u, w\rangle$.

Proof

We note that $l_0 r$ is even and $a_0 \in \mathbb{Z}$. If r is even, then $a \in \mathbb{Z}$. If r is odd, then l_0 is even. Hence $l_0 a \in \mathbb{Z}$. Then $\langle u, w\rangle = (la_0 - l_0 a)r$ is divisible by r. □

LEMMA 2.5

(1) *We have that*

$$\dim\bigl(\mathcal{M}_H(v)^{\mu ss} \setminus \mathcal{M}_H(v)^{ss}\bigr) \leq \langle v^2\rangle - 1. \tag{2.4}$$

(2) *Assume that $\langle v^2\rangle > 0$. Then*

$$\dim\bigl(\mathcal{M}_H(v)^{\mu ss} \setminus \mathcal{M}_H(v)^{s}\bigr) \leq \langle v^2\rangle - 1. \tag{2.5}$$

In particular, if $\mathcal{M}_H(v)^{\mu ss} \neq \emptyset$, then $\mathcal{M}_H(v)^{s} \neq \emptyset$ and $\dim \mathcal{M}_H(v)^{\mu ss} = \langle v^2\rangle$.

Proof
By Lemma 1.5, it is sufficient to prove (1). Let F be a μ-semistable sheaf of $v(F)=v$. We assume that F is not semistable. Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F \tag{2.6}$$

be the Harder–Narasimhan filtration of F. We set

$$v_i := v(F_i/F_{i-1}) = l_i(r+\xi) + a_i \varrho_X, \quad 1 \le i \le s. \tag{2.7}$$

Since $\chi(F_i/F_{i-1})/\operatorname{rk}(F_i/F_{i-1}) > \chi(F_{i+1}/F_i)/\operatorname{rk}(F_{i+1}/F_i)$, we get that

$$\frac{a_0}{l_0} \ge \frac{a_1}{l_1} > \frac{a_2}{l_2} > \cdots > \frac{a_s}{l_s}, \tag{2.8}$$

where the leftmost inequality is a consequence of $\langle w^2\rangle = 0$, $\langle v_i^2\rangle \ge 0$, and (2.3). Let $\mathcal{F}^{\mathrm{HN}}(v_1, v_2, \ldots, v_s)$ be the substack of $\mathcal{M}_H(v)^{\mu\mathrm{ss}}$ whose element E has the Harder–Narasimhan filtration of the above type. We will prove that $\dim \mathcal{F}^{\mathrm{HN}}(v_1, v_2, \ldots, v_s) \le \langle v^2\rangle - 1$. Since

$$\operatorname{Hom}\bigl(F_i/F_{i-1}, F_j/F_{j-1}(K_X)\bigr) = 0$$

for $i<j$, [9, Lemma 5.3] implies that

$$\dim \mathcal{F}^{\mathrm{HN}}(v_1, v_2, \ldots, v_s) = \sum_{i=1}^{s} \dim \mathcal{M}_H(v_i)^{\mathrm{ss}} + \sum_{i<j} \langle v_j, v_i\rangle. \tag{2.9}$$

For $i<j$, by using Lemma 2.2 and (2.8), we see that

$$\begin{aligned} \langle v_i, v_j\rangle &= l_i l_j(\xi^2) - (l_i a_j + l_j a_i) r \\ &= l_i l_j(\xi^2) - 2 l_j a_i r + (a_i l_j - a_j l_i) r \\ &= l_j\bigl(l_i(\xi^2) - 2a_i r\bigr) + (a_i l_j - a_j l_i) r \\ &\ge (a_i l_j - a_j l_i) r \ge r/2 \ge 1, \end{aligned} \tag{2.10}$$

where the inequality $r \ge 2$ comes from our assumption for case A. Hence if $\langle v_i^2\rangle > 0$ for all i, then, by using Lemma 1.5, we see that

$$\dim \mathcal{F}^{\mathrm{HN}}(v_1, v_2, \ldots, v_s) = \langle v^2\rangle - \sum_{i<j} \langle v_i, v_j\rangle \le \langle v^2\rangle - 1. \tag{2.11}$$

Assume that $\langle v_i^2\rangle = 0$; that is, $v_i = l_i' w$, $l_i' \in \mathbb{Z}$. Then $i=1$ and $r \mid \langle v_j, w\rangle$ by Lemma 2.4. Hence

$$\begin{aligned} \langle v_1, v_j\rangle - l_1' &= l_1'\bigl(\langle w, v_j\rangle - 1\bigr) \\ &\ge l_1'(r-1) > 0. \end{aligned} \tag{2.12}$$

In this case, by using Proposition 1.9, we see that

$$\dim \mathcal{F}^{\mathrm{HN}}(v_1, v_2, \ldots, v_s) \le \langle v^2\rangle - \Bigl(\sum_{i<j} \langle v_i, v_j\rangle - l_1'\Bigr) \le \langle v^2\rangle - 1. \tag{2.13}$$

Hence we get our lemma. □

PROPOSITION 2.6

Assume that $\langle v^2 \rangle > 0$. *Then for a general* H,

$$\dim\bigl(\mathcal{M}_H(v)^{\mathrm{s}} \setminus \mathcal{M}_H(v)^{\mu\mathrm{s}}\bigr) < \langle v^2 \rangle. \tag{2.14}$$

In particular, $\mathcal{M}_H(v)^{\mu\mathrm{s}} \neq \emptyset$. *Moreover, there is a* μ*-stable locally free sheaf in each irreducible component.*

Proof

Let E be a stable sheaf with $v(E) = v$, and let E_1 be a μ-stable subsheaf of E such that E/E_1 is torsion-free. We set

$$\begin{aligned} v_1 &:= v(E_1) = (l_1 r, l_1 \xi, a_1), \\ v_2 &:= v(E/E_1) = (l_2 r, l_2 \xi, a_2). \end{aligned} \tag{2.15}$$

Since $\chi(E_1)/\operatorname{rk} E_1 < \chi(E)/\operatorname{rk} E$, we get $\langle v(E_1)^2 \rangle > 0$ and

$$\frac{a_1}{l_1} < \frac{a_2}{l_2}. \tag{2.16}$$

Let $J(v_1, v_2)$ be the substack of $\mathcal{M}_H(v)^{\mathrm{s}}$ consisting of E which has a subsheaf $E_1 \subset E$. We will use Lemma 0.6 to estimate $\dim J(v_1, v_2)$. By [9, Lemma 3.1], $\dim \operatorname{Hom}(E_1, (E/E_1)(K_X)) \leq l_2/l_1$. We will bound the dimension of the substack

$$\begin{aligned} \mathcal{N}_n(v_1, v_2) := \bigl\{(E_1, E_2) \in \mathcal{M}_H(v_1)^{\mu\mathrm{s}} \times \mathcal{M}_H(v_2)^{\mu\mathrm{ss}} \bigm| \dim(E_1^{\vee\vee}/E_1) = n, \\ \dim \operatorname{Hom}\bigl(E_1, E_2(K_X)\bigr) \neq 0\bigr\}. \end{aligned} \tag{2.17}$$

For a fixed $E_2 \in \mathcal{M}_H(v_2)^{\mu\mathrm{ss}}$,

$$\#\bigl\{E_1^{\vee\vee} \bigm| E_1 \in \mathcal{M}_H(v_1)^{\mu\mathrm{s}}, \operatorname{Hom}\bigl(E_1, E_2(K_X)\bigr) \neq 0\bigr\} < \infty. \tag{2.18}$$

Indeed the double dual of the graded object associated to the Jordan–Hölder filtration with respect to μ-stability is well defined and $E_1^{\vee\vee}$ must be one of these stable factors. For a locally free sheaf, let $\operatorname{Quot}^n_{F/X}$ be the quot-scheme parameterizing all quotients $F \to A$ such that A is a 0-dimensional sheaf of $\chi(A) = n$. In [19, Theorem 0.4, Section 5], we computed the number of rational points of $\operatorname{Quot}^n_{F/X}$ over finite fields, which implies that

$$\dim \operatorname{Quot}^n_{F/X} = (\operatorname{rk} F + 1)n. \tag{2.19}$$

Since $E_1 \in \mathcal{M}_H(v_1)^{\mu\mathrm{s}}$ is simple, we see that

$$\begin{aligned} &\dim\bigl\{E_1 \in \mathcal{M}_H(v_1)^{\mu\mathrm{s}} \mid \dim(E_1^{\vee\vee}/E_1) = n, \operatorname{Hom}\bigl(E_1, E_2(K_X)\bigr) \neq 0\bigr\} \\ &\quad \leq (\operatorname{rk} v_1 + 1)n - 1. \end{aligned} \tag{2.20}$$

Since $E_1^{\vee\vee}$ is μ-stable, Lemma 2.2 implies that $\langle v(E_1^{\vee\vee})^2 \rangle \geq 0$. Then we get

$$\dim \mathcal{M}_H(v_1)^{\mathrm{ss}} = \langle v_1^2 \rangle = 2l_1 r n + \bigl\langle v(E_1^{\vee\vee})^2 \bigr\rangle \geq 2l_1 r n. \tag{2.21}$$

Since $r \geq 2$, we get

$$\begin{aligned} \dim \mathcal{N}_n(v_1, v_2) &\leq \dim \mathcal{M}_H(v_1)^{\mu\mathrm{s}} + \dim \mathcal{M}_H(v_2)^{\mu\mathrm{ss}} - \bigl((l_1 r - 1)n + 1\bigr) \\ &\leq \dim \mathcal{M}_H(v_1)^{\mu\mathrm{s}} + \dim \mathcal{M}_H(v_2)^{\mu\mathrm{ss}} - 2 \end{aligned} \tag{2.22}$$

if $n>0$. If $n=0$, then the same inequality also holds, since $\langle v_1^2\rangle > 0 = 2l_1rn$. Moreover, if $\langle v_2^2\rangle = 0$, then by Lemma 2.2, $\operatorname{Hom}(E_1,E_2)\neq 0$ implies that $v(E_1^{\vee\vee})\in \mathbb{Q}v_2$, which shows that $l_1\in l_0\mathbb{Z}$. Therefore $\mathcal{N}_n(v_1,v_2)=\emptyset$ unless $l_1\in l_0\mathbb{Z}$.

If $\langle v_2^2\rangle>0$, then Lemma 2.5 implies that $\dim\mathcal{M}_H(v_2)^{\mu ss}=\langle v_2^2\rangle$. We also have $\dim\mathcal{M}_H(v_1)^{\mu s}=\langle v_1^2\rangle$ by $\langle v_1^2\rangle>0$. Hence Lemma 0.6 and (2.22) imply that

$$\begin{aligned}\dim\mathcal{M}_H(v)^{\mathrm{s}}-\dim J(v_1,v_2)&=\min\Big(\langle v_1,v_2\rangle-\frac{l_2}{l_1}+2,\langle v_1,v_2\rangle\Big)\\ &=l_1\frac{\langle v_2^2\rangle}{2l_2}+l_2\frac{\langle v_1^2\rangle}{2l_1}-\max\Big(\frac{l_2}{l_1}-2,0\Big)>0.\end{aligned}\tag{2.23}$$

We next treat the case where $\langle v_2^2\rangle=0$. Then $v_2=l_2'w, l_2'\in\mathbb{Z}$. By Proposition 1.9, $\dim\mathcal{M}_H(v_2)^{\mu ss}\le\langle v_2^2\rangle+l_2'$. If $l_1\in l_0\mathbb{Z}$, then $l_2/l_1\le l_2'$. In this case, by using (2.22) and Lemma 2.4, we see that

$$\begin{aligned}\dim\mathcal{M}_H(v)^{\mathrm{s}}-\dim J(v_1,v_2)&\ge\langle v_1,v_2\rangle-l_2/l_1-l_2'+2\\ &\ge l_2'\big(\langle v_1,w\rangle-2\big)+2>0.\end{aligned}\tag{2.24}$$

If $l_1\notin l_0\mathbb{Z}$, then since $\mathcal{N}_n(v_1,v_2)=\emptyset$, we see that

$$\dim\mathcal{M}_H(v)^{\mathrm{s}}-\dim J(v_1,v_2)=l_2'\big(\langle v_1,w\rangle-1\big)>0\tag{2.25}$$

by Lemma 2.4.

We will prove that there is a μ-stable locally free sheaf. Let $\mathcal{M}_H(v)^{\mathrm{nlf}}$ be the closed substack of $\mathcal{M}_H(v)^{\mu s}$ consisting of nonlocally free sheaves. By (2.19), we have

$$\begin{aligned}\dim\mathcal{M}_H(v)^{\mathrm{nlf}}&\le\max_{b>0}\big(\dim\mathcal{M}_H(v+b\varrho_X)^{\mu s}+(rl+1)b\big)\\ &\le\langle v^2\rangle+\delta-(rl-1),\end{aligned}\tag{2.26}$$

where

$$\delta=\begin{cases}\frac{l}{l_0}, & v+b\varrho_X=\frac{l}{l_0}w,\ell(w)=2,\\ [\frac{l}{2l_0}], & v+b\varrho_X=\frac{l}{l_0}w,\ell(w)=1,\\ 0, & \text{otherwise.}\end{cases}\tag{2.27}$$

If $\ell(w)=1$, then

$$(rl-1)-\delta\ge\frac{l}{l_0}(rl_0-1/2)-1>0,$$

since rl_0 is even. If $\ell(w)=2$, then

$$(rl-1)-\delta\ge\frac{l}{l_0}(rl_0-1)-1>0,$$

unless $rl_0=2$, $l=l_0$. In this case, we see that $r=1$ and $w=2e^{\xi}$. Since $u:=(1,\xi,\frac{(\xi^2)+1}{2})$ is a Mukai vector satisfying $\langle u^2\rangle=-1$, this case does not occur. Therefore $\dim\mathcal{M}_H(v)^{\mathrm{nlf}}<\langle v^2\rangle$ and the last claim holds. □

2.2. Case B

Assume that $r \mid (\xi^2)+1$. Then $v_0 := (r, \xi, a_0)$ is a primitive Mukai vector with $\langle v_0^2 \rangle = -1$, where $a_0 := \frac{(\xi^2)+1}{2r} \in \mathbb{Z} + \frac{1}{2}$. We take a general polarization H with respect to v_0. Let F be the μ-stable locally free sheaf with $v(F) = v_0$.

LEMMA 2.7

If there is a μ-stable sheaf E with $v(E) = lv_0 + b\varrho_X$, then $\langle v(E)^2 \rangle \geq l^2$ or $E \cong F, F(K_X)$.

Proof

If $\operatorname{rk} E = r$, then $l = 1$ and $b \leq 0$, which implies the claim. Assume that $\operatorname{rk} E > r$. By the μ-stability of E, F, $\operatorname{Hom}(F, E) = \operatorname{Hom}(E, F(K_X)) = 0$. Hence $0 \geq \chi(F, E) = -\langle v_0, v(E) \rangle = l + br$. Since $\langle v(E)^2 \rangle = l(-l - 2br)$, we get the claim. □

REMARK 2.2

Since $l = \gcd(\operatorname{rk} E, c_1(E)) \in \mathbb{Z}$, we have $v(E), lv_0 \in v(K(X))$, which implies that $b \in \mathbb{Z}$.

