Relative trace formulas for unitary hyperbolic spaces

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To Professor Takayuki Oda

Abstract We develop relative trace formulas of unitary hyperbolic spaces for split rank 1 unitary groups over totally real number fields.

Contents

1. Introduction

Let E/F be a CM extension of a totally real number field F. We consider the unitary group $G=U(\mathbf{h})$ of a nondegenerate Hermitian space $(V,\mathbf{h})$ over E of dimension $m\geq 4$ such that the signature of $\mathbf{h}$ is $(1-,(m-1)+)$ at all the Archimedean places of F. Let $\ell\in V$ be a vector such that $\mathbf{h}[\ell]=+1$, and let H be the stabilizer of $E\ell$ in G. An automorphic cuspidal representation π of $G(\mathbb{A}_F)$ is said to be $H(\mathbb{A}_F)$-*distinguished* if π contains a cusp form φ on $G(\mathbb{A}_F)$ whose H-period integral $\mathcal{P}_H(\varphi)=\int_{H(F)\backslash H(\mathbb{A}_F)}\varphi(h)\,\mathrm{d}h$ is not zero, where $\mathrm{d}h$ is the Tamagawa measure on $H(\mathbb{A}_F)$.

There are several reasons to believe the existence of a functorial transfer from a class of $\mathrm{GL}_2(\mathbb{A}_F)$-distinguished cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_E)$ to the set of $H(\mathbb{A}_F)$-distinguished cuspidal representations of $G(\mathbb{A}_F)$ (see [14], [5]). To realize such a transfer between the sets of distinguished automorphic representations on different groups, one uses a comparison of relative trace formulas

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whose efficiency has been confirmed in many cases since its cultivation by Jacquet (see [10], [12], [13]; see also Flicker [6]). In this article, we fully develop a relative trace formula for the symmetric space $H\backslash G$ to serve as one of the main tools for constructing the transfer mentioned above, including all the associated technicalities necessary to carry out the comparison. Let us review its deduction in a slightly simplified setting.

We choose a maximal $\mathfrak{o}_E$-lattice $\mathcal{L} \subset V$ and fix maximal compact subgroups $\mathcal{U}_v \subset G(F_v)$ by taking the stabilizer of $\mathcal{L}$ in $G(F_v)$ at all finite places v of F. We start with a decomposable function

$$\Phi(g) = \prod_v \Phi_v(g_v), \quad g \in G(\mathbb{A}_F),$$

on the adèle group $G(\mathbb{A}_F)$, whose non-Archimedean factors $\Phi_v(g_v)$ are left $H(F_v)$-invariant smooth functions on $G(F_v)$ such that Φ_v is of compact support modulo $H(F_v)$ and coincides with the characteristic function of $H(F_v)\mathcal{U}_v$ for almost all v's. Using the local harmonic analysis on $H(F_v)\backslash G(F_v)$, we explicitly construct a wide and flexible enough class of Archimedean factors $\Phi_v(g_v)$ with no support condition but with some weak gauge-estimate on $G(F_v)$ instead. This weaker support condition on Φ is the main feature of our version of the relative trace formula. For such Φ's, we show that the Poincaré series

$$\mathbf{\Phi}(g) = \sum_{\gamma \in H(F)\backslash G(F)} \Phi(\gamma g), \quad g \in G(\mathbb{A}_F),$$

converges absolutely and locally uniformly. To attain this, we construct a majorant of $\mathbf{\Phi}(g)$, adopting a method developed in [29] and [19] to the adelic setting. The relative trace formula is obtained by computing the $H(\mathbb{A}_F)$-period integral $\mathcal{P}_H(\mathbf{\Phi})$ in two ways, leading to its two different expressions, the spectral side and the geometric side. The only serious issue to be settled here is the absolute convergence of the expressions. On the spectral side, the problem is already treated by Lapid [16] in a wide setting. As with Lapid, in our case we also need an estimation of the unitary Eisenstein series on a Siegel domain of $G(\mathbb{A}_F)$ uniform in the spectral parameter. Such an estimate, stated in Lemma 6.5, becomes available by a modification of the proof in [7]; for our purpose, we need to attain the best possible exponent of the norm $\|g\|$ in the majorant. From the gauge-estimate of $\mathbf{\Phi}$, we have $\mathbf{\Phi} \in L^{2+\epsilon}(G(F)\backslash G(\mathbb{A}_F))$ for some $\epsilon > 0$ (see Lemma 6.3), which yields the spectral expansion of $\mathbf{\Phi}$ by automorphic forms. The resolution involves the $H(\mathbb{A}_F)$-period of the unitary Eisensteins; in Section 5.9, we compute them very explicitly by employing Shintani's method. The nonexistence of $H(\mathbb{A}_F)$-distinguished residual forms other than constants is also proved. Invoking a weak version of Weyl's law on the locally symmetric manifold of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ (see [2]) and the uniform estimate of Eisenstein series mentioned above as well as a similar uniform estimate for cusp forms, we prove that the spectral resolution of $\mathbf{\Phi}(g)$ converges absolutely and locally uniformly on $G(\mathbb{A}_F)$ (see Proposition 6.7). The argument also shows that $\mathcal{P}_H(\mathbf{\Phi})$ is computable by taking the $H(\mathbb{A}_F)$-period integral of each component in the spectral resolution. The process provides us with the spectral side (see Proposition 6.8). In Section 7.1, we classify double cosets $H(F)\gamma H(F) \subset G(F)$ in terms of the number $\mathrm{N}^\flat(\gamma) = \mathrm{N}_{E/F}\mathbf{h}(\gamma^{-1}\ell, \ell)$. The geometric side is obtained by the familiar unfolding procedure (see [11]); our gauge-estimate of $\boldsymbol{\Phi}$ (see Lemma 6.1) ensures the absolute convergence of the geometric side, which is necessary to apply Fubini's theorem. As a consequence, the geometric side is expressed as a linear combination of $\Phi(e)$ and the sum of adelic orbital integrals $J(\gamma, \Phi) = \int_{H_\gamma(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \Phi(\gamma h)\, \mathrm{d}O_\gamma(h)$ over double cosets $H(F)\gamma H(F)$ different from $H(F)$. The upshot of Sections 6 and 7 is the relative trace formula enunciated in Theorem 7.4. The integral $J(\gamma, \Phi)$ is an Euler product of similar local orbital integrals $J_v(\gamma, \Phi_v)$ over all places v of F. In Section 8, we study the germ expansion of non-Archimedean local orbital integrals $J_v(\gamma, \Phi_v)$ for the regular coset $H(F_v)\gamma H(F_v)$ near a singular coset; this kind of theory is crucial to realize the transfer of orbital integrals in the comparison of trace formulas. In Section 9, instead of developing a similar germ theory, we compute all the Archimedean orbital integrals $J_v(\gamma, \Phi_v)$ directly in terms of Gaussian hypergeometric series.

The first three sections after the Introduction are preliminaries. In Section 2, we introduce basic notation and symbols which are valid throughout the article. In Section 3, we recall the harmonic analysis of complex hyperbolic spaces following [4] and [25]. In Section 4, we prove the necessary property of the Poisson integrals and the normalized local intertwining operators. The holomorphy of these operators in the closure of positive chamber plays a pivotal role both in the computation of Eisenstein periods (see Section 5.9) and in the uniform estimate of unitary Eisenstein series (see Lemma 6.5). This article has two companion works [27] and [28]. In [27], we proved the fundamental lemma. In [28], we will complete the comparison of relative trace formulas.

Basic notation. Let $\mathbb{N}$ denote the set of positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a compact interval $I \subset \mathbb{R}$ and $\delta > 0$, set $\mathcal{T}_{\delta,I} = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \geq \delta, \mathrm{Re}(z) \in I\}$. For a number field or a non-Archimedean local field F, the maximal order of F is denoted by $\mathfrak{o}_F$. For any totally disconnected topological space X, $\mathcal{S}(X)$ denotes the $\mathbb{C}$-vector space of all those locally constant complex-valued functions on X of compact support. For a set X and its subset Y, the symbol $\mathbb{1}_Y$ stands for the characteristic function of Y on X.

2. Preliminaries

2.1

Let F be a field of characteristic 0, and let $E = F[\sqrt{\theta}]$ be a quadratic étale F-algebra. The Galois conjugate over F of an element $\alpha \in E$ is denoted by $\bar{\alpha}$. Let V be an E-module of E-rank m, and let $\mathbf{h} : V \times V \to F$ be a nondegenerate Hermitian form on V such that $\mathbf{h}(\alpha x, \beta y) = \alpha\bar{\beta}\mathbf{h}(x, y)$ for $x, y \in V$ and $\alpha, \beta \in E$. For $x \in V$, the value $\mathbf{h}(x, x)$ is also denoted by $\mathbf{h}[x]$. As in Section 1, let

$$G = \left\{g \in \mathrm{GL}_E(V) \mid \mathbf{h}(gx, gy) = \mathbf{h}(x, y) \text{ for all } x, y \in V\right\}$$

be the unitary group of $(V, \mathbf{h})$. We assume that a pair of F-isotropic vectors e, e' such that $\mathbf{h}(e, e') = 1$ is given, and we set $V_1 = (Ee + Ee')^{\perp}$. Let G_1 be the unitary group of V_1. Let P be the stabilizer in G of the submodule Ee, and let P_1 be the stabilizer of the vector e. Then P is an F-parabolic subgroup of G, and P is a semidirect product of P_1 and an F-torus whose set of F-points consists of all the elements $[\tau] \in G$ with $\tau \in E^{\times}$, defined by

$$[\tau]e = \tau e, \qquad [\tau]e' = \bar{\tau}^{-1}e', \qquad [\tau]|V_1 = \mathrm{id}.$$

For $g_1 \in G_1(F)$, let $\mathsf{m}[g_1]$ denote the element of $G(F)$ which acts on V_1 by g_1 and on $Ee + Ee'$ by the identity; we define M_1 to be the F-subgroup whose set of F-points is $\{\mathsf{m}[g_1] \mid g_1 \in G_1(F)\}$. For $X \in V_1$ and $b \in F$, define $\mathsf{n}[X; b] \in G(F)$ as

$$\mathsf{n}[X;b]e = e, \qquad \mathsf{n}[X;b]y = y - \mathbf{h}(y, X)e \quad (y \in V_1),$$
$$\mathsf{n}[X;b]e' = e' + X + \left(-2^{-1}\mathbf{h}[X] + \sqrt{\theta}b\right)e.$$

Let N be the unipotent radical of P; then $N(F) = \{\mathsf{n}[X;b] \mid X \in V_1, b \in F\}$ and $P_1 = M_1 N$. If a vector $\ell \in V$ such that

$$\mathbf{h}[\ell] = 1 \qquad \text{and} \qquad \mathbf{h}(\ell, e) = 1$$

is given, we define H to be the stabilizer of the submodule $E\ell \subset V$ and H_0 to be that of the vector ℓ. Then H is a symmetric subgroup of G obtained as the fixator of the inner automorphism $g \mapsto s_\ell g s_\ell^{-1}$, where s_ℓ is the reflection of V such that $s_\ell(\ell) = -\ell$.

2.2. Gauge-forms

Let $t \in F$. We will regard $\mathbf{\Sigma}(t) = \{x \in V - \{0\} \mid \mathbf{h}[x] = t\}$ as an F-variety by identifying it with $\mathbf{\Sigma}_Q(t)$ (see Lemma A.1), where Q denotes the F-quadratic form $\mathbf{h}[x]$ on $V \cong F^{2m}$. Let ω_V be the gauge-form on V defined as $\omega_V = \det(\mathbf{h}(\xi_i, \xi_j)) \times \prod_{j=1}^{m} \frac{\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j}{2\sqrt{\theta}}$ for any E-basis $\{\xi_j\}$ of V, where z_j denotes the E-coordinate functions on V dual to $\{\xi_j\}$. By fixing an F-point $\xi \in \mathbf{\Sigma}(t)$, we have an F-isomorphism $G(\xi)\backslash G \cong \mathbf{\Sigma}(t)$ sending a coset $G(\xi)g$ to the vector $g^{-1}\xi$, where $G(\xi)$ denotes the stabilizer of ξ in G. Let $\phi : V - \{0\} \to F$ be the F-morphism defined by $\phi(x) = \mathbf{h}[x]$. From Lemma A.1, there exists a unique G-invariant gauge-form $\omega_{\mathbf{\Sigma}(t)}$ on $\mathbf{\Sigma}(t)$ such that $\omega_V = \omega_{\mathbf{\Sigma}(t)} \wedge \phi^*(\mathrm{d}t)$, where t is the coordinate of F. We fix an F-rational left-invariant gauge-form ω_G on G once and for all. Then we take the unique gauge-form $\omega_{G(\xi)}$ on $G(\xi)$ so that ω_G, $\omega_{G(\xi)}$, and $\omega_{\mathbf{\Sigma}(t)}$ match together algebraically in the sense of [30, p. 24]. In this way, we fix ω_{P_1} on $P_1 = G(e)$ and ω_{H_0} on $H_0 = G(\ell)$. Let $\omega_{P_1\backslash G}$ and $\omega_{H_0\backslash G}$ be the gauge-forms on $P_1\backslash G \cong \mathbf{\Sigma}(0)$ and $H_0\backslash G \cong \mathbf{\Sigma}(1)$ corresponding to $\omega_{\mathbf{\Sigma}(0)}$ and $\omega_{\mathbf{\Sigma}(1)}$, respectively. Since $P = \{[\tau] \mid \tau \in E^{\times}\}P_1$, there exists a left P-invariant gauge-form ω_P such that $\omega_P = \frac{\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}}{2\sqrt{\theta}\mathrm{N}_{E/F}(\tau)} \wedge \omega_{P_1}$. Let T be the F-torus whose set of F-points is $E^1 = \{\alpha \in E \mid \mathrm{N}_{E/F}(\alpha) = 1\}$. Since $H \cong T \times H_0$, we define an H-invariant gauge-form ω_H on H by taking the wedge product of pullbacks of ω_T and ω_{H_0}, where ω_T is the gauge-form of T such that $\frac{\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}}{2\sqrt{\theta}\mathrm{N}_{E/F}(\alpha)} = \omega_T \wedge \frac{\mathrm{d}t}{t}$ with $t = \mathrm{N}_{E/F}(\alpha)$.

It is not difficult to see that $PH = \{g \in G \mid \mathbf{h}(g^{-1}e, \ell) \neq 0\}$, which shows that PH is a Zariski-open subset of G. Set $H_P = P \cap H$. It turns out that the restriction of the map $[\tau]\mathsf{m}[g_1]\mathsf{n}[X;b] \mapsto (\tau, g_1)$ from P to $E^\times \times G_1$ induces an isomorphism $H_P \cong E^1 \times G_1$. Since $PH \cong P \times H_P\backslash H$, we have an H-invariant gauge-form $\omega_{H_P\backslash H}$ on $H_P\backslash H$ such that $\omega_G|PH$ corresponds to $\omega_P \wedge \omega_{H_P\backslash H}$ on $P \times (H_P\backslash H)$. We endow H_P with the left-invariant gauge-form ω_{H_P} such that ω_H, ω_{H_P}, and $\omega_{H_P\backslash H}$ match together algebraically.

2.3. Local Tamagawa measures

Let F be a local field of characteristic 0, and let $|\cdot|_F$ be the normalized valuation of F. Given a gauge-form ω_X on a smooth F-variety X, the usual process (see [30, Section 2.2]) yields a measure $|\omega_X|_F$ on the F-points $X(F)$. For example, the Haar measure $|\mathrm{d}t|_F$ on F is such that $\int_{\mathfrak{o}_F} |\mathrm{d}t|_F = 1$ if F is non-Archimedean, and $\int_0^1 |\mathrm{d}t|_F = 1$ if $F = \mathbb{R}$. On the spaces such as $G(F)$, $H(F)$, $\mathbf{\Sigma}(t)$, $P_1(F)$, and $H_P(F)\backslash H(F)$, we put the (left-)invariant measures obtained from the gauge-forms fixed in Section 2.2.

Let $\mathcal{D}(P(F)\backslash G(F))$ be the space of all those continuous functions $\phi : G(F) \to \mathbb{C}$ such that $\phi([\tau]p_1 g) = |\mathrm{N}_{E/F}(\tau)|_F^{m-1}\phi(g)$ for all $\tau \in E^\times$, $p_1 \in P_1(F)$. Then we have a (continuous) linear functional $\mu_{P\backslash G} : \mathcal{D}(P(F)\backslash G(F)) \to \mathbb{C}$ satisfying the relation

$$\oint_{P(F)\backslash G(F)} \Big(\int_{P(F)} f(pg)|\omega_P|_F(p) \Big) \, \mathrm{d}\mu_{P\backslash G}(g) = \int_{G(F)} f(g)|\omega_G|_F, \quad f \in C_{\mathrm{c}}\big(G(F)\big), \tag{2.1}$$

where $\oint_{P(F)\backslash G(F)} \phi(g)\, \mathrm{d}\mu_{P\backslash G}(g)$ denotes the value $\mu_{P\backslash G}(\phi)$ for $\phi \in \mathcal{D}(P(F)\backslash G(F))$. Note that the modulus character of $P(F)$ is given by $\delta_{P(F)}([\tau]p_1) = |\tau\bar{\tau}|_F^{m-1}$ ($\tau \in E^\times$, $p_1 \in P_1(F)$).

LEMMA 2.1

For any $\phi \in \mathcal{D}(P(F)\backslash G(F))$, we have

$$\mu_{P\backslash G}(\phi) = \int_{H_P(F)\backslash H(F)} \phi |\omega_{H_P\backslash H}|_F.$$

Proof

This follows from $\omega_G|PH = \omega_P \wedge \omega_{H_P\backslash H}$ and (2.1). □

3. Local harmonic analysis at Archimedean places

We let $F = \mathbb{R}$ and $E = F[\sqrt{\theta}] = \mathbb{C}$, and we identify an $\mathbb{R}$-algebraic group with its $\mathbb{R}$-points; thus $G = G(\mathbb{R})$, $H = H(\mathbb{R})$, and so on. We assume that $\mathrm{sgn}(\mathbf{h}) = (1-, (m-1)+)$, and we set $\ell^- = \ell - e$. Let $\mathcal{U}$ be the stabilizer of $\mathbb{C}\ell^-$. Since $\mathbf{h}[\ell^-] = -1$ and $\mathbf{h}(\ell, \ell^-) = 0$, $\mathcal{U}$ is a maximal compact subgroup of G and $\mathcal{U}_H = \mathcal{U} \cap H$ is a maximal compact subgroup of H. Fix an orthonormal basis $\{\ell_j\}_{j=1}^{m-1}$

of $(\ell^-)^\perp$ such that $\ell_{m-1}=\ell$, and set $\ell_m=\ell^-$; by means of the basis $\{\ell_j\}_{j=1}^m$ of V, we identify G with $U(m-1,1)$. The minimal majorant of $\mathbf{h}$ is given by $\|Z\|_{\mathcal{U}}=\{\sum_{j=1}^m|\mathbf{h}(Z,\ell_j)|^2\}^{1/2}$ $(Z\in V)$. Moreover, $H\cong U(m-2,1)\times U(1)$ and $\mathcal{U}\cong\{\mathrm{diag}(k_1,k_2)\mid(k_1,k_2)\in U(m-1)\times U(1)\}$. For a dominant weight $l=\{l(j)\}_{1\le j\le m-1}$ $(l(j)\in\mathbb{Z},\ l(j)\ge l(j+1))$ and $c\in\mathbb{Z}$, let $(\tau_{(l;c)},W_{(l;c)})$ be a unitary $\mathcal{U}=U(m-1)\times U(1)$-module obtained as the tensor product of an irreducible representation of $U(m-1)$ with highest weight l and the character $z\mapsto z^c$ of $U(1)$. First we recall some integration formulas, which are more or less well known; we need to determine the normalizing constants therein.

LEMMA 3.1

We have

$$\mu_{P\backslash G}(\phi)=|\sqrt{\theta}|^{1-m}\pi^{m-1}\Gamma(m-1)^{-1}\int_{\mathcal{U}}\phi(k)\,\mathrm{d}k,\quad \phi\in\mathcal{D}(P\backslash G),$$

where $\mathrm{d}k$ *is the Haar measure on* $\mathcal{U}$ *with total volume* 1.

Proof

There exists a constant $A>0$ such that

$$\mu_{P\backslash G}(\phi)=A\int_{\mathcal{U}}\phi(k)\,\mathrm{d}k \tag{3.1}$$

for all $\phi\in\mathcal{D}(P\backslash G)$. Fix $\epsilon>0$, and set

$$\phi(g)=\int_{\mathbb{C}^\times}\exp\bigl(-\epsilon\|g^{-1}[\tau]^{-1}e\|_{\mathcal{U}}^2\bigr)|\tau|_{\mathbb{C}}^{-(m-1)}\Bigl|\frac{\mathrm{d}\tau\wedge\mathrm{d}\bar\tau}{2\sqrt{\theta}\tau\bar\tau}\Bigr|_{\mathbb{R}},\quad g\in G.$$

By the polar coordinates $\tau=re^{i\varphi}$, we have $t=\mathrm{N}_{E/F}(\tau)=r^2$ and $\frac{\mathrm{d}\tau\wedge\bar\tau}{2\sqrt{\theta}\tau\bar\tau}=\mathrm{d}^\times t\wedge\frac{\mathrm{d}\varphi}{2i\sqrt{\theta}}$. Since $\|Z\|_{\mathcal{U}}$ is $\mathcal{U}$-invariant and $\|e\|_{\mathcal{U}}^2=2$, it is easy to see that $\int_{\mathcal{U}}\phi(k)\,\mathrm{d}k=|\sqrt{\theta}|^{-1}2^{1-m}\pi\epsilon^{1-m}\Gamma(m-1)$. On the other hand, if we set $\varphi(t)=\int_{\mathbf{\Sigma}(t)}\exp(-\epsilon\|Z\|^2)|\omega_{\mathbf{\Sigma}(t)}|_{\mathbb{R}}$ for $t\in\mathbb{R}$, then

$$\begin{aligned}\int_{\mathbb{R}}\varphi(t)e^{2\pi it\tau}\,\mathrm{d}t&=\int_V\exp\bigl(-\epsilon\|Z\|^2\bigr)e^{2\pi i\tau\mathbf{h}[Z]}|\omega_V|_{\mathbb{R}}\\&=|\sqrt{\theta}|^{-m}\frac{\pi^m}{(\epsilon-2\pi i\tau)^{m-1}(\epsilon+2\pi i\tau)}.\end{aligned}$$

By the Fourier inversion, $\varphi(0)$ equals

$$\begin{aligned}\int_{\mathbb{R}}\frac{|\sqrt{\theta}|^{-m}\pi^m}{(\epsilon-2\pi i\tau)^{m-1}(\epsilon+2\pi i\tau)}\,\mathrm{d}\tau&=2\pi i\,\mathrm{Res}_{\tau=-\frac{\epsilon}{2\pi i}}\frac{|\sqrt{\theta}|^{-m}\pi^m}{(\epsilon-2\pi i\tau)^{m-1}(\epsilon+2\pi i\tau)}\\&=2^{1-m}\pi^m|\sqrt{\theta}|^{-m}\epsilon^{1-m}.\end{aligned}$$

Thus

$$\begin{aligned}\mu_{P\backslash G}(\phi)&=\int_{P_1\backslash G}\exp\bigl(-\epsilon\|g^{-1}e\|_{\mathcal{U}}^2\bigr)|\omega_{P_1\backslash G}|_{\mathbb{R}}\\&=\int_{\mathbf{\Sigma}(0)}\exp\bigl(-\epsilon\|Z\|^2\bigr)|\omega_{\mathbf{\Sigma}(0)}|_{\mathbb{R}}=\varphi(0)=2^{1-m}\pi^m|\sqrt{\theta}|^{-m}\epsilon^{1-m}.\end{aligned}$$

From the identity (3.1), we have $A=\pi^{m-1}\Gamma(m-1)^{-1}|\sqrt{\theta}|^{1-m}$. ☐

Let $A=\{a^{(t)} \mid t\in\mathbb{R}\}$ be a 1-parameter subgroup of G defined as

$$a^{(t)}\ell=(\cosh t)\ell+(\sinh t)\ell^-, \qquad a^{(t)}\ell^-=(\cosh t)\ell^-+(\sinh t)\ell,$$

$$a^{(t)}|(\mathbb{C}\ell+\mathbb{C}\ell^-)^\perp=\mathrm{id},$$

and let $Z_{\mathcal{U}\cap H}(A)$ be the centralizer of A in $\mathcal{U}\cap H$.

LEMMA 3.2

The map $H\times[0,+\infty)\times\mathcal{U}\to G$ sending (h,t,k) to $ha^{(t)}k$ induces a homeomorphism from the quotient space $H\times[0,+\infty)\times\mathcal{U}/\sim$ onto G, where $\sim$ is an equivalence relation on $H\times[0,+\infty)\times\mathcal{U}$ such that $(h,t,k)\sim(h_1,t_1,k_1)$ if and only if $t=t_1=0$, $hk=h_1k_1$ or $t=t_1\neq 0$, $(hm,m^{-1}k)=(h_1,k_1)$ with some $m\in Z_{\mathcal{U}\cap H}(A)$. In particular, $G-H\mathcal{U}$ is a disjoint union of $Ha^{(t)}\mathcal{U}$ $(t>0)$. We have the integration formula

$$\int_{H\backslash G} f(g)|\omega_{H\backslash G}|_{\mathbb{R}}=C_G\int_0^\infty\int_{\mathcal{U}} f(a^{(t)}k)(\cosh t)^{2m-3}(\sinh t)\,\mathrm{d}t\,\mathrm{d}k,$$

where $C_G=|\sqrt{\theta}|^{1-m}4\pi^{m-1}\Gamma(m-1)^{-1}$.

Proof

This follows from [9, Part II, Theorems 2.4, 2.5] except the value of C_G. By the basis $\{\ell_j\}_{j=1}^m$ fixed at the beginning of Section 3, we write a general point of V as $Z=\sum_{j=1}^m z_j\ell_j$. Set $\xi_{2j-1}=\mathrm{Re}(z_j)$ and $\xi_{2j}=\mathrm{Im}(z_j)$ $(1\le j\le m)$. Then $|\omega_V|_{\mathbb{R}}=(\sqrt{|\theta|})^{-m}\prod_{j=1}^{2m}\mathrm{d}\xi_j$. The $\mathcal{U}$-orbit of ℓ_{m-1} (resp., ℓ_m) coincides with the unit sphere $S^{2m-3}\subset\ell_m^\perp$ (resp., $S^1\ell_m$). If we set $(\eta_j)=k\ell_{m-1}$ and $k\ell_m=e^{\varphi i}\ell_m$ for $k\in\mathcal{U}$, then the relation $Z=rka^{(t)}\ell$ $(r>0,\ k\in\mathcal{U},\ t\in\mathbb{R})$ can be written as

$$\xi_j=r\cosh t\eta_j \quad (1\le j\le 2m-2),$$

$$\xi_{2m-1}=-r\sinh t\cos\varphi, \qquad \xi_{2m}=-r\sinh t\sin\varphi.$$

From these,

$$\mathrm{d}\xi_1\wedge\cdots\wedge\mathrm{d}\xi_{2m}=-r^{2m-1}(\cosh t)^{2m-3}\sinh t\,\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\varphi\wedge\mathrm{d}\eta$$

is obtained, where $\mathrm{d}\eta=\sum_{j=1}^{2m-2}(-1)^j\eta_j\,\mathrm{d}\eta_1\wedge\cdots\wedge\widehat{\mathrm{d}\eta_j}\wedge\cdots\mathrm{d}\eta_{2m-2}$ is the gauge-form on the $(2m-3)$-dimensional Euclidean sphere S^{2m-3}. Hence $\omega_{\mathbf{\Sigma}(1)}=(\sqrt{|\theta|})^{-m}(\cosh t)^{2m-3}\sinh t\,\mathrm{d}t\wedge\mathrm{d}\varphi\wedge\mathrm{d}\eta$. Since $\omega_{H_0\cap\mathcal{U}\backslash\mathcal{U}}=\mathrm{d}\eta$ is the gauge-form of the manifold $H_0\cap\mathcal{U}\backslash\mathcal{U}\cong S^{2m-3}$ and $\mathrm{vol}(S^{2m-3},|\mathrm{d}\eta|_{\mathbb{R}})=2\pi^{m-1}\Gamma(m-1)^{-1}$, we have

$$\begin{aligned}|\omega_{H\backslash G}|_{\mathbb{R}}&=\frac{|\omega_{\mathbf{\Sigma}(1)}|_{\mathbb{R}}}{|2\sqrt{\theta}|_{\mathbb{R}}^{-1}|\mathrm{d}\varphi|_{\mathbb{R}}}\\&=\big(\sqrt{|\theta|}\big)^{1-m}4\pi^{m-1}\Gamma(m-1)^{-1}(\cosh t)^{2m-3}\sinh t|\mathrm{d}t|_{\mathbb{R}}\,\mathrm{d}\dot{k},\end{aligned}$$

where $\mathrm{d}\dot{k}$ is the $\mathcal{U}$-invariant measure on $H_0\cap\mathcal{U}\backslash\mathcal{U}$ with total measure 1. This gives us C_G. □

3.1. Spherical function of the principal series

Fix $d\in\mathbb{N}_0$. We put

$$c_d(s)=\Gamma(2s)\Gamma\Big(\frac{2s+m-1}{2}+d\Big)^{-1}\Gamma\Big(\frac{2s-m+3}{2}-d\Big)^{-1},$$

viewing this as a meromorphic function on $\mathbb{C}$. We write (τ_d, W_d) in place of $(\tau_{(d,0,\dots,0,-d;0)}, W_{(d,0,\dots,-d;0)})$. Then $\dim_{\mathbb{C}} W_d^{\mathcal{U}\cap H}=1$; we fix a unit vector $\vartheta_d\in W_d^{\mathcal{U}\cap H}$ once and for all. Let $C^\infty(H\backslash G;\tau_d)$ denote the space of all the C^∞-functions $f:G\to W_d$ such that

$$f(hgk)=\tau_d(k)^{-1}f(g)\quad \text{for all } (h,g,k)\in H\times G\times\mathcal{U}. \tag{3.2}$$

LEMMA 3.3

For $s\in\mathbb{C}$ outside the poles of $\Gamma(s+\frac{m-1}{2}+d)\Gamma(s-\frac{m-3}{2}-d)$, there exists a unique function $\Phi_d(s)\in C^\infty(H\backslash G;\tau_d)$ such that $\Phi_d(s;e)=\vartheta_d$ satisfying the Casimir eigenequation

$$\mathcal{C}_G\Phi_d(s)=2^{-1}\big\{(2s)^2-(m-1)^2\big\}\Phi_d(s).$$

We have $\Phi_d(s;a^{(t)})=\phi_d(s,t)\vartheta_d$ for all $t\in\mathbb{R}$ with

$$\begin{aligned}\phi_d(s;t)&=(\cosh t)^{-2s-m+1}\\ &\quad\times{}_2F_1\Big(\frac{2s+m-1}{2}+d,\frac{2s-m+3}{2}-d;1;\tanh^2 t\Big).\end{aligned}\tag{3.3}$$

Proof

The existence follows from [9, Part II, Theorem 6.2]. Let $f\in C^\infty(H\backslash G;\tau_d)$. From (3.2), there exists a C^∞-function $\phi(t)$ in $t>0$ such that $f(a^{(t)})=\phi(t)\vartheta_d$. Then from [24, Proposition 7.1], the Casimir eigenequation $\mathcal{C}_G f=2^{-1}\{(2s)^2-(m-1)^2\}f$ yields

$$\begin{aligned}&\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}+\Big(\frac{1}{\tanh t}+(2m-3)\tanh t\Big)\frac{\mathrm{d}\phi}{\mathrm{d}t}+\frac{4d(d+m-2)}{\cosh^2 t}\phi\\ &\quad=\big\{(2s)^2-(m-1)^2\big\}\phi.\end{aligned}\tag{3.4}$$

By setting $w=(\cosh t)^{\nu+m-1}\phi(t)$, $z=\tanh^2 t$, this is transformed to the Gaussian hypergeometric equation $z(1-z)w''+\{c-(a+b+1)z\}w'-abz=0$ $(0<z<1)$ with

$$(a,b,c)=\Big(\frac{2s+m-1}{2}+d,\frac{2s-m+3}{2}-d,1\Big),$$

which admits the unique smooth solution on $|z|<1$ such that $w(0)=1$. □

3.1.1

For $s\in\mathbb{C}$ outside the pole divisor of $\{sc_d(s)\}^{-1}$, let $\Psi_d(s)$ be the unique W_d-valued smooth function on $G-H\mathcal{U}$ having the equivariance

$$\Psi_d(s;hgk)=\tau_d(k)^{-1}\Psi_d(s;g)\quad\text{for all } (h,k)\in H\times\mathcal{U}, \tag{3.5}$$

whose radial part is given by $\Psi_d(s;a^{(t)})=\psi_d(s;t)\vartheta_d$ $(t>0)$ with

$$\begin{aligned}\psi_d(s;t)&=\frac{-1}{2C_G}\frac{1}{sc_d(s)}(\cosh t)^{-2s-m+1}\\ &\quad\times{}_2F_1\Big(\frac{2s+m-1}{2}+d,\frac{2s-m+3}{2}-d;2s+1;\frac{1}{\cosh^2 t}\Big).\end{aligned}\tag{3.6}$$

By the last formula on page 47 of [17], we have the relation

$$\Phi_d(s;g)=-2C_G c_d(s)c_d(-s)s\big\{\Psi_d(s;g)-\Psi_d(-s;g)\big\},\quad g\in G-H\mathcal{U}.\tag{3.7}$$

LEMMA 3.4

There exists $N>0$ such that, for any compact interval $I\subset(-1,+\infty)$ and $\delta>0$,

$$\big\|\Psi_d(s;a_v^{(t)})\big\|\le\big(1+\big|\mathrm{Im}(s)\big|\big)^N(\cosh t)^{-2\,\mathrm{Re}(s)-m+1},\quad s\in\mathcal{T}_{I,\delta},t\ge\delta.$$

Proof

Set $a_s=s+\frac{m-1}{2}+d$, and $b_s=s-\frac{m-3}{2}-d$. From [17, p. 54], we have the integral representation

$$\psi_d(s;t)=-C_G^{-1}(\cosh t)^{-(2s+m-1)}\int_0^1 x^{b_s-1}\mathcal{F}_s(t,x)\,\mathrm{d}x,\quad \mathrm{Re}(s)>\frac{m-3}{2}+d,$$

with $\mathcal{F}_s(t,x)=(1-x)^{a_s-1}\{1-x(\cosh t)^{-2}\}^{-a_s}$. Let $n\in\mathbb{N}$ be such that $n>\frac{m-3}{2}+d$. We argue as in [25, Lemma 9] to obtain

$$\begin{aligned}&\int_0^1 x^{b_s-1}\mathcal{F}_s(t,x)\,\mathrm{d}t\\ &\quad=\sum_{k=1}^{n}\Big\{\prod_{j=1}^{k}\frac{1}{b_s+j-1}\Big\}\frac{\mathcal{F}_s^{(k-1)}(t,2^{-1})}{2^{b_s+k-1}}\\ &\qquad+\Big\{\prod_{j=1}^{n}\frac{1}{b_s+j-1}\Big\}\int_0^{1/2}x^{b_s+n-1}\mathcal{F}_s^{(n)}(t,x)\,\mathrm{d}x+\int_{1/2}^1 x^{b_s-1}\mathcal{F}_s(t,x)\,\mathrm{d}x,\end{aligned}\tag{3.8}$$

where $\mathcal{F}_s^{(j)}(t,x)=\frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathcal{F}_s(t,x)$. Since $|\mathcal{F}_s(t,x)|\ll(1-x)^{\mathrm{Re}(a_s)-1}$ for $(s,x)\in\mathcal{T}_{I,\delta\times}\times[1/2,1]$ uniformly in $t\ge\delta$, the third term of (3.8) is absolutely convergent for $\mathrm{Re}(s)>-\frac{m-1}{2}-d$ and is bounded by a constant uniformly in $s\in\mathcal{T}_{I,\delta}$ and $t\ge\delta$. From

$$\begin{aligned}\mathcal{F}_s^{(k)}(t,x)&=\sum_{j=0}^{k}\binom{k}{j}\Big\{\prod_{\alpha=0}^{j}(a_s-1+\alpha)\Big\}\Big\{\prod_{\beta=0}^{k-j-1}(a_s+\beta)\Big\}\\ &\quad\times(\cosh^2 t)^{j-k}(1-x)^{a_s-j-1}\Big(1-\frac{x}{\cosh^2 t}\Big)^{a_s+j+k},\end{aligned}$$

we have $|\mathcal{F}_s^{(j)}(t,x)|\ll(1+|\mathrm{Im}(s)|)^j$ for $(s,x)\in\mathcal{T}_{I,\delta}\times[0,1/2]$ uniformly in $t\ge\delta$, from which the second term of (3.8) is seen to be absolutely convergent for $\mathrm{Re}(s)>-n+\frac{m-3}{2}+d$ and to be bounded by $O((1+|\mathrm{Im}(s)|)^n)$ on $s\in\mathcal{T}_{I,\delta}$ uniformly in $t\ge\delta$. The first term of (3.8) is evidently holomorphic on $\mathcal{T}_{I,\delta}$ and is bounded uniformly on $s\in\mathcal{F}_{I,\delta}$, $t\ge\delta$. □

It turns out that $t \mapsto \Psi_d(s; a^{(t)})$ is the unique solution to (3.4) on $t > 0$ satisfying the asymptotic conditions

$$\Psi_d(s; a^{(t)}) = 2C_G{}^{-1} \log t + O(1) \quad (t \to +0), \tag{3.9}$$

$$\Psi_d(s; a^{(t)}) = O(e^{-2\,\mathrm{Re}(s) - m + 1}) \quad (t \to +\infty). \tag{3.10}$$

In particular, the function $\Psi_d(s; g)$ in g has logarithmic singularities along $H\mathcal{U}$.

LEMMA 3.5

Let $\phi \in C^\infty(H\backslash G, \tau_d)$ be a function with the majorization $\sum_{j=0}^{2} \|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\phi(a^{(t)})\| \ll e^{2\delta t}$ for $t \in \mathbb{R}_+$ with some constant $\delta > 0$. If $\mathrm{Re}(s) > \frac{m-1}{2} + \delta$, we have the formula

$$\int_{H\backslash G} \big(\Psi_d^{(s)}(g) \mid \big[\mathcal{C}_G - 2^{-1}\big\{(2\bar{s})^2 + (m-1)^2\big\}\big]\phi(g)\big) |\omega_{H\backslash G}|_{\mathbb{R}} = 2\big(\vartheta_d|\phi(e)\big),$$

whose left-hand side converges absolutely.

Proof

We argue exactly in the same way as [20, Proposition 23] by using (3.9), (3.10), and Lemma 3.2 to get the conclusion. □

3.1.2

Let $\mathcal{A}$ be the space of all those even entire functions $\alpha(s)$ such that, for any $c_1 < c_2$ and for any $N > 0$, the estimate $|\alpha(\sigma + it)| \ll (1 + |t|)^{-N}$ $(t \in \mathbb{R})$ holds uniformly in $\sigma \in [c_1, c_2]$. For $\alpha \in \mathcal{A}$, we introduce the α-smoothing of $\Psi_d(s)$ by the contour integral

$$\hat{\Psi}_d(\alpha; g) = \frac{1}{2\pi i} \int_{(\sigma)} \Psi_d(s; g)\alpha(s) s \,\mathrm{d}s, \quad g \in G - H\mathcal{U}, \tag{3.11}$$

with $\sigma > \frac{m-3}{2} + d$.

LEMMA 3.6

On any compact subset U of $G - H\mathcal{U}$, the integral (3.11) converges uniformly and absolutely, defining a C^∞-function on $G - H\mathcal{U}$ which is locally square-integrable on G; it has a unique C^∞-extension to the whole group G.

Proof

Fix $\delta > 0$ and a compact interval $I \subset (\frac{m-3}{2} + d, +\infty)$. From Lemma 3.4, there exists a constant $N \in \mathbb{N}$ such that $\|\Psi_d(s; g)\| \ll (1 + |\mathrm{Im}(s)|)^N$ for $g \in U$, $s \in \mathcal{T}_{I,\delta}$. Since $|\alpha(s)| \ll (1 + |\mathrm{Im}(s)|)^{-N-3}$ $(s \in \mathcal{T}_{I,\delta})$, the integral (3.11) is absolutely convergent uniformly in $g \in U$. In particular, $g \mapsto \hat{\Psi}_d(\alpha; g)$ is continuous on $G - H\mathcal{U}$. From the first part of Lemma 3.2, for any relatively compact open set $U \subset G$, there exists $U_0 \subset H$ such that $U \subset U_0\{a^{(t)} \mid t \geq 0\}\mathcal{U}$. By Lemma 3.4, we have

$$\begin{aligned}\int_{U_0}|\omega_H|_{\mathbb{R}}\int_{\mathcal{U}}\mathrm{d}k\int_0^\infty\big\|\hat{\Psi}_d(\alpha;ha^{(t)}k)\big\|(\cosh t)^{2m-3}\sinh t\,\mathrm{d}t\\ \ll\Big\{\frac{1}{2\pi}\int_{(\sigma)}\big(1+\big|\mathrm{Im}(s)\big|\big)^N\big|\alpha(s)\big||s\,\mathrm{d}s|\Big\}\\ \times\Big\{\int_0^\infty(\cosh t)^{-(2\sigma+m-1)}(\cosh t)^{2m-3}\sinh t\,\mathrm{d}t\Big\},\end{aligned}$$

whose majorant is convergent. Thus $\hat{\Psi}_d(\alpha;g)$ is integrable on U; since $\mathrm{vol}(U)<\infty$, it becomes square-integrable on U also. Therefore, the function $\hat{\Psi}_d(\alpha;g)$ defines a W_d-valued distribution on the Riemannian manifold $G/\mathcal{U}$. Let $f:G\to W_d$ be a smooth compactly supported function such that $f(gk)=\tau_d(k)^{-1}f(g)$ for all $k\in\mathcal{U}$. Applying Lemma 3.5 to the H-invariant function $\phi(g)=\int_H f(hg)|\omega_H|_{\mathbb{R}}$, we have

$$\begin{aligned}\int_G\big(\hat{\Psi}_d(\alpha;g)|\mathcal{C}_G f(g)\big)|\omega_G|_{\mathbb{R}}=\int_G\big(\Psi_d(\alpha_1;g)|f(g)\big)|\omega_G|_{\mathbb{R}}\\ +\Big\{\frac{1}{2\pi i}\int_{(\sigma)}\alpha(s)s\,\mathrm{d}s\Big\}\Big\{\int_H\big(\vartheta_d|f(h)\big)|\omega_H|_{\mathbb{R}}\Big\},\end{aligned}$$

where $\alpha_n(s)=2^{-n}\{(2s)^2-(m-1)^2\}^n\alpha(s)$ for $n\in\mathbb{N}$. By shifting the contour (σ) to $(-\sigma)$ and then by using the relation $\alpha(s)=\alpha(-s)$, we easily see that $\int_{(\sigma)}\alpha(s)s\,\mathrm{d}s=0$. Thus we have the distributional differential equation $\Delta_d^n\hat{\Psi}_d(\alpha)=\hat{\Psi}_d(\alpha_n)$ on the manifold $G/\mathcal{U}$, where Δ_d is the elliptic differential operator induced from $\mathcal{C}_G$ on the distributional sections of the C^∞-vector bundle $G\times_{\mathcal{U},\tau_d}W_d\to G/\mathcal{U}$. From the argument above, we have $\Delta_d^n\hat{\Psi}_d(\alpha)\in L^2(U)$ $(\forall n\in\mathbb{N})$ for any open relatively compact $\mathcal{U}$-invariant set $U\subset G$. By a form of Sobolev's lemma, we conclude that $\hat{\Psi}_d(\alpha)$ is represented by a C^∞-section of the bundle $G\times_{\mathcal{U},\tau_d}W_d\to G/\mathcal{U}$. □

For $d,j\in\mathbb{N}_0$, let $f_{d,j}\in C^\infty(H\backslash G,\tau_d)$ be the unique function determined by

$$\begin{aligned}f_{d,j}(a^{(t)})=(\cosh^2 t)^{-(m-2+d-j)}\\ \times{}_2F_1\Big(m-2+2d-j,-j;m-2+2d-2j;\frac{1}{\cosh^2 t}\Big)\vartheta_d.\end{aligned}$$

We remark that this is a polynomial of $(\cosh^2 t)^{-1}$, which for $j=0$ is simply $f_{d,0}(a^{(t)})=(\cosh^2 t)^{-(m-2+d)}\vartheta_d$.

LEMMA 3.7

For any $t>0$, we have

$$\begin{aligned}\hat{\Psi}_d(\alpha,a^{(t)})=\frac{-1}{C_G}\Big\{\frac{1}{8\pi i}\int_{i\mathbb{R}}\Phi_d(s;a^{(t)})\alpha(s)\frac{\mathrm{d}s}{|c_d(s)|^2}\\ +\sum_{\substack{j\in\mathbb{Z}\\ 0\le j\le d+\frac{m-3}{2}}}\frac{(-1)^j}{j!}\frac{\Gamma(2d+m-2-j)}{\Gamma(2d+m-3-2j)}\\ \times f_{d,j}(a^{(t)})\alpha\Big(\frac{m-3}{2}+d-j\Big)\Big\}.\end{aligned}$$

Proof

This is the same as in [25, Proposition 12]. The relation (3.7) is used in the proof. □

LEMMA 3.8

For any $R>0$, we have

$$\left\|\hat{\Psi}_d(\alpha; a^{(t)})\right\| \ll (\cosh t)^{-R}, \quad t \in \mathbb{R}. \tag{3.12}$$

Proof

Let $2\sigma-(m-1)>R$. Then from (3.11) and Lemma 3.4,

$$\left\|\hat{\Psi}_d(\alpha; a^{(t)})\right\| \ll (\cosh t)^{-R} \frac{1}{2\pi} \int_{(\sigma)} \left(1+\left|\mathrm{Im}(s)\right|\right)^N \left|\alpha(s)\right| \left|s\,\mathrm{d}s\right|$$

for all $t \geq 1$. From the estimate $|\alpha(s)| = O((1+|\mathrm{Im}(s)|)^{-N-3})$, the integral converges. Thus (3.12) is obtained at least for $t \geq 1$. Since $\|\hat{\Psi}_d(\alpha; a^{(t)})\|$ is continuous on $\mathbb{R}$ from Lemma 3.6, the estimate on $t \geq 1$ is extended to $\mathbb{R}$ with a possibly larger implied constant. □

REMARK

The representation $I(|\cdot|_{\mathbb{R}}^{\nu}) = \mathrm{Ind}_P^G(|\cdot|_{\mathbb{C}}^{\nu} \otimes \mathbf{1}_{M_1})$ is called the *principal series* of $H\backslash G$ (see [4]). The discrete spectrum of $L^2(H\backslash G)$ is completely described by the functions $f_{d,j}$ $(d, j \in \mathbb{N}_0)$, which belong to $L^2(H\backslash G)$ if and only if $0 \leq j < \frac{m-3}{2} + d$.