LEMMA 2.8 (see [22, Lemma 4.4])

Let v be an arbitrary Mukai vector of $\operatorname{rk} v > 0$. Let $\mathcal{M}_H(v)^{\mu ss}$ be the moduli stack of μ-semistable sheaves E of $v(E) = v$, and let $\mathcal{M}_H(v)^{p\mu ss}$ be the closed substack of $\mathcal{M}_H(v)^{\mu ss}$ consisting of properly μ-semistable sheaves. We assume that $\langle v^2 \rangle \geq l^2$. Then

(1)

$$\langle v^2 \rangle - \dim \mathcal{M}_H(v)^{p\mu ss} \geq \frac{\langle v^2 \rangle}{2l} - \frac{l}{2} + 1 \tag{2.28}$$

unless $r = 1$ and $l = 2$;

(2) *if $\mathcal{M}_H(v)^{\mu ss}$ is not empty, then there is a μ-stable locally free sheaf E of $v(E) = v$ in each irreducible component, unless $v = (2, 0, -1)e^{\xi}$;*

(3) *if $v = (2, 0, -1)e^{\xi}$, then there is an irreducible component of $\mathcal{M}_H(v, L)^{ss}$ containing μ-stable locally free sheaves.*

Proof

(1) By Lemma 0.5, we get that

$$\dim \mathcal{M}_H(v)^{\mu ss} \geq \langle v^2 \rangle. \tag{2.29}$$

We will show that

$$\dim \mathcal{M}_H(v)^{p\mu ss} \leq \langle v^2 \rangle - \Big(\frac{\langle v^2 \rangle}{2l} - \frac{l}{2} + 1 \Big). \tag{2.30}$$

For this purpose, we estimate the moduli number of Jordan–Hölder filtrations. Let E be a μ-semistable sheaf of $v(E) = v$, and let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_t = E \tag{2.31}$$

be a Jordan–Hölder filtration of E with respect to μ-stability. We set $E_i := F_i/F_{i-1}$. We also set

$$v(E) := lv_0 + a\varrho_X, \qquad v(E_i) := l_i v_0 + a_i \varrho_X.$$

By Lemma 2.7, $\langle v_i^2 \rangle \neq 0$. Hence $\dim \mathcal{M}_H(v_i)^{\mu s} = \langle v_i^2 \rangle$. Let $J(v_1, v_2, \dots, v_t)$ be the substack of $\mathcal{M}_H(v)^{\mu ss}$ such that $E \in \mathcal{M}_H(v)^{\mu ss}$ has a filtration (2.31). By using Lemma 0.6 successively, we see that

$$\begin{aligned} &\dim J(v_1, v_2, \dots, v_t) \\ &\leq \sum_i \dim \mathcal{M}_H(v_i)^{\mu s} + \sum_{i<j} \big(\dim \operatorname{Ext}^1(E_j, E_i) - \dim \operatorname{Hom}(E_j, E_i)\big) \\ &= -\chi(E, E) + \sum_{i>j} \chi(E_j, E_i) + \sum_{i<j} \dim \operatorname{Ext}^2(E_j, E_i). \end{aligned} \tag{2.32}$$

Since $\langle v(E_i), v(E_j) \rangle = -l_i l_j - r(l_i a_j + l_j a_i)$, we see that

$$\sum_{i>j} \chi(E_j, E_i) = -\sum_{i>j} \big\langle v(E_j), v(E_i) \big\rangle = -\sum_i \frac{(l - l_i)\langle v(E_i)^2 \rangle}{2l_i}. \tag{2.33}$$

We set $\max_i\{l_i\} = (l-k)$. If $\langle v(E_i)^2 \rangle = -1$ for all i, then $v(E_i) = v_0$ for all i. As $\langle v(E)^2 \rangle \geq l^2$, this is impossible. So there is an integer i_0 such that $\langle v(E_{i_0})^2 \rangle \geq 0$. Since $(l-k) + (t-1) \leq \sum_i l_i = l$, we obtain that $t \leq k+1$. Since $l - l_i - k \geq 0$ and $\langle v(E_i)^2 \rangle \geq -1$, we get that

$$\begin{aligned} \sum_{i>j} \big\langle v(E_j), v(E_i) \big\rangle &= k \sum_i \frac{\langle v(E_i)^2 \rangle}{2l_i} + \sum_i \frac{(l - l_i - k)\langle v(E_i)^2 \rangle}{2l_i} \\ &= k \frac{\langle v(E)^2 \rangle}{2l} + \sum_i \frac{(l - l_i - k)\langle v(E_i)^2 \rangle}{2l_i} \\ &\geq k \frac{\langle v(E)^2 \rangle}{2l} - \sum_{i \neq i_0} \frac{(l - l_i - k)}{2} \\ &\geq k \frac{\langle v(E)^2 \rangle}{2l} - \frac{(l-1-k)k}{2}. \end{aligned}$$

Assume that $\operatorname{Ext}^2(E_j, E_i) = 0$ for some $i < j$. Then we get that

$$\sum_{i<j} \dim \operatorname{Ext}^2(E_j, E_i) \leq \frac{(k+1)k}{2} - 1.$$

Then the moduli number of these filtrations is bounded by

$$\langle v^2 \rangle - k\frac{\langle v^2 \rangle}{2l} + \frac{(l-1-k)k}{2} + \frac{(k+1)k}{2} - 1 \leq \langle v^2 \rangle - 1 - k\Big(\frac{\langle v^2 \rangle}{2l} - \frac{l}{2}\Big). \tag{2.34}$$

Therefore we get a desired estimate for this case.

Assume that $\operatorname{Ext}^2(E_j, E_i) \neq 0$ for all $i < j$. Then $E_i^{\vee\vee} \cong E_j^{\vee\vee}(K_X)$. In particular, $l_i = l_j$ for all $i < j$. Suppose first that $l_i \geq 2$ for all i. Then $l_i = l - k$

implies that $k \le l-2$ and

$$\sum_{i>j} \langle v_i, v_j \rangle = k\frac{\langle v^2 \rangle}{2l}.$$

Since $4 \le l_1 + l_2 = 2(l-k) \le \sum_i l_i = l$, we have $2 \le l/2 \le k$. Hence the dimension of these filtrations is bounded by

$$\begin{aligned} \langle v^2 \rangle - k\frac{\langle v^2 \rangle}{2l} + \frac{(k+1)k}{2} &\le \langle v^2 \rangle - k\Big(\frac{\langle v^2 \rangle}{2l} - \frac{l-1}{2}\Big) \\ &\le \langle v^2 \rangle - \Big(\frac{\langle v^2 \rangle}{2l} - \frac{l}{2} + 1\Big). \end{aligned} \tag{2.35}$$

If instead $l_i = 1$ for all i, then we must have $t = 2$, and hence $l = 2$. Indeed, if we have $h < i < j$, then $E_h^{\vee\vee} \cong E_i^{\vee\vee}(K_X)$, $E_h^{\vee\vee} \cong E_j^{\vee\vee}(K_X)$, and $E_i^{\vee\vee} \cong E_j^{\vee\vee}(K_X)$, from which it follows that $E_i^{\vee\vee} \cong E_i^{\vee\vee}(K_X)$. As $\operatorname{rk} E_i$ is odd, this is impossible, so $t = 2$ as claimed, and $l = 2$ follows from this and $l_i = 1$.

Assume that $r > 1$. Then a general member $E_1 \in \mathcal{M}_H(v_1)^{\mu s}$ is locally free by a similar estimate to (2.26). If there is a nonzero homomorphism $\phi : E_1 \to E_2(K_X)$, then ϕ is injective and $\operatorname{coker} \phi$ is 0-dimensional by the μ-stability of E_1 and E_2. Since E_1 is locally free, $E_1 \cong E_2(K_X)$. In particular $v_1 = v_2$ and $v = 2v_1$. Since $\langle v^2 \rangle \ge l^2$, $\langle v_1^2 \rangle = \langle v_2^2 \rangle > 0$. Then for a general locally free sheaf E_1, we have $\operatorname{Hom}(E_1, E_2(K_X)) = 0$. Therefore for a general filtration, we have $\operatorname{Ext}^2(E_2, E_1) = 0$, which shows that the same estimate of (2.34) holds. Therefore (1) holds.

(2) The existence of a locally free sheaf follows from Lemma 2.7 and the last paragraph of the proof of Proposition 2.6, unless $r = 1$ and $l = 2$. So we assume that $r = 1$ and $l = 2$. This case is treated by Kim [8]. For completeness, we give a different argument. If E_2 is not locally free or $\det E_1 = \det E_2$, then $\operatorname{Ext}^2(E_2, E_1) = 0$ for a general filtration, and hence the same estimate of (2.34) holds. On the other hand, if E_2 is locally free and $E_1 = I_Z \otimes E_2(K_X)$ (which implies that E is not locally free), then we only have the estimate

$$\langle v^2 \rangle - k\frac{\langle v^2 \rangle}{2l} + \frac{(l-1-k)k}{2} + \frac{(k+1)k}{2} \le \langle v^2 \rangle - \Big(\frac{\langle v^2 \rangle}{2l} - \frac{l}{2}\Big). \tag{2.36}$$

In this case, if $\langle v^2 \rangle > 4$, then there is a μ-stable locally free sheaf.

(3) We set $v := (2, 0, -1)$. We have that $\mathcal{M}_H(v, 0)^{\mathrm{ss}}$ contains a μ-stable locally free sheaf by the proof of (2). Indeed, we have $\det E_1 = \det E_2$, which shows (2.34). We next treat $\mathcal{M}_H(v, K_X)^{\mathrm{ss}}$. By the proof of (2), it is sufficient to construct a μ-semistable locally free sheaf E of $v(E) = v$ and $\det E = \mathcal{O}_X(K_X)$. Indeed, for an irreducible component containing a locally free sheaf, E_1 is a locally free sheaf and there is an ideal sheaf of two points with $E_2 = E_1(K_X) \otimes I_{Z'}$, which shows (2.34).

For the ideal sheaf I_Z of two points, we have

$$\operatorname{Hom}\big(I_Z(K_X), \mathcal{O}_X\big) = \operatorname{Ext}^2\big(I_Z(K_X), \mathcal{O}_X\big) = 0.$$

Hence $\dim \operatorname{Ext}^1(I_Z(K_X),\mathcal{O}_X)=1$. We take a nontrivial extension

$$0\to\mathcal{O}_X\to E\to I_Z(K_X)\to 0. \tag{2.37}$$

Since $\operatorname{Ext}^1(I_W(K_X),\mathcal{O}_X)=0$ for $c_2(I_W)=0,1$, if E is not locally free, then $E^{\vee\vee}\cong\mathcal{O}_X\oplus\mathcal{O}_X(K_X)$, which shows that the exact sequence (2.37) splits. Therefore E is locally free. □

REMARK 2.3

For surjective homomorphisms $\phi_1:\mathcal{O}_X\to k_x\oplus k_y$ and $\phi_2:\mathcal{O}_X(K_X)\to k_x\oplus k_y$, the kernel E of

$$\mathcal{O}_X\oplus\mathcal{O}_X(K_X)\overset{(\phi_1,\phi_2)}{\longrightarrow}k_x\oplus k_y$$

is a stable nonlocally free sheaf. Then they form an irreducible component of $\mathcal{M}_H(v,K_X)^{\mathrm{ss}}$ consisting of nonlocally free sheaves. Therefore $\mathcal{M}_H(v,K_X)^{\mathrm{ss}}$ has at least two irreducible components. Combining [24, Remark 4.1], the $\mathcal{M}_H(2v_0,L)^{\mathrm{ss}}$ are reducible if $\langle v_0^2\rangle=1$ and $L\equiv K_X \mod 2$.

By the reflection associated to v_0 (see [24]), we get the following result.

PROPOSITION 2.9

Assume that $2br-l>0$; that is, $\langle(lv_0-b\varrho_X)^2\rangle=l(2br-l)>0$. Then

$$\mathcal{M}_H(lv_0-b\varrho_X)^{\mathrm{ss}}\cong\mathcal{M}_H\big((2br-l)v_0^{\vee}-b\varrho_X\big)^{\mathrm{ss}}.$$

2.3. Case C

Assume that there is a stable sheaf E_0 such that $v(E_0)=(r,\xi,a_0)$ and $\langle v(E_0)^2\rangle=-2$. Thus we assume that r is even, $r\mid(\xi^2)/2+1$, and $\xi\equiv D+\frac{r}{2}K_X \mod 2$, where D is a nodal cycle (see Theorem 1.1). We set $v_0=(r,\xi,a_0)$. As in the proof of Lemma 2.7, we have the following.

LEMMA 2.10

If $\mathcal{M}_H(v)^{\mu\mathrm{s}}\neq\emptyset$, then $\langle v^2\rangle\geq 2l^2$ or $v=v_0$.

PROPOSITION 2.11

Assume that $\langle v^2\rangle\geq 2l^2$. Then $\mathcal{M}_H(v)^{\mu\mathrm{s}}\neq\emptyset$. Moreover, each irreducible component contains a μ-stable locally free sheaf.

Proof

By Lemma 0.5, we get that

$$\dim\mathcal{M}_H(v)^{\mu\mathrm{ss}}\geq\langle v^2\rangle. \tag{2.38}$$

We will show that

$$\dim\mathcal{M}_H(v)^{\mathrm{p}\mu\mathrm{ss}}\leq\langle v^2\rangle-\Big(\frac{\langle v^2\rangle}{2l}-l+1\Big). \tag{2.39}$$

For this purpose, we estimate the moduli number of Jordan–Hölder filtrations. Let E be a μ-semistable sheaf of $v(E)=v$, and let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_t = E \tag{2.40}$$

be a Jordan–Hölder filtration of E with respect to μ-stability. We set $E_i := F_i/F_{i-1}$. We also set

$$v(E) := lv_0 + a\varrho_X, \qquad v(E_i) := l_i v_0 + a_i \varrho_X.$$

By Lemma 2.10, $\langle v_i^2 \rangle \neq 0$. Hence

$$\dim \mathcal{M}_H(v_i)^{\mu s} = \begin{cases} \langle v_i^2 \rangle, & \text{if } v_i \neq v_0, \\ \langle v_i^2 \rangle + 1 = -1, & \text{if } v_i = v_0. \end{cases} \tag{2.41}$$

Since $\langle v^2 \rangle > 0$, there is an integer i_0 such that $v_i \neq v_0$. Let $J(v_1, v_2, \ldots, v_t)$ be the substack of $\mathcal{M}_H(v)^{\mu ss}$ such that $E \in \mathcal{M}_H(v)^{\mu ss}$ has a filtration (2.40). By using [9, Lemma 5.2] successively, we see that

$$\begin{aligned} &\dim J(v_1, v_2, \ldots, v_t) \\ &\leq \sum_i \dim \mathcal{M}_H(v_i)^{\mu s} + \sum_{i<j} \big(\dim \operatorname{Ext}^1(E_j, E_i) - \dim \operatorname{Hom}(E_j, E_i)\big) \\ &\leq -\chi(E,E) + \sum_{i>j} \chi(E_j, E_i) + \sum_{i<j} \dim \operatorname{Ext}^2(E_j, E_i) + (t-1). \end{aligned} \tag{2.42}$$

By the same computation of [22, Lemma 4.4], we get the desired estimate. Hence the existence of a locally free sheaf follows by Lemma 2.10 and the last paragraph of the proof of Proposition 2.6. □

By the (-2)-reflection associated to v_0, we also get the following.

PROPOSITION 2.12
Assume that $br - l > 0$; that is, $\langle (lv_0 - b\varrho_X)^2 \rangle = 2l(br-l) > 0$. Then

$$\mathcal{M}_H(lv_0 - b\varrho_X)^{ss} \cong \mathcal{M}_H\big((br-l)v_0^{\vee} - b\varrho_X\big)^{ss}.$$

3. Moduli spaces on elliptic surfaces

3.1. f-semistability

In this section, we study moduli spaces of semistable sheaves on elliptic surfaces. Then we apply the results to the moduli spaces on Enriques surfaces, since Enriques surfaces have elliptic fibrations. Since we use the Bogomolov inequality, we assume that the characteristic of k is zero or the Bogomolov inequality holds. In particular, we can apply the results for moduli spaces on Enriques surfaces by Theorem 1.1.

Let $\pi : X \to C$ be an elliptic surface such that every fiber is irreducible. Let f be a fiber of π. We have a homomorphism

$$\begin{array}{rccc} \tau: & K(X) & \to & \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}, \\ & E & \mapsto & \big(\operatorname{rk} E, c_1(E), \chi(E)\big). \end{array} \tag{3.1}$$

We set $K(X)_{\mathrm{top}} := K(X)/\ker\tau$. For $\mathbf{e}\in K(X)_{\mathrm{top}}$, let $\mathcal{M}(\mathbf{e})$ be the moduli stack of coherent sheaves E whose topological invariants are $\mathbf{e}$. Let $\mathcal{M}_H(\mathbf{e})^{\mathrm{ss}}$ (resp., $\mathcal{M}_H(\mathbf{e})^{\mathrm{s}}$) be the substack of $\mathcal{M}(\mathbf{e})$ consisting of semistable sheaves (resp., stable sheaves). Let E be a torsion-free sheaf on X. We denote by $\mathbf{e}\in K(X)$ the class of E in $K(X)$. Let H be an ample divisor on X, and set $H_f := H+nf$, where n is a sufficiently large integer depending on $\mathbf{e}$. Let D be a curve on X such that $(D,f)=0$. For a coherent sheaf F on D, we set

$$\begin{aligned}\deg(F) &:= \chi(F)-\chi(\mathcal{O}_D)=\chi(F),\\ \deg_E(F) &:= \deg(E^\vee\otimes F)=\mathrm{rk}(E)\deg F-\big(c_1(E),c_1(F)\big)=\chi(E,F).\end{aligned}\tag{3.2}$$

DEFINITION 3.1

For a coherent sheaf E, we set

$$\Delta(E) := 2\,\mathrm{rk}\,E c_2(E)-(\mathrm{rk}\,E-1)\big(c_1(E)^2\big).$$

DEFINITION 3.2

(1) A torsion-free sheaf E is *f-semistable* if for all subsheaves $F\neq 0$ of E,

$$\frac{(c_1(F),f)}{\mathrm{rk}\,F}\le\frac{(c_1(E),f)}{\mathrm{rk}\,E}.$$

If the inequality is strict for all subsheaves F of E with $0<\mathrm{rk}\,F<\mathrm{rk}\,E$, then E is f-stable.