4. Local harmonic analysis at non-Archimedean places

In this section, we let F be a local non-Archimedean field of characteristic 0. The normalized valuation of F is denoted by $|\cdot|_F$. We fix a prime element ϖ of F, and we set $q = |\varpi|_F^{-1}$. Put $|\alpha|_E = |\alpha\bar{\alpha}|_F$ for $\alpha \in E$. If E is a field, let $\varepsilon_{E/F}$ denote the quadratic character of $F^\times$ trivial on $\mathrm{N}_{E/F}(E^\times)$; if E is not a field, set $\varepsilon_{E/F} = 1$. We identify an F-algebraic group with its F-points; thus $G = G(F)$, $H = H(F)$, and so on. We assume that $\mathrm{rank}_E(V) \geq 4$, and we fix a maximal $\mathfrak{o}_E$-lattice $\mathcal{L}$ in $(V, \mathbf{h})$. Set $\mathcal{U}$ to be the stabilizer of $\mathcal{L}$ in G. The aim of this section is to prepare necessary ingredients for the local harmonic analysis of $H\backslash G$.

4.1. The Poisson integrals

We set $\mathbb{X}_F = \mathbb{C}/2\pi\sqrt{-1}(\log q)^{-1}\mathbb{Z}$. For any quasicharacter χ of $E^\times$ and any irreducible smooth representation σ of M_1, we consider the normalized induced module $\mathrm{Ind}_P^G(\chi \otimes \sigma)$ of G.

LEMMA 4.1

The representation $\pi = \mathrm{Ind}_P^G(\chi \otimes \sigma)$ is H-distinguished (i.e., $\mathrm{Hom}_H(\pi, \mathbb{C}) \neq \{0\}$) only if σ is the trivial representation of M_1 and $\chi|E^1 = 1$.

Proof

This is proved by the same argument as in [27, Lemma 18]. □

For a quasicharacter η of $F^\times$, set

$$I(\eta) = \mathrm{Ind}_P^G(\eta \circ \mathrm{N}_{E/F} \otimes \mathbf{1}_{M_1}).$$

For a unitary character η of $F^\times$ and $\nu \in \mathbb{X}_F$, the Poisson integral is defined by

$$\big\langle \Xi\big(\eta|\cdot|_F^\nu\big), f\big\rangle = \oint_{P\backslash G} Y\big(\eta|\cdot|_F^\nu; g\big) f(g)\,\mathrm{d}\mu_{P\backslash G}, \quad f \in I\big(\eta|\cdot|_F^\nu\big), \tag{4.1}$$

where

$$Y\big(\eta|\cdot|_F^\nu; g\big) = \begin{cases} (\eta|\cdot|_F^{\nu-(m-1)/2})(\mathrm{N}_{E/F}(\mathbf{h}(g\ell, e)), & g \in PH, \\ 0, & g \in G - PH. \end{cases}$$

If $\mathrm{Re}(\nu) \geq \frac{m-1}{2}$, then the integral (4.1) converges absolutely, defining an H-invariant $\mathbb{C}$-linear form $\Xi(\eta|\cdot|_F^\nu) : I(\eta|\cdot|_F^\nu) \to \mathbb{C}$ because the function $Y(\eta|\cdot|_F^\nu)$ with $\mathrm{Re}(\nu) \geq \frac{m-1}{2}$ is continuous on G and is right H-invariant. By Bernstein's theorem (see [8, Section 12.2]), there exists a polynomial $R_\eta(z) \in \mathbb{C}[z]$ such that $\nu \mapsto R_\eta(q^\nu)\Xi(\eta|\cdot|_F^\nu)$ extends to an entire family of H-invariant functionals on $I(\eta|\cdot|_F^\nu)$ ($\nu \in \mathbb{X}_F$). We define the normalized H-invariant functional $\Xi^0(\eta|\cdot|_F^\nu)$ on $I(\eta|\cdot|_F^\nu)$ by setting

$$\Xi^0\big(\eta|\cdot|_F^\nu\big) = \frac{L(2\nu+1, \eta^2\varepsilon_{E/F}^m)}{L(\nu - \frac{m-3}{2}, \eta \circ \mathrm{N}_{E/F})} L(m-1, \varepsilon_{E/F}^{m-1}) \Xi\big(\eta|\cdot|_F^\nu\big)$$

for all unitary characters η of $F^\times$ and $\nu \in \mathbb{X}_F$.

LEMMA 4.2

For any flat section $f^{(\nu)}$ of $I_v(\eta|\cdot|_F^\nu)$ over $\nu \in \mathbb{X}_F$, the function $\langle \Xi^0(\eta|\cdot|_F^\nu), f^{(\nu)}\rangle$ is holomorphic on $\mathrm{Re}(\nu) \geq 0$.

Proof

For $\phi \in \mathcal{S}(V - \{0\})$, set

$$\tilde{\phi}^{(\nu)}(g) = \int_{E^\times} \phi(g^{-1}\tau e)\eta(\tau\bar{\tau})|\tau|_E^{\nu+(m-1)/2}\Big|\frac{\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}}{2\sqrt{\theta}\tau\bar{\tau}}\Big|_F, \quad g \in G.$$

Then $\tilde{\phi}^{(\nu)}$ is a holomorphic section of $I(\eta|\cdot|_F^\nu)$. Given a flat section $f^{(\nu)}$, there exists a finite collection of holomorphic functions $c_j(\nu)$ and functions $\phi_j \in \mathcal{S}(V - \{0\})$ such that $f^{(\nu)} = \sum_j c_j(\nu)\tilde{\phi}_j^{(\nu)}$ for all $\mathrm{Re}(\nu) \geq 0$. We have

$$\begin{aligned}\big\langle \Xi^0\big(\eta|\cdot|_F^\nu\big), \tilde{\phi}^{(\nu)}\big\rangle &= \int_{P_1\backslash G} Y\big(\eta|\cdot|_F^\nu; g\big)\phi(g^{-1}e)\Big|\frac{\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}}{2\sqrt{\theta}\tau\bar{\tau}}\Big|_F \\ &= \int_{\mathbf{\Sigma}(0)} \big(\eta|\cdot|^{\nu-(m-1)/2}\big)\big(\mathrm{N}_{E/F}\mathbf{h}(\ell,\xi)\big)\phi(\xi)|\omega_{\mathbf{\Sigma}(0)}|_F.\end{aligned}$$

For a general element $\xi \in \mathbf{\Sigma}(0)$, we set $\xi = z\ell + Z$ with $z \in E$ and $Z \in \ell^\perp$. Then $\mathbf{h}[Z] = -z\bar{z}$. Let $\mathbf{\Sigma}'(t)$ denote the hyperboloid $\mathbf{h}[Z] = t$ in $\ell^\perp$, and let $\omega_{\mathbf{\Sigma}'(t)}$ be the gauge-form on $\mathbf{\Sigma}'(t)$ as in Section 2.2. As seen from Lemma A.1, $\omega_{\mathbf{\Sigma}(0)}$ is decomposed to $\omega_{\mathbf{\Sigma}'(-z\bar{z})} \wedge \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{2\sqrt{\theta}}$. The function $\phi(z\ell + Z)$ can be expressed as a finite sum of decomposable functions $\phi_1(z)\phi_2(Z)$ ($\phi_1\in\mathcal{S}(E)$, $\phi_2\in\mathcal{S}(\ell^\perp)$). For our purpose, it is harmless to assume that $\phi(z\ell+Z)$ itself is one of these decomposable functions. Thus

$$\begin{aligned}\langle\Xi^0(\eta|\cdot|_F^\nu),\tilde\phi^{(\nu)}\rangle &= \int_{E^\times}(\eta|\cdot|^{\nu-(m-1)/2})(z\bar z)\phi_1(z)\\ &\quad\times\Big\{\int_{\mathbf{\Sigma}'(-z\bar z)}\phi_2(Z)|\omega_{\mathbf{\Sigma}'(-z\bar z)}|_F\Big\}\Big|\frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{2\sqrt\theta}\Big|_F\\ &= \int_{\mathrm{N}_{E/F}(E^\times)}(\eta|\cdot|^{\nu-(m-1)/2})(t)\Phi_1(t)\Phi_2(t)|\mathrm{d}t|_F,\end{aligned}$$

where we set

$$\Phi_1(t)=\int_{E(t)}\phi_1(z)|\omega_{E(t)}|_F,\qquad \Phi_2(t)=\int_{\mathbf{\Sigma}'(-t)}\phi_2(Z)|\omega_{\mathbf{\Sigma}'(-t)}|_F,\quad t\in\mathrm{N}_{E/F}(E),$$

with $E(t)$ denoting the fiber $\mathrm{N}_{E/F}^{-1}(t)$ and $\omega_{E(t)}$ denoting a gauge-form on $E(t)$. Suppose that E is a field. From Lemma A.2(1), $\Phi_1(t)$ is a restriction to $\mathrm{N}_{E/F}(E)$ of a Schwartz–Bruhat function on F, and there is a constant C such that $\Psi_2(t)=\Phi_2(t)-C\varepsilon_{E/F}^{m-1}(t)|t|_F^{m-2}$ is a restriction of a Schwartz–Bruhat function on F. Let $Z(\chi;\varphi)$ denote the (analytic continuation of the) Tate zeta integral $\int_{\mathrm{N}_{E/F}(E^\times)}\chi(t)\varphi(t)|\mathrm{d}t|_F$ for a character χ of $F^\times$ and $\varphi\in\mathcal{S}(F)$. Then we have that $\langle\Xi(\eta|\cdot|_F^\nu),f^{(\nu)}\rangle$ is a sum of zeta integrals like $Z(\eta|\cdot|_F^{\nu-(m-1)/2};\Phi_1\Psi_2)$ and $CZ(\eta|\cdot|_F^{\nu+(m-3)/2};\Phi_1)$. The latter term is holomorphic on $\mathrm{Re}(\nu)\ge 0$. The former one has the same singularity as the analytic continuation of the integral

$$\begin{aligned}&\int_{\mathfrak{o}_F\cap\mathrm{N}_{E/F}(E^\times)}\eta(t)|t|_F^{\nu-(m-1)/2}|\mathrm{d}t|_F\\ &\quad=\mathrm{vol}\big(\mathrm{N}_{E/F}(\mathfrak{o}_E^\times);|\mathrm{d}t|_F\big)L\Big(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F}\Big),\\ &\qquad\mathrm{Re}(\nu)>\frac{m-3}{2},\end{aligned}$$

which is canceled by the normalizing factor. Suppose that E is isomorphic to $F\oplus F$. In this case $N_{E/F}(E)=F$ and, from Lemma A.2(1), there exists a constant C_1 such that $\Phi_1(t)-C_1\,\mathrm{ord}_F(t)$ extends to a Schwartz–Bruhat function on F. We argue similarly to show that $\langle\Xi(\eta|\cdot|_F^\nu),f^{(\nu)}\rangle$ has a meromorphic continuation to $\mathbb{C}$ whose singularity on $\mathrm{Re}(\nu)\ge 0$ is the same as the analytic continuation of the integral

$$\int_{\mathfrak{o}_F-\{0\}}\eta(t)|t|_F^{\nu-(m-1)/2}\mathrm{ord}_F(t)|\mathrm{d}t|_F=\mathrm{vol}\big(\mathfrak{o}_F^\times;|\mathrm{d}t|_F\big)\frac{q^{-(\nu-(m-3)/2)}\eta(\varpi)}{(1-q^{-(\nu-(m-3)/2)}\eta(\varpi))^2},$$
$$\mathrm{Re}(\nu)>\frac{m-3}{2},$$

which coincides with $q^{-(\nu-(m-3)/2)}L(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F})$ up to a constant factor and is canceled by the normalizing factor. $\square$

LEMMA 4.3

Suppose that $2 \in \mathfrak{o}_F^\times$, that E is not a ramified extension of F, and that $\mathcal{L}$ is self-dual. Let η be an unramified unitary character, and let $f_0^{(\nu)}$ be the $\mathcal{U}$-invariant vector of $I(\eta|\cdot|_F^\nu)$ such that $f_0^{(\nu)}(k)=1$ for all $k\in\mathcal{U}$. Then

$$\langle \Xi^0(\eta|\cdot|_F^\nu), f_0^{(\nu)}\rangle = 1.$$

Proof

Let ϕ_0 be the characteristic function of $\mathcal{L}_{\rm prim}$, and let $\tilde{\phi}_0^{(\nu)}$ be the corresponding holomorphic section of $I(\eta|\cdot|_F^\nu)$ as in the proof of Lemma 4.2. We have $\phi_0([\tau]ke)\neq 0$ for some $k\in\mathcal{U}$ if and only if $\tau\in\mathfrak{o}_E^\times$, where $\mathfrak{o}_E=\mathfrak{o}_F[\sqrt{\theta}]$. Thus

$$\tilde{\phi}_0^{(\nu)}(k)=\int_{\mathfrak{o}_E^\times}\phi_0(ke)\Big|\frac{d\tau\wedge d\bar{\tau}}{2\sqrt{\theta}\tau\bar{\tau}}\Big|_F=\mathrm{vol}(\mathfrak{o}_E^\times),\quad k\in\mathcal{U},$$

with $\mathrm{vol}(\mathfrak{o}_E^\times)=1-q^{-2}$ if E is an unramified field extension and $\mathrm{vol}(\mathfrak{o}_E^\times)=(1-q^{-1})^2$ if $E\cong F\oplus F$. Combining this with the obvious P-equivariance $\tilde{\phi}_0^{(\nu)}([\tau]p_1g)=(\eta|\cdot|_F^{\nu+(m-1)/2})(\tau\bar{\tau})\tilde{\phi}_0^{(\nu)}(g)$ $([\tau]p_1\in P)$, we have the equality $\tilde{\phi}_0^{(\nu)}=\mathrm{vol}(\mathfrak{o}_E^\times)f_0^{(\nu)}$. Thus, from $\omega_P=\omega_{P_1}\wedge\frac{d\tau\wedge d\bar{\tau}}{2\sqrt{\theta}\tau\bar{\tau}}$,

$$\begin{aligned}
\langle \Xi(\eta|\cdot|_F^\nu), f_0^{(\nu)}\rangle &= \mathrm{vol}(\mathfrak{o}_E^\times)\oint_{P\backslash G} Y(\eta|\cdot|_F^\nu;g)\tilde{\phi}_0^{(\nu)}(g)d\mu_{P\backslash G}\\
&=\mathrm{vol}(\mathfrak{o}_E^\times)\oint_{P\backslash G}\int_{E^\times} Y(\eta|\cdot|_F^\nu;[\tau]g)\phi_0(\tau^{-1}g^{-1}e)\\
&\quad\times|\tau\bar{\tau}|_F^{m-1}\Big|\frac{d\tau\wedge d\bar{\tau}}{2\sqrt{\theta}\tau\bar{\tau}}\Big|_F d\mu_{P\backslash G}\\
&=\mathrm{vol}(\mathfrak{o}_E^\times)\int_{P_1\backslash G} Y(\eta|\cdot|_F^\nu;g)\phi_0(g^{-1}e)|\omega_{P_1\backslash G}|_F\\
&=\mathrm{vol}(\mathfrak{o}_E^\times)\int_{\mathbf{\Sigma}(0)\cap\mathcal{L}_{\rm prim}}(\eta|\cdot|_F^{\nu+(m-1)/2})\big(\mathrm{N}_{E/F}(\mathbf{h}(\xi,e))\big)|\omega_{\mathbf{\Sigma}(0)}|_F.
\end{aligned}$$

From the proof of [27, Lemma 20] and from [27, Remark, Section 5.3], the last integral is evaluated to be

$$\mathrm{vol}(\mathfrak{o}_E^\times)^{-1}L(m-1,\varepsilon_{E/F}^{m-1})^{-1}\frac{L(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F})}{L(2\nu+1,\eta^2\varepsilon_{E/F}^m)}.$$

□

4.2. Normalized intertwining operator

Let η be a unitary character of $F^\times$, and let $\nu\in\mathbb{X}_F$. The normalized intertwining operator $R(\eta,\nu):I(\eta|\cdot|_F^\nu)\to I(\bar{\eta}|\cdot|_F^{-\nu})$ is defined by an analytic continuation of the absolute convergent integral

$$[R(\eta,\nu)f](g)=\frac{L(\nu+\frac{m-1}{2},\eta\circ\mathrm{N}_{E/F})L(2\nu+1,\varepsilon_{E/F}^m\eta^2)}{L(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F})L(2\nu,\varepsilon_{E/F}^m\eta^2)}\int_{\bar{N}}f(\bar{n}g)\,d\bar{n},$$

$$f\in I(\eta|\cdot|_F^\nu),$$

for $\mathrm{Re}(\nu) > \frac{m-1}{2}$, where $\bar{N}$ is the unipotent radical of the opposite of P and $\mathrm{d}\bar{n}$ is the Haar measure on $\bar{N}$ such that $\mathrm{vol}(\bar{N} \cap \mathcal{U}) = 1$.

LEMMA 4.4

Under the same setting and assumptions as in Lemma 4.3, we have $R(\eta,\nu) f_0^{(\nu)} = f_0^{(\nu)}$ *for all* $\mathrm{Re}(\nu) \geq 0$.

Proof

This is more or less a standard fact. Here is a brief sketch. Let $\mathcal{L} = \sum_{j=1}^{l} (\mathfrak{o}_E e_j + \mathfrak{o}_E e'_j) + \mathcal{L}_0$ be the Witt decomposition; thus $V_0 = \mathcal{L}_0 \otimes_{\mathfrak{o}_E} E$ is an anisotropic Hermitian space with $\dim_E V_0$ being 0 or 1. Let B be the F-Borel subgroup of G stabilizing the corresponding maximal isotropic flag. We may assume that $e_1 = e$ and $e'_1 = e'$. In particular, $B \subset P$ and B has the Levi factor isomorphic to $(E^\times)^l \times U(V_0)$. It is easy to see that $I(\eta|\cdot|_F^\nu) \subset \mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_l; \mathbf{1}_{U(V_0)})$ with $\chi_1 = \eta \circ \mathrm{N}_{E/F}|\cdot|_E^\nu$, $\chi_j = |\cdot|_E^{-\frac{m+1-2j}{2}}$ $(2 \leq j \leq l)$. Our operator is obtained as a restriction of the standard intertwining operator $\tilde{R}_w(\chi)$ from $\mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_l; \mathbf{1}_{U(V_0)})$ to $\mathrm{Ind}_B^G(w(\chi_1, \chi_2, \ldots, \chi_l); \mathbf{1}_{U(V_0)})$ with a particular Weyl group element w. Let $\tilde{f}_0^{(\chi)}$ be the element of $\mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_l; \mathbf{1}_{U(V_0)})$ extending $f_0^{(\nu)}$. Then $\tilde{R}_w(\chi) f_0^{(\chi)} = 1$ is confirmed by the Gindikin–Karpelevich formula. Since $R(\eta,\nu) f_0^{(\nu)} = \tilde{R}(\chi) \tilde{f}_0^{(\chi)}$, we are done. □

Since our G is not quasisplit in general, the analytical properties of $R(\eta,\nu)$ do not seem obvious from the published works. Here, we provide what we need in the proofs of Lemmas 5.1 and 5.2.

PROPOSITION 4.5

For any flat section $f^{(\nu)}$ *of* $I(\eta|\cdot|_F^\nu)$ *and* $g \in G$, *the function* $\nu \mapsto [R(\eta,\nu) f](g)$ *is holomorphic on* $\mathrm{Re}(\nu) \geq 0$.

Proof

Suppose that E is a field. As in the proof of Lemma 4.2, we may suppose that $f^{(\nu)} = \tilde{\phi}^{(\nu)}$ with some $\phi \in \mathcal{S}(V - \{0\})$. There exists a positive constant C such that

$$\int_{\bar{N}} f^{(\nu)}(\bar{n})\,\mathrm{d}\bar{n} = C \int_{\bar{N}} \int_{E^\times} \phi(\tau \bar{n}^{-1} e) \eta(\tau\bar{\tau}) |\tau|_E^{\nu+(m-1)/2} \left| \frac{\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}}{2\sqrt{\theta}\tau\bar{\tau}} \right|_F |\omega_{\bar{N}}|_F.$$

The mapping $p : (\tau, \bar{n}) \mapsto Z = \tau \bar{n}^{-1} e$ from $E^\times \times \bar{N}$ to $\mathbf{\Sigma}(0)$ is an injective morphism whose image is Zariski dense. Since $\omega_{\mathbf{\Sigma}(0)}$ is proportional to $(\tau\bar{\tau})^{-(m-1)} \times p^* \frac{\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau}}{2\sqrt{\theta}\tau\bar{\tau}} \wedge p^* \omega_{\bar{N}}$, the integral coincides with $\mathfrak{I}_\nu(0)$ up to a positive constant, where

$$\mathfrak{I}_\nu(t) = \int_{\mathbf{\Sigma}(t)} \phi(Z) \big(\eta|\cdot|_F^{\nu-(m-1)/2}\big) \big(\mathrm{N}_{E/F} \mathbf{h}(Z, e')\big) |\omega_{\mathbf{\Sigma}(t)}|_F, \quad t \in F. \tag{4.2}$$

Thus the desired holomorphy follows from the next lemma. □

LEMMA 4.6

(1) *If* $\mathrm{Re}(\nu) > \frac{m-3}{2}$, *for each* $t \in F$, *the integral (4.2) converges absolutely, defining a holomorphic function on that region. For each* $\mathrm{Re}(\nu) > \frac{m-3}{2}$, *the function* $t \mapsto \mathfrak{I}_\nu(t)$ *is continuous and integrable on* F.

(2) *The function* $\nu \mapsto L(\nu - \frac{m-3}{2}, \eta \circ \mathrm{N}_{E/F})^{-1} L(2\nu, \varepsilon_{E/F}^m \eta^2)^{-1} \mathfrak{I}_\nu(0)$ *has a holomorphic extension to* $\mathbb{X}_F$.

Proof

Write a general point $Z \in V$ in the form $Z = ze + we' + Z_0$ with $z, w \in E$ and $Z_0 \in V_1$. We may further assume that $\phi(Z)$ is of the form $\phi_+(z)\phi_-(w)\phi_0(Z_0)$ with $\phi_+, \phi_- \in \mathcal{S}(E)$ and $\phi_0 \in \mathcal{S}(V_1)$. Thus the Fourier transform of $\mathfrak{I}_\nu(t)$ ($\mathrm{Re}(\nu) > \frac{m-3}{2}$) becomes

$$\begin{aligned}\int_F \mathfrak{I}_\nu(t)\psi(\tau t)|\mathrm{d}t|_F &= \int_V \phi(Z)\big(\eta|\cdot|_F^{\nu-(m-1)/2}\big)\big(\mathrm{N}_{E/F}\mathbf{h}(Z,e')\big)\psi\big(\tau\mathbf{h}[Z]\big)\,\mathrm{d}Z \\ &= \hat{\mathcal{M}}_{\phi_0}(\tau) I(\phi_+,\phi_-;\nu,\tau),\end{aligned}$$

where, for $\tau \in F$,

$$\hat{\mathcal{M}}_{\phi_0}(\tau) = \int_{V_1} \phi_0(Z_0)\psi\big(\tau\mathbf{h}[Z_0]\big)|\omega_{V_1}|_F,$$

$$\begin{aligned}I(\phi_+\phi_-;\nu,\tau) = \int_{E^2} &\phi_+(z)\phi_-(w)\big(\eta|\cdot|_F^{\nu-(m-1)/2}\big)(z\bar{z}) \\ &\times \psi\big(\tau(z\bar{w}+w\bar{z})\big)\Big|\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{2\sqrt{\theta}}\Big|_F\Big|\frac{\mathrm{d}w\wedge\mathrm{d}\bar{w}}{2\sqrt{\theta}}\Big|_F.\end{aligned}$$

Let $\mathcal{F}\alpha(\xi) = \int_E \alpha(w)\psi(\xi\bar{w}+w\bar{\xi})|\frac{\mathrm{d}w\wedge\mathrm{d}\bar{w}}{2\sqrt{\theta}}|_F$ be the Fourier transform of $\alpha \in \mathcal{S}(E)$. Then

$$I(\phi_+,\phi_-;\nu,\tau) = \int_E \phi_+(z)[\mathcal{F}\phi_-](\tau z)\big(\eta|\cdot|_F^{\nu-(m-1)/2}\big)(z\bar{z})\Big|\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{2\sqrt{\theta}}\Big|_F.$$

By setting $\phi'_+(z) = \phi_+(z) - \phi_+(0)\delta(z \in \mathfrak{o}_E)$, we write this integral as a sum of $I(\phi'_+,\phi_-;\nu,\tau)$ and

$$\phi_+(0)\int_{\mathfrak{o}_E}[\mathcal{F}\phi_-](\tau z)\big(\eta|\cdot|_F^{\nu-(m-1)/2}\big)(z\bar{z})\Big|\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{2\sqrt{\theta}}\Big|_F. \tag{4.3}$$

There exist constants $c_1 < c_2$ such that $\phi'_+(z) = 0$ if $|z|_E < c_1$ and $[\mathcal{F}\phi_-](\tau z) = 0$ if $|z|_E \le c_2|\tau|_E^{-1}$. Thus the integration domain of $I(\phi'_+,\phi_-;\nu,\tau)$ can be restricted to the annulus $c_1 \le |z|_E \le c_2|\tau|_E^{-1}$, which is empty if $|\tau|_E > c_2/c_1$. From this, $I(\phi'_+,\phi_-;\nu,\tau)$ is absolutely convergent for all ν and defines a Schwartz–Bruhat function in $\tau \in F$ depending holomorphically on ν and, moreover, whose value at $\tau = 0$ is the Tate zeta integral $\mathcal{F}\phi_-(0)Z_E(\phi'_+;\eta|\cdot|_F^{\nu-(m-1)/2}\circ\mathrm{N}_{E/F})$. (In this proof, for $\alpha \in \mathcal{S}(E)$ and a quasicharacter χ of $E^\times$, we define the Tate zeta integral by $Z_E(\alpha,\chi) = \int_{E^\times}\alpha(z)\chi(z)|\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{2\sqrt{\theta}}|_F$.) In other words, there exists a function $J_\nu \in \mathcal{S}(F^\times)$ depending holomorphically on $\nu \in \mathbb{X}_F$ such that

$$(4.4)\quad I(\phi'_+,\phi_-;\nu,\tau)=\mathcal{F}\phi_-(0)Z_E\big(\phi'_+;\eta|\cdot|_F^{\nu-(m-1)/2}\circ \mathrm{N}_{E/F}\big)\,\delta(\tau\in\mathfrak{o}_F)+J_\nu(\tau),\qquad \tau\in F,\nu\in\mathbb{X}_F.$$

Let us examine the integral (4.3). By the variable change $z\to\tau^{-1}z$, it becomes

$$\phi_+(0)\big(\eta^2|\cdot|_F^{-2\nu+(m-3)}\big)(\tau)\int_{\tau^{-1}\mathfrak{o}_E}[\mathcal{F}\phi_-](z)\big(\eta|\cdot|_F^{\nu-(m-1)/2}\big)(z\bar z)\Big|\frac{\mathrm{d}z\wedge\mathrm{d}\bar z}{2\sqrt{\theta}}\Big|_F.$$

If we complete the integration domain to E, the Tate zeta integral emerges. Hence (4.3) is absolutely convergent on $\mathrm{Re}(\nu)>\frac{m-3}{2}$ and is written in the form

$$\phi_+(0)\big(\eta^2|\cdot|_F^{-2\nu+(m-3)}\big)(\tau)\big\{Z_E\big(\mathcal{F}\phi_-;\eta|\cdot|_F^{\nu-(m-1)/2}\circ\mathrm{N}_{E/F}\big)+R_\nu(\tau)\big\}$$

on the region $\mathrm{Re}(\nu)>\frac{m-3}{2}$, where $R_\nu(\tau)$, given by an integral on some annulus $|\tau|_F^{-1}<|z|_E<C$, is a Schwartz–Bruhat function on $F^\times$ holomorphic in ν on the whole space $\mathbb{X}_F$. By the Tate theory, $L(s+1,\eta\circ\mathrm{N}_{E/F})^{-1}Z_E(\alpha;\eta|\cdot|_F^s\circ\mathrm{N}_{E/F})$ has a holomorphic extension to $\mathbb{X}_F$ for any $\alpha\in\mathcal{S}(E)$. Summing up the argument thus far, we see that $\mathfrak{I}_\nu(\tau)$ is integrable on F if $\mathrm{Re}(\nu)>\frac{m-3}{2}$, and we obtain the identity

$$(4.5)\quad \begin{aligned}&L\Big(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F}\Big)^{-1}\hat{\mathfrak{I}}_\nu(\tau)\\&=\hat{\mathcal{M}}_{\phi_0}(\tau)\big\{\beta_1(\nu)\,\delta(\tau\in\mathfrak{o}_F)+\beta_2(\nu)\big(\eta^2|\cdot|_F^{-2\nu+(m-3)}\big)(\tau)+\alpha_\nu(\tau)\big\},\\&\mathrm{Re}(\nu)>\frac{m-3}{2}\end{aligned}$$

with some holomorphic functions $\beta_1(\nu)$, $\beta_2(\nu)$ on $\mathbb{X}_F$ and some holomorphic family of Schwartz–Bruhat functions $\alpha_\nu\in\mathcal{S}(F^\times)$. The integral $\hat{\mathcal{M}}_{\phi_0}(\tau)$ is the Fourier transform of the function $\mathcal{M}_{\phi_0}$ recalled in Section A.1. From [21, Proposition 4.4], it is of the form

$$(4.6)\quad \hat{\mathcal{M}}_{\phi_0}(\tau)=C_1\,\delta(\tau\notin\mathfrak{o}_F)\varepsilon_{E/F}^m(\tau)|\tau|_F^{-(m-2)}+C_2\delta(\tau\in\mathfrak{o}_F)+\gamma(\tau)$$

with some constants C_1, C_2 and $\gamma\in\mathcal{S}(F^\times)$. If the Fourier inversion formula can be applied, we have

$$\begin{aligned}&L\Big(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F}\Big)^{-1}\mathfrak{I}_\nu(0)\\&=\int_F\hat{\mathcal{M}}_{\phi_0}(\tau)L\Big(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F}\Big)^{-1}I(\phi_+,\phi_-;\nu,\tau)|\mathrm{d}\tau|_F\\&=\int_F\big\{C_1\,\delta(\tau\notin\mathfrak{o}_F)\varepsilon_{E/F}^m(\tau)|\tau|_F^{-(m-2)}+C_2\delta(\tau\in\mathfrak{o}_F)+\gamma(\tau)\big\}\\&\quad\times\big\{\beta_1(\nu)\,\delta(\tau\in\mathfrak{o}_F)+\beta_2(\nu)\big(\eta^2|\cdot|_F^{-2\nu+(m-3)}\big)(\tau)+\alpha_\nu(\tau)\big\}|\mathrm{d}\tau|_F.\end{aligned}$$

To justify this computation, we have to confirm the absolute convergence of the integral, which we can write as a sum of integrals of functions from $\mathcal{S}(F^\times)$ producing holomorphic terms and the following two integrals:

$$C_1\beta_2(\nu)\int_{F-\mathfrak{o}_F}(\varepsilon_{E/F}^m\eta^2)(\tau)|\tau|_F^{-2\nu-1}|\mathrm{d}\tau|_F,$$

$$C_2\beta_2(\nu)\int_{\mathfrak{o}_F}\eta^2(\tau)|\tau|_F^{-2\nu+m-3}|\mathrm{d}\tau|_F.$$

The first integral is absolutely convergent on $\mathrm{Re}(\nu)>0$ and has a holomorphic continuation to $\mathbb{X}_F$ when multiplied by $L(2\nu,\varepsilon_{E/F}^m\eta^2)^{-1}$. The second integral is absolutely convergent on $\mathrm{Re}(\nu)<\frac{m-2}{2}$, defining a holomorphic function on that region. Consequently, at least on the region $\frac{m-3}{2}<\mathrm{Re}(\nu)<\frac{m-2}{2}$, the function $\mathfrak{I}_\nu(\tau)$ is integrable on F and we can apply the Fourier inversion formula to obtain an expression of $L(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F})^{-1}L(2\nu,\varepsilon_{E/F}^m\eta^2)^{-1}\mathfrak{I}_\nu(0)$, which admits a holomorphic extension to the left half-plane $\mathrm{Re}(\nu)<\frac{m-2}{2}$. □

5. Periods of automorphic forms

From now on, we work on a global setting. For a number field K, we will use the following notation throughout this article. Let Σ_{fin}^K and Σ_∞^K denote the set of finite places of K and the set of infinite places of K, respectively. We set $\Sigma^K=\Sigma_{\mathrm{fin}}^K\cup\Sigma_\infty^K$. For $v\in\Sigma^K$, let K_v be the completion of K at v, and let $|\cdot|_{K_v}$ be the normalized valuation of K_v. When $v\in\Sigma_{\mathrm{fin}}^K$, q_{K_v} denotes the order of the residue field of K_v. The modulus of an idèle $a\in\mathbb{A}_K^\times$ is denoted by $|a|_K=\prod_v|a_v|_{K_v}$. Let $K_\infty=K\otimes_{\mathbb{Q}}\mathbb{R}$, and let $\mathbb{A}_K^\infty$ denote the finite adèles of K; thus $\mathbb{A}_K=K_\infty\times\mathbb{A}_K^\infty$.

5.1

Let E/F be a CM extension of a totally real number field F of degree d_F; thus, E is a totally imaginary number field of degree $2d_F$. We fix an element $\theta\in F$ such that $E=F[\sqrt{\theta}]$ once and for all. By the class field theory, the extension E/F yields a quadratic idèle class character $\varepsilon_{E/F}$ of $F^\times$ trivial on the norms $\mathrm{N}_{E/F}(\mathbb{A}_E^\times)$. Let $(V,\mathbf{h})$ be a nondegenerate Hermitian space of dimension m, and let $G=U(\mathbf{h})$ be its unitary group as in Section 2.1. We set $\Sigma_\infty=\Sigma_\infty^F$ and $\Sigma_{\mathrm{fin}}=\Sigma_{\mathrm{fin}}^F$. For $v\in\Sigma^F$, the F_v-algebra $E\otimes_F F_v$ is denoted by E_v. When $v\in\Sigma_{\mathrm{fin}}$, set $\mathfrak{o}_{E_v}=\mathfrak{o}_E\otimes_{\mathfrak{o}_F}\mathfrak{o}_{F_v}$. For any $v\in\Sigma^F$, the Hermitian form on the E_v-module $V_v=V\otimes_E E_v$ induced from $\mathbf{h}$ by the scalar extension is denoted by $\mathbf{h}_v$. From now on, we keep the following assumptions:

(i) $m\geq 4$;

(ii) for any $v\in\Sigma_\infty$, the Hermitian form $\mathbf{h}_v$ on $V_v\cong\mathbb{C}^m$ has exactly one negative eigenvalue.

From these, by the Hasse–Minkowski theorem, the maximal totally isotropic subspace of V is 1-dimensional. Thus the F-algebraic group G is of F-rank 1. We can find a pair of isotropic vectors e, e' in V such that $\mathbf{h}(e,e')=1$ and the orthogonal $(Ee+Ee')^\perp$ is anisotropic. We fix such a pair of vectors e, e' and a vector $\ell\in V$ satisfying $\mathbf{h}[\ell]=1$ and $\mathbf{h}(\ell,e)=1$ once and for all, and define objects V_1, G_1, P_1, M_1, and N_1 as in Section 2.1.

5.2. Compact subgroups

Set $\ell^- = \ell - e$. For $v \in \Sigma_\infty$, let $\mathcal{U}_v$ be the stabilizer in $G(F_v)$ of $E_v\ell^- \subset V(F_v)$. Since $\mathbf{h}[\ell^-] = -1$, $\mathcal{U}_v$ is a maximal compact subgroup of $G(F_v) \cong U(m-1,1)$. Due to $\mathbf{h}_v(\ell, \ell^-) = 0$, $\mathcal{U}_{H,v} = \mathcal{U}_v \cap H(F_v)$ becomes a maximal compact subgroup of $H(F_v)$. We fix an $\mathfrak{o}_E$-lattice $\mathcal{L}_1 \subset V_1$ which is maximal in the sense of [22] once and for all, and we set $\mathcal{L} = \mathfrak{o}_E e + \mathfrak{o}_E e' + \mathcal{L}_1$. For $v \in \Sigma_{\mathrm{fin}}$, let $\mathcal{U}_v$ be the stabilizer of $\mathcal{L}_v = \mathcal{L} \otimes_{\mathfrak{o}_F} \mathfrak{o}_{F_v}$ in $G(F_v)$; then $\mathcal{U}_v$ is a maximal compact subgroup of $G(F_v)$. We have the Iwasawa decomposition $G(F_v) = P(F_v)\mathcal{U}_v$ for all $v \in \Sigma^F$.

5.3. Global Tamagawa measures

Following [30], we define a Haar measure $|\mathrm{d}^\times t|_{\mathbb{A}}$ on $\mathbb{A}_F^\times$ by $|\mathrm{d}^\times t|_{\mathbb{A}} = |D_{F/\mathbb{Q}}|^{-1/2} \times (\mathrm{Res}_{s=1}\, \zeta_F(s))^{-1} \prod_{v \in \Sigma^F} \zeta_{F_v}(1) |\frac{\mathrm{d}t}{t}|_{F_v}$, where $\zeta_F(s)$ denotes the completed Dedekind zeta function of F and $\zeta_{F_v}(s)$ denotes its local v-factor. For any smooth F-variety X, its gauge-form ω_X, and a set of convergence factors $\{\lambda_v\}_{v \in \Sigma_{\mathrm{fin}}}$ for X, we define a measure $|\omega_X|_{\mathbb{A}^\infty}$ on the finite adèle points $X(\mathbb{A}_F^\infty)$ as the restricted product of $|\omega_X|_{F_v}^* = \lambda_v^{-1} |\omega_X|_{F_v}$ (see [30, Section 2.3]). Let $|\omega_X|_{F_\infty}$ on $X(F_\infty)$ be the product measure of $|\omega_X|_{F_v}$ over $v \in \Sigma_\infty$. We define a measure $|\omega_X|_{\mathbb{A}}$ on the adèle points $X(\mathbb{A}_F)$ by taking the product of $|\omega_X|_{F_\infty}$ and $|\omega_X|_{\mathbb{A}^\infty}$ multiplied by $|D_{F/\mathbb{Q}}|^{-\dim(X)/2}$, and we set $\mathrm{vol}(X) = \mathrm{vol}(X(\mathbb{A}_F); |\omega_X|_{\mathbb{A}})$. If $(V_i, \mathbf{h}_i)$ $(1 \le i \le r)$ is a finite collection of nondegenerate Hermitian spaces over E, as a set of convergence factors for $U = \prod_{i=1}^r U(\mathbf{h}_i)$, we always take $\{L(1, \varepsilon_{E_v/F_v})^{-r}\}_v$. Then for any left-invariant gauge-form ω_U on U, we have $\mathrm{vol}(U) = \{2L(1, \varepsilon_{E/F})\}^r$. Indeed, in [30, Section 4.4], it is shown that $\mathrm{vol}(SU(\mathbf{h}_i)) = 1$ if we take the convergence factor $\{1\}_v$. From [30, Section 3.7(c)], $\mathrm{vol}(E^1) = 2L(1, \varepsilon_{E/F})$ for the convergence factor $\{L(1, \varepsilon_{E_v/F_v})^{-1}\}_v$. Since $SU(\mathbf{h}_i)$ is a normal subgroup of $U(\mathbf{h}_i)$ with $SU(\mathbf{h}_i) \backslash U(\mathbf{h}_i) \cong T$, we apply [30, Theorem 2.4.4] to see that $\mathrm{vol}(U(\mathbf{h}_i)) = 2L(1, \varepsilon_{E/F})$. In this way, we fix Haar measures $|\omega_G|_{\mathbb{A}}$, $|\omega_{H_0}|_{\mathbb{A}}$, $|\omega_H|_{\mathbb{A}}$, $|\omega_{H_P}|_{\mathbb{A}}$, and $|\omega_{G_1}|_{\mathbb{A}}$ on $G(\mathbb{A}_F)$, $H_0(\mathbb{A}_F)$, $H(\mathbb{A}_F)$, $H_P(\mathbb{A}_F)$, and $G_1(\mathbb{A}_F)$, respectively. If U' is an F-subgroup which is also a direct product of unitary groups and if gauge-forms ω_U, $\omega_{U'}$, and $\omega_{U' \backslash U}$ on U, U', and $U' \backslash U$, respectively, are given as matching together algebraically, then we apply [30, Theorem 2.4.3] to endow U/U' with a U-invariant gauge-form to define a $U(\mathbb{A}_F)$-invariant measure $|\omega_{U/U'}|_{\mathbb{A}}$ on $(U' \backslash U)(\mathbb{A}_F)$. In particular, for the hyperboloid $\mathbf{\Sigma}(t)$ (see Section 2.2) and $H_P \backslash H$, the set of convergence factors is $\{1\}_v$.

5.4. Siegel domain and norm functions

Set $\mathcal{U} = \prod_{v \in \Sigma^F} \mathcal{U}_v$ and $\mathcal{U}_\infty = \prod_{v \in \Sigma_\infty} \mathcal{U}_v$. Viewing them as subgroups of $G(\mathbb{A}_F)$, we have the Iwasawa decomposition $G(\mathbb{A}_F) = P(\mathbb{A}_F)\mathcal{U}$. For $g \in G(\mathbb{A}_F)$, we set $a(g) = |\tau|_E$ by decomposing $g = [\tau]\mathsf{m}[g_1]nk$ with $g_1 \in G_1(\mathbb{A}_F)$, $\tau \in \mathbb{A}_E^\times$, $n \in N(\mathbb{A}_F)$, and $k \in \mathcal{U}$. For a real $\tau > 0$, define $\underline{\tau} \in \mathbb{A}_E^\times$ as $\underline{\tau}_w = \tau^{1/d_F}$ $(w \in \Sigma_\infty^E)$, $\underline{\tau}_w = 1$ $(w \in \Sigma_{\mathrm{fin}}^E)$. Then we have $|\underline{\tau}|_E = \tau^2$ and $\underline{\tau} \in \mathbb{A}_F^\times$. For $t_0 > 0$, set $\mathfrak{a}_G^+(t_0) = \{[\underline{\tau}] \mid \tau \ge t_0\}$. Any subset $\mathfrak{S}_G \subset G(\mathbb{A}_F)$ of the form $\mathcal{N}\mathfrak{a}_G^+(t_0)\mathcal{U}$ with a relatively compact subset $\mathcal{N}$ of $P(\mathbb{A}_F)^1 = \{[u]p_1 \mid u \in \mathbb{A}_E^1, p_1 \in P_1(\mathbb{A}_F)\}$ and $t_0 > 0$ is called a *Siegel domain* of $G(\mathbb{A}_F)$ with respect to P and $\mathcal{U}$. For a given

Siegel domain $\mathfrak{S}_G = \mathcal{N}\mathfrak{a}_G^+(t_0)\mathcal{U}$ and $\gamma \in G(F)$, we set $\mathfrak{S}_G^\gamma = \gamma^{-1}\mathcal{N}\mathfrak{a}_G^+(t_0)\gamma\mathcal{U}$. Let $\rho : \mathrm{GL}_E(V) \to \mathrm{GL}_m$ be an E-isomorphism. For $v \in \Sigma$, we define a norm function on $G(F_v)$ by $\|g_v\|_{G(F_v),\rho} = \sup\{|\rho(g)_{ij}|_{E_v} \mid 1 \le i,j \le n\}$ if $v \in \Sigma_{\mathrm{fin}}$ and by $\|g_v\|_{G(F_v),\rho} = \|\rho(g)\|_{\mathrm{HS}}$, the Hilbert–Schmidt norm of $\rho(g) \in \mathrm{GL}_m(\mathbb{C})$, if $v \in \Sigma_\infty$. For $g = (g_v) \in G(\mathbb{A}_F)$, the product $\|g\|_{G,\rho} = \prod_{v\in\Sigma} \|g_v\|_{G(F_v),\rho}$, which makes sense because $\|g_v\|_{G(F_v),\rho} = 1$ for almost all v, is called the *norm* of g with respect to ρ. If ρ' is another E-isomorphism like ρ, then the corresponding norm functions are comparable; that is, $\|g\|_{G,\rho} \asymp \|g\|_{G,\rho'}$ on $G(\mathbb{A}_F)$. We fix ρ once and for all, and omit the subscript ρ from $\|\cdot\|_{G,\rho}$. For $t = (t_v)_{v\in\Sigma_\infty} \in \mathbb{R}^{\Sigma_\infty}$, we set $\mathsf{a}(t) = (a_v^{(t_v)})_{v\in\Sigma_\infty} \in G(F_\infty)$, where $a_v^{(t)}$ denotes the 1-parameter subgroup $a^{(t)}$ of $G(F_v)$ introduced in Section 3 (see Lemma 3.2 above). The following easily confirmed relations are frequently used:

$$\big\|[\underline{\tau}]\big\|_G \asymp \tau + \tau^{-1}, \qquad a(\underline{\tau}) = \tau^2,$$
$$\big\|\mathsf{a}(t)\big\|_G \asymp \prod_{v\in\Sigma_\infty} \cosh t_v \quad \text{for } \tau > 0 \text{ and } t \in \mathbb{R}^{\Sigma_\infty}.$$

5.5. Eisenstein series

Let σ be an irreducible automorphic representation of $G_1(\mathbb{A}_F)$. Since G_1 is F-anisotropic, σ is cuspidal. Let $\chi = \bigotimes_v \chi_v$ be a unitary idèle class character of $E^\times$ trivial on $\{\underline{\tau} \mid \tau > 0\}$. We define $V(\sigma,\chi)$ to be the space of all the smooth functions $f : M_1(F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ satisfying $f([\tau]g) = \chi(\tau)f(g)$ for all $\tau \in \mathbb{A}_E^\times$ and such that the function $g_1 \mapsto f(\mathsf{m}[g_1]k)$ in $g_1 \in G_1(\mathbb{A}_F)$ belongs to the space of σ for all $k \in \mathcal{U}$. Then we define the representation $I(\sigma,\chi,\nu)$ of $G(\mathbb{A}_F)$ by

$$\big[I(\sigma,\chi,\nu;g)f\big](x) = \big(a(xg)a(x)^{-1}\big)^{\nu+\rho_G} f(xg), \quad g \in G(\mathbb{A}_F),$$

with $\rho_G = \frac{m-1}{2}$. For a family of dominant weights $\Lambda = \{(l_v;c_v)\}_{v\in\Sigma_\infty}$, let $(\tau_\Lambda, W(\Lambda))$ be the external tensor product over $v \in \Sigma_\infty$ of the representations of $\mathcal{U}_v$ on $W_{(l_v;c_v)}$ (see Section 3); that is, $W(\Lambda) = \boxtimes_{v\in\Sigma_\infty} W_{(l_v;c_v)}$.