(2) Let $\mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}$ be the substack of $\mathcal{M}(\mathbf{e})$ consisting of f-semistable sheaves E.

REMARK 3.1

(1) The f-semistability of E is equivalent to the semistability of the restriction $E\otimes k(\eta)$ of E to the generic fiber; f-semistability is an open condition. Indeed, E is f-semistable if and only if $E_{|\pi^{-1}(t)}$ is semistable for a point $t\in C$, which is an open condition.

(2) $\mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}$ is not bounded in general. For a positive number B, let $\mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}_B$ be the open substack of $\mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}$ consisting of E such that for any subsheaf F of E,

$$\frac{(c_1(F),H)}{\mathrm{rk}\,F}\le\frac{(c_1(E),H)}{\mathrm{rk}\,E}+B.$$

Then $\mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}_B$ is bounded (see [11], [12]) and $\mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}=\bigcup_B\mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}_B$.

LEMMA 3.1

We set

$$N(H,\mathbf{e}) := \frac{(H,f)^2(\mathrm{rk}\,\mathbf{e})^2\Delta(\mathbf{e})-2(H^2)}{4(H,f)}.\tag{3.3}$$

Then we have the following results.

(1) *For rational numbers n_1,n_2 with $n_1,n_2>N(H,\mathbf{e})$, $H+n_1f$ and $H+n_2f$ are not separated by a wall with respect to $\mathbf{e}$. Thus $\mathcal{M}_{H+n_1f}(\mathbf{e})^{\mathrm{ss}}=\mathcal{M}_{H+n_2f}(\mathbf{e})^{\mathrm{ss}}$.*

(2) *A torsion-free sheaf E is semistable with respect to $H+nf$ $(n>N(H,\mathbf{e}))$ if and only if E is f-semistable and for any subsheaf $F\neq 0$ with $(\operatorname{rk} E c_1(F)-\operatorname{rk} F c_1(E), f)=0$,*

$$\frac{\chi(F(kH))}{\operatorname{rk} F}\leq\frac{\chi(E(kH))}{\operatorname{rk} E}\quad (k\gg 0). \tag{3.4}$$

Proof
(1) Assume that there is a wall between $H+n_1 f$ and $H+n_2 f$ $(N(H,\mathbf{e})<n_1<n_2)$. Then by [20, Definition 2.1], there is an exact sequence

$$0\to E_1\to E\to E_2\to 0 \tag{3.5}$$

such that

(i) E_1 and E_2 are $(H+\lambda f)$-semistable, and
(ii) $\xi:=\operatorname{rk} E_2 c_1(E_1)-\operatorname{rk} E_1 c_1(E_2)\in(H+\lambda f)^\perp$ $(n_1\leq\lambda\leq n_2)$.

We express ξ as $\xi=xH+yf+D$ $(D\in H^\perp\cap f^\perp)$. Then $y=-\frac{((H^2)+\lambda(H,f))}{(f,H)}x$. Since $x(H,f)=(\xi,f)$ is an integer, if $x\neq 0$, then we get

$$\begin{aligned}(\xi^2)&=-x^2\big((H^2)+2(H,f)\lambda\big)+(D^2)\\&\leq-\frac{((H^2)+2(H,f)\lambda)}{(H,f)^2}\\&<-\frac{(\operatorname{rk}\mathbf{e})^2\Delta(\mathbf{e})}{2}.\end{aligned} \tag{3.6}$$

By [20, Lemma 2.1], we see that $-(\xi^2)\leq(\operatorname{rk}\mathbf{e})^2\Delta(\mathbf{e})/2$ (see also [24, Section 5]). Hence we get $x=0$. Thus there is no wall between $H+n_1 f$ and $H+n_2 f$ $(n_1<n_2)$.

(2) Let E be a torsion-free sheaf with a topological invariant $\mathbf{e}$. By (1), the following conditions are equivalent.

(a) For a rational number $n>N(H,\mathbf{e})$,

$$\frac{(c_1(F),H+nf)}{\operatorname{rk} F}\leq\frac{(c_1(E),H+nf)}{\operatorname{rk} E} \tag{3.7}$$

for all subsheaves $F\neq 0$ of E; that is, E is μ-semistable with respect to $H+nf$.

(b) For any rational number $n>N(H,\mathbf{e})$,

$$\frac{(c_1(F),H+nf)}{\operatorname{rk} F}\leq\frac{(c_1(E),H+nf)}{\operatorname{rk} E} \tag{3.8}$$

for all subsheaves $F\neq 0$ of E.

(c) We have that

$$\frac{(c_1(F),f)}{\operatorname{rk} F}<\frac{(c_1(E),f)}{\operatorname{rk} E} \tag{3.9}$$

or

$$\frac{(c_1(F),f)}{\operatorname{rk} F}=\frac{(c_1(E),f)}{\operatorname{rk} E},\qquad\frac{(c_1(F),H)}{\operatorname{rk} F}\leq\frac{(c_1(E),H)}{\operatorname{rk} E} \tag{3.10}$$

for all subsheaves $F\neq 0$ of E.

Obviously (b) $\Leftrightarrow$ (a) and (b) $\Rightarrow$ (c) hold. We prove that (c) implies (a). For a torsion-free sheaf E satisfying (c), let $N(E)(> N(H,\mathbf{e}))$ be a rational number satisfying

$$N(E) > \big(\operatorname{rk} E c_1(F) - \operatorname{rk} F c_1(E), H\big) \tag{3.11}$$

for all subsheaves F of E. Let F be a subsheaf of E. By (c), it is sufficient to prove that $(\operatorname{rk} E c_1(F) - \operatorname{rk} F c_1(E), H + nf) < 0$ for $n \geq N(E)$ if $(\operatorname{rk} E c_1(F) - \operatorname{rk} F c_1(E), f) < 0$. By (3.11),

$$\begin{aligned}
&\big(\operatorname{rk} E c_1(F) - \operatorname{rk} F c_1(E), H + nf\big)\\
&\quad = \big(\operatorname{rk} E c_1(F) - \operatorname{rk} F c_1(E), H\big) + n\big(\operatorname{rk} E c_1(F) - \operatorname{rk} F c_1(E), f\big)\\
&\quad < N(E) - n \leq 0.
\end{aligned} \tag{3.12}$$

Therefore (a) holds. Hence the claim holds. □

DEFINITION 3.3

A torsion-free sheaf E is semistable with respect to H_f if E is semistable with respect to $H + nf$ for all $n \gg 0$.

PROPOSITION 3.2

Let n be a rational number with $n > N(H,\mathbf{e}) + 2$. Assume that $(\epsilon, x) \in (H^\perp \cap f^\perp)_{\mathbb{Q}} \times \mathbb{Q}$ satisfies $|(\epsilon^2)| < (H,f)$ and $|x| < 1$. Then $\mathcal{M}_{(H+\epsilon)+(n+x)f}(\mathbf{e})^{\mathrm{ss}} = \mathcal{M}_{(H+\epsilon)_f}(\mathbf{e})^{\mathrm{ss}}$. Thus for a chamber $\mathcal{C}$ with $H_f \in \overline{\mathcal{C}}$, there is $\epsilon \in (H^\perp \cap f^\perp)_{\mathbb{Q}}$ such that $(H+\epsilon)_f \in \mathcal{C}$.

Proof

Since $n > N(H,\mathbf{e}) + 2$, we get

$$n + x > N(H,\mathbf{e}) + 2 - |x| > N(H,\mathbf{e}) + 1 > N(H+\epsilon,\mathbf{e}).$$

Applying Lemma 3.1(1), we get $\mathcal{M}_{(H+\epsilon)+(n+x)f}(\mathbf{e})^{\mathrm{ss}} = \mathcal{M}_{(H+\epsilon)_f}(\mathbf{e})^{\mathrm{ss}}$. □

LEMMA 3.3 (Bogomolov inequality)

If $\mathcal{M}_f(\mathbf{e})^{\mathrm{ss}} \neq \emptyset$, then $\Delta(\mathbf{e}) \geq 0$.

Proof

If E is an f-stable sheaf E, then it is H_f-stable, and hence $\Delta(E) \geq 0$ by the Bogomolov inequality.

We next treat the general case. We note that $E \in \mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}$ is a successive extension of f-stable sheaves E_i with $(c_1(E_i), f)/\operatorname{rk} E_i = (c_1(E), f)/\operatorname{rk} E$. For an extension

$$0 \to E_1 \to E \to E_2 \to 0$$

of $E_i \in \mathcal{M}_f(\mathbf{e}_i)^{\mathrm{ss}}$ with $(c_1(E_1), f)/\operatorname{rk} E_1 = (c_1(E_2), f)/\operatorname{rk} E_2$, we have

$$\begin{aligned}\Delta(E) &= \operatorname{rk} E \frac{\Delta(E_1)}{\operatorname{rk} E_1} + \operatorname{rk} E \frac{\Delta(E_2)}{\operatorname{rk} E_2} - \frac{((\operatorname{rk} E_1 c_1(E_2) - \operatorname{rk} E_2 c_1(E_1))^2)}{\operatorname{rk} E_1 \operatorname{rk} E_2} \\ &\geq \operatorname{rk} E \frac{\Delta(E_1)}{\operatorname{rk} E_1} + \operatorname{rk} E \frac{\Delta(E_2)}{\operatorname{rk} E_2}.\end{aligned} \tag{3.13}$$

Hence by the induction of $\operatorname{rk} E$, we get the claim. □

DEFINITION 3.4

We set

$$\mathcal{N} := \mathrm{NS}_{\mathrm{f}}(X)/\big(\mathbb{Q}f \cap \mathrm{NS}_{\mathrm{f}}(X)\big). \tag{3.14}$$

Let $\mathcal{V}$ be the set of $(D, n) \in \mathcal{N} \times \mathbb{Z}$ such that

$$\begin{aligned} D &= \operatorname{rk} E c_1(E_1) - \operatorname{rk} E_1 c_1(E) \mod \mathbb{Q}f, \\ n &= \Delta(E_1), \end{aligned} \tag{3.15}$$

where E_1 is a subsheaf of $E \in \mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}$ satisfying

(1) E/E_1 is a torsion-free sheaf, and
(2) $(\operatorname{rk} E c_1(E_1) - \operatorname{rk} E_1 c_1(E), f) = 0$.

LEMMA 3.4

(1) *$\mathcal{V}$ is a finite set.*
(2) *There is a small neighborhood U of $\mathrm{NS}(X)_{\mathbb{R}} \cap H^{\perp} \cap f^{\perp}$ such that for $\epsilon \in U \cap \mathrm{NS}(X)_{\mathbb{Q}}$ and any $(D, n) \in \mathcal{V}$,*

$$(D, H + \epsilon) \geq 0 \Longrightarrow (D, H) \geq 0.$$

Proof

(1) We note that E_1 and $E_2 := E/E_1$ are f-semistable sheaves. Hence $\Delta(E_1)$, $\Delta(E_2) \geq 0$ by Lemma 3.3. Since $\operatorname{rk} E_2 c_1(E_1) - \operatorname{rk} E_1 c_1(E_2) = \operatorname{rk} E c_1(E_1) - \operatorname{rk} E_1 c_1(E)$, (3.13) implies that

$$\begin{aligned} (\operatorname{rk} E)^2 \Delta(E) &\geq -(D^2) \geq 0, \\ \Delta(E) &\geq \Delta(E_1) = n \geq 0. \end{aligned} \tag{3.16}$$

Since $\mathcal{N} \cap f^{\perp}$ is negative definite, the choice of D is finite. Therefore $\mathcal{V}$ is a finite set.

(2) This is obvious. □

DEFINITION 3.5

Let $V_{B,H}$ be the set of $\tau \in \tau(K(X))$, where there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{3.17}$$

such that $E \in \mathcal{M}_f(\mathbf{e})^{ss}_B$, E_2 is a torsion-free sheaf, and E_1 satisfies

$$\begin{aligned} \tau(E_1) &= \tau, \\ \big(\operatorname{rk} E c_1(E_1) - \operatorname{rk} E_1 c_1(E), f\big) &= 0, \\ \big(\operatorname{rk} E c_1(E_1) - \operatorname{rk} E_1 c_1(E), H\big) &\geq 0. \end{aligned} \tag{3.18}$$

LEMMA 3.5

We have that $V_{B,H}$ is a finite set.

Proof

By

$$B(\operatorname{rk} E)^2 > B \operatorname{rk} E \operatorname{rk} E_1 \geq \big(\operatorname{rk} E c_1(E_1) - \operatorname{rk} E_1 c_1(E), H\big) \geq 0 \tag{3.19}$$

and Lemma 3.4, we get the claim. □

PROPOSITION 3.6

We can take $\epsilon \in (H^\perp \cap f^\perp)_\mathbb{Q}$ satisfying the following properties:

(i) $|(\epsilon^2)| < (H, f)$;

(ii) *for $E \in \mathcal{M}_f(H, \mathbf{e})^{ss}_B$, let*

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{3.20}$$

be the Harder–Narasimhan filtration with respect to $(H + \epsilon) + nf$, where $n > N(H, \mathbf{e}) + 1$; then

(1) $E_i := F_i/F_{i-1}$ *are semistable sheaves with respect to $(H + \epsilon)_f$,*

(2) $(\operatorname{rk} E c_1(E_i) - \operatorname{rk} E_i c_1(E), f) = 0$,

(3) *$H + \epsilon$ is general with respect to $\tau(E_i)$ for all i.*

Proof

For $E \in \mathcal{M}_f(\mathbf{e})^{ss}$, let $N(E)$ be a number such that

$$N(E) > \big(\operatorname{rk}(E) c_1(F) - \operatorname{rk}(F) c_1(E), H + \epsilon\big) \tag{3.21}$$

for all subsheaves F of E. Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{3.22}$$

be the Harder–Narasimhan filtration with respect to $(H + \epsilon) + N(E)f$. We set

$$D_i := (\operatorname{rk} E) c_1(F_i) - (\operatorname{rk} F_i) c_1(E) \in \operatorname{NS}(X).$$

Then $(D_i, H+\epsilon+N(E)f) \geq 0$ for all i. If $(D_i, f) < 0$, then by (3.21) and $(D_i, f) \in \mathbb{Z}$, we have

$$\big(D_i, H + \epsilon + N(E)f\big) < N(E) + N(E)(D_i, f) \leq 0.$$

Therefore $(D_i, f) \geq 0$. On the other hand, $E \in \mathcal{M}_f(\mathbf{e})^{ss}$ implies that $(D_i, f) \leq 0$. Therefore we have $(D_i, f) = 0$, which implies that $(\overline{D_i}, \Delta(F_i)) \in \mathcal{V}$, where $\overline{D_i} = D_i \mod \mathbb{Q}f$. Since E is f-semistable, every F_i is f-semistable. We set $E_i :=$

F_i/F_{i-1}. Then the E_i's are f-semistable, and hence $\Delta(E_i) \geq 0$. By using (3.13) and $(\operatorname{rk} E c_1(E_i) - \operatorname{rk} E_i c_1(E), f) = 0$, we see that $\Delta(E_i) < \Delta(E)$. Hence

$$N(H+\epsilon, E_i) < N(H+\epsilon, \mathbf{e}) < N(H, \mathbf{e}) + 1.$$

Applying Lemma 3.1 to E_i, the E_i's are semistable with respect to $(H+\epsilon)+nf$ for all $n > N(H, \mathbf{e}) + 1$. Since

$$\left(\frac{c_1(E_i)}{\operatorname{rk} E_i} - \frac{c_1(E_{i+1})}{\operatorname{rk} E_{i+1}}, H+\epsilon+nf\right) \tag{3.23}$$

is independent of n, (3.22) is the Harder–Narasimhan filtration with respect to $(H+\epsilon)+nf$ for all $n > N(H, \mathbf{e}) + 1$.