Let $f : G(\mathbb{A}_F) \to W(\Lambda)$ be a smooth function satisfying

$$f\big([\tau]p_1gk\big) = \chi(\tau)\tau_\Lambda(k_\infty)^{-1}f(g), \quad \tau \in \mathbb{A}_E^\times, p_1 \in M_1(F)N(\mathbb{A}_F), k \in \mathcal{U}, \tag{5.1}$$

such that, for any $w \in W(\Lambda)^*$ and for any $k \in \mathcal{U}$, the function $\langle w^*, f(\mathsf{m}[g_1]k)\rangle$ on $g_1 \in G_1(\mathbb{A}_F)$ belongs to the space of σ. For any $\nu \in \mathbb{C}$, the function $f^{(\nu)}(g) = a(g)^{\nu+\rho_G} f(g)$ can be viewed as an element of the intertwining space $\mathrm{Hom}_{\mathcal{U}_\infty}(\tau_\Lambda^*, I(\sigma,\chi,\nu))$. The $W(\Lambda)$-valued Eisenstein series

$$E(f^{(\nu)};g) = \sum_{\gamma\in P(F)\backslash G(F)} f^{(\nu)}(\gamma g), \quad g \in G(\mathbb{A}_F), \tag{5.2}$$

convergent when $\mathrm{Re}(\nu) > \rho_G$, has a meromorphic continuation to $\mathbb{C}$ holomorphic on the imaginary axis $\mathrm{Re}(\nu) = 0$.

5.5.1. The distinguished Eisenstein series

Let Y_F be the set of unitary idèle class characters of $F^\times$ trivial on $\{\underline{\tau} \mid \tau > 0\}$. For $\eta \in Y_F$, put $(I(\eta|\cdot|_F^\nu), V(\eta)) = (I(\mathbf{1}, \eta\circ \mathrm{N}_{E/F}), V(\mathbf{1}, \eta\circ\mathrm{N}_{E/F}))$, where $\mathbf{1}$ denotes the trivial representation of $G_1(\mathbb{A}_F)$ on the constant functions. For $\eta \in Y_F$ and $v \in \Sigma_\infty$, let $b_v(\eta) \in \mathbb{R}$ be the unique real number such that $\eta_v(x) = |x|_{F_v}^{ib_v(\eta)}$ for all positive $x \in F_v^\times$. Set

$$Q_\infty(\eta|\cdot|_F^\nu) = \prod_{v\in\Sigma_\infty} \bigl(1 + |\nu + ib_v(\eta)|^2\bigr), \quad \nu \in \mathbb{C}.$$

We call the vector $\nu_\infty(\eta|\cdot|_F^\nu) = \{\nu + ib_v(\eta)\}_{v\in\Sigma_\infty}$ the *Archimedean spectral parameter* of the principal series $I(\eta|\cdot|_F^\nu) = \bigotimes_v I(\eta_v|\cdot|_{F_v}^\nu)$. Given $\mathfrak{d} = \{d_v\}_{v\in\Sigma_\infty} \in \mathbb{N}_0^{\Sigma_\infty}$, let $(\tau_\mathfrak{d}, W(\mathfrak{d}))$ denote the irreducible unitary $\mathcal{U}_\infty$-module obtained as the external tensor product of $\mathcal{U}_v$-modules (τ_{d_v}, W_{d_v}) (see Section 3.1) over all $v \in \Sigma_\infty$. Set $\mathcal{H}_\mathfrak{d}(\eta) = \mathrm{Hom}_{\mathcal{U}_\infty}(W(\mathfrak{d})^*, V(\eta))$ identified with a subspace of $W(\mathfrak{d})$-valued smooth functions on $G(\mathbb{A}_F)$. Let $\hat{\mathcal{H}}_\mathfrak{d}(\eta)$ be the Hilbert space completion of $\mathcal{H}_\mathfrak{d}(\eta)$ by the Hermitian inner product

$$(f|f_1)_{\eta,\mathfrak{d}} = \int_{\mathcal{U}} \bigl(f(k)|f_1(k)\bigr)\,\mathrm{d}k, \quad f, f_1 \in \mathcal{H}_\mathfrak{d}(\eta),$$

with $\mathrm{d}k$ the probability Haar measure on $\mathcal{U}$, and fix an orthonormal basis $\mathcal{B}_\mathfrak{d}(\eta)$ of $\hat{\mathcal{H}}_\mathfrak{d}(\eta)$ contained in $\mathcal{H}_\mathfrak{d}(\eta)$ and consisting of decomposable functions. The Eisenstein series $E(f^{(\nu)})$ with $f \in \mathcal{H}_\mathfrak{d}(\eta)$ play an important role; they are referred to as the *distinguished Eisenstein series*. It is well known that there exists a meromorphic function $m_G(\eta,\nu) : I(\eta|\cdot|_F^\nu) \to I(\bar\eta|\cdot|_F^{-\nu})$ such that the constant term of $E(f^{(\nu)})$ along the parabolic P is

$$\int_{N(F)\backslash N(\mathbb{A}_F)} E(f^{(\nu)}, ng)\,\mathrm{d}n = f^{(\nu)}(g) + \bigl(m_G(\eta,\nu)f\bigr)^{(-\nu)}(g), \quad g \in G(\mathbb{A}_F), \tag{5.3}$$

where $\mathrm{d}n$ is the Haar measure on $N(\mathbb{A}_F)$ with $\mathrm{vol}(N(F)\backslash N(\mathbb{A}_F)) = 1$. From Lemma 4.4 and [3, Theorem 8.2], there exists a positive constant $C > 0$ such that

$$\begin{aligned}\bigl[m_G(\eta,\nu)f\bigr](g) = C\,&\frac{L^\infty(\nu - \frac{m-3}{2}, \eta\circ \mathrm{N}_{E/F})L^\infty(2\nu, \varepsilon_{E/F}^m\eta^2)}{L^\infty(\nu + \frac{m-1}{2}, \eta\circ \mathrm{N}_{E/F})L^\infty(2\nu+1, \varepsilon_{E/F}^m\eta^2)}\\ &\times \Bigl\{\prod_{v\in\Sigma_{\mathrm{fin}}} \bigl[R_v(\eta,\nu)(f_v)\bigr](g_v)\Bigr\}\\ &\times \prod_{v\in\Sigma_\infty} \frac{2^{-2(\nu+ib_v(\eta))}\Gamma(2(\nu+ib_v(\eta)))\Gamma(\nu+ib_v(\eta)-\frac{m-3}{2})^2}{\Gamma(\nu+ib_v(\eta)-\frac{m-3}{2}-d_v)^2\Gamma(\nu+ib_v(\eta)+\frac{m-1}{2}+d_v)^2}\\ &\times \Bigl\{\bigotimes_{v\in\Sigma_\infty} f_v(g_v)\Bigr\}\end{aligned} \tag{5.4}$$

for any decomposable elements $f = \bigotimes_v f_v \in \mathcal{H}_\mathfrak{d}(\eta)$, where $R_v(\eta_v,\nu) : I(\eta_v|\cdot|_{F_v}^\nu) \to I(\bar\eta_v|\cdot|_{F_v}^{-\nu})$ is the normalized intertwining operator studied in Section 4.2.

LEMMA 5.1
The distinguished Eisenstein series $E(f^{(\nu)})$ is holomorphic on $\mathrm{Re}(\nu) \geq 0$ except

for a possible pole at $\nu=\frac{m-1}{2}$. The pole occurs only when η is trivial or $\varepsilon_{E/F}$ and $d_v=0$ for all $v\in\Sigma_\infty$.

Proof
Suppose that $\eta=1$ or $\varepsilon_{E/F}$. From (5.4) and by Proposition 4.5, $m_G(\nu;f)$ is holomorphic on $\mathrm{Re}(\nu)\geq 0$ except at $\nu=\frac{m-1}{2}$. □

LEMMA 5.2
For any $\delta>0$ and a compact interval $I\subset(0,+\infty)$, there exists a constant $N>0$ such that

$$\left\|m_G(\eta,\nu)f\right\|_{\eta,\mathfrak{d}}\ll Q_\infty\big(\eta|\cdot|_F^{\nu}\big)^N,\quad f\in\mathcal{B}_{\mathfrak{d}}(\eta),\nu\in\mathcal{T}_{\delta,I},\eta\in Y_F.$$

There exists a constant $N_0>0$ such that

$$\left|\big(m_G(\eta,\nu)m_G'(\eta,\nu)f|f\big)_{\eta,\mathfrak{d}}\right|\ll Q_\infty\big(\eta|\cdot|_F^{\nu}\big)^{N_0},\quad f\in\mathcal{B}_{\mathfrak{d}}(\eta),\nu\in i\mathbb{R},\eta\in Y_F,$$

where $m_G'(\eta,\nu)$ denotes the derivative of $m_G(\eta,\nu)\in\mathrm{End}_{\mathbb{C}}(V(\eta))$ with respect to ν.

Proof
In (5.4), the factor $[R(\eta_v,\nu)f_v](g_v)$, which is identically 1 except for a finite number of v's, is bounded on $\mathcal{T}_{\delta,I}$ because it is holomorphic on $\mathcal{T}_{\delta,I}$ (see Proposition 4.5) and $\log q_v$-periodic in $\mathrm{Im}(\nu)$; the gamma factor is also bounded on $\mathcal{T}_{\delta,I}$ uniformly in η by Stirling's formula. The L-values in the denominator are bounded from below by a constant uniformly in $z\in\mathcal{T}_{\delta,I}$ and η, due to the absolute convergence of the Euler products. By the convexity bound of $L^\infty(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F})$ and $L^\infty(2\nu,\varepsilon_{E/F}^m\eta^2)$ in the numerator, we have the polynomial bound

$$\left\|m_G(\eta,\nu)f\right\|_{\eta,\mathfrak{d}}\ll Q_\infty\big(\eta|\cdot|_F^{\nu}\big)^N,\quad \nu\in\mathcal{T}_{\delta,I},\eta\in Y_F,\tag{5.5}$$

with some $N>0$. For the logarithmic derivative, we argue as in [7, Proposition 2]; as an ingredient, we need a polynomial bound of $L^\infty(\pm\nu+\frac{m-1}{2},\eta\circ\mathrm{N}_{E/F})^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}L^\infty(\pm\nu+\frac{m-1}{2},\eta\circ\mathrm{N}_{E/F})$ and $L^\infty(1\pm 2\nu,\eta^2\varepsilon_{E/F}^m)^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}L^\infty(1\pm 2\nu,\eta^2\varepsilon_{E/F}^m)$ uniform in $\eta\in Y_F$. Since $I\subset[0,\frac{m+1}{2})$ implies that $\mathrm{Re}(-\nu+\frac{m-1}{2})>1$ for all $\nu\in i\mathbb{R}$, the former one is bounded, due to the convergence of the Euler product. For the latter one, which is more delicate because it involves values of Hecke's L-function at the boundary of the critical strip, we apply [1] to have its uniform majorant of the form $Q_\infty(\eta|\cdot|_F^{\nu})^N$ with some $N>0$. □

5.6. Smoothed Eisenstein series

From (i) and (ii) in Section 5.1, $\dim_F(\ell^\perp)=2(m-1)\geq 6$ and the signature of $\ell^\perp$ at all $v\in\Sigma_\infty$ is $(1-,(m-2)+)$. By the same reasoning as in Section 5.1, we fix a pair of F-isotropic vectors e_H, e_H' both orthogonal to ℓ such that $\mathbf{h}(e_H,e_H')=1$ once and for all. Then P_H, the stabilizer of Ee_H in H, is an F-parabolic subgroup of H. We fix a maximal $\mathfrak{o}_E$-lattice $\mathcal{L}_H$ which contains $\mathfrak{o}_Ee_H+\mathfrak{o}_Ee_H'$ as an $\mathfrak{o}_E$-direct summand. For any $v\in\Sigma_{\mathrm{fin}}$, let $\mathcal{U}_{H,v}$ be the stabilizer of $\mathcal{L}_{H,v}=\mathcal{L}_H\otimes_{\mathfrak{o}_E}\mathfrak{o}_{E,v}$ in $H(F_v)$. For $v\in\Sigma_\infty$, set $\mathcal{U}_{H,v}=\mathcal{U}_v\cap H(F_v)$. We define $\mathcal{U}_H$ to be the direct product of $\mathcal{U}_{H,v}$ over all $v \in \Sigma^F$. The $\mathcal{U}_H$-spherical Eisenstein series on $H(\mathbb{A}_F)$ is defined by the meromorphic continuation to $\mathbb{C}$ of the absolutely convergent series

$$E_H(z;h) = \sum_{\delta \in P_H(F)\backslash H(F)} a_H(\delta h)^{z+\rho_H}, \quad \mathrm{Re}(z) > \rho_H, h \in H(\mathbb{A}_F),$$

where $\rho_H = \frac{m-2}{2}$, and $a_H : H(\mathbb{A}_F) \to \mathbb{R}_+$ is defined by the Iwasawa decomposition $H(\mathbb{A}_F) = P_H(\mathbb{A}_F)\mathcal{U}_H$ in the same way as $a : G(\mathbb{A}_F) \to \mathbb{R}_+$. Let $m_H(z)$ be the m-function for $E_H(z)$ describing the constant term along P_H, and let r_H denote the residue of $m_H(z)$ at the simple pole $z = \rho_H$. To regularize divergent integrals on $H(F)\backslash H(\mathbb{A}_F)$, following [23] and [31], we use the smoothed Eisenstein series on $H(\mathbb{A}_F)$ defined by

$$\mathcal{E}_{\beta,\lambda}(h) = \frac{r_H^{-1}}{2\pi i} \int_{(\sigma)} \frac{\beta(z)}{\lambda - z} E_H(z;h)\,\mathrm{d}z, \quad h \in H(\mathbb{A}_F), \mathrm{Re}(\lambda) > \rho_H, \tag{5.6}$$

with (σ) a vertical contour $\mathrm{Re}(z) = \sigma$ such that $\rho_H < \sigma < \mathrm{Re}(\lambda)$, where $\beta(z)$ is an entire function such that $\beta(\rho_H) = 1$ and $\sup\{|\beta(\sigma + it)|(1 + |t|)^N \mid t \in \mathbb{R}, \sigma \in [c_1, c_2]\} < \infty$ for all $N \in \mathbb{N}$ and real numbers $c_1 < c_2$.

LEMMA 5.3

The integral (5.6) converges absolutely, defining a holomorphic function on $\mathrm{Re}(\lambda) > \rho_H$. *For any* $\epsilon > 0$ *and any Siegel domain* $\mathfrak{S}_H$ *of* $H(\mathbb{A}_F)$ *defined by* P_H *and* $\mathcal{U}_H$*, there exists a constant* $C > 0$ *such that*

$$\left|(\lambda - \rho_H)\mathcal{E}_{\beta,\lambda}(h)\right| \le C a_H(h)^{-\mathrm{Re}(\lambda)+\rho_H}, \quad h \in \mathfrak{S}_H, \mathrm{Re}(\lambda) \in (\rho_H, \rho_H + \epsilon). \tag{5.7}$$

Moreover, for all $h \in H(\mathbb{A}_F)$*, we have the pointwise convergence*

$$\lim_{\lambda \to \rho_H + 0} (\lambda - \rho_H)\mathcal{E}_{\beta,\lambda}(h) = 1.$$

Proof

Let $\delta > 0$, and let $I \subset (0, +\infty)$ be a compact interval. We need a uniform estimate like [7, Corollary 2] for our Eisenstein series $E_H(z)$, which is induced from cuspidal because P_H/N_H is F-anisotropic. Although the setting of [7] does not cover our case in a strict sense, the argument in [7, Section 5.3] can be modified to be applied to $E_H(z)$. The crucial point is the estimation of the L^2-norm of the truncated Eisenstein series $\|\Lambda^T E_H(z)\|_2$ which, by the Maass–Selberg relation, boils down to an upper bound of the function $m_H(z)$ in the vertical strip $\mathcal{T}_{\delta,I}$ (cf. [7, Proposition 2]). In our case, since I is in $\mathrm{Re}(z) > 0$, the Maass–Selberg relation takes the form

$$\begin{aligned} C\left\|\Lambda^T E_H(z)\right\|_2 &= \frac{e^{2xT} - e^{-2xT}}{2x} + \frac{e^{-2xT}}{2x}\left(1 - \left|m_H(z)\right|^2\right) \\ &\quad + \frac{m_H(\bar{z})e^{2iyT} - m_H(z)e^{-2iyT}}{iy}, \\ &\quad z = x + iy \in \mathcal{T}_{\delta,I}, \end{aligned}$$

with a constant $C>0$ and no logarithmic derivative of $m_H(z)$ involved. To estimate this, the first assertion of Lemma 5.2 (applied to $E_H(z)$ and $m_H(z)$) is enough. The remaining part of [7, Section 5.3] goes through as it is. Consequently, for any element D of the universal enveloping algebra of $N_H(F_\infty)$, there exists $N>0$ such that

$$\left|\left[E_H(z)*D\right](h)\right| \ll \left(1+\left|\mathrm{Im}(z)\right|\right)^N a_H(h)^N, \quad z\in\mathcal{T}_{\delta,I}, h\in\mathfrak{S}_H.$$

By this, from [18, Lemma I.2.10], we can deduce the following estimate for the nonconstant term of the Eisenstein series $E_H^*(z;h)=E_H(z;h)-\{a_H(h)^{z+\rho_H}+m_H(z)a_H(h)^{-z+\rho_H}\}$:

$$\text{(5.8)} \qquad \left|E_H^*(z;h)\right| \ll_{N_1} \left(1+\left|\mathrm{Im}(z)\right|\right)^N a_H(h)^{-N_1}, \quad z\in\mathcal{T}_{\delta,I}, h\in\mathfrak{S}_H,$$

where (and below) $N_1>0$ is an arbitrary large number. We have

$$\mathcal{E}_{\beta,\lambda}(h)=I_+(\lambda,h)+I^*(\lambda,h)+I_-(\lambda,h)$$

for $h\in\mathfrak{S}_H$ with

$$I_+(\lambda,h)=\frac{r_H^{-1}}{2\pi i}\int_{(\sigma)}\frac{\beta(z)}{\lambda-z}a_H(h)^{z+\rho_H}\,\mathrm{d}z,$$

$$I^*(\lambda,h)=\frac{r_H^{-1}}{2\pi i}\int_{(\sigma)}\frac{\beta(z)}{\lambda-z}E_H^*(z;h)\,\mathrm{d}z,$$

$$I_-(\lambda,h)=\frac{r_H^{-1}}{2\pi i}\int_{(\sigma)}\frac{\beta(z)}{\lambda-z}m_H(z)a_H(h)^{-z+\rho_H}\,\mathrm{d}z,$$

where $\rho_H<\sigma<\mathrm{Re}(\lambda)$. The integral $I_+(\lambda,h)$ can be estimated as $|I_+(\lambda,h)|\ll a_H(h)^{-N_1}$ on $\mathfrak{S}_H$ by shifting the contour (σ) far to the left. From (5.8), the contour (σ) in $I^*(\lambda,h)$ can be shifted to any vertical line in the half-plane $\mathrm{Re}(z)>0$; by this, we have a holomorphic continuation of $I^*(\lambda,h)$ to $\mathrm{Re}(\lambda)>0$ with the estimation $|I^*(\lambda,h)|\ll a_H(h)^{-N_1}$ on $\mathfrak{S}_H$. In these estimations for $I_+(\lambda,h)$ and $I^*(\lambda,h)$, the implied constants are taken to be uniform for λ lying in the strip $\rho_H<\mathrm{Re}(\lambda)\le\rho_H+\epsilon$. By shifting the contour in $I_-(\lambda,h)$ far to the right (beyond λ) and accounting for the residue at $z=\lambda$, we have the expression

$$\text{(5.9)} \quad I_-(\lambda,h)=\frac{\beta(\rho_H)}{\lambda-\rho_H}a_H(h)^{-\lambda+\rho_H}+\frac{r_H^{-1}}{2\pi i}\int_{(\sigma_1)}\frac{\beta(z)}{\lambda-z}m_H(z)a_H(h)^{-z+\rho_H}\,\mathrm{d}z,$$

whose second term is holomorphic on the half-plane $\mathrm{Re}(\lambda)<\sigma_1$ and is estimated by $a_H(h)^{-\sigma_1+\rho_H}$. This completes the proof of (5.7). Let us show the second assertion. We already see that $I_+(\lambda,h)+I^*(\lambda,h)$ is holomorphic at $\lambda=\rho_H$. By (5.9), we have a meromorphic continuation of $I_-(\lambda,h)$ around $\lambda=\rho_H$ and $\lim_{\lambda\to\rho_H+0}(\lambda-\rho_H)I_-(\lambda,h)=\lim_{\lambda\to\rho_H+0}(\lambda-\rho_H)r_H^{-1}m_H(\lambda)=1$. □

5.7. The H-periods of automorphic forms

When we consider the automorphic forms on $G(\mathbb{A}_F)$, they are always required to be $\mathcal{U}$-finite under the right-translation.

Let (τ, W) be a finite-dimensional continuous representation of $\mathcal{U}_\infty$. A function $\varphi : G(\mathbb{A}_F) \to W$ is called a *W-valued automorphic form* if it has the $\mathcal{U}_\infty$-equivariance $\varphi(gk_\infty) = \tau(k_\infty)^{-1}\varphi(g)$ for all $k_\infty \in \mathcal{U}_\infty$ and if, for any $w^* \in W^*$, the coefficient $\langle w^*, \varphi(g)\rangle$ is an automorphic form in the usual sense (see [18, Section I.2.17]). For such φ, if the integral

$$\mathcal{P}_H(\varphi) = \int_{H(F)\backslash H(\mathbb{A}_F)} \varphi(h)|\omega_H|_{\mathbb{A}}$$

is absolutely convergent, it is called the *H-period integral* of φ. By the $\mathcal{U}_\infty$-equivariance of φ, we have $\mathcal{P}_H(\varphi) \in W^{H(\mathbb{A}_F)\cap \mathcal{U}_\infty}$.

LEMMA 5.4
Let $\mathfrak{S}_H \subset H(\mathbb{A}_F)$ be a Siegel domain with respect to P_H and $\mathcal{U}_H$. Let $s \in \mathbb{R}$, and let ξ be a smooth $\mathbb{C}$-valued function on $H(F)\backslash H(\mathbb{A}_F)$ such that $|\xi(h)| \le B_\xi \|h\|_G^s$ on $\mathfrak{S}_H$ for a constant $B_\xi > 0$. Let φ a W-valued automorphic form such that $\|\varphi(g)\| \le B_\varphi \|g\|_G^r$ on $G(\mathbb{A}_F)$ with some constant $B_\varphi > 0$ and $r > 0$. Let $U \subset G(\mathbb{A}_F)$ be a compact set. If $s + r < 2(m-2)$, then

$$\int_{h\in H(F)\backslash H(\mathbb{A}_F)} \big\|\xi(h)\varphi\big(h\mathsf{a}(t)g\big)\big\| |\omega_H|_{\mathbb{A}} \le B_\xi B_\varphi C_0 \prod_{v\in\Sigma_\infty} (\cosh t_v)^r, \quad t \in \mathbb{R}^{\Sigma_\infty}, g \in U,$$

with a constant C_0 only dependent on r and s. In particular, the integral $\int_{H(F)\backslash H(\mathbb{A}_F)} \xi(h)\varphi(h)|\omega_H|_{\mathbb{A}}$ converges absolutely. The H-period integral $\mathcal{P}_H(\varphi)$ converges absolutely if $r < 2(m-2)$.

Proof
Fix $\gamma_0 \in G(F)$ such that $\gamma_0 e_H = e$ and $\gamma_0 e'_H = e'$. Then $P_H = \gamma_0^{-1} P \gamma_0 \cap H$. We can choose a Siegel domain $\mathfrak{S}_G$ satisfying $\mathfrak{S}_H \subset H(\mathbb{A}_F) \cap \mathfrak{S}_G^{\gamma_0}$. We have

$$\begin{aligned}\big\|\xi(h)\varphi\big(h\mathsf{a}(t)g\big)\big\| &\le B_\xi B_\varphi \|h\|_G^s \big\|h\mathsf{a}(t)g\big\|_G^r \\ &\le B_\xi B_\varphi \|h\|_G^{s+r}\big\|\mathsf{a}(t)\big\|_G^r, \quad h \in \mathfrak{S}_H, t \in \mathbb{R}^{\Sigma_\infty}, g \in U.\end{aligned}$$

Since $a(\gamma_0 h \gamma_0^{-1}) \asymp a_H(h)$,

$$\|h\|_G \asymp \|\gamma_0 h \gamma_0^{-1}\|_G \asymp a_G(\gamma_0 h\gamma_0^{-1})^{1/2} \asymp a_H(h)^{1/2}$$

for $h \in \mathfrak{S}_H$. Hence

$$\int_{\mathfrak{S}_H} \|h\|_G^{s+r}|\omega_H|_{\mathbb{A}} \ll \int_{t_0}^\infty \tau^{s+r}\tau^{-2(m-2)}\, \mathrm{d}^\times \tau,$$

whose majorant is convergent for $s + r - 2(m-2) < 0$. Since $\|\mathsf{a}(t)\|_G = \prod_{v\in\Sigma_\infty} \|a_v^{(t_v)}\|_{G(E_v)} \asymp \prod_{v\in\Sigma_\infty} \cosh t_v$, we are done. □

5.8. Eisenstein periods

From Lemma 5.4 applied to $\varphi = E(f^{(\nu)})$ with $r = 2(|\operatorname{Re}(\nu)| + \rho_G)$, we have the following.

COROLLARY 5.5

Let ξ be a smooth function on $H(\mathbb{A}_F)$ such that $|\xi(h)| \ll a(h)^\beta$ on $\mathfrak{S}_H$ with some $\beta \in \mathbb{R}$. If ν is a regular point for the Eisenstein series $E(f^{(\nu)})$ such that $\beta + |\operatorname{Re}(\nu)| < \frac{m-3}{2}$, the integral $\int_{H(F)\backslash H(\mathbb{A}_F)} \xi(h)E(f^{(\nu)};h)|\omega_H|_{\mathbb{A}}$ converges absolutely. The H-period integral $\mathcal{P}_H(E(f^{(\nu)}))$ converges absolutely for $\nu \in i\mathbb{R}$.

Proof

Since $m \geq 4$, for the constant function $\xi(h) = 1$ (with $\beta = 0$), the convergence region $|\operatorname{Re}(\nu)| < \frac{m-3}{2}$ contains $i\mathbb{R}$. □

The triple (χ, σ, Λ) is called *distinguished* if the following conditions are satisfied.

(i) We have that σ is the trivial representation of $G_1(\mathbb{A}_F)$.

(ii) There is a unitary idèle class character η of $F^\times$ such that $\chi = \eta \circ \mathrm{N}_{E/F}$.

(iii) For all $v \in \Sigma_\infty$, $c_v = 0$. There exists $\mathfrak{d} = \{d_v\}_{v\in\Sigma_\infty} \in \mathbb{N}_0^{\Sigma_\infty}$ such that the dominant weight $l_v = \{l_v(j)\}_{1\leq j\leq m-1}$ is given by $l_v(1) = d_v$, $l_v(j) = 0$ $(1 < j < m-1)$, and $l_v(m-1) = -d_v$. If this is the case, we write $\Lambda = \Lambda_{\mathfrak{d}}$.

THEOREM 5.6

(1) *Let $\nu \in i\mathbb{R}$. The H-period integral $\mathcal{P}_H(E(f^{(\nu)}))$ is zero unless (χ, σ, Λ) is distinguished. Suppose that (χ, σ, Λ) is distinguished and that f is a pure tensor $\bigotimes f_v$ with $f_v \in I_v(\eta_v)$ if $v \in \Sigma_{\mathrm{fin}}$ and $f_v \in \operatorname{Hom}_{\mathcal{U}_v}(W_{d_v}^*, I_v(\eta_v))$ if $v \in \Sigma_\infty$, where $\Lambda = \Lambda_{\mathfrak{d}}$ with $\mathfrak{d} = \{d_v\}_{v\in\Sigma_\infty} \in \mathbb{N}_0^{\Sigma_\infty}$. Then we have*

$$
\begin{aligned}
\mathcal{P}_H\big(E(f^{(\nu)})\big) &= \frac{4|D_{F/\mathbb{Q}}|^{-2(m-1)}L(1,\varepsilon_{E/F})^2}{L^\infty(m-1,\varepsilon_{E/F}^{m-1})}\,\frac{L^\infty(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F})}{L^\infty(2\nu+1,\eta^2\varepsilon_{E/F}^m)} \\
&\quad\times \prod_{v\in\Sigma_{\mathrm{fin}}} \Xi^0\big(\eta_v|\cdot|_{F_v}^\nu; f_v\big) \\
&\quad\times \Big\{ \bigotimes_{v\in\Sigma_\infty} \frac{2^{2-m}\pi|\sqrt{\theta}|_{E_v}^{(1-m)/2}\Gamma_{\mathbb{C}}(\nu+ib_v(\eta)-\frac{m-3}{2})^2}{\Gamma_{\mathbb{C}}(\nu+ib_v(\eta)-\frac{m-3}{2}-d_v)\Gamma_{\mathbb{C}}(\nu+ib_v(\eta)+\frac{m-1}{2}+d_v)} \\
&\quad\times \operatorname{Pr}_v\big(f_v(1)\big)\Big\},
\end{aligned}
\tag{5.10}
$$

where $\operatorname{Pr}_v : W_{d_v} \to W_{d_v}^{\mathcal{U}_v\cap H(F_v)}$ denotes the orthogonal projector.

(2) *Let $\nu = r > 0$ be a pole of the Eisenstein series $E(f^{(\nu)})$, and set $\varphi_r = \operatorname{Res}_{\nu=r} E(f^{(\nu)})$. We have $\mathcal{P}_H(\varphi_r) = 0$ unless (χ, σ, Λ) is distinguished and $r = \frac{m-1}{2}$.*

5.9. Proof of Theorem 5.6

For any Paley–Wiener function $\alpha(\nu)$ such that $\alpha(\nu) = \alpha(-\nu)$, we define the wave packet of α by

$$
\hat{E}(\alpha, f; g) = \frac{1}{2\pi}\int_{\mathbb{R}} E(f^{(it)}; g)\alpha(it)\,\mathrm{d}t, \quad g \in G(\mathbb{A}_F), \tag{5.11}
$$

and consider the integral

$$I(\lambda) = \int_{H(F)\backslash H(\mathbb{A}_F)} \mathcal{E}_{\beta,\lambda}(h)\hat{E}(\alpha, f; h)|\omega_H|_{\mathbb{A}}.$$

Since $|\alpha(it)| = O((1+|t|)^{-N})$ $(|t| \to +\infty)$ for any $N > 0$, $|\hat{E}(\alpha, f; g)| \ll a(g)^{\rho_G}$ on $\mathfrak{S}_G$. Hence from Lemma 5.3 and Corollary 5.5, the integral $I(\lambda)$ converges absolutely when $\mathrm{Re}(\lambda) > 1/2$, defining a holomorphic function. We will compute $\lim_{\lambda \to \rho_H + 0} I(\lambda)$. From [18, Proposition IV.1.11], the Eisenstein series $E(f^{(\nu)})$ has a finite number of poles s_j $(1 \le j \le r)$ in $\mathrm{Re}(\nu) \ge 0$, which are all simple and on the interval $(0, \rho_G]$. Let ϕ_j denote the residue of $E(f^{(\nu)})$ at $\nu = s_j$. Then ϕ_j is an L^2-automorphic form on $G(\mathbb{A}_F)$ with the estimation $\|\phi_j(g)\| \ll a(g)^{-\nu_j+\rho_G}$ on a Siegel domain $\mathfrak{S}_G$ of $G(\mathbb{A}_F)$. By shifting the contour in (5.11) to the convergence region $\sigma > \rho_G$, we have

$$\hat{E}(\alpha, f; g) = \frac{1}{2\pi i}\int_{(\sigma)} E(f^{(\nu)}; g)\alpha(\nu)\,\mathrm{d}\nu - \sum_j \phi_j(g)\alpha(s_j).$$

By plugging this and changing the order of integrals, we have

$$I(\lambda) = \frac{1}{2\pi i}\int_{(\sigma)} J(\lambda, \nu)\alpha(\nu)\,\mathrm{d}\nu - \sum_j \alpha(s_j) R_j(\lambda), \tag{5.12}$$

where

$$J(\lambda, \nu) = \int_{H(F)\backslash H(\mathbb{A}_F)} E(f^{(\nu)}; h)\mathcal{E}_{\beta,\lambda}(h)|\omega_H|_{\mathbb{A}},$$

$$R_j(\lambda) = \int_{H(F)\backslash H(\mathbb{A}_F)} \mathcal{E}_{\beta,\lambda}(h)\phi_j(h)|\omega_H|_{\mathbb{A}}.$$

Fubini's theorem can be applied to obtain (5.12) since the integrals $J(\lambda, \nu)$ and $R_j(\lambda)$ are seen to be absolutely convergent for $\mathrm{Re}(\lambda) > \sigma + 1/2$ from Lemmas 5.3 and 5.4. Let $\mathrm{Re}(\nu) > \rho_G$ and $\mathrm{Re}(\lambda) > \mathrm{Re}(\nu) + 1/2$. Then from the series expression (5.2), we have

$$\begin{aligned} J(\lambda, \nu) &= \int_{H(F)\backslash H(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash G(F)} f^{(\nu)}(\gamma h)\mathcal{E}_{\beta,\lambda}(h)|\omega_H|_{\mathbb{A}} \\ &= \sum_{\delta \in P(F)\backslash G(F)/H(F)} \int_{H(F)\backslash H(\mathbb{A}_F)} \sum_{\gamma \in H_\delta(F)\backslash H(F)} f^{(\nu)}(\delta\gamma h)\mathcal{E}_{\beta,\lambda}(\gamma h)|\omega_H|_{\mathbb{A}} \\ &= \sum_{\delta \in P(F)\backslash G(F)/H(F)} \int_{H_\delta(F)\backslash H(\mathbb{A}_F)} f^{(\nu)}(\delta h)\mathcal{E}_{\beta,\lambda}(h)|\omega_H|_{\mathbb{A}}, \end{aligned}$$

where $H_\delta = H \cap \delta^{-1}P\delta$. Let $\gamma_0 \in G(F)$ be as in the proof of Lemma 5.4. Then $P(F)\backslash G(F)/H(F) = \{e, \gamma_0\}$. Hence $J(\lambda, \nu) = J_e(\lambda, \nu) + J_{\gamma_0}(\lambda, \nu)$ with $J_\delta(\lambda, \nu)$ denoting the integrals in the last displayed formula. Let us examine the integral $J_{\gamma_0}(\lambda, \nu)$. Let $P_{H,1}$ be the stabilizer in P_H of the vector e_H. Set $[\tau]_H = \gamma_0^{-1}[\tau]\gamma_0$ for $\tau \in \mathbb{A}_E^\times$. By the Iwasawa decomposition $H(\mathbb{A}_F) = \{[\tau]_H \mid \tau \in \mathbb{A}_E^\times\} P_{H,1}(\mathbb{A}_F) \times \mathcal{U}_H$,

$$\begin{aligned}J_{\gamma_0}(\lambda,\nu) &= \int_{P_H(F)\backslash H(\mathbb{A}_F)} \mathcal{E}_{\beta,\lambda}(h) f^{(\nu)}(\gamma_0 h)|\omega_H|_{\mathbb{A}} \\ &= \int_{E^\times\backslash\mathbb{A}_E^\times}\int_{P_{H,1}(F)\backslash P_{H,1}(\mathbb{A}_F)}\int_{\mathcal{U}_H}\mathcal{E}_{\beta,\lambda}\big(l[\tau]_H\big) \\ &\quad\times f^{(\nu)}\big(\gamma_0 n l[\tau]_H k\big)|\tau|_E^{-2\rho_H}\Big|\frac{\mathrm{d}\tau\wedge\mathrm{d}\bar\tau}{2\sqrt{\theta}\tau\bar\tau}\Big|_{\mathbb{A}}\mathrm{d}l\,\mathrm{d}k,\end{aligned}$$

where $\mathrm{d}l$ and $\mathrm{d}k$ are some Haar measures on $P_{H,1}(\mathbb{A}_F)$ and $\mathcal{U}_H$, respectively. We may assume that $P_{H,1}(F)\backslash P_{H,1}(\mathbb{A}_F)$ has volume 1. Since $\gamma_0^{-1}P\gamma_0\cap H = P_H$ and $\gamma_0^{-1}P_1\gamma_0\cap H = P_{H,1}$, we have

$$\begin{aligned}J_{\gamma_0}(\lambda,\nu) &= \int_{E^\times\backslash\mathbb{A}_E^\times}\Big\{\int_{P_{H,1}(F)\backslash P_{H,1}(\mathbb{A}_F)}\mathcal{E}_{\beta,\lambda}\big(l[\tau]_H\big)\,\mathrm{d}l\Big\} \\ &\quad\times\Big\{\int_{\mathcal{U}_H} f^{(\nu)}\big([\tau]\gamma_0 k\big)\,\mathrm{d}k\Big\}|\tau|_E^{-2\rho_H}\Big|\frac{\mathrm{d}\tau\wedge\mathrm{d}\bar\tau}{2\sqrt{\theta}\tau\bar\tau}\Big|_{\mathbb{A}} \\ &= \int_{E^\times\backslash\mathbb{A}_E^\times}\Big\{\int_{P_{H,1}(F)\backslash P_{H,1}(\mathbb{A}_F)}\mathcal{E}_{\beta,\lambda}\big(l[\tau]_H\big)\,\mathrm{d}l\Big\} \\ &\quad\times\Big\{\int_{\mathcal{U}_H} f^{(\nu)}(\gamma_0 k)\,\mathrm{d}k\Big\}|\tau|_E^{\nu+\rho_G-2\rho_H}\Big|\frac{\mathrm{d}\tau\wedge\mathrm{d}\bar\tau}{2\sqrt{\theta}\tau\bar\tau}\Big|_{\mathbb{A}}.\end{aligned}$$

From (5.3),

$$\int_{P_{H,1}(F)\backslash P_{H,1}(\mathbb{A}_F)}\mathcal{E}_{\beta,\lambda}\big(l[\tau]_H\big)\,\mathrm{d}l = \frac{r_H^{-1}}{2\pi i}\int_{(c)}\frac{\beta(z)}{\lambda-z}\big(|\tau|_E^{z+\rho_H}+m_H(z)|\tau|_E^{-z+\rho_H}\big)\,\mathrm{d}z.$$

Substituting this into the last expression of $J_{\gamma_0}(\lambda,\nu)$, we have the formula

$$J_{\gamma_0}(\lambda,\nu) = r_H^{-1}\Big(\int_{\mathcal{U}_H} f^{(\nu)}(\gamma_0 k)\,\mathrm{d}k\Big)\big(\mathfrak{t}_1(\lambda,\nu)+\mathfrak{t}_2(\lambda,\nu)\big) \tag{5.13}$$

with

$$\begin{aligned}\mathfrak{t}_1(\lambda,\nu) &= \int_{E^\times\backslash\mathbb{A}_E^\times}\Big\{\frac{1}{2\pi i}\int_{(c)}\frac{\beta(z)}{\lambda-z}|\tau|_E^{z+\nu+\rho_G-\rho_H}\,\mathrm{d}z\Big\}\Big|\frac{\mathrm{d}\tau\wedge\mathrm{d}\bar\tau}{2\sqrt{\theta}\tau\bar\tau}\Big|_{\mathbb{A}}, \\ \mathfrak{t}_2(\lambda,\nu) &= \int_{E^\times\backslash\mathbb{A}_E^\times}\Big\{\frac{1}{2\pi i}\int_{(c)}\frac{\beta(z)}{\lambda-z}m_H(z)|\tau|_E^{-z+\nu+\rho_G-\rho_H}\,\mathrm{d}z\Big\}\Big|\frac{\mathrm{d}\tau\wedge\mathrm{d}\bar\tau}{2\sqrt{\theta}\tau\bar\tau}\Big|_{\mathbb{A}},\end{aligned}$$

which are evaluated by the following lemma. Note that $\rho_G-\rho_H = 1/2$.

LEMMA 5.7

Let $\mathrm{Re}(\nu) > \rho_G$ *and* $\mathrm{Re}\,\lambda > \mathrm{Re}\,\nu + 1/2$. *Then,*

$$\begin{aligned}\mathfrak{t}_1(\lambda,\nu) &= \mathrm{vol}(E^\times\backslash\mathbb{A}_E^1)\frac{\beta(-\nu-1/2)}{\lambda+(\nu+1/2)}, \\ \mathfrak{t}_2(\lambda,\nu) &= \mathrm{vol}(E^\times\backslash\mathbb{A}_E^1)\frac{\beta(\nu+1/2)m_H(\nu+1/2)}{\lambda-(\nu+1/2)}.\end{aligned}$$

Proof

Let $c' < \mathrm{Re}(\nu)+1/2 < c$ and $1/2 < c' < \rho_H$. From Lemma 5.1, the only singularity of the function $m_H(z)$ on $\mathrm{Re}(z) > 1/2$ is the simple pole at $z=\rho_H$. We have

$$\begin{aligned}
&\mathrm{vol}(E^\times\backslash\mathbb{A}_E^1)^{-1}\mathfrak{t}_2(\lambda,\nu)\\
&\quad=\int_0^{+\infty}\Big\{\frac{1}{2\pi i}\int_{(c)}\frac{\beta(z)}{\lambda-z}m_H(z)t^{-z+\nu+1/2}\,\mathrm{d}z\Big\}\,\mathrm{d}^\times t\\
&\quad=\int_0^{1}\Big(\frac{1}{2\pi i}\int_{(c')}\frac{\beta(z)}{\lambda-z}m_H(z)t^{-z+\nu+1/2}\,\mathrm{d}z+\frac{r_H\beta(\rho_H)}{\lambda-\rho_H}t^{-\rho_H+\nu+1/2}\Big)\,\mathrm{d}^\times t\\
&\qquad+\int_1^{+\infty}\Big(\frac{1}{2\pi i}\int_{(c)}\frac{\beta(z)}{\lambda-z}m_H(z)t^{-z+\nu+1/2}\,\mathrm{d}z\Big)\,\mathrm{d}^\times t\\
&\quad=\frac{1}{2\pi i}\int_{(c')}\frac{\beta(z)}{\lambda-z}\frac{-m_H(z)}{z-(\nu+1/2)}\,\mathrm{d}z+\frac{r_H\beta(\rho_H)}{\lambda-\rho_H}\frac{1}{-\rho_H+\nu+1/2}\\
&\qquad+\frac{1}{2\pi i}\int_{(c)}\frac{\beta(z)}{\lambda-z}\frac{m_H(z)}{z-(\nu+1/2)}\,\mathrm{d}z\\
&\quad=(\mathrm{Res}_{z=\nu+1/2}+\mathrm{Res}_{z=\rho_H})\Big(\frac{\beta(z)}{\lambda-z}\frac{m_H(z)}{z-(\nu+1/2)}\Big)+\frac{r_H\beta(\rho_H)}{\lambda-\rho_H}\frac{1}{-\rho_H+\nu+1/2}\\
&\quad=\frac{\beta(\nu+1/2)}{\lambda-(\nu+1/2)}m_H(\nu+1/2).
\end{aligned}$$

Since $\beta(\rho_H)=1$, we are done. The computation of $\mathfrak{t}_1(\lambda,\nu)$ is similar. □

From (5.13) and Lemma 5.7, the term $J_{\gamma_0}(\lambda,\nu)$ is evidently meromorphic in $\lambda\in\mathbb{C}$, and for $\mathrm{Re}\,\nu>\rho_G$,

$$\lim_{\lambda\to\rho_H+0}(\lambda-\rho_H)J_{\gamma_0}(\lambda,\nu)=0 \tag{5.14}$$

uniformly in $\mathrm{Im}(\nu)$. Let us examine the integral $J_e(\lambda,\nu)$. Since G_1 is F-anisotropic (from the assumption in Section 5.1) and $H_P\cong E^1\times G_1$, the factor space $H_P(F)\backslash H_P(\mathbb{A}_F)$ is compact. We have

$$J_e(\lambda,\nu)=\int_{H_P(F)\backslash H(\mathbb{A}_F)}\mathcal{E}_{\beta,\lambda}(h)f^{(\nu)}(h)|\omega_H|_{\mathbb{A}}.$$

LEMMA 5.8

If $\mathrm{Re}(\nu)>\rho_G$, *then* $\int_{H_P(F)\backslash H(\mathbb{A}_F)}\|f^{(\nu)}(h)\||\omega_H|_{\mathbb{A}}<+\infty$. *The integral*

$$\boldsymbol{\Xi}(f^{(\nu)})=\int_{H_P(F)\backslash H(\mathbb{A}_F)}f^{(\nu)}(h)|\omega_H|_{\mathbb{A}}$$

is zero unless (χ,σ,Λ) *is distinguished.*

Proof

Since G_1 is F-anisotropic, $f^{(\nu)}(\mathsf{m}[g_1]uk)$ is bounded for $(g_1,u,k)\in G_1(\mathbb{A}_F)\times N(\mathbb{A}_F)\times\mathcal{U}$. From this remark, we have a constant $C>0$ such that $\|f^{(\nu)}(g)\|\le Ca(g)^{\mathrm{Re}(\nu)+\rho_G}$ for all $g\in G(\mathbb{A}_F)$. Since the function $a(g)$ is left $H_P(\mathbb{A}_F)$-invariant,

$$\int_{H_P(F)\backslash H(\mathbb{A}_F)} a(h)^{\mathrm{Re}(\nu)+\rho_G}|\omega_H|_{\mathbb{A}}$$
$$= \mathrm{vol}\big(H_P(F)\backslash H_P(\mathbb{A}_F)\big)\int_{H_P(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} a(h)^{\mathrm{Re}(\nu)+\rho_G}|\omega_{H_P\backslash H}|_{\mathbb{A}}.$$