By Lemma 3.4(2), we get $(D_i, H) \geq 0$. Thus $\tau(F_i) \in V_{B,H}$. Since $V_{B,H}$ is a finite set, we can take ϵ such that $H+\epsilon$ is general for all $(r_2, \xi_2, \chi_2) - (r_1, \xi_1, \chi_1)$, where $(r_i, \xi_i, \chi_i) \in V_{B,H}$ $(i = 1, 2)$ satisfies $r_2 > r_1$. Therefore $H+\epsilon$ is general with respect to $\tau(E_i)$. □

3.2. Some estimates on substacks

LEMMA 3.7

For $E \in \mathcal{M}_f(\mathbf{e})^{\mathrm{ss}}$, there is an exact sequence

$$0 \to \widetilde{E} \to E \to F \to 0 \tag{3.24}$$

such that

(i) $\widetilde{E}_{|D}$ *is a stable purely* 1*-dimensional sheaf for every fiber* D *with reduced structure,*

(ii) F *is a purely* 1*-dimensional sheaf supported on fibers, and*

(iii) $\operatorname{Hom}(E', F) = 0$ *if* E' *is a coherent sheaf of rank* r *on* X *such that* $E'_{|D}$ *is a semistable sheaf of degree* $(c_1(E), D)$ *for every* D.

By these properties, $\widetilde{E}$ *and* F *are uniquely determined by* E.

Proof

If $E_{|D}$ is not purely 1-dimensional or purely 1-dimensional but not semistable, then we take a surjective homomorphism $\phi : E \to E_{|D} \to G$ such that G is a semistable 1-dimensional sheaf with $\deg_E(G) < 0$. We set $E' := \ker \phi$. Then E' is an f-semistable sheaf with $\operatorname{rk} E' = \operatorname{rk} E$ and $(c_1(E'), D) = (c_1(E), D)$. If $E'_{|D}$ is not semistable, then we continue the same procedure. Since

$$0 \leq \Delta(E') = \Delta(E) + 2 \deg_E G < \Delta(E),$$

we finally get a desired subsheaf $\widetilde{E}$ of E. We set $F := E/\widetilde{E}$. Since F is a successive extension of semistable 1-dimensional sheaves G with $\deg_E(G) < 0$, we have $\operatorname{Hom}(E', F) = 0$. □

For the quotient F of E in (3.24), let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F \tag{3.25}$$

be the Harder–Narasimhan filtration of F with respect to H. We remark that semistability is independent of the choice of H by the irreducibility of fibers of π. Then

$$\operatorname{Hom}\big(F_i/F_{i-1}, F_j/F_{j-1}(K_X)\big) = 0, \quad i < j. \tag{3.26}$$

By the construction of F, $\deg_E(F_i/F_{i-1}) < 0$ for all i. In particular,

$$\operatorname{Hom}\big(\widetilde{E}, F_i/F_{i-1}(K_X)\big) = 0 \tag{3.27}$$

for all i. Let

$$F_{i-1} = F_{i-1,0} \subset F_{i-1,1} \subset \cdots \subset F_{i-1,n(i-1)} = F_i = F_{i,0} \tag{3.28}$$

be a filtration of F_i such that $F_{i,j}/F_{i,j-1}$ are stable sheaves; thus, F_i/F_{i-1} is S-equivalent to $\bigoplus_{j=1}^{n(i-1)} F_{i-1,j}/F_{i-1,j-1}$. We set $E_{i,j} := \ker(E \to F/F_{i,j})$. Then we have a filtration

$$0 \subset \widetilde{E} = E_{0,0} \subset E_{0,1} \subset \cdots \subset E_{0,n(0)} = E_{1,0} \subset E_{1,1} \subset \cdots \subset E_{s-1,n(s-1)} = E_s = E$$

such that $E_{i,j}/E_{i,j-1} \cong F_{i,j}/F_{i,j-1}$. By Lemma 3.9, $F_{i,j}/F_{i,j-1}$ are stable sheaves on a reduced and irreducible divisor D_{ij}. Since the $E_{i,j}$'s are torsion-free and $E_{i,j} \to F_{i,j}/F_{i,j-1}$ are surjective, we have

$$\operatorname{rk} E = \operatorname{rk} E_{i,j} \geq \operatorname{rk} F_{i,j}/F_{i,j-1}.$$

Let $\mathbf{f}_i \in K(X)_{\text{top}}$ be the class of F_i/F_{i-1}, and let $\widetilde{\mathbf{e}} \in K(X)_{\text{top}}$ be the class of $\widetilde{E}$. We set

$$(r_{ij} D_{ij}, d_{ij}) := \big(c_1(F_{ij}/F_{i,j-1}), \chi(F_{ij}/F_{i,j-1})\big),$$

where $r_{ij}, d_{ij} \in \mathbb{Z}$ and $\gcd(r_{ij}, d_{ij}) = 1$. Then

$$\begin{aligned} 0 &< r_{ij} \leq r, \\ -\deg_E(F_{i,j}/F_{i,j-1}) &= r_{ij}\big(c_1(E), D_{ij}\big) - r d_{ij} > 0, \\ \Delta(E) &= 2\sum_{i,j}\big(r_{ij}\big(c_1(E), D_{ij}\big) - r d_{ij}\big) + \Delta(\widetilde{E}). \end{aligned} \tag{3.29}$$

Hence we see that the choice of $\mathbf{f}_i$ is finite.

PROPOSITION 3.8

Let $\mathcal{F}(\widetilde{\mathbf{e}}, \mathbf{f}_1, \ldots, \mathbf{f}_s)$ be the stack of filtrations

$$0 \subset \widetilde{E} = \widetilde{F}_0 \subset \widetilde{F}_1 \subset \widetilde{F}_2 \subset \cdots \subset \widetilde{F}_{s-1} \subset \widetilde{F}_s = E \tag{3.30}$$

such that $\widetilde{E} \in \mathcal{M}_f(\widetilde{\mathbf{e}})^{\text{ss}}$ satisfies Lemma 3.7(i) and $\widetilde{F}_i/\widetilde{F}_{i-1} \in \mathcal{M}_H(\mathbf{f}_i)^{\text{ss}}$ for all i. Then

$$\dim \mathcal{F}(\widetilde{\mathbf{e}}, \mathbf{f}_1, \ldots, \mathbf{f}_s) = -\sum_i \chi(\mathbf{f}_i, \widetilde{\mathbf{e}}) + \dim \mathcal{M}_f(\widetilde{\mathbf{e}})^{\text{ss}} + \sum_i \mathcal{M}_H(\mathbf{f}_i)^{\text{ss}}. \tag{3.31}$$

Proof
By (3.26), (3.27), and the Serre duality, we have

$$\begin{aligned}\operatorname{Ext}^2(F_j/F_{j-1}, F_i/F_{i-1}) = 0, \quad i < j,\\ \operatorname{Ext}^2(F_i/F_{i-1}, \widetilde{E}) = 0, \quad 1 \le i \le s.\end{aligned} \tag{3.32}$$

Then the proof of [9, Lemma 5.3] implies that

$$\begin{aligned}&\dim \mathcal{F}(\widetilde{\mathbf{e}}, \mathbf{f}_1, \dots, \mathbf{f}_s)\\ &= -\sum_i \chi(\mathbf{f}_i, \widetilde{\mathbf{e}}) - \sum_{i<j} \chi(\mathbf{f}_j, \mathbf{f}_i) + \dim \mathcal{M}_f(\widetilde{\mathbf{e}})^{\mathrm{ss}} + \sum_i \mathcal{M}_H(\mathbf{f}_i)^{\mathrm{ss}}\\ &= -\sum_i \chi(\mathbf{f}_i, \widetilde{\mathbf{e}}) + \dim \mathcal{M}_f(\widetilde{\mathbf{e}})^{\mathrm{ss}} + \sum_i \mathcal{M}_H(\mathbf{f}_i)^{\mathrm{ss}}.\end{aligned} \tag{3.33}$$

□

LEMMA 3.9
Let D be a reduced and irreducible curve on X with $(D^2) = 0$. For an element $G_1 \in K(X)$ with $\operatorname{rk} G_1 > 0$, let E be a G_1-twisted stable purely 1-dimensional sheaf such that $\operatorname{Div}(E) = rD$ and $\chi(E) = d$. Then E is a stable sheaf on D. In particular, $\gcd(r, d) = 1$.

Proof
We note that $\mathcal{O}_D(D)$ is a numerically trivial line bundle on D. Let T be the torsion submodule of $E_{|D}$. Then $E' := E_{|D}/T$ has the Harder–Narasimhan filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E'.$$

We set $(c_1(F_i/F_{i-1}), \chi(F_i/F_{i-1})) := (r_i D, d_i)$. Then $r_i \in \mathbb{Z}_{>0}$, $\sum_{i=1}^s r_i \le r$ and

$$\frac{d_1}{r_1} > \frac{d_2}{r_2} > \cdots > \frac{d_s}{r_s}.$$

By the G_1-twisted stability of E, we have

$$\frac{(\operatorname{rk} G_1)d - (c_1(G_1), rD)}{(rD, H)} \le \frac{(\operatorname{rk} G_1)d_s - (c_1(G_1), r_s D)}{(r_s D, H)},$$

where the inequality is strict unless $E \cong F_s/F_{s-1}$. Hence $d/r \le d_s/r_s$.

There is a positive integer k such that E is an $\mathcal{O}_{(k+1)D}$-module and $E(-kD) \overset{kD}{\to} E$ is nonzero. Then we have a nonzero homomorphism $E_{|D}(-kD) \to E$. Then $\operatorname{Hom}(F_i/F_{i-1}(-kD), E) \ne 0$ for some i, which implies that $d_i/r_i \le d/r$. Then $i = s$ and $d/r = d_s/r_s$. By the stability of E, $E \to F_s/F_{s-1}$ is an isomorphism. In particular, E is a stable sheaf on D. Since D is a reduced and irreducible curve of $g(D) = 1$, there is an elliptic surface X' with a section such that D is a fiber. We set $m := \gcd(r, d)$ and $(r', d') := (r/m, d/m)$. For a general polarization H' on X', we consider the moduli space $Y := M_{H'}(0, r'f, d')$ of stable sheaves F of dimension 1 on X' whose Chern character is $(0, r'f, d')$, where f is a fiber. Let $\mathcal{E}$ be a universal family. Then by the general theory of Fourier–Mukai transforms (see [27, Proof of Lemma 2.3.6]), we see that E is a successive extension of $\mathcal{E}_{|X'\times\{y\}}$ $(y \in Y)$. Since E is stable, $m = 1$. Therefore $\gcd(r, d) = 1$. □

COROLLARY 3.10

For $E \in \mathcal{M}_H(0, rf, d)^{ss}$, we have a decomposition $E \cong \bigoplus_i E_i$ such that $D_i := \operatorname{Supp}(E_i)$ are fibers of π with reduced scheme structure, the E_i's are successive extensions of stable sheaves E_{ij} on D_i with $\frac{\chi(E_{ij})}{\operatorname{rk}(E_{ij})(D_i,H)} = \frac{d}{r(f,H)}$, and $D_i \cap D_j = \emptyset$ for $i \neq j$.

LEMMA 3.11

Let D be a reduced and irreducible curve on X with $(D^2) = 0$. For a torsion-free sheaf E on X, $E_{|nD}$ is semistable if and only if $E_{|D}$ is semistable. Moreover, if $E_{|D}$ is semistable, then E is locally free in a neighborhood of D.

Proof

We have a filtration

$$0 \subset E(-nD) \subset E\big(-(n-1)D\big) \subset E\big(-(n-2)D\big) \subset \cdots \subset E(-D) \subset E. \tag{3.34}$$

We set $L := \mathcal{O}_D(-D)$. Then $E(-kD)/E(-(k+1)D) \cong E_{|D} \otimes L^{\otimes k}$ and $\chi(E_{|D} \otimes L^{\otimes k}) = \chi(E_{|D})$ for $0 < k < (n-1)$. Hence $E_{|nD}$ is semistable if and only if $E_{|D}$ is semistable. If $E_{|D}$ is semistable, then $E_{|D}$ is purely 1-dimensional, which shows that E is locally free in a neighborhood of D. □

3.3. For the case of an unnodal Enriques surface

Let X be an unnodal Enriques surface. Let $U := \mathbb{Z}e_1 + \mathbb{Z}e_2$ be a hyperbolic sublattice of the lattice $H^2(X, \mathbb{Z})_f$. We assume that e_1, e_2 are effective and that $|2e_1|$ gives an elliptic fibration $\pi : X \to \mathbb{P}^1$. Since X is unnodal, every fiber of π is irreducible. Let $2\Pi_1, 2\Pi_2$ be the multiple fibers of π. Let $\eta \in \mathbb{P}^1$ be the generic point of $\mathbb{P}^1$. Let $u := (r, de_2, 0)$ be a primitive and isotropic Mukai vector. We note that r is even. We assume that $\gcd(r, d) = 1$. For a Mukai vector $v \in (0, re_1, d)^\perp$, we can write

$$v = lu + ne_1 + \delta + a\varrho_X,$$

where $l, n, a \in \mathbb{Z}$ and $\delta \in U^\perp$. If v is primitive and $\ell(v) = 2$, then $2 \mid l, 2 \mid n, 2 \mid \delta$ and $2 \nmid a$. We can easily show the following claims.

LEMMA 3.12

Let $v_i := l_i u + n_i e_1 + \delta_i + a_i \varrho_X$ $(i = 1, 2)$ be two primitive Mukai vectors with $l_i, n_i, a_i \in \mathbb{Z}$ and $\delta_i \in U^\perp$.

(1) *We have that*

$$\langle v_1, v_2 \rangle = \frac{l_2}{2l_1}\langle v_1^2 \rangle + \frac{l_1}{2l_2}\langle v_2^2 \rangle - \frac{1}{2l_1 l_2}\big\langle (l_2\delta_1 - l_1\delta_2)^2 \big\rangle. \tag{3.35}$$

(2) *If* $\ell(v_1) = 2$, *then*

$$\langle v_1, v_2\rangle = (l_1 n_2 + l_2 n_1)d + (\delta_1, \delta_2) - (l_1 a_2 + l_2 a_1)r \in 2\mathbb{Z}. \tag{3.36}$$

Moreover, if $\ell(v_2) = 2$ *also holds, then* $\langle v_1, v_2\rangle \in 4\mathbb{Z}$.

Let E be a f-stable sheaf with $v(E) = lu + ne_1 + \delta + a\varrho_X$, where $l, n, a \in \mathbb{Z}$ and $\delta \in U^\perp$. Since the f-stability implies the H_f-stability, $\operatorname{rk} E = lr$ is even, and X is unnodal, we have $\langle v(E)^2\rangle \geq 0$. Then as in the proof of Lemma 3.3, by using Lemma 3.12, we get the following inequality.

LEMMA 3.13
If $\mathcal{M}_f(v)^{\mathrm{ss}} \neq \emptyset$, *then* $\langle v^2\rangle \geq 0$.

PROPOSITION 3.14
Assume that r *is even and* $(r, d) = 1$. *We set*

$$v := lu + ne_1 + \delta + a\varrho_X, \quad l, n, a \in \mathbb{Z}, \delta \in U^\perp.$$

(1) *Assume that* $\langle v^2\rangle > 0$. *Then* $\mathcal{M}_{H_f}(v)^{\mathrm{ss}}$ *is an open and dense substack of* $\mathcal{M}_f(v)^{\mathrm{ss}}$. *In particular,* $\dim \mathcal{M}_f(v)^{\mathrm{ss}} = \langle v^2\rangle$.
(2) *Assume that* $\langle v^2\rangle = 0$. *Then* $\dim \mathcal{M}_f(v)^{\mathrm{ss}} \leq [\frac{l}{2}]$.

Proof
It is sufficient to prove the claim for bounded substacks $\mathcal{M}_f(v)^{\mathrm{ss}}_B$, $B \in \mathbb{Q}$. Replacing H_f by $(H+\epsilon)_f$ in Proposition 3.6, we may assume that (1), (2), and (3) in Proposition 3.6 hold for the Harder–Narasimhan filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F \tag{3.37}$$

of $F \in \mathcal{M}_f(v)^{\mathrm{ss}}_B$. Indeed, if $\mathcal{M}_{(H+\epsilon)_f}(v)^{\mathrm{ss}}$ is dense, then since Proposition 3.2 implies that $\mathcal{M}_{(H+\epsilon)_f}(v)^{\mathrm{ss}} \subset \mathcal{M}_{H_f}(v)^{\mathrm{ss}}$, $\mathcal{M}_{H_f}(v)^{\mathrm{ss}}$ is also dense.