The volume factor is finite since H_P is F-anisotropic. To have the first assertion of the lemma, it suffices to show that

$$\Xi_v = \int_{H_P(F_v)\backslash H(F_v)} a(g_v)^{\mathrm{Re}(\nu)+\rho_G}|\omega_{H_P\backslash H}|_{F_v} < +\infty$$

and $\prod_v \Xi_v < +\infty$. From Lemma 2.1, we have

$$\Xi_v = \oint_{P(F_v)\backslash G(F_v)} Y\big(|\cdot|_{F_v}^{\mathrm{Re}(\nu)}; g_v\big)a(g_v)^{\mathrm{Re}(\nu)+\rho_G}\,\mathrm{d}\mu_{P\backslash G}(g_v)$$

with $Y(|\cdot|_{F_v}^{\nu}; g_v)$ being continuous on $G(F_v)$ for $\mathrm{Re}(\nu) \geq \rho_G$; thus $\Xi_v < +\infty$. There exists a finite subset $S \subset \Sigma^F$ such that Ξ_v $(v \notin S)$ is given by Lemma 4.3. The convergence of $\prod_v \Xi_v$ follows from that of the Euler product

$$\prod_{v\notin S} \frac{\zeta_{E_v}(\mathrm{Re}(\nu) - \frac{m-3}{2})}{L(2\,\mathrm{Re}(\nu)+1, \varepsilon_{E_v/F_v}^{m})} L(m-1, \varepsilon_{E_v/F_v}^{m-1})^{-1}.$$

To prove the remaining half of the lemma, we write

$$\int_{H_P(F)\backslash H(\mathbb{A}_F)} f^{(\nu)}(h)|\omega_H|_{\mathbb{A}}$$
$$= \int_{h\in H_P(\mathbb{A}_F)\backslash H(\mathbb{A}_F)}\Big(\int_{l\in H_P(F)\backslash H_P(\mathbb{A}_F)} f^{(\nu)}(lh)|\omega_{H_P}|_{\mathbb{A}}\Big)|\omega_{H_P\backslash H}|_{\mathbb{A}}.$$

With $\mathrm{d}^1\tau$ and $\mathrm{d}g_1$ being some Haar measures on $\mathbb{A}_E^1$ and on $G_1(\mathbb{A}_F)$, respectively, the integral in the bracket equals

$$\Big(\int_{E^\times\backslash\mathbb{A}_E^1} \chi(\tau)|\tau|_E^{\nu+\rho_G}\,\mathrm{d}^\times\tau\Big)\Big(\int_{G_1(F)\backslash G_1(\mathbb{A}_F)} f^{(\nu)}\big(\mathsf{m}[g_1]h\big)\,\mathrm{d}g_1\Big),$$

which vanishes unless $\chi|\mathbb{A}_E^1$ is trivial and $g_1 \mapsto f^{(\nu)}(\mathsf{m}[g_1]h)$ is a constant. The restriction $\chi|\mathbb{A}_E^1$ is trivial if and only if $\chi = \eta\circ \mathrm{N}_{E/F}$ with some $\eta \in Y_F$. Thus the first two conditions for (χ,σ,Λ) to be distinguished follow. The constraint on Λ comes from the condition $W_{l_v}^{\mathcal{U}_v\cap H(F_v)} \neq \{0\}$. □

LEMMA 5.9

Let $\mathrm{Re}(\nu) > \rho_G$*. Then, the integral* $J_e(\lambda,\nu)$ *converges absolutely for* $\mathrm{Re}(\lambda) > \rho_G$ *and*

$$\lim_{\lambda\to\rho_H+0}(\lambda-\rho_H)J_e(\lambda,\nu) = \boldsymbol{\Xi}(f^{(\nu)})$$

uniformly with respect to $\mathrm{Im}(\nu)$.

Proof

We have

$$\begin{aligned}&\big|(\lambda-\rho_H)J_e(\lambda,\nu)-\boldsymbol{\Xi}(f^{(\nu)})\big|\\&\qquad\le C\int_{H_P(F)\backslash H(\mathbb{A}_F)}\big|(\lambda-\rho_H)\mathcal{E}_{\beta,\lambda}(h)-1\big|a(h)^{\mathrm{Re}(\nu)+\rho_G}|\omega_H|_{\mathbb{A}}\end{aligned}$$

with a constant $C>0$ independent of ν and λ. Fix a small $\epsilon>0$. From Lemma 5.3, there exists a constant $C_0>0$ such that $\sup_{\rho_H<\lambda<\rho_H+\epsilon}|(\lambda-\rho_H)\mathcal{E}_{\beta,\lambda}(h)|\le C_0$ on a Siegel set $\mathfrak{S}_H$; since the left-hand side is an $H(F)$-invariant function, the same inequality is valid for any $h\in H(\mathbb{A}_F)$. Hence

$$\big|(\lambda-\rho_H)\mathcal{E}_{\beta,\lambda}(h)-1\big|a(h)^{\mathrm{Re}(\nu)+\rho_H}\le (C_0+1)a(h)^{\mathrm{Re}(\nu)+\rho_H},\qquad h\in H(\mathbb{A}_F),\lambda\in(\rho_H,\rho_H+\epsilon].$$

From Lemma 5.8, the right-hand side of this inequality is integrable on $H_P(F)\backslash H(\mathbb{A}_F)$. Thus by the Lebesgue dominated convergence theorem, we are done because $\lim_{\lambda\to\rho_H+0}(\lambda-\rho_H)\mathcal{E}_{\beta,\lambda}(h)=1$ for any $h\in H(\mathbb{A}_F)$. □

LEMMA 5.10

Let $\sigma>\rho_G$. Then

$$\begin{aligned}&\frac{1}{2\pi i}\int_{i\mathbb{R}}\mathcal{P}_H\big(E(f^{(\nu)})\big)\alpha(\nu)\,\mathrm{d}\nu\\&\qquad=\frac{1}{2\pi i}\int_{(\sigma)}\boldsymbol{\Xi}(f^{(\nu)})\alpha(\nu)\,\mathrm{d}\nu-\sum_j\alpha(s_j)\mathcal{P}_H(\phi_j).\end{aligned}\tag{5.15}$$

Proof

By Lemma 5.3 and Corollary 5.5, the left-hand side of (5.15) coincides with the limit $\lim_{\lambda\to\rho_H+0}(\lambda-\rho_H)I(\lambda)$, which in turn is evaluated as in the right-hand side of (5.15) by means of (5.12), (5.14), and Lemma 5.9. □

If (χ,σ,Λ) is not distinguished, then $\boldsymbol{\Xi}(f^{(\nu)})=0$ for all $\mathrm{Re}(\nu)>\rho_G$ from Lemma 5.8. By letting $\alpha(s)$ vary, from the formula (5.15), we obtain $\mathcal{P}_H(E(f^{(\nu)}))=0$ for all $\nu\in i\mathbb{R}$ and $\mathcal{P}_H(\phi_j)=0$ for all j. In the rest of the proof, we assume that (χ,σ,Λ) is distinguished. Let $\eta\in Y_F$ be such that $\chi=\eta\circ\mathrm{N}_{E/F}$. In this case, from the proof of Lemma 5.8,

$$\boldsymbol{\Xi}(f^{(\nu)})=\mathrm{vol}(H_P)|D_{F/\mathbb{Q}}|^{-\dim(H_P\backslash H)/2}\bigotimes_v\int_{H_P(F_v)\backslash H(F_v)}f_v^{(\nu)}(h)|\omega_{H_P\backslash H}|_{F_v}^*$$

for $\mathrm{Re}\,\nu>\rho_G$. Here the factor $|D_{F/\mathbb{Q}}|^{-\dim(H_P\backslash H)/2}$ arises from the definition of the global Tamagawa measure. Since $H_P\cong E^1\times U(V_1)$, $|\omega_{H_P\backslash H}|_{F_v}^*=|\omega_{H_P\backslash H}|_{F_v}$, $\mathrm{vol}(H_P)=\{2L(1,\varepsilon_{E/F})\}^2$ and $\dim(H_P\backslash H)=m^2-(m-2)^2=4(m-1)$ (see Section 5.3). By means of Lemma 4.3, the product becomes

$$\begin{aligned}&L^\infty(m-1,\varepsilon_{E/F}^{m-1})^{-1}\frac{L^\infty(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F})}{L^\infty(2\nu+1,\eta^2\varepsilon_{E/F}^m)}\\&\qquad\times\Big\{\prod_{v\in\Sigma_{\mathrm{fin}}}\Xi^0\big(\eta_v|\cdot|_{F_v}^\nu;f_v\big)\Big\}\bigotimes_{v\in\Sigma_\infty}\Xi_v(f_v^{(\nu)})\end{aligned}$$

with

$$\Xi_v(f_v^{(\nu)}) = \int_{H_P(F_v)\backslash H(F_v)} f_v^{(\nu)}(h)|\omega_{H_P\backslash H}|_{F_v} \in W_{l_v} \quad (v \in \Sigma_\infty).$$

LEMMA 5.11

For $\mathrm{Re}(\nu) > \rho_G$,

$$\Xi_v(f_v^{(\nu)}) = \frac{|\sqrt{\theta}|_{E_v}^{(1-m)/2}\pi^{m-1}\Gamma(\nu+ib_v(\eta)-\frac{m-3}{2})^2}{\Gamma(\nu+ib_v(\eta)-\frac{m-3}{2}-d_v)\Gamma(\nu+ib_v(\eta)+\frac{m-1}{2}+d_v)}\mathrm{Pr}_v\big(f_v(1)\big).$$

Proof

Without loss of generality, we may assume that $b_v(\eta)=0$. From Lemma 2.1, we see that the integral $\Xi_v(f_v^{(\nu)})$ coincides with the Poisson integral $\langle \Xi(\eta_v|\cdot|_{F_v}^{\nu}), f_v^{(\nu)}\rangle$ defined by (4.1), for any $f_v^{(\nu)} \in (W_{d_v}\otimes_{\mathbb{C}} I(\eta_v|\cdot|_{F_v}^{\nu}))^{\mathcal{U}_v}$. Set $\Phi_v^{(\nu)}(g) = \Xi_v(R(g)f_v^{(\nu)})$ for $g\in G(F_v)$. Then $\Phi_v^{(\nu)}: G(F_v)\to W_{l_v}$ is a spherical function studied in Section 3.1. Since $\Phi_v^{(\nu)}(1)=\Xi_v(f_v^{(\nu)})$, we have $\Phi_v^{(\nu)}(a_v^{(t)}) = \Xi_v(f_v^{(\nu)})\phi_{d_v}(s;t)\vartheta_{d_v}$ with $\phi_{d_v}(s;t)$ given by (3.3), which, by [17, p. 47], equals

$$(\cosh t)^{2\nu-m+1}{}_2F_1\Big(\frac{-2\nu-m+3}{2}+d_v, \frac{-2\nu+m-1}{2}-d_v; 1; \tanh^2 t\Big).$$

On the one hand, by the formula in [17, p. 40], we have

$$\begin{aligned}\lim_{t\to+\infty} t^{-2\nu+m-1}\Phi_v^{(\nu)}(a_v^{(t)}) &= \Xi_v(f_v^{(\nu)})2^{-2\nu+m-1}\\ &\quad\times {}_2F_1\Big(\frac{-2\nu-m+3}{2}+d_v, \frac{-2\nu+m-1}{2}-d_v; 1; 1\Big)\\ &= \Xi_v(f_v^{(\nu)})\frac{2^{-2\nu+m-1}\Gamma(2\nu)}{\Gamma(\nu+\frac{m-1}{2}+d_v)\Gamma(\nu-\frac{m-3}{2}-d_v)}\end{aligned} \tag{5.16}$$

for $\mathrm{Re}(\nu)\gg 0$. On the other hand, by the same way as in [9, Proposition 7.7], we have

$$\lim_{t\to\infty} t^{-2\nu+m-1}\big(\Phi_v^{(\nu)}(a_v^{(t)})|\vartheta_{d_v}\big) = \int_{\bar{N}(F_v)} \big(f_v^{(\nu)}(\bar{n})|\vartheta_{d_v}\big)|\omega_{\bar{N}}|_{F_v}, \quad \mathrm{Re}(\nu)\gg 0,$$

where $\omega_{\bar{N}}$ is the unique gauge-form on $\bar{N}$ such that $\omega_G|P\bar{N} = \omega_P\wedge\omega_{\bar{N}}$ on the Zariski-open set $P\bar{N}\cong P\times\bar{N}$ of G. From [3, Theorem 8.2] after adjusting for the difference of the normalization of measures by Lemma 3.1, the last integral is evaluated as

$$|\sqrt{\theta}|_{E_v}^{(1-m)/2}\pi^{m-1}\frac{2^{-2\nu+m-1}\Gamma(2\nu)\Gamma(\nu-\frac{m-3}{2})^2}{\Gamma(\nu-\frac{m-3}{2}-d_v)^2\Gamma(\nu+\frac{m-1}{2}+d_v)^2}. \tag{5.17}$$

By equating (5.16) with (5.17), we have the desired expression for $(\Xi_v(f_v^{(\nu)})|\vartheta_{d_v})$ on $\mathrm{Re}(\nu)\gg 0$. $\square$

Consequently, we see that $\boldsymbol{\Xi}(f^{(\nu)})$ with $\mathrm{Re}(\nu)>\rho_G$ equals the right-hand side of (5.10). By this expression, $\boldsymbol{\Xi}(f^{(\nu)})$ has a meromorphic continuation on $\mathbb{C}$. The singularity on $\mathrm{Re}(\nu)\geq 0$ arises from the completed L-function $L^\infty(\nu-\frac{m-3}{2},\eta\circ$

$\mathrm{N}_{E/F})\prod_{v\in\Sigma_\infty}\Gamma_{\mathbb{C}}(\nu-\frac{m-3}{2})$, which is holomorphic except for simple poles at $\nu=\frac{m-1}{2},\frac{m-3}{2}$ (only when $\eta=1$ or $\varepsilon_{E/F}$). Moreover, we have

$$\left\|\mathbf{\Xi}(f^{(\nu)})\right\| \ll \frac{|L^{\infty}(\nu-\frac{m-3}{2},\eta\circ \mathrm{N}_{E/F})|}{|L^{\infty}(2\nu+1,\eta^2\varepsilon_{E/F}^m)|}\prod_{v\in\Sigma_{\mathrm{fin}}}\left|\Xi^0\left(\eta_v|\cdot|_{F_v}^{\nu};f_v\right)\right|$$
$$\times\Big\{\prod_{v\in\Sigma_\infty}\frac{|\Gamma_{\mathbb{C}}(\nu+ib(\eta_v)-\frac{m-3}{2})^2|}{|\Gamma_{\mathbb{C}}(\nu+ib(\eta_v)-\frac{m-3}{2}-d_v)\Gamma_{\mathbb{C}}(\nu+ib_v(\eta)+\frac{m-1}{2}+d_v)|}\Big\}$$

for $\mathrm{Re}(\nu)\ge 0$. Take any $\mathcal{T}_{I,\delta}$ in $\mathrm{Re}(\nu)\ge 0$. By Stirling's formula, the gamma factor turns out be $O(1)$ on $\mathcal{T}_{I,\delta}$. The product of normalized Poisson integrals is also $O(1)$ on $\mathcal{T}_{I,\delta}$ (because it is periodic in $\mathrm{Im}(\nu)$). From the convexity bound, we have $|L^\infty(\nu-\frac{m-3}{2},\eta\circ\mathrm{N}_{E/F})|\ll(1+|\mathrm{Im}(\nu)|)^{N_1}$ on $\mathcal{T}_{I,\delta}$ with some constant $N_1>0$. From [1], we have a polynomial bound $|L^\infty(2\nu+1,\eta^2\varepsilon_{E/F}^m)|^{-1}\ll(1+|\mathrm{Im}(\nu)|)^{N_2}$ on $\mathcal{T}_{I,\delta}$. Thus, we have a polynomial bound of $\mathbf{\Xi}(f^{(\nu)})$ on $\mathcal{T}_{I,\delta}$. By shifting the contour from (σ) back to the imaginary axis, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(\sigma)}\mathbf{\Xi}(f^{(\nu)})\alpha(\nu)\,\mathrm{d}\nu\\ &\quad=\frac{1}{2\pi i}\int_{i\mathbb{R}}\mathbf{\Xi}(f^{(\nu)})\alpha(\nu)\,\mathrm{d}\nu\\ &\qquad+\alpha\Big(\frac{m-1}{2}\Big)\mathrm{Res}_{\nu=\frac{m-1}{2}}\mathbf{\Xi}(f^{(\nu)})+\alpha\Big(\frac{m-3}{2}\Big)\mathrm{Res}_{\nu=\frac{m-3}{2}}\mathbf{\Xi}(f^{(\nu)}).\end{aligned}$$

Comparing this with Lemma 5.10, we obtain

$$\mathcal{P}_H\big(E(f^{(\nu)})\big)=\mathbf{\Xi}(f^{(\nu)})\quad(\nu\in i\mathbb{R}),$$
$$\mathcal{P}_H(\phi_{\frac{m-1}{2}})=\mathrm{Res}_{\nu=\frac{m-1}{2}}\mathbf{\Xi}(f^{(\nu)}),\qquad\mathcal{P}_H(\phi_{\frac{m-3}{2}})=\mathrm{Res}_{\nu=\frac{m-3}{2}}\mathbf{\Xi}(f^{(\nu)})$$

with $\phi_s=\mathrm{Res}_{\nu=s}E(f^{(\nu)})$, and $\mathcal{P}_H(\phi_j)=0$ for a residual form ϕ_j (if any) other than $\phi_{\frac{m-1}{2}}$ and $\phi_{\frac{m-3}{2}}$. Actually, Lemma 5.1 shows that no such ϕ_j exists other than $\phi_{\frac{m-1}{2}}$.

6. The spectral side

6.1. A majorant

Let $\mathcal{V}=\prod_{v\in\Sigma_{\mathrm{fin}}}\mathcal{V}_v$ be a compact open subset of $G(\mathbb{A}_F^\infty)$ such that $\mathcal{V}_v=\mathcal{U}_v$ for almost all v. Let $\phi_{\mathcal{V}_v}$ be the characteristic function of $H(F_v)\mathcal{V}_v$ on $G(F_v)$, and set $\phi_{\mathcal{V}}(g_{\mathrm{fin}})=\prod_{v\in\Sigma_{\mathrm{fin}}}\phi_{\mathcal{V}_v}(g_v)$ for $g_{\mathrm{fin}}\in G(\mathbb{A}_F^\infty)$. For $N\in\mathbb{N}$ and $v\in\Sigma_\infty$, let us define a function $\phi_v^{(N)}:G(F_v)\to\mathbb{R}_+$ by

$$\phi_v^{(N)}(g_v)=\|g_v^{-1}\ell\|_{\mathcal{U}_v}^{-N},\quad g\in G(F_v),$$

where $\|\cdot\|_{\mathcal{U}_v}$ is the minimal majorant of $\mathbf{h}_v$ with respect to $\mathcal{U}_v$. The function $\phi_{\mathcal{V}}^{(N)}:G(\mathbb{A}_F)\to\mathbb{R}_+$ defined by

$$\phi_{\mathcal{V}}^{(N)}(g)=\Big\{\prod_{v\in\Sigma_\infty}\phi_v^{(N)}(g_v)\Big\}\phi_{\mathcal{V}}(g_{\mathrm{fin}}),\quad g=g_\infty g_{\mathrm{fin}}\in G(\mathbb{A}_F),$$

is left $H(\mathbb{A}_F)$-invariant and is smooth on $G(\mathbb{A}_F)$. Thus the sum

$$\mathbf{\Phi}_{\mathcal{V}}^{(N)}(g) = \sum_{\gamma \in H(F)\backslash G(F)} \phi_{\mathcal{V}}^{(N)}(\gamma g), \quad g \in G(\mathbb{A}_F),$$

is well defined.

LEMMA 6.1

Let $N > 2(m-1)$. The series $\mathbf{\Phi}_{\mathcal{V}}^{(N)}$ converges absolutely and normally on $G(\mathbb{A}_F)$ defining a left $G(F)$-invariant continuous function. For any Siegel domain $\mathfrak{S}_G \subset G(\mathbb{A}_F)$ with respect to P and $\mathcal{U}$, we have the estimation

$$\mathbf{\Phi}_{\mathcal{V}}^{(N)}(g) \ll a(g), \quad g \in \mathfrak{S}_G.$$

If $N > 4m-6$, then the function $\mathbf{\Phi}_{\mathcal{V}}^{(N)}$ belongs to $L^l(G(F)\backslash G(\mathbb{A}_F))$ for all $1 \leq l < m-1$.