By the boundedness of $\mathcal{M}_f(v)^{\mathrm{ss}}_B$ (or Lemma 3.5), the choice of $v(F_i/F_{i-1})$ $(1 \leq i \leq s)$ is finite. We set

$$v_i := v(F_i/F_{i-1}) = l_i u + n_i e_1 + \delta_i + a_i \varrho_X, \quad 1 \leq i \leq s. \tag{3.38}$$

By Lemma 3.13, $\langle v_i^2\rangle \geq 0$ for all i. Let $\mathcal{F}^{\mathrm{HN}}(v_1, v_2, \ldots, v_s)$ be the substack of $\mathcal{M}_H(v)^{\mu\mathrm{ss}}$ whose element E has the Harder–Narasimhan filtration of the above type.

(1) Assume that $\langle v^2\rangle > 0$. We will prove that

$$\dim \mathcal{F}^{\mathrm{HN}}(v_1, v_2, \ldots, v_s) < \langle v^2\rangle. \tag{3.39}$$

Since $\operatorname{Hom}(F_i/F_{i-1}, F_j/F_{j-1}(K_X)) = 0$ for $i<j$, [9, Lemma 5.3] implies that

$$\begin{aligned}&\dim \mathcal{F}^{\mathrm{HN}}(v_1, v_2, \dots, v_s)\\ &= \sum_{i=1}^{s} \dim \mathcal{M}_H(v_i)^{\mathrm{ss}} + \sum_{i<j} \langle v_j, v_i\rangle \\ &= \langle v^2\rangle - \Big(\sum_{i<j}\langle v_j, v_i\rangle + \sum_{i=1}^{s}\big(\langle v_i^2\rangle - \dim \mathcal{M}_H(v_i)^{\mathrm{ss}}\big)\Big).\end{aligned} \tag{3.40}$$

If v_i is isotropic, then we write $v_i = k_i u_i$, where u_i is primitive and $k_i \in \mathbb{Z}_{>0}$. By Proposition 1.9, $\dim \mathcal{M}_H(v_i)^{\mathrm{ss}} \le k_i/2, k_i$ according as $\ell(u_i) = 1,2$. If there is v_p with $\langle v_p^2\rangle > 0$, then Lemma 3.12 implies that $\langle v_p, v_i\rangle = k_i\langle v_p, u_i\rangle \ge k_i, 2k_i$ according as $\ell(u_i) = 1,2$. Hence (3.39) holds. Assume that all v_i are isotropic. By Lemma 3.12, $\langle v, v_i\rangle > 0$ for all i. Hence for all i, there is a positive integer $n(i)$ such that $\langle v_i, v_{n(i)}\rangle > 0$. We set $\epsilon := \ell(u_i) + \ell(u_j) - 2$. Then

$$2^{\epsilon} k_i k_j - \dim \mathcal{M}_H(v_i)^{\mathrm{ss}} - \dim \mathcal{M}_H(v_j)^{\mathrm{ss}} > 0.$$

Hence

$$\sum_{i \in \{i \mid i<n(i)\}} \langle v_i, v_{n(i)}\rangle > \sum_i \dim \mathcal{M}_H(v_i)^{\mathrm{ss}}.$$

Therefore (3.39) holds.

(2) Assume that $\langle v^2\rangle = 0$. By Lemma 3.13, Lemma 3.12(1), and $\langle v^2\rangle = \sum_i \langle v_i^2\rangle + \sum_{i\neq j}\langle v_i, v_j\rangle$, we see that $\langle v_i, v_j\rangle = 0$ for $i \neq j$, every v_i in (3.38) is isotropic, and $\dim \mathcal{M}_H(v_i)^{\mathrm{ss}} \le [k_i/2], k_i$ according as $\ell(u_i) = 1,2$. If $\ell(u_i) = 2$, then $l_i \ge 2k_i$. Hence $\dim \mathcal{M}_H(v_i)^{\mathrm{ss}} \le l_i/2$ for all i. Therefore

$$\dim \mathcal{F}^{\mathrm{HN}}(v_1, v_2, \dots, v_s) = \sum_i \dim \mathcal{M}_H(v_i)^{\mathrm{ss}} \le \frac{l}{2}.$$

□

DEFINITION 3.6

(1) Let $\mathcal{M}_H(v)^{\mathrm{ss}}_*$ be the open substack of $\mathcal{M}_H(v)^{\mathrm{ss}}$ consisting of E such that $E_{|\pi^{-1}(t)}$ is semistable for all $t \in \mathbb{P}^1$. By Lemma 3.11, $\mathcal{M}_H(v)^{\mathrm{ss}}$ consists of E such that $E_{|\pi^{-1}(t)_{\mathrm{red}}}$ is semistable for all $t \in \mathbb{P}^1$, where $\pi^{-1}(t)_{\mathrm{red}}$ is the reduced part of $\pi^{-1}(t)$.

(2) Let $\mathcal{M}_f(v)^{\mathrm{ss}}_*$ be the open substack of $\mathcal{M}_f(v)^{\mathrm{ss}}$ consisting of E such that $E_{|\pi^{-1}(t)_{\mathrm{red}}}$ is semistable for all $t \in \mathbb{P}^1$.

PROPOSITION 3.15

We set $v := lu + ne_1 + \delta + a\varrho_X$, where $l, n, a \in \mathbb{Z}$, $l>0$, and $\delta \in U^{\perp}$. Then $\mathcal{M}_f(v)^{\mathrm{ss}}_*$ is an open and dense substack of $\mathcal{M}_f(v)^{\mathrm{ss}}$.

Proof

For $E \in \mathcal{M}_f(v)^{\mathrm{ss}}$, we have the filtration (3.24). For the filtration (3.25), we set

$$v_i := v(F_i/F_{i-1}) = k_i(0, r_i e_1, d_i),$$

where $(0, r_i e_1, d_i)$ are primitive. By Proposition 3.8,

$$\begin{aligned}
&\langle v^2\rangle - \dim \mathcal{F}(\tilde{v}, v_1, \dots, v_s)\\
&= \sum_i lk_i(r_i d - r d_i) - \sum_i \dim \mathcal{M}_H(v_i)^{\mathrm{ss}} - \bigl(\dim \mathcal{M}_f(\tilde{v})^{\mathrm{ss}} - \langle \tilde{v}^2\rangle\bigr).
\end{aligned} \tag{3.41}$$

We first assume that $\tilde{v}$ is isotropic. Then $\dim \mathcal{M}_f(\tilde{v})^{\mathrm{ss}} - \langle \tilde{v}^2\rangle \le [\frac{l}{2}]$ by Proposition 3.14(2). If $\ell((0, r_i e_1, d_i)) = 2$, then r_i is even. In this case, we have $(r_i d - r d_i) \in 2\mathbb{Z}$. Hence

$$\begin{aligned}
&lk_i(r_i d - r d_i) - \dim \mathcal{M}_H(v_i)^{\mathrm{ss}} - \bigl(\dim \mathcal{M}_f(\tilde{v})^{\mathrm{ss}} - \langle \tilde{v}^2\rangle\bigr)\\
&\ge \min\Bigl\{lk_i - \Bigl[\frac{k_i}{2}\Bigr] - \Bigl[\frac{l}{2}\Bigr], 2lk_i - k_i - \Bigl[\frac{l}{2}\Bigr]\Bigr\}\\
&> 0.
\end{aligned} \tag{3.42}$$

If $\tilde{v}$ is not isotropic, then by using Proposition 3.14(1), we get $\langle v^2\rangle - \dim \mathcal{F}(\tilde{v}, v_1, \dots, v_s) > 0$. Therefore our claim holds. □

By the proof of Proposition 3.15, we can compute the boundary components of $\mathcal{M}_f(v)^{\mathrm{ss}}_*$. Indeed, we see that

$$lk_i(r_i d - r d_i) - \dim \mathcal{M}_H(v_i)^{\mathrm{ss}} - \bigl(\dim \mathcal{M}_f(\tilde{v})^{\mathrm{ss}} - \langle \tilde{v}^2\rangle\bigr) = 1 \tag{3.43}$$

implies that $(l, k_i) = (1,1), (1,2), (2,1)$. If $\langle v^2\rangle - \dim \mathcal{F}(\tilde{v}, v_1, \dots, v_s) = 1$, then we see that $s = 1$ and $r_i d - r d_i = 1$ for $\ell(v_i) = 1$ and $s = 1$ and $r_i d - r d_i = 2$ for $\ell(v_i) = 2$. Thus a general member E of $\mathcal{M}_f(v)^{\mathrm{ss}} \setminus \mathcal{M}_f(v)^{\mathrm{ss}}_*$ fits in an extension

$$0 \to E' \to E \to F \to 0, \tag{3.44}$$

where $E' \in \mathcal{M}_f(\widetilde{v}_1)^{\mathrm{ss}}$ and $F \in \mathcal{M}_H(v_1)^{\mathrm{ss}}$.

Assume that $l = 1$. Then $\dim \mathcal{M}_f(\widetilde{v})^{\mathrm{ss}} = 0$, and hence we only need to consider E fitting in (3.44). We take integers (p, q) such that $0 < p \le r$ and $pd - rq = 1$, and we set $u_1 := (0, pe_1, q)$. Let $\mathcal{F}(v - u_1, u_1)^{\mathrm{s}}$ be the open substack parameterizing torsion-free sheaves E fitting in the extension (3.44) such that $\mathrm{Div}(F) = p\Pi_i$. Then it defines a divisor $\mathcal{D}_i$ on $\mathcal{M}_f(v)^{\mathrm{ss}}$.

We note that $\mathcal{M}_H(2u_1, pf)^{\mathrm{s}}$ consists of stable locally free sheaves of rank p and degree $2q$ on a smooth fiber f. Let $\mathcal{F}(v - 2u_1, 2u_1)^{\mathrm{s}}$ be the open substack parameterizing torsion-free sheaves E fitting in the extension (3.44) such that $\mathrm{Div}(F) = pf$. Then it defines a divisor $\mathcal{D}_3$ on $\mathcal{M}_f(v)^{\mathrm{ss}}$.

We set $u_2 := (0, 2p'e_1, 2q')$, where $0 < p' \le r$, $(p', q') = (p \pm r/2, q \pm d/2)$. Then u_2 is a primitive Mukai vector with $\langle v, u_2\rangle = (2p')d - r(2q') = 2$. We note that $\mathcal{M}_H(u_2, p'f)^{\mathrm{s}}$ consists of stable locally free sheaves of rank p' and degree $2q'$ on a smooth fiber. Let $\mathcal{F}(v - u_2, u_2)^{\mathrm{s}}$ be the open substack parameterizing torsion-free sheaves E fitting in the extension (3.44) such that $\mathrm{Div}(F) = p'f$. Then it defines a divisor $\mathcal{D}_4$ on $\mathcal{M}_f(v)^{\mathrm{ss}}$.

Since $l=1$, we have $\mathcal{M}_f(v)^{\mathrm{ss}} = \mathcal{M}_{H_f}(v)^{\mathrm{ss}}$, and hence

$$\mathcal{M}_{H_f}(v)^{\mathrm{ss}} = \mathcal{M}_{H_f}(v)^{\mathrm{ss}}_* \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \tag{3.45}$$

up to codimension 2.

EXAMPLE 3.1

For $v=(2, e_2+ne_1+\delta, a)$ with $u=(2,e_2,0)$, we have

$$(3.46)\qquad \begin{aligned} &(p,q)=(1,0),\\ &(p',q')=(p,q)+\frac{1}{2}(r,d)=\left(2,\frac{1}{2}\right). \end{aligned}$$

In particular, $u_1=(0,e_1,0)$ and $u_2=(0,4e_1,1)$. As we will see in the next section, (3.45) holds without removing codimension 2 subsets.

4. Irreducibility

4.1. Unnodal case

Assume that X is an unnodal Enriques surface and that f is a smooth fiber of the elliptic fibration $\pi: X \to \mathbb{P}^1$ defined by $|2e_1|$. Let $v=(r,\xi,\frac{s}{2})$ be a primitive Mukai vector such that r is even. Then for $L \in \mathrm{NS}(X)$ with $[L \mod K_X]=\xi$, we have an equality of "Hodge polynomials" of the stacks defined in [24] (see [28, Proposition 2.4, Theorem 2.6])

$$e\Big(\mathcal{M}_H\Big(v, L+\frac{r}{2}K_X\Big)^{\mathrm{ss}}\Big) = e\big(\mathcal{M}_{H_f}(v', L'+K_X)^{\mathrm{ss}}\big),$$

where $H_f = H+nf\ (n \gg 0)$, $v'=(2,\zeta,\frac{s'}{2})$, $[L' \mod K_X]=\zeta$, $L \equiv L' \mod 2$, $\langle v^2\rangle = \langle v'^2\rangle$, and $\zeta=0$ if $\ell(v)=2$. Assume that $\ell(v)=\ell(v')=2$. Then $v'=(2,0,-2n)$ for some $n \in \mathbb{Z}$. We set $v''=(4,2(e_2+(n+1)e_1),1)$. Then $e(\mathcal{M}_{H_f}(v',L'+K_X)^{\mathrm{ss}}) = e(\mathcal{M}_{H_f}(v'',L'')^{\mathrm{ss}})$, where $L' \equiv L'' \mod 2$. In order to prove the irreducibility of $\mathcal{M}_H(v,L)^{\mathrm{ss}}$, it is sufficient to prove the irreducibility for the following two cases:

(1) $v=(2,e_2+ne_1+\delta,a)$,
(2) $v=(4,2(e_2+(n+1)e_1),1)$.

In particular, $u=(2,e_2,0)$ in the notation of Section 3.3. We note that $M_H(0,f,1)$ is a fine moduli space, that it is isomorphic to X, and that it parameterizes torsion-free sheaves of rank 1 on a reduced and irreducible fiber $\pi^{-1}(t)$ and stable vector bundles of rank 2 and degree 1 on Π_i. Thus

$$M_H(0,f,1) = \bigcup_{t \in \mathbb{P}^1_0} \overline{\mathrm{Pic}}^1_{\pi^{-1}(t)} \cup M_{\Pi_1}(2,1) \cup M_{\Pi_2}(2,1),$$

where π has reduced fibers over $\mathbb{P}^1_0$, $\overline{\mathrm{Pic}}^1_{\pi^{-1}(t)}$ are the compactified Jacobians of degree 1, and $M_{\Pi_i}(2,1)$ are the moduli spaces of stable vector bundles of rank 2 and degree 1 on Π_i. We take an identification $M_H(0,f,1) \cong X$. Let $\mathcal{E}$ be a universal family on $X \times X$. By [2],

$$\mathcal{E}_{|X\times\{x\}} \otimes K_X \cong \mathcal{E}_{|X\times\{x\}}, \quad x \in X, \tag{4.1}$$

and

$$\begin{array}{rccc} \Phi^{\mathcal{E}}_{X\to X}: & \mathbf{D}(X) & \to & \mathbf{D}(X), \\ & E & \mapsto & \mathbf{R}p_{2*}\big(p_1^*(E)\otimes\mathcal{E}\big) \end{array} \tag{4.2}$$

is an equivalence, that is, a Fourier–Mukai transform, where $p_i : X \times X \to X$ $(i=1,2)$ are projections. We consider a contravariant Fourier–Mukai transform

$$\begin{array}{rccc} \Psi: & \mathbf{D}(X) & \to & \mathbf{D}(X), \\ & E & \mapsto & \mathbf{R}\operatorname{Hom}_{p_2}\big(p_1^*(E),\mathcal{E}\big). \end{array} \tag{4.3}$$

Since $\Psi(\mathcal{O}_X)$ is a line bundle, replacing the universal family, we may assume that $\Psi(\mathcal{O}_X) = \mathcal{O}_X$. We set $\Psi^i(E) := H^0(\Psi(E)[i]) \in \operatorname{Coh}(X)$.

LEMMA 4.1

We have $\Psi(0,0,1) = (0,2e_1,1)$, $\Psi(0,4e_1,1) = (0,-2e_1,1)$, *and* $\Psi(0,e_1,0) = (0,-e_1,0)$.

Proof

We note that $\Psi(k_x) = \mathcal{E}_{|\{x\}\times X}[-2]$. Hence $c_1(\Psi(k_x)) = 2e_1$. Since $1 = \chi(\mathcal{O}_X, k_x) = -\langle \Psi(k_x), \Psi(\mathcal{O}_X)\rangle$, we have $\Psi(0,0,1) = (0,2e_1,1)$. Since $\Psi(\mathcal{E}_{|X\times\{x\}}) = k_x[-2]$, we have $\Psi(0,2e_1,1) = (0,0,1)$. Since

$$\begin{aligned} (0,4e_1,1) &= 2(0,2e_1,1) - (0,0,1), \\ 2(0,e_1,0) &= (0,2e_1,1) - (0,0,1), \end{aligned} \tag{4.4}$$

we have

$$\begin{aligned} \Psi(0,4e_1,1) &= (0,-2e_1,1), \\ \Psi(0,e_1,0) &= (0,-e_1,0). \end{aligned} \tag{4.5}$$

□

For cases (1) and (2), by [27, Proposition 3.4.5], Ψ induces an isomorphism

$$\Psi : \mathcal{M}_{H_f}(v)^{\mathrm{ss}} \to \mathcal{M}_{H'}^{G'}(w)^{\mathrm{ss}}, \tag{4.6}$$

where $w = (0,\xi,a)$ with $(\xi, e_1) = 1, 2$, $\mathcal{M}_{H'}^{G'}(w)^{\mathrm{ss}}$ is the moduli stack of G'-twisted semistable sheaves, and $H' \in \mathrm{NS}(X)_{\mathbb{Q}}$ and $G' \in K(X)$ depend on the choice of H and v.

REMARK 4.1

Since H is a general polarization, $\mathcal{M}_{H_f}^{G}(v)^{\mathrm{ss}}$ is independent of the choice of G. Hence we do not need to consider twisted semistability.

We have a support map

$$\begin{array}{rccc} \varphi: & \mathcal{M}_{H'}^{G'}(w,L)^{\mathrm{ss}} & \to & |L|, \\ & E & \mapsto & \det E. \end{array} \tag{4.7}$$

LEMMA 4.2

We have that $\mathcal{M}_H(v)^{ss}_*$ is isomorphic to the open substack $\mathcal{M}^{G'}_{H'}(w)^{ss}_*$ of $\mathcal{M}^{G'}_{H'}(w)^{ss}$ consisting of G'-twisted semistable sheaves E such that $\operatorname{Div}(E)$ is flat over $\mathbb{P}^1$.

Proof

As we remarked, Ψ induces an isomorphism $\mathcal{M}_{H_f}(v)^{ss} \to \mathcal{M}^{G'}_{H'}(w)^{ss}$. In particular, $\Psi(E)[1] = \Psi^1(E)$ is a sheaf for $E \in \mathcal{M}_{H_f}(v)^{ss}$ (see [27, Proposition 3.4.5]). By $\Psi^2(E) = 0$ and $\operatorname{rk}\Psi(E) = 0$, $\Psi(E)[1]$ is represented by a two-term complex of locally free sheaves of the same rank, and we have

$$\begin{aligned} \operatorname{Supp}(\Psi^1(E)) &= \{x \in X \mid \operatorname{Ext}^1(E, \mathcal{E}_{|X\times\{x\}}) \neq 0\} \\ &= \{x \in X \mid \operatorname{Hom}(E, \mathcal{E}_{|X\times\{x\}}) \neq 0\}. \end{aligned} \tag{4.8}$$

For $E \in \mathcal{M}_H(v)^{ss}_*$, the semistability of $E_{|\pi^{-1}(t)}$ implies that $E_{|\pi^{-1}(t)}$ is a successive extension of $\mathcal{E}_{|X\times\{x\}}$ ($x \in \pi^{-1}(t)$). Hence $\operatorname{Hom}(E, \mathcal{E}_{|X\times\{x\}}) = 0$ for a general point of $x \in \pi^{-1}(t)$. Then by (4.8), $\operatorname{Supp}(\Psi^1(E))$ does not contain a fiber, which implies that $\operatorname{Div}(\Psi^1(E))$ is flat over $\mathbb{P}^1$.

Conversely for $L \in \mathcal{M}^{G'}_{H'}(w)^{ss}_*$, $L^* := L^\vee[1]$ is a purely 1-dimensional sheaf on X (see Lemma 0.4). Hence

$$E := \operatorname{Ext}^1_{p_1}(p_2^*(L), \mathcal{E}) = p_{1*}(p_2^*(L^*) \otimes \mathcal{E})$$

is a locally free sheaf on X such that $E_{|\pi^{-1}(t)}$ is semistable for all $t \in \mathbb{P}^1$. Indeed, if there is a quotient $E_{|\pi^{-1}(t)} \to F$ with $v(F) = (0, ae_1, b)$, $a > 2b$, then $\Psi(F)[1]$ is a torsion sheaf on $\pi^{-1}(t)$ with an injective homomorphism $\Psi^1(F) \to \Psi^1(E) = L$, which is a contradiction. Therefore the claim holds. □

For case (1), we have $w = (0, \xi, a)$ with $(\xi, e_1) = 1$. We take $L \in \operatorname{NS}(X)$ with $[L \bmod K_X] = \xi$. Then for $E \in \mathcal{M}^{G'}_{H'}(w, L)^{ss}_*$, $\operatorname{Div}(E)$ is integral. Let $|L|_*$ be the open subscheme parameterizing integral curves. Then $\varphi : \mathcal{M}^{G'}_{H'}(w, L)^{ss}_* \to |L|_*$ is a family of compactified Jacobians over $|L|_*$. By [1], all fibers are irreducible of dimension $(L^2)/2$. Hence $\mathcal{M}^{G'}_{H'}(w, L)^{ss}_*$ is irreducible. Thus, by Proposition 3.15, we have the following.

PROPOSITION 4.3

We have that $\mathcal{M}^{G'}_{H'}(w, L)^{ss}$ is irreducible.

We note that $|L| \setminus |L|_*$ is a Cartier divisor consisting of three irreducible components:

$$\begin{aligned} \Gamma_i &:= \{D \in |L| \mid \Pi_i \subset D\} \quad (i = 1, 2), \\ \Gamma' &:= \{D \in |L| \mid \pi^{-1}(t) \subset D, t \in \mathbb{P}^1\}. \end{aligned} \tag{4.9}$$

Hence $\mathcal{M}_{H_f}(v)^{ss} \setminus \mathcal{M}_{H_f}(v)^{ss}_*$ is also a Cartier divisor. Then (3.45) holds without removing codimension 2 subsets.

REMARK 4.2

Set-theoretically, we have $\varphi^{-1}(\Gamma_i) = \Psi(\mathcal{D}_i)$ $(i = 1,2)$ and $\varphi^{-1}(\Gamma') = \Psi(\mathcal{D}_3) \cup \Psi(\mathcal{D}_4)$.

Let u_1 and u_2 be the Mukai vectors in Example 3.1. From the exact sequence

$$0 \to \widetilde{E} \to E \to F \to 0$$

in (3.44) with $F \in \mathcal{M}_H(u_1, p\Pi_i)^{\mathrm{s}} \cup \mathcal{M}_H(2u_1, pf)^{\mathrm{s}} \cup \mathcal{M}_H(u_2, p'f)^{\mathrm{s}}$ and $\widetilde{E} \in \mathcal{M}_f(v - v(F))_*^{\mathrm{ss}}$,

$$\operatorname{Ext}^2(F, \mathcal{E}_{|X\times\{x\}}) = \operatorname{Hom}(\mathcal{E}_{|X\times\{x\}}, F)^\vee = 0 \tag{4.10}$$

for all $x \in X$ by (4.1), $\frac{1}{2} = \frac{d}{r} > \frac{q}{p} = 0$, and $\frac{1}{2} = \frac{d}{r} > \frac{q'}{p'} = \frac{1}{4}$. Since $\operatorname{Hom}(F, \mathcal{E}_{|X\times\{x\}}) = 0$ for a general $x \in X$, we see that

$$\Psi(\widetilde{E})[1], \Psi(E)[1], \Psi(F)[1] \in \operatorname{Coh}(X),$$

and we have an exact sequence

$$0 \to \Psi^1(F) \to \Psi^1(E) \to \Psi^1(\widetilde{E}) \to 0.$$

By using Lemma 4.1, we have the following description of the boundary divisors.

(i) For a general member $E \in \Psi(\mathcal{D}_i)$ $(i = 1,2)$, $\operatorname{Div}(E) = \Pi_i + C$, where C is flat over π.

(ii) For a general member $E \in \Psi(\mathcal{D}_i)$ $(i = 3,4)$, $\operatorname{Div}(E) = f + C$, where C is flat over π.

By this description of the boundary, we have the following claim.

PROPOSITION 4.4

We set

$$|L|_{\mathrm{nr}} := \{D \in |L| \mid D \text{ is not reduced}\}$$

and $\mathcal{M}_{H'}^{G'}(w,L)_{\mathrm{nr}}^{\mathrm{ss}} = \varphi^{-1}(|L|_{\mathrm{nr}})$. *Then* $\operatorname{codim}_{\mathcal{M}_{H'}^{G'}(w,L)^{\mathrm{ss}}}(\mathcal{M}_{H'}^{G'}(w,L)_{\mathrm{nr}}^{\mathrm{ss}}) \geq 2$.

For a general G'', we also have a morphism

$$\psi : \mathcal{M}_{H'}^{G''}(w,L)^{\mathrm{ss}} \to |L|. \tag{4.11}$$

COROLLARY 4.5 (see [17, Assumption 2.16])

For the morphism (4.11), we also have

$$\operatorname{codim}_{\mathcal{M}_{H'}^{G''}(w,L)^{\mathrm{ss}}} \psi^{-1}(|L|_{nr}) \geq 2.$$

Proof

We note that G''-twisted semistability of E is independent of the choice of G'' if $\operatorname{Div}(E)$ is irreducible. Since $\mathcal{M}_{H'}^{G''}(w,L)^{\mathrm{ss}}$ and $\mathcal{M}_{H'}^{G'}(w,L)^{\mathrm{ss}}$ are irreducible with torsion canonical bundles, we have a birational map

$$\xi : \mathcal{M}_{H'}^{G''}(w,L)^{\mathrm{ss}} \cdots \to \mathcal{M}_{H'}^{G'}(w,L)^{\mathrm{ss}}$$

such that ξ induces an isomorphism in codimension 1 and $\psi = \varphi \circ \xi$. Hence we have a one-to-one correspondence of divisors. Therefore the claim holds. $\square$

We next treat case (2). Since $\Psi(v) = 2\Psi((2, e_2 + (n+1)e_1, 0)) + (0, 2e_1, 1)$ by Lemma 4.1, we set $w = (0, 2\xi, a)$ with $\gcd(2, a) = 1$. Let L be a divisor with $[L \bmod K_X] = \xi$. We will prove the irreducibility of $\mathcal{M}_{H'}^{G'}(w, 2L)^{\mathrm{ss}}$ and $\mathcal{M}_{H'}^{G'}(w, 2L + K_X)^{\mathrm{ss}}$. In order to prove the irreducibility of $\mathcal{M}_{H'}^{G'}(w, 2L)^{\mathrm{ss}}$, we consider the support map $\varphi : \mathcal{M}_{H'}^{G'}(w, 2L)^{\mathrm{ss}} \to |2L|$. We set

$$N_1 := \{D \in |2L| \mid D = 2C\},$$
$$N_2 := \{D \in |2L| \mid D = C_1 + C_2, (C_1, e_1) = (C_2, e_1) = 1, C_1 \neq C_2\}, \tag{4.12}$$

and

$$\mathcal{M}_i := \{E \in \mathcal{M}_{H'}^{G'}(w, 2L)_*^{\mathrm{ss}} \mid \mathrm{Div}(E) \in N_i\}. \tag{4.13}$$

LEMMA 4.6
We have $\dim \mathcal{M}_1 \leq \frac{7}{2}(L^2) - 1$.

Proof
Let $E \in \mathcal{M}_{H'}^{G'}(w, 2L)^{\mathrm{ss}}$ satisfy $\mathrm{Div}(E) = 2C$, $C \in |L + \epsilon K_X|$ $(\epsilon = 0, 1)$. Assume that E is not an $\mathcal{O}_C$-module. Then we have an exact sequence

$$E(-C) \xrightarrow{\psi} E \to E_{|C} \to 0$$

with $\operatorname{im} \psi \neq 0$. Since E is pure, $\operatorname{im} \psi$ is purely 1-dimensional. For the 0-dimensional submodule T of $E_{|C}$, we set $E_0 = E_{|C}/T$ and $E_1 = \ker(E \to E_0)$. Then we have an exact sequence

$$0 \to E_1 \to E \to E_0 \to 0 \tag{4.14}$$

and an injective homomorphism $E_0(-C) \cong \operatorname{im} \psi \subset E_1$. Since $\mathrm{Div}(E_0(-C)) = \mathrm{Div}(\operatorname{im} \psi) \neq 0$, $\mathrm{Div}(E_0) = \mathrm{Div}(E_0(-C)) = C$ and $\mathrm{Div}(E_1) = C$. In particular, E_0 and E_1 are $\mathcal{O}_C$-modules. If C is smooth, the locus of E fitting into (4.14) is of dimension $2(C^2) = 4g(C) - 4$ by [6, Proposition 2.1], where we used Remark 0.2. Since $\dim |C| = (C^2)/2$, $\mathcal{M}_1$ is of dimension $5(L^2)/2$ in a neighborhood of E. Assume that C is singular. We set $v(E_i) = v_i := (0, C, a_i)$. Since $v(E_0(-C)) = (0, C, a_0 - (C^2))$, we have

$$v(E_1) = v_1 = (0, C, a_0 - (C^2) + k), \quad k \geq 0.$$

Since $2a_0 - (C^2) + k = a$ is odd, $k \geq 1$. Since C is an integral curve, and E_0 and E_1 are torsion-free sheaves of rank 1 on C, they are stable sheaves on C. We set

$$n := \chi(E_0) - \chi(E_1) = (C^2) - k.$$

Let P be a smooth point of C. Then E_1 and E_2 are locally free $\mathcal{O}_C$-modules in a neighborhood of P. Hence

$$\begin{aligned} \dim \operatorname{Hom}(E_1, E_0(K_X)) &\leq \dim \operatorname{Hom}(E_1, E_0(K_X - (n+1)P)) + n + 1 \\ &= (C^2) - k + 1 \leq (C^2). \end{aligned}$$

We set

$$M_i := \{E \in \mathcal{M}_{H'}^{G'}(v_i, C)_*^{\mathrm{ss}} \mid \mathrm{Div}(E) \text{ is singular}\}. \tag{4.15}$$

Since all fibers of $\varphi : \mathcal{M}_{H'}^{G'}(v_i, C)_*^{\mathrm{ss}} \to |C|_*$ are of dimension $(C^2)/2$ by [1], we get

$$\dim M_0 \times_{|C|} M_1 \leq (C^2)/2 - 1 + 2\big((C^2)/2\big), \tag{4.16}$$

and hence

$$\dim M_0 \times_{|C|} M_1 + \langle v_0, v_1 \rangle + (C^2) \leq \frac{7}{2}(C^2) - 1, \tag{4.17}$$

which shows the locus of E fitting into (4.14) such that $E_i \in M_i$ is at most of dimension $7(C^2)/2 - 1$ by Lemma 0.6.

Assume that E is an $\mathcal{O}_C$-module. If C is smooth, then E is a stable locally free sheaf of rank 2 on C. Hence the dimension of $\mathcal{M}_1$ is $\dim |C| + 4(g(C) - 1) = 5(C^2)/2$ in a neighborhood of E. We assume that C is singular. We set $a = 2k + 1$. Let $\mathcal{O}_C(1)$ be a line bundle of degree 1 on C. Since $\chi(E(-k)) = 1$, we have a homomorphism $\mathcal{O}_C(k) \to E$. Hence we get an exact sequence

$$0 \to E_1 \to E \to E_0 \to 0 \tag{4.18}$$

such that E_0 and E_1 are torsion-free of rank 1 and E_1 contains $\mathcal{O}_C(k)$. We set $v(E_i) = v_i = (0, C, a_i)$. Then

$$\begin{aligned} a_1 &= \chi(E_1) = \chi\big(\mathcal{O}_C(k)\big) + l \quad (l \geq 0), \\ a_0 &= a - a_1 = 2k + 1 - \chi\big(\mathcal{O}_C(k)\big) - l. \end{aligned} \tag{4.19}$$

Since $\chi(\mathcal{O}_C(k)) = k - \frac{(C^2)}{2}$, we have

$$\chi(E_0) - \chi(E_1) = (2k + 1) - 2\chi\big(\mathcal{O}_C(k)\big) - 2l = 1 + (C^2) - 2l.$$

Hence $\dim \mathrm{Hom}(E_1, E_0(K_X)) \leq (C^2) + 2 - 2l$. If $l \geq 1$, then

$$\dim M_0 \times_{|C|} M_1 + \langle v_0, v_1 \rangle + (C^2) \leq \frac{7}{2}(C^2) - 1. \tag{4.20}$$

If $l = 0$, then $E_1 = \mathcal{O}_C(k)$. Hence

$$\begin{aligned} -1 + \dim M_0 + \langle v_0, v_1 \rangle + (C^2) + 2 &\leq -1 + (C^2) - 1 + 2(C^2) + 2 \\ &= 3(C^2). \end{aligned} \tag{4.21}$$

Since $(C^2)/2 \geq 1$, $7(C^2)/2 - 1 - 3(C^2) \geq 0$. Hence the locus of E fitting into (4.18) is at most of dimension $7(C^2)/2 - 1$ by Lemma 0.6. Therefore we get the claim. □

LEMMA 4.7

We have $\dim \mathcal{M}_2 \leq 4(L^2) - 2$.

Proof

For $E \in \mathcal{M}_2$, we have an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0, \tag{4.22}$$

where $\operatorname{Div} E = C_1 + C_2$, $\operatorname{Div}(E_1) = C_1$, and $\operatorname{Div}(E_2) = C_2$. By the flatness of $\operatorname{Div}(E)$, C_1 and C_2 are integral curves and $C_1 \neq C_2$. By the stability of E, we have

$$\frac{a}{(2L,H)} \leq \frac{\chi(E_2)}{(C_2,H)}, \qquad \frac{\chi(E_2)-(C_1,C_2)}{(C_2,H)} \leq \frac{a}{(2L,H)}. \tag{4.23}$$

We set $v_i := v(E_i) = (0, C_i, a_i)$. By (4.23), the choice of v_i is finite. Since $(C_i, e_1) = 1$, the v_i's are primitive with $\ell(v_i) = 1$. Hence $\dim \mathcal{M}_{H'}^{G'}(v_i)^{\mathrm{ss}} = \langle v_i^2 \rangle = (C_i^2)$. Let $J(v_1, v_2)$ be the substack of $\mathcal{M}_2$ such that $E_i \in \mathcal{M}_{H'}^{G'}(v_i, C_i)^{\mathrm{ss}}$. Since C_1 and C_2 are different integral curves, we have $\operatorname{Hom}(E_1, E_2(K_X)) = 0$, and hence

$$\begin{aligned} \dim J(v_1,v_2) &= \dim \mathcal{M}_{H'}^{G'}(v_1,C_1)^{\mathrm{ss}} + \dim \mathcal{M}_{H'}^{G'}(v_2,C_2)^{\mathrm{ss}} + (C_1,C_2) \\ &= (C_1^2) + (C_2^2) + (C_1,C_2) = 4(L^2) - (C_1,C_2). \end{aligned} \tag{4.24}$$

Since X is unnodal, we have $(C_1^2), (C_2^2) \geq 0$ (see Section 0.1). If (C_1^2), $(C_2^2) > 0$, then $(C_1,C_2)^2 \geq (C_1^2)(C_2^2) \geq 4$. Hence $(C_1,C_2) \geq 2$. If one of $(C_i^2) = 0$, then $(C_1,C_2) = (2L,C_i) - (C_i^2) = 2(L,C_i) \geq 2$. Therefore $\dim J(v_1,v_2) \leq 4(L^2) - 2$. □

We set $\mathcal{M}_{H'}^{G'}(w,2L)_0^{\mathrm{ss}} := \mathcal{M}_{H'}^{G'}(w,2L)_*^{\mathrm{ss}} \setminus (\mathcal{M}_1 \cup \mathcal{M}_2)$.

PROPOSITION 4.8

We have that $\mathcal{M}_{H'}^{G'}(w,2L)_0^{\mathrm{ss}}$ is an open and dense substack of $\mathcal{M}_{H'}^{G'}(w,2L)^{\mathrm{ss}}$. In particular, it is irreducible.

Proof

By Lemma 4.6 and Lemma 4.7, we get the first assertion. For an integral curve $C \in |2L|$, $\varphi^{-1}(C)$ is irreducible by [1]. Hence the irreducibility follows. □

We next show the irreducibility of $\mathcal{M}_{H'}^{G'}(w,2L+K_X)^{\mathrm{ss}}$. We note that $C_1 \neq C_2$ for any $C_1 + C_2 \in |2L+K_X|$. So we do not need to worry about nonreduced curves in this case, and the analogue of Lemma 4.7 is enough. Hence we get the following.

PROPOSITION 4.9

We have that $\mathcal{M}_{H'}^{G'}(w,2L+K_X)^{\mathrm{ss}}$ is irreducible.

By Propositions 4.3, 4.8, and 4.9, Theorem 0.2 holds if X is unnodal.

REMARK 4.3

For $w = (0, 2\xi, 2b)$, we define $\mathcal{M}_i (\subset \mathcal{M}_{H'}^{G'}(w,2L)_*^{\mathrm{ss}})$ in a similar way. Then we see that $\dim \mathcal{M}_1 \leq \frac{7}{2}(L^2)$ and $\dim \mathcal{M}_2 \leq 4(L^2) - 2$. Hence $\mathcal{M}_{H'}^{G'}(w,2L)^{\mathrm{ss}}$ and $\mathcal{M}_{H'}^{G'}(w,2L+K_X)^{\mathrm{ss}}$ are irreducible. In particular, $\mathcal{M}_H(2v,L)^{\mathrm{ss}}$ is irreducible, where v is primitive, $\operatorname{rk} v$ is even, and $\ell(v) = 1$.

REMARK 4.4

For a divisor $D' := 2^m D + K_X$ such that $D(\in \mathrm{NS}(X))$ is primitive, D' is not linearly equivalent to pC, $p \geq 2$. Hence we also see that $\mathcal{M}_H(2^m v, L)^{\mathrm{ss}}$ is irreducible, if v is primitive and $L \neq 2L'$, $L' \in \mathrm{NS}(X)$.

4.2. General cases

We will treat a general case by the arguments in [28, Section 3]. Let $(\mathcal{X}, \mathcal{H}) \to S$ be a general deformation of (X, H) such that a general member is not nodal and $(\mathcal{X}_0, \mathcal{H}_0) = (X, H)$ $(0 \in S)$. Let $\mathcal{L}$ be a family of divisors such that $\mathcal{L}_0 = L \in \mathrm{NS}(X)$. Then we have a family of moduli spaces of semistable sheaves $\phi : M_{(\mathcal{X},\mathcal{H})}(v, \mathcal{L}) \to S$. Since $\mathrm{Pic}(\mathcal{X}_s) = H^2(\mathcal{X}_s, \mathbb{Z})$ is locally constant, we may assume that $\mathcal{H}_s$ is general with respect to v for all $s \in S$. Let M_0 be the open subscheme of $M_{(\mathcal{X},\mathcal{H})}(v)$ such that $E \in \mathrm{Coh}(\mathcal{X}_s)$ belongs to M_0 if and only if $\mathrm{Hom}(E, E(K_{\mathcal{X}_s})) = 0$. Then M_0 is smooth over S. By Lemma 1.3, M_0 is a dense subscheme of $M_{(\mathcal{X},\mathcal{H})}(v, \mathcal{L})$. Since $(M_0)_s$ is irreducible for any unnodal surface $\mathcal{X}_s$, M_0 is irreducible, which implies that $M_{(\mathcal{X},\mathcal{H})}(v, \mathcal{L})$ is also irreducible. By the Zariski connectedness theorem, all fibers are connected. In particular, $M_H(v, L)$ is connected. If $\langle v^2 \rangle \geq 4$, then $M_H(v, L)$ is irreducible by Lemmas 1.3 and 1.5(1). Therefore we get the following.

THEOREM 4.10

Let $v = (r, \xi, a)$ be a primitive Mukai vector such that r is even. Then $\mathcal{M}_H(v, L)^{\mathrm{ss}}$ is connected for a general H. Moreover, if $\langle v^2 \rangle \geq 4$, then $\mathcal{M}_H(v, L)^{\mathrm{ss}}$ is irreducible for a general H.

REMARK 4.5

Let v be a primitive Mukai vector. By [24, Remark 4.1], the proof of [28, Theorem 2.6], and [28, Remark 2.19], we have

$$e\bigl(\mathcal{M}_H(mv)^{\mathrm{ss}}\bigr) = e\bigl(\mathcal{M}_H(mw)^{\mathrm{ss}}\bigr),$$

where $w = (1, 0, -\frac{s}{2})$ if $\mathrm{rk}\, v$ is odd, and $w = (2, \xi_i, -\frac{s}{2})$ $(1 \leq i \leq 2^{10})$ with

$$\{\xi_i \mod 2 \mid 1 \leq i \leq 2^{10}\} = \mathrm{NS}_{\mathrm{f}}(X) \otimes \mathbb{Z}/2\mathbb{Z} = (\mathbb{Z}/2\mathbb{Z})^{\oplus 10}$$

if $\mathrm{rk}\, v$ is even. By the works of Gieseker and Li [4] or O'Grady [15], moduli stacks are asymptotically irreducible.

If $\mathrm{rk}\, w = 1$, then there is $N_1(m)$ such that $\mathcal{M}_H(mw, L)^{\mathrm{ss}}$ is irreducible for $\langle w^2 \rangle \geq N_1(m)$. Assume that $\mathrm{rk}\, w = 2$. For each $(m \,\mathrm{rk}\, w, m\xi_i)$, there is $N(m, \xi_i)$ such that $\mathcal{M}_H(mw, L)^{\mathrm{ss}}$ is irreducible if $\langle w^2 \rangle \geq N(m, \xi_i)$. We set $N_2(m) := \max_i N(m, \xi_i)$. Then $\mathcal{M}_H(mv, L)^{\mathrm{ss}}$ is irreducible if $\langle v^2 \rangle \geq N(m) := \max\{N_1(m), N_2(m)\}$.

Appendix

A.1 Relative Fourier–Mukai transforms

Let $\pi : X \to C$ be an elliptic surface, and let f be a smooth fiber of π. Let $\mathbf{e} \in K(X)_{\mathrm{top}}$ be the class of a coherent sheaf E with $\operatorname{rk} E = r$ and $(c_1(E), f) = d$. Assume that $\gcd(r, d) = 1$. Then $\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}} = \mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{s}}$ is smooth of dimension $-\chi(\mathbf{e}, \mathbf{e}) + p_g$, where $p_g := \dim H^2(X, \mathcal{O}_X)$ is the geometric genus of X. In this case, Bridgeland [2, Theorem 1.1] showed that a suitable relative Fourier–Mukai transform induces a birational map between $M_{H_f}(\mathbf{e})$ and the moduli of stable sheaves of rank 1. In this section, we slightly refine the correspondence and compare the Picard groups. We assume that every fiber is irreducible. Then, as we will easily see from Proposition 3.8, the birational map is a regular map up to codimension 1 for many cases; however, it is not if $r = 2$. In order to treat this case, we use a composition of a Fourier–Mukai transform and the taking dual functor as in (4.3).

We first assume that there are no multiple fibers. Then we have a refinement of Proposition 3.8. Let

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F$$

be the filtration in (3.25). Since there are no multiple fibers, we can set

$$\big(c_1(F_i/F_{i-1}), \chi(F_i/F_{i-1})\big) := l_i(r_i f, d_i),$$

where $l_i, r_i, d_i \in \mathbb{Z}$, $\gcd(r_i, d_i) = 1$, and $l_i > 0$. We set $\mu_{\min}(E) := d_s/r_s$. We define $\mathcal{F}(\widetilde{\mathbf{e}}, \mathbf{f}_1, \ldots, \mathbf{f}_s)$ as in Proposition 3.8. Then we have the following.

PROPOSITION A.1

We have $\operatorname{codim} \mathcal{F}(\widetilde{\mathbf{e}}, \mathbf{f}_1, \ldots, \mathbf{f}_s) = \sum_i l_i((r_i d - r d_i) - 1)$.

Proof

We note that $E \otimes K_X \cong E$ for $E \in \mathcal{M}_H((0, r_i f, d_i))^{\mathrm{ss}} = \mathcal{M}_H((0, r_i f, d_i))^{\mathrm{s}}$ and $\dim \mathcal{M}_H((0, r_i f, d_i))^{\mathrm{ss}} = 1$. Then we see that every member of $\mathcal{M}_H(\mathbf{f}_i)^{\mathrm{ss}}$ is generated by elements of $\mathcal{M}_H((0, r_i f, d_i))^{\mathrm{s}}$. Hence we get $\dim \mathcal{M}_H(\mathbf{f}_i)^{\mathrm{ss}} = l_i$. Then by Proposition 3.8,

$$\dim \mathcal{F}(\widetilde{\mathbf{e}}, \mathbf{f}_1, \ldots, \mathbf{f}_s) = \sum_i l_i(r_i d - r d_i) + \dim \mathcal{M}_{H_f}(\widetilde{\mathbf{e}})^{\mathrm{ss}} + \sum_i l_i. \tag{A.1}$$

Since $\chi(\widetilde{\mathbf{e}}, \mathbf{f}_i) = \chi(\mathbf{f}_i, \widetilde{\mathbf{e}})$, we get

$$\begin{aligned} \dim \mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}} &= -\chi(\mathbf{e}, \mathbf{e}) + p_g \\ &= -\chi(\widetilde{\mathbf{e}}, \widetilde{\mathbf{e}}) - \sum_i 2\chi(\widetilde{\mathbf{e}}, \mathbf{f}_i) + p_g \\ &= \dim \mathcal{M}_{H_f}(\widetilde{\mathbf{e}})^{\mathrm{ss}} + 2\sum_i l_i(r_i d - r d_i). \end{aligned} \tag{A.2}$$

Hence the claim holds. □

Let $Y := M_H(0, r'f, d')$ be a fine moduli space of stable sheaves on X, and let $\mathbf{P}$ be a universal family on $X \times Y$. Then Y has an elliptic fibration $\pi' : Y \to C$. We also denote a smooth fiber of π' by f; $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ denote the projections. We consider a contravariant functor

$$\text{(A.3)} \qquad \begin{array}{rccc} \Phi_{X\to Y}^{\mathbf{P}} \circ D_X : & \mathbf{D}(X) & \to & \mathbf{D}(Y), \\ & E & \mapsto & \mathbf{R}\operatorname{Hom}_{p_Y}\big(p_X^*(E), \mathbf{P}\big). \end{array}$$

By the Grothendieck–Serre duality, $D_Y \circ \Phi_{X\to Y}^{\mathbf{P}} \cong \Phi_{X\to Y}^{\mathbf{P}^\vee[2]\otimes p_X^*(\omega_X)} \circ D_X$. Hence $\Phi_{Y\to X}^{\mathbf{P}} \circ D_Y$ is the inverse of $\Phi_{X\to Y}^{\mathbf{P}} \circ D_X$. Assume that $r'd - rd' > 0$. Then we have the following.

LEMMA A.2

For $E \in \mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}$, $\Phi_{X\to Y}^{\mathbf{P}}(E^\vee)[1] \in \operatorname{Coh}(Y)$. Moreover, if $\mu_{\min}(E) \geq d'/r'$, then $\Phi_{X\to Y}^{\mathbf{P}}(E^\vee)[1]$ is torsion-free. In particular, $\Phi_{X\to Y}^{\mathbf{P}}(E^\vee)[1]$ is f-semistable.

Proof

We note that

$$\mathbf{P}_{|X\times\{y\}} \otimes K_X \cong \mathbf{P}_{|X\times\{y\}}$$

for all $y \in Y$ by the general theory of Fourier–Mukai transforms (see [2]). By the Serre duality and the torsion-freeness of E,

$$\operatorname{Ext}^2(E, \mathbf{P}_{|X\times\{y\}}) = \operatorname{Hom}(\mathbf{P}_{|X\times\{y\}}, E)^\vee = 0.$$

Hence $H^2(\Phi_{X\to Y}^{\mathbf{P}}(E^\vee)) = 0$. We note that $\operatorname{Hom}(E, \mathbf{P}_{|X\times\{y\}}) = 0$ if $E_{|\pi^{-1}(\pi'(y))}$ is semistable. Since $E_{|f}$ is semistable for a general fiber of π, $\operatorname{Hom}(E, \mathbf{P}_{|X\times\{y\}}) = 0$ for a general $y \in Y$. Since $H^0(\Phi_{X\to Y}^{\mathbf{P}}(E^\vee))$ is torsion-free, $H^0(\Phi_{X\to Y}^{\mathbf{P}}(E^\vee)) = 0$. Therefore $\Phi_{X\to Y}^{\mathbf{P}}(E^\vee)[1] \in \operatorname{Coh}(Y)$.

Assume that $\mu_{\min}(E) \geq d'/r'$. If $\operatorname{Hom}(E, \mathbf{P}_{|X\times\{y\}}) \neq 0$, then F_s/F_{s-1} in (3.25) is a semistable sheaf with $\mu(F_s/F_{s-1}) = d'/r'$ and we have a surjective homomorphism $F_s/F_{s-1} \to \mathbf{P}_{|X\times\{y\}}$. Assume that F_s/F_{s-1} is S-equivalent to $\bigoplus_{i=1}^k E_i$, where the E_i's are stable 1-dimensional sheaves with $\mu(E_i) = d'/r'$. Then $\mathbf{P}_{|X\times\{y\}} \in \{E_1, \ldots, E_k\}$. Therefore $H^1(\Phi_{X\to Y}^{\mathbf{P}}(E^\vee))$ is torsion-free (see [24, Lemma 2.6]). Since semistability is preserved under any Fourier–Mukai transform on an elliptic curve, $H^1(\Phi_{X\to Y}^{\mathbf{P}}(E^\vee))$ is f-semistable (see Remark 3.1). □

PROPOSITION A.3

Let $\mathbf{e}' \in K(Y)$ be the class of an ideal sheaf $I_Z \in \operatorname{Hilb}_Y^b$. Assume that $r \geq 2$. Then there is a (contravariant) Fourier–Mukai transform $\mathbf{D}(X) \to \mathbf{D}(Y)$ which induces an isomorphism

$$\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}} \setminus \mathcal{Z} \to \mathcal{M}_{H'_f}(\mathbf{e}')^{\mathrm{ss}} \setminus \mathcal{Z}',$$

where $2b = -\chi(\mathbf{e}, \mathbf{e}) + \chi(\mathcal{O}_X)$, $\mathcal{Z} \subset \mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}$, and $\mathcal{Z}' \subset \mathcal{M}_{H'_f}(\mathbf{e}')^{\mathrm{ss}}$ are closed substacks with

$$\dim \mathcal{Z}, \dim \mathcal{Z}' \leq \dim \mathcal{M}_{H'_f}(\mathbf{e})^{\mathrm{ss}} - 2.$$

Proof
Let (p,q) be a pair of integers such that $dp-rq=1$ and $0<p<r$. We first assume that $p\le r/2$. If two integers x,y satisfy $dx-ry=m$ $(m=1,2)$, then $(x,y)=m(p,q)+n(r,d)$, where $n\in\mathbb{Z}$. If $0<x\le r$, then we get $n=0$; that is, $(x,y)=m(p,q)$. If a pair (x,y) of integers satisfy $x>0$ and $y/x<d/r$, then we have $y/x\le q/p$ or $x\ge r+p>r$ by [22, Lemma 5.1].

For the filtration (3.25), we have $0<r_i\le r$ and $d_i/r_i<d/r$. Hence we have

$$\frac{q}{p}\ge\frac{d_1}{r_1}>\frac{d_2}{r_2}>\cdots>\frac{d_s}{r_s}. \tag{A.4}$$

Assume that

$$\operatorname{codim}\mathcal{F}(\widetilde{\mathbf{e}},\mathbf{f}_1,\dots,\mathbf{f}_s)=\sum_i l_i\big((r_id-rd_i)-1\big)\le 1. \tag{A.5}$$

Then $r_id-rd_i=1,2$ and $0<r_i\le r$ imply that $(r_i,d_i)=(p,q),(2p,2q)$. We set

$$\mathcal{Z}:=\big\{E\in\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}\bigm|\mu_{\min}(E)<q/p\big\}.$$

Then $\dim\mathcal{Z}\le\dim\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}-2$ by Proposition A.1. For $(r',d')=(p,q)$, we consider the Fourier–Mukai transform (A.3). Then

$$\operatorname{rk}\Phi_{X\to Y}^{\mathbf{P}[1]}(\mathbf{e}^\vee)=-\chi(\mathbf{e},\mathbf{P}_{|X\times\{y\}})=pd-rq=1. \tag{A.6}$$

Hence we may assume that $\Phi_{X\to Y}^{\mathbf{P}[1]}(\mathbf{e}^\vee)=\mathbf{e}'$. Then

$$r=\chi(\mathbf{e}^\vee,k_x)=\chi\big(\mathbf{e}',\mathbf{P}_{|\{x\}\times Y}[1]\big).$$

Hence we have $\tau(\mathbf{P}_{|\{x\}\times Y})=(0,r'f,-r)$. We set

$$\mathcal{Z}':=\big\{E\in\mathcal{M}_{H_f}(\mathbf{e}')^{\mathrm{ss}}\bigm|\mu_{\min}(E)<-r/p\big\}.$$

Since $r/p\ge 2$, we have $\dim\mathcal{Z}'\le\dim\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}-2$ by Proposition A.1. Hence $\Phi_{X\to Y}^{\mathbf{P}[1]}\circ D_X$ induces an isomorphism

$$\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}\setminus\mathcal{Z}\to\mathcal{M}_{H_f'}(\mathbf{e}')^{\mathrm{ss}}\setminus\mathcal{Z}'$$

by Lemma A.2. We next assume that $p>r/2$. In this case, we have $r\ge 3$. Then the closed substack W of $\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}$ consisting of nonlocally free sheaves is of codimension $r-1\ge 2$. For the topological invariant $\mathbf{e}^\vee$, $(x,y)=(r-p,q-d)$ satisfies $0<r-p<r/2$ and $(-d)x-ry=1$. Applying the first part, we have a similar isomorphism. Thus $\Phi_{X\to Y}^{\mathbf{P}^\vee[2]}$ induces an isomorphism

$$\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}\setminus\mathcal{Z}\to\mathcal{M}_{H_f'}(\mathbf{e}')^{\mathrm{ss}}\setminus\mathcal{Z}',$$

where $(r',d')=(r-p,d-q)$, $\mathcal{Z}$ consists of E which is nonlocally free or $\mu_{\min}(E^\vee)<-(d-q)/(r-p)=-d'/r'$, and $\mathcal{Z}'$ consists of F with $\mu_{\min}(F)\le -r/(r-p)=-r/r'$. Indeed, we note that $\mathbf{Q}:=\mathbf{P}^\vee[1]$ is the universal family on $M_H(0,r'f,-d')\times X$ by Lemma 0.4(2) and the irreducibility of fibers of π. Since $\Phi_{X\to Y}^{\mathbf{P}^\vee[2]}(E)=\Phi_{X\to Y}^{\mathbf{Q}[1]}\circ D_X(E^\vee)$, we get the isomorphism, where we use $\mu_{\min}(F)\ne -r/r'$ for the local freeness of E. $\square$

For $\mathbf{e}\in K(X)_{\rm top}$, we set

$$K(X)_{\mathbf{e}}:=\{\alpha\in K(X)\mid \chi(\alpha,\mathbf{e})=0\}.$$

We have a homomorphism

$$\begin{array}{rccc} \theta_{\mathbf{e}}: & K(X)_{\mathbf{e}} & \to & \operatorname{Pic}(M_{H_f}(\mathbf{e})), \\ & \alpha & \mapsto & \det p_!(\mathcal{E}\otimes p_X^*(\alpha^\vee)), \end{array} \tag{A.7}$$

where $\mathcal{E}$ is a universal family. We note that $\theta_{\mathbf{e}}$ can be defined even if there is no universal family by using a family on a quot-scheme. For the Fourier–Mukai transform Φ in Proposition A.3, we have a commutative diagram

$$\begin{array}{ccc} K(X)_{\mathbf{e}} & \xrightarrow{\Phi} & K(Y)_{\mathbf{e}'} \\ {\scriptstyle\theta_{\mathbf{e}}}\downarrow & & \downarrow{\scriptstyle\theta_{\mathbf{e}'}} \\ \operatorname{Pic}(M_{H_f}(\mathbf{e})) & = & \operatorname{Pic}(M_{H'_f}(\mathbf{e}')) \end{array} \tag{A.8}$$

COROLLARY A.4

Assume that $\dim M_{H_f}(\mathbf{e})\geq 4+q(X)$ *and* $k=\mathbb{C}$. *Then we have an exact sequence*

$$0\longrightarrow \ker\tau\longrightarrow K(X)_{\mathbf{e}}\xrightarrow{\theta_{\mathbf{e}}}\operatorname{Pic}(M_{H_f}(\mathbf{e}))/\operatorname{Pic}(\operatorname{Alb}(M_{H_f}(\mathbf{e})))\longrightarrow 0.$$

Proof

By Proposition A.3 and (A.8), it is sufficient to prove the claim for $\mathbf{e}'$. In this case, we note that $\pi_1(\operatorname{Hilb}_Y^b)\cong\pi_1(Y)$ (see [16, Section 1]), $H^1(\operatorname{Hilb}_Y^b,\mathbb{Z})=H^1(Y,\mathbb{Z})$, and

$$H^2(\operatorname{Hilb}_Y^b,\mathbb{Z})=H^2(Y,\mathbb{Z})\oplus\bigwedge^2 H^1(Y,\mathbb{Z})\oplus\mathbb{Z}\delta,$$

where 2δ is the exceptional divisor of the Hilbert–Chow map. Then it is easy to see that the claim holds (see [21, Section 3.2]). $\square$

REMARK A.1

Assume that $\dim\mathcal{M}_{H_f}(\mathbf{e})^{\rm s}>\dim\operatorname{Pic}^0(X)$. If $r'>r$ and $dr'-rd'=1$, then $\Phi_{X\to Y}^{\mathbf{P}[1]}(E^\vee)$ is not torsion-free for any $E\in\mathcal{M}_{H_f}(\mathbf{e})^{\rm ss}$. Indeed there is a quotient $E\to F$ such that $\tau(F)=(0,r_0f,d_0)$, where $dr_0-rd_0=1$ and $0<r_0\leq r$. Since $d_0/r_0<d'/r'$, we see that $\Phi^{\mathbf{P}[1]}(F^\vee)$ is a torsion subsheaf of $\Phi_{X\to Y}^{\mathbf{P}[1]}(E^\vee)$.

We slightly generalize Proposition A.3 to the case where there is a multiple fiber. Let mf_0 be a multiple fiber of π. Let F be a semistable sheaf on mf_0, and set $\tau(F)=l_i(0,r_if_0,d_i)$. Then

$$\dim\mathcal{M}_H(0,l_ir_if_0,l_id_i)^{\rm ss}\leq[l_im_i/m],\quad m_i:=\gcd(r_i,m)$$

by Remark 1.5. Since $(c_1(E),f_0)=(c_1(E),f)/m=d/m$ for $E\in\mathcal{M}_{H_f}(\mathbf{e})^{\rm ss}$, we have $\chi(E,F)=rd_i-r_id/m$. Let (p,q) be a pair of integers such that $dp-rq=1$ and $0<p<r$. Assume that $mp\leq r$. If $r_id/m-rd_i=1$ $(0<r_i\leq r)$, then $(r_i,d_i)=(mp,q)$. If $r_id/m-rd_i=2$ $(0<r_i\leq r)$ and $m\mid r_i$, then we also have

$(r_i, d_i) = (2mp, 2q)$ by $0 < 2p, r_i/m \leq r$. If $r_i d/m - r d_i \geq 3$, then

$$l_i(r_i d/m - r d_i) - l_i \geq 2l_i \geq 2.$$

If $r_i d/m - r d_i = 2$ and $m_i < m$ (i.e., $m \nmid r_i$), then we have

$$l_i(r_i d/m - r d_i) - l_i m_i/m \geq 3l_i/2.$$

Therefore there is a closed substack $\mathcal{Z}$ of $\mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}}$ such that $\Phi_{X\to Y}^{\mathbf{P}[1]}(E)$ is torsion-free for $E \in \mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}} \setminus \mathcal{Z}$ and $\dim \mathcal{Z} \leq \dim \mathcal{M}_{H_f}(\mathbf{e})^{\mathrm{ss}} - 2$, where $Y = M_H(0, pf, q)$. Then we also see that Proposition A.3 holds if $mp \leq r$ for all multiple fibers mf_0. In particular, for an unnodal Enriques surface, Proposition A.3 holds.

Let X be an unnodal Enriques surface. As we noted in Remark A.1, a relative Fourier–Mukai transform does not preserve stability for any member of $\mathcal{M}_{H_f}(v)^{\mathrm{ss}}$ in general. However, we have the following result.

PROPOSITION A.5

For $Y := M_H(0, r'(2e_1), d')$ $(\gcd(2r', d') = 1)$, let $\Phi_{X\to Y}^{\mathbf{P}} : \mathbf{D}(X) \to \mathbf{D}(Y)$ be the Fourier–Mukai transform by a universal family $\mathbf{P}$. Then for the Mukai vector $v = lu + ne_1 + \delta + a\varrho_X$ in Section 3.3, $\Phi_{X\to Y}^{\mathbf{P}}$ preserves stability for a general member of $\mathcal{M}_{H_f}(v)^{\mathrm{ss}}$.

Proof

By Propositions 3.15 and 3.14, $\mathcal{M}_f(v)_*^{\mathrm{ss}}$ is an open and dense substack of $\mathcal{M}_{H_f}(v)^{\mathrm{ss}}$. We note that $E \in \mathcal{M}_f(v)_*^{\mathrm{ss}}$ is a locally free sheaf by Definition 3.6. Let C be the reduced subscheme of $\pi^{-1}(t)$. Then $E_{|C}$ is a semistable locally free sheaf and $\mathbf{P}_{|X\times\{y\}}$ a stable sheaf on C, where $\pi'(y) = t$. If $\chi(E \otimes \mathbf{P}_{|X\times\{y\}}) = 0$, then Lemma 4.2 implies that $\Psi(E^\vee)[1] = \Phi_{X\to Y}^{\mathbf{P}[1]}(E)$ is a 1-dimensional stable sheaf. If $\chi(E^\vee, \mathbf{P}_{|X\times\{y\}}) < 0$, then $\mathrm{Hom}(E^\vee, \mathbf{P}_{|X\times\{y\}}) = 0$ for all $y \in Y$, which implies that $\Phi_{X\to Y}^{\mathbf{P}[1]}(E)$ is a locally free sheaf. By the proof of Lemma A.2, $\Phi_{X\to Y}^{\mathbf{P}[1]}(E)$ is f-semistable. Applying Proposition 3.14, $\Phi_{X\to Y}^{\mathbf{P}[1]}(E)$ is a stable sheaf for a general E.

Assume that $\chi(E^\vee, \mathbf{P}_{|X\times\{y\}}) > 0$. Since $\mathrm{Ext}^1(E^\vee, \mathbf{P}_{|X\times\{y\}}) = \mathrm{Hom}(E, (\mathbf{P}_{|X\times\{y\}})^\vee[1])^\vee = 0$, $\Phi_{X\to Y}^{\mathbf{P}}(E)$ is a locally free sheaf. By using Propositions 3.15 and 3.14 again, we see that $\Phi_{X\to Y}^{\mathbf{P}}(E)$ is a stable sheaf for a general $E \in \mathcal{M}_{H_f}(v)^{\mathrm{ss}}$. □

In particular, if r is even and $\gcd(r/2, (c_1, e_1)) = 1$, then for $Y = M_H(0, r'(2e_1), d')$ with $d'r/2 - r'(c_1, e_1) = 1$, $\Phi_{X\to Y}^{\mathbf{P}}$ induces a birational map

$$M_{H_f}\Big(r, c_1, \frac{s}{2}\Big) \cdots \to M_{H_f}\Big(2, \zeta, \frac{s'}{2}\Big),$$

where $\zeta \in \mathrm{Pic}(X)$ and $(\zeta^2) - 2s' = (c_1^2) - rs$.

REMARK A.2
If $r = 4$, then Nuer [14, Section 6] constructed birational maps of the moduli spaces by using (-1)-reflections.

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Department of Mathematics, Faculty of Science, Kobe University, Kobe, Japan; yoshioka@math.kobe-u.ac.jp