Proof

Let $\mathcal{N} = \mathcal{N}_\infty \mathcal{N}_{\mathrm{fin}}$ be a relatively compact open neighborhood of the identity in $G(\mathbb{A}_F)$ with $\mathcal{N}_\infty$ and $\mathcal{N}_{\mathrm{fin}}$ being open in $G(F_\infty)$ and $G(\mathbb{A}_F^\infty)$, respectively. We may suppose that $\mathcal{N}_{\mathrm{fin}}$ is a subgroup such that $\mathcal{V}\mathcal{N}_{\mathrm{fin}} = \mathcal{V}$. Since $\mathcal{N}_\infty = \prod_{v \in \Sigma_\infty} \mathcal{N}_v$ is relatively compact, we have

$$\|(gy)^{-1}\ell\|_{\mathcal{U}_v} \leq B_v \|g^{-1}\ell\|_{\mathcal{U}_v}, \quad (g, y) \in G(F_v) \times \mathcal{N}_v,$$

with $B_v = \sup_{y \in \mathcal{N}_v} \{\sum_j \|(y^{-1})^* v_j\|_{\mathcal{U}_v}^2\}^{1/2}$, where $\{v_j\}$ is an orthonormal basis of $V(F_v)$ with respect to the Hermitian inner product associated to the norm $\|\cdot\|_{\mathcal{U}_v}$, and where $(y^{-1})^*$ denotes the Hermitian adjoint of the operator $y^{-1} \in \mathrm{End}_{\mathbb{C}}(V(F_v))$. Thus, if we set $B = \prod_v B_v^N$, then

$$\phi_{\mathcal{V}}^{(N)}(g) \leq B \phi_{\mathcal{V}}^{(N)}(gy), \quad (g, y) \in G(\mathbb{A}_F) \times \mathcal{N}.$$

From this,

$$\begin{aligned}
&\mathbf{\Phi}_{\mathcal{V}}^{(N)}(g) B^{-1} \mathrm{vol}(\mathcal{N}) \\
&\quad \leq \int_{y \in \mathcal{N}} \sum_{\gamma \in H(F)\backslash G(F)} \phi_{\mathcal{V}}^{(N)}(\gamma g y) |\omega_G|_{\mathbb{A}} \\
&\quad \leq \sum_{\gamma \in H(F)\backslash G(F)} \int_{y \in H(F)\backslash G(\mathbb{A})} \sum_{\delta \in H(F)} \mathbb{1}_{\mathcal{N}}(g^{-1}\gamma^{-1}\delta^{-1} y) \phi_{\mathcal{V}}^{(N)}(y) |\omega_G|_{\mathbb{A}} \\
&\quad \leq \int_{y \in H(F)\backslash G(\mathbb{A}_F)} \Big\{ \sum_{\gamma \in G(F)} \mathbb{1}_{\mathcal{N}}(g^{-1}\gamma y) \Big\} \phi_{\mathcal{V}}^{(N)}(y) |\omega_G|_{\mathbb{A}}
\end{aligned}$$

with $\mathbb{1}_{\mathcal{N}}$ the characteristic function of $\mathcal{N}$ on $G(\mathbb{A}_F)$. Let U be an arbitrary compact set in $G(\mathbb{A}_F)$. Since $\sum_{\gamma \in G(F)} \mathbb{1}_{\mathcal{N}}(g^{-1}\gamma y) \leq \#(G(F) \cap U\mathcal{N}\mathcal{N}^{-1}U^{-1})(< +\infty)$ for $g \in U$ and $y \in G(\mathbb{A}_F)$, the last integral is majorized uniformly in $g \in U$ by

$$\int_{H(F)\backslash G(\mathbb{A}_F)} \phi_{\mathcal{V}}^{(N)}(y)|\omega_G|_{\mathbb{A}},$$

which is the product of $\mathrm{vol}(H(F)\backslash H(\mathbb{A}_F))|D_{F/\mathbb{Q}}|^{-\dim(H\backslash G)/2}$ and

$$\mathrm{vol}\big(H(\mathbb{A}_F^\infty)\backslash H(\mathbb{A}_F^\infty)\mathcal{V}; |\omega_{H\backslash G}|_{\mathbb{A}^\infty}\big)\Big\{\prod_{v\in\Sigma_\infty}\int_{y_v\in H(F_v)\backslash G(F_v)} \|y_v^{-1}\ell\|_{\mathcal{U}_v}^{-N}|\omega_{H\backslash G}|_{F_v}\Big\}.$$

Since $\mathcal{V}_v$ is compact and $\mathcal{V}_v = \mathcal{U}_v$ for almost all v, the first factor is finite. For $v \in \Sigma_\infty$, the integral

$$\int_{y_v\in H(F_v)\backslash G(F_v)} \|y_v^{-1}\ell\|_{\mathcal{U}_v}^{-N}|\omega_{H\backslash G}|_{F_v}$$
$$= C_G\int_0^\infty (2\cosh 2t)^{-N/2}(\cosh t)^{2m-3}\sinh t\,\mathrm{d}t$$

is convergent if $N > 2(m-1)$. This settles the first assertion of the lemma. From Lemma 6.2(1), we have

$$\mathbf{\Phi}_{\mathcal{V}}^{(N)}(g) \ll a(g)^{m-1}\int_{H(F)\backslash G(F)g\mathcal{N}} \phi_{\mathcal{V}}^{(N)}(y)\,|\omega_G|_{\mathbb{A}}, \quad g\in\mathfrak{S}_G.$$

Let $\phi_{\mathcal{V}}^{(N)}(y) \neq 0$; then $y = h\mathsf{a}(t)u$ with some $h \in H(\mathbb{A}_F)$, $t \in (\mathbb{R}_+)^{\Sigma_\infty}$, and $u \in \mathcal{U}_\infty\mathcal{V}$. Hence $\|y\|_G \ll \|h\|_G\|\mathsf{a}(t)\|_G$. If $y \in G(F)g\mathcal{N}$, then from Lemma 6.2(2), we have $\|y\|_G^{-1} \ll a(g)^{-1/2}$. Hence

$$\int_{H(F)\backslash G(F)g\mathcal{N}} \phi_{\mathcal{V}}^{(N)}(y)|\omega_G|_{\mathbb{A}}$$
$$\ll \int_{t\in\mathbb{R}_+^{\Sigma_\infty}} I\big(\|\mathsf{a}(t)\|_G^{-1}a(g)^{1/2}\big)\prod_{v\in\Sigma_\infty}\phi_v^{(N)}(a_v^{(t_v)})(\cosh t_v)^{2m-3}\sinh t_v\,\mathrm{d}t_v$$

with $I(T) = \int_{\substack{h\in\mathfrak{S}_H\\ \|h\|_G\geq T}}|\omega_H|_{\mathbb{A}}$. Since $I(\|\mathsf{a}(t)\|_G^{-1}a(g)^{1/2}) \ll \|\mathsf{a}(t)\|_G^{2(m-2)}a(g)^{-(m-2)}$ by Lemma 6.2 (3) and since $\|\mathsf{a}(t)\|_G \ll \prod_{v\in\Sigma_\infty} e^{t_v}$, the t-integral is majorized by

$$a(g)^{-(m-2)}\prod_v\int_0^\infty e^{2t_v(m-2)}(\cosh 2t_v)^{-N/2}(\cosh t_v)^{2m-3}\sinh t_v\,\mathrm{d}t_v.$$

If $N > 4m-6$, then the t_v-integrals are convergent. Therefore,

$$\Phi(g) \ll a(g)^{m-1}\times a(g)^{-(m-2)} = a(g)$$

for $g \in \mathfrak{S}_G$. By the Iwasawa decomposition, we have

$$\int_{\mathfrak{S}_G}\big|\mathbf{\Phi}_{\mathcal{V}}^{(N)}(g)\big|^l|\omega_G|_{\mathbb{A}} \ll \int_{t_0}^\infty a\big(\underline{[r]}\big)^l r^{-2(m-1)}\frac{\mathrm{d}r}{r} = \int_{t_0}^\infty r^{2l-2(m-1)}\frac{\mathrm{d}r}{r}.$$

The last r-integral is finite if $l < m-1$. □

LEMMA 6.2

(1) *Let $\mathcal{N}$ be a relatively compact open set of $G(\mathbb{A}_F)$. Then*

$$\sum_{\gamma\in G(F)}\mathbb{1}_{\mathcal{N}}(g^{-1}\gamma y) \ll a(g)^{m-1}\mathbb{1}_{G(F)g\mathcal{N}}(y), \quad g\in\mathfrak{S}_G, y\in G(\mathbb{A}_F).$$

(2) *Let* $\|\cdot\|_G$ *be the norm function on* $G(\mathbb{A}_F)$. *Then*

$$\|\gamma g y\|_G^{-1} \ll a(g)^{-1/2}, \quad \gamma \in G(F), g \in \mathfrak{S}_G, y \in \mathcal{N}.$$

(3) *Let* $\mathfrak{S}_H$ *be a Siegel domain of* $H(\mathbb{A}_F)$ *with respect to* P_H *and* $\mathcal{U}_H$. *Then*

$$\int_{\substack{h\in\mathfrak{S}_H\\ \|h\|_G\geq T}} |\omega_H|_{\mathbb{A}} \ll T^{-2(m-2)}, \quad T>1.$$

Proof
We prove (1) and (2) in the same way as [26, Lemma 3.3]. Let $\gamma_0 \in G(F)$ be the element introduced in the proof of Lemma 5.4, and set $[\tau]_H = \gamma_0^{-1}[\tau]\gamma_0$ for $\tau \in \mathbb{A}_E^\times$ as in Section 5.9. A general element of $\mathfrak{S}_H$ is of the form $h = y[\underline{\tau}]_H k$ with $k \in \mathcal{U}_H$ and $y \in P_H(\mathbb{A}_F)$ lying in a fixed compact set and $\tau \in (t_0, +\infty)$. Thus $\|h\|_G \asymp \|[\underline{\tau}]_H\|_G = \|\gamma_0^{-1}[\underline{\tau}]\gamma_0\|_G \asymp \|[\underline{\tau}]\|_G \asymp \tau$. From this, we have

$$\int_{\substack{h\in\mathfrak{S}_H\\ \|h\|_G\geq T}} |\omega_H|_{\mathbb{A}} \ll \int_T^\infty \tau^{-2(m-2)}\frac{\mathrm{d}\tau}{\tau} \ll T^{-2(m-2)}.$$

□

6.2. Test functions

Let $\mathfrak{d} = \{d_v\}_{v\in\Sigma_\infty} \in \mathbb{N}_0^{\Sigma_\infty}$, and let $(\tau_{\mathfrak{d}}, W(\mathfrak{d}))$ be the unitary representation of $\mathcal{U}_\infty$ defined in Section 5.5. As before, we fix a unit vector $\vartheta_{d_v} \in W_{d_v}^{\mathcal{U}_v\cap H(F_v)}$ for each $v \in \Sigma_\infty$, and we set $\vartheta(\mathfrak{d}) = \bigotimes_{v\in\Sigma_\infty}\vartheta_{d_v}$. Recall $\mathcal{A}$ (see Section 3.1.2). For a decomposable function $\alpha = \bigotimes_{v\in\Sigma_\infty}\alpha_v \in \otimes_{v\in\Sigma_\infty}\mathcal{A}$, let us define a function $\hat{\Psi}_{\mathfrak{d}}(\alpha) : G(F_\infty) \to W(\mathfrak{d}) \cong \bigotimes_{v\in\Sigma_\infty} W_{d_v}$ by setting

$$\hat{\Psi}_{\mathfrak{d}}(\alpha; g_\infty) = \bigotimes_{v\in\Sigma_\infty}\hat{\Psi}_{d_v}(\alpha_v; g_v), \quad g_\infty = (g_v)_v \in G(F_\infty).$$

A function $\phi \in \mathcal{S}(G(\mathbb{A}_F^\infty))$ is called *decomposable* if there exists a family of functions $\phi_v \in \mathcal{S}(G(F_v))$ with $v \in \Sigma_{\mathrm{fin}}$ such that $\phi_v = \mathbb{1}_{\mathcal{U}_v}$ for almost all v and

$$\phi(g_{\mathrm{fin}}) = \prod_{v\in\Sigma_{\mathrm{fin}}}\phi_v(g_v), \quad g_{\mathrm{fin}} = (g_v) \in G(\mathbb{A}_F^\infty).$$

For a decomposable function ϕ, we define a smooth function $\Phi_{\mathfrak{d}}(\alpha,\phi) : G(\mathbb{A}_F) \to W(\mathfrak{d})$ by setting

$$\Phi_{\mathfrak{d}}(\alpha,\phi; g) = \hat{\Psi}_{\mathfrak{d}}(\alpha; g_\infty)\prod_{v\in\Sigma_{\mathrm{fin}}}\phi_v^H(g_v), \quad g \in G(\mathbb{A}_F),$$

where

$$\phi_v^H(g_v) = \mathrm{vol}\big(H(F_v)\cap\mathcal{U}_v; |\omega_H|_{F_v}^*\big)^{-1}\int_{H(F_v)}\phi_v(h_v g_v)|\omega_H|_{F_v}^*, \quad g_v \in G(F_v).$$

Note that $\phi_v^H = \phi_{\mathcal{U}_v}$ if $\phi_v = \mathbb{1}_{\mathcal{U}_v}$. From the construction, the function $\Phi = \Phi_{\mathfrak{d}}(\alpha,\phi)$ has the equivariance

$$\Phi(hgk_\infty) = \tau_{\mathfrak{d}}(k_\infty)^{-1}\Phi(g), \quad h \in H(\mathbb{A}_F), g \in G(\mathbb{A}_F), k_\infty \in \mathcal{U}_\infty. \tag{6.1}$$

6.3. The relative kernel functions

Let $\Phi = \Phi_{\mathfrak{d}}(\alpha, \phi)$ be as in Section 6.2. From (6.1), the summation

$$\mathbf{\Phi}(g) = (-1)^{d_F} \mathrm{vol}\big(H(\mathbb{A}_F^\infty) \cap \mathcal{U}; |\omega_H|_{\mathbb{A}^\infty}\big) \sum_{\gamma \in H(F)\backslash G(F)} \Phi(\gamma g), \quad g \in G(\mathbb{A}_F),$$

makes sense if absolutely convergent. When the dependence on the data $(\mathfrak{d}, \alpha, \phi)$ matters, we write $\mathbf{\Phi}_{\mathfrak{d}}(\alpha, \phi; g)$ in place of $\mathbf{\Phi}(g)$.

LEMMA 6.3

The series $\mathbf{\Phi}(g) = \mathbf{\Phi}_{\mathfrak{d}}(\alpha, \phi; g)$ converges absolutely and normally on $G(\mathbb{A}_F)$, defining a continuous $W(\mathfrak{d})$-valued left $G(F)$-invariant function. For any $l \in [1, m-1)$, we have $\mathbf{\Phi} \in (L^l(G(F)\backslash G(\mathbb{A}_F)) \otimes_{\mathbb{C}} W(\mathfrak{d}))^{\mathcal{U}_\infty}$.

Proof

For $v \in \Sigma_{\mathrm{fin}}$, set $\mathcal{V}_v = \mathrm{Supp}(\phi_v)$. Then there exists a constant $B_v > 0$ such that $|\phi_v^H(g_v)| \le B_v \phi_{\mathcal{V}_v}(g_v)$ for all $g_v \in G(F_v)$. For a place $v \in \Sigma_{\mathrm{fin}}$ such that $\phi_v = \mathbb{1}_{\mathcal{U}_v}$, we can take $B_v = 1$. Fix an integer $N > 4m - 6$. For $v \in \Sigma_\infty$, Lemma 3.8 yields a constant $B_v > 0$ such that $\|\hat{\Psi}_{d_v}(\alpha_v; g_v)\| \le B_v \|g_v^{-1}\ell\|_{\mathcal{U}_v}^{-N}$ for all $g_v \in G(F_v)$. Taking the product of local estimations, we have $\|\Phi(g)\| \le B\phi_{\mathcal{V}}^{(N)}(g)$ for all $g \in G(\mathbb{A}_F)$ with $B = \prod_v B_v$. Then we apply Lemma 6.1 to complete the proof. $\square$

6.4. Spectral expansion of the relative kernel function

For any unitary representation π of $G(\mathbb{A}_F)$, we set $\pi[\mathfrak{d}] = (\pi \otimes_{\mathbb{C}} W(\mathfrak{d}))^{\mathcal{U}_\infty}$, viewing this as a closed subspace of the Hilbert space $\pi \otimes_{\mathbb{C}} W(\mathfrak{d})$. In particular, $L^2(G(F)\backslash G(\mathbb{A}_F))[\mathfrak{d}]$ is identified with the space of equivalence classes of measurable functions $\varphi : G(F)\backslash G(\mathbb{A}_F) \to W(\mathfrak{d})$ such that $\varphi(gk_\infty) = \tau_{\mathfrak{d}}(k_\infty)^{-1}\varphi(g)$ for all $k_\infty \in \mathcal{U}_\infty$ and $\int_{G(F)\backslash G(\mathbb{A}_F)} \|\varphi(g)\|^2 |\omega_G|_{\mathbb{A}} < +\infty$. Let $\Pi_{\mathrm{dis}}(G)$ (resp., $\Pi_{\mathrm{cus}}(G)$) be the set of all the irreducible closed subrepresentations of $L^2(G(F)\backslash G(\mathbb{A}_F))$ (resp., $L^2_{\mathrm{cus}}(G(F)\backslash G(\mathbb{A}_F))$). For $\pi \in \Pi_{\mathrm{dis}}(G)$, the space $\pi[\mathfrak{d}]$ is identified with the space of $W(\mathfrak{d})$-valued L^2-automorphic forms (see Section 5.7) whose coefficients generate π; we fix an orthonormal basis $\mathcal{B}_{\mathfrak{d}}(\pi)$ of $\pi[\mathfrak{d}]$ once and for all.

LEMMA 6.4

Let $\varphi : G(\mathbb{A}_F) \to W(\mathfrak{d})$ be a smooth function which satisfies the following conditions.

(i) *We have $\varphi(\gamma g k_\infty) = \tau_{\mathfrak{d}}(k_\infty)^{-1}\varphi(g)$ for all $\gamma \in G(F)$ and $k_\infty \in \mathcal{U}_\infty$.*

(ii) *There exists a family of complex numbers $\{\nu_v\}_{v \in \Sigma_\infty}$ such that*

$$\varphi * \mathcal{C}_{G(F_v)} = 2^{-1}\big\{(2\nu_v)^2 - (m-1)^2\big\}\varphi \quad \text{for all } v \in \Sigma_\infty.$$

(iii) *For any $\epsilon > 0$ and for any $D \in U(\mathfrak{g}_\infty)$, the majorization $\|[\varphi * D](g)\| \ll_\epsilon a(g)^{(m-1)/2+\epsilon}$ on $g \in \mathfrak{S}_G$ holds.*

Then the integral $\langle \boldsymbol{\Phi}|\varphi\rangle = \int_{G(F)\backslash G(\mathbb{A}_F)} (\boldsymbol{\Phi}(g)|\varphi(g))|\omega_G|_{\mathbb{A}}$ *converges absolutely and*

$$\langle \boldsymbol{\Phi}|\varphi\rangle = \Big\{ \prod_{v\in\Sigma_\infty} \alpha_v(\nu_v)\Big\}\big(\vartheta(\mathfrak{d})|\mathcal{P}_H(\varphi*\bar{\phi})\big).$$

Proof

Set

$$\varphi^H(g_\infty) = \int_{H(\mathbb{A}_F^\infty)\backslash G(\mathbb{A}_F^\infty)} \bar{\phi}^H(g_{\mathrm{fin}})\Big\{\int_{H(F)\backslash H(\mathbb{A}_F)} \varphi(hg_\infty g_{\mathrm{fin}})|\omega_H|_{\mathbb{A}}\Big\}|\omega_{H\backslash G}|_{\mathbb{A}^\infty}, \quad g_\infty \in G(F_\infty),$$

where $\phi^H(g_{\mathrm{fin}})$ is the product of $\phi_v^H(g_v)$ over all $v\in\Sigma_{\mathrm{fin}}$. From condition (iii), by Lemma 5.4, we see that the integral converges absolutely and has the majorization

$$\sum_{j=0}^{2}\Big\|\frac{\mathrm{d}^j}{\mathrm{d}t_w^j}\varphi^H\big(\mathsf{a}(t)\big)\Big\| \ll_\epsilon \prod_{v\in\Sigma_\infty}(\cosh t_v)^{m-1+\epsilon}, \quad t\in\mathbb{R}^{\Sigma_\infty}, w\in\Sigma_\infty. \tag{6.2}$$

Moreover $\varphi^H(g_\infty)$, when viewed as a function in $g_v \in G(F_v)$, belongs to $C^\infty(H(F_v)\backslash G(F_v),\tau_{d_v})$ for all $v\in\Sigma_\infty$. We have

$$\begin{aligned}
&(-1)^{d_F}\mathrm{vol}\big(H(\mathbb{A}_F^\infty)\cap\mathcal{U};|\omega_H|_{\mathbb{A}^\infty}\big)^{-1}\langle\boldsymbol{\Phi}|\varphi\rangle\\
&= \int_{H(F)\backslash G(\mathbb{A}_F)} \big(\Phi(g)|\varphi(g)\big)|\omega_G|_{\mathbb{A}}\\
&= \int_{H(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \Big(\Phi(g)\,\big|\, \int_{H(F)\backslash H(\mathbb{A}_F)} \varphi(hg)|\omega_H|_{\mathbb{A}}\Big)|\omega_{H\backslash G}|_{\mathbb{A}}\\
&= \int_{H(F_\infty)\backslash G(F_\infty)} \big(\hat{\Psi}_{\mathfrak{d}}(\alpha;g_\infty)|\varphi^H(g_\infty)\big)|\omega_{H\backslash G}|_{F_\infty}\\
&= \Big(\frac{1}{2\pi i}\Big)^{d_F}\int_{\mathbb{L}(\sigma)}\Big\{\int_{H(F_\infty)\backslash G(F_\infty)} \big(\Psi_{\mathfrak{d}}(\mathbf{s};g_\infty)|\varphi^H(g_\infty)\big)|\omega_{H\backslash G}|_{F_\infty}\Big\}\\
&\quad\times\alpha(\mathbf{s})\,\mathrm{d}\mu_\infty(\mathbf{s}),
\end{aligned}\tag{6.3}$$

where $\Psi_{\mathfrak{d}}(\mathbf{s};g_\infty) = \bigotimes_{v\in\Sigma_\infty}\Psi_{d_v}(s_v;g_v)$, $\alpha(\mathbf{s}) = \prod_{v\in\Sigma_\infty}\alpha_v(s_v)$, and the outer integral in the last line is the multidimensional contour integral on $\mathbb{L}(\sigma) = \{\mathbf{s}\in\mathbb{C}^{\Sigma_\infty} \mid \mathrm{Re}(s_v) = \sigma\}$ with respect to the differential form $\mathrm{d}\mu_\infty(\mathbf{s}) = \prod_{v\in\Sigma_\infty} s_v\,\mathrm{d}s_v$. Due to the estimation (6.2) combined with Lemma 3.4, we have a constant $N>0$ such that

$$\big|\big(\Psi_{\mathfrak{d}}(\mathbf{s};\mathsf{a}(t))|\varphi^H(\mathsf{a}(t))\big)\big| \ll_\epsilon \prod_{v\in\Sigma_\infty}\big(1+|\mathrm{Im}(s_v)|\big)^N(\cosh t_v)^{-2\,\mathrm{Re}(s_v)+\epsilon}, \quad t\in\mathbb{R}^{\Sigma_\infty}.$$

By this,

$$\int_{\mathbb{L}(\sigma)}\Big\{\int_{H(F_\infty)\backslash G(F_\infty)}\big|\big(\Psi_{\mathfrak{d}}(\mathbf{s};g_\infty)|\varphi^H(g_\infty)\big)\big|\Big\}|\alpha(\mathbf{s})||\mathrm{d}\mu_\infty(\mathbf{s})|$$

$$\ll \int_{\mathbb{L}(\sigma)} \Big\{ \int_{\mathbb{R}_+^{\Sigma_\infty}} \big|\big(\Psi_{\mathfrak{d}}(\mathbf{s}; \mathsf{a}(t)) \big| \varphi^H(\mathsf{a}(t))\big)\big| \prod_{v \in \Sigma_\infty} (\cosh t_v)^{2m-3} (\sinh t_v)\, \mathrm{d}t_v \Big\}$$
$$\times |\alpha(\mathbf{s})| |\mathrm{d}\mu_\infty(\mathbf{s})|$$
$$\ll_\epsilon \Big\{ \int_{\mathbb{R}_+^{\Sigma_\infty}} \prod_{v \in \Sigma_\infty} (\cosh t_v)^{-2\sigma+2m-3+\epsilon} (\sinh t_v)\, \mathrm{d}t_v \Big\}$$
$$\times \Big\{ \int_{\mathbb{L}(\sigma)} \big(1 + |\mathrm{Im}(s_v)|\big)^N |\alpha(\mathbf{s})| |\mathrm{d}\mu_\infty(\mathbf{s})| \Big\}.$$

In the last line, the t_v-integrals are convergent if $\sigma > m-1$ and the $\mathbf{s}$-integral is also convergent due to the majorization $|\alpha_v(s_v)| \ll (1 + |\mathrm{Im}(s_v)|)^{-N-2}$ on $\mathrm{Re}(s_v) = \sigma$. Thus all the equalities occurring in the displayed formula (6.3) are justified by Fubini's theorem. By (6.2), we can apply Lemma 3.5 successively to obtain

$$\int_{H(F_\infty)\backslash G(F_\infty)} \Big(\Psi_{\mathfrak{d}}(\mathbf{s}; g_\infty) | \prod_{v \in \Sigma_\infty} \big[\mathcal{C}_{G(F_v)} - 2^{-1}\{(2s_v)^2 - (m-1)^2\}\big] \varphi^H(g_\infty)\Big)$$
$$\times |\omega_{H\backslash G}|_{F_\infty}$$
$$= 2^{d_F} \big(\vartheta(\mathfrak{d}) | \varphi^H(e)\big).$$

By condition (ii), this yields

$$\int_{H(F_\infty)\backslash G(F_\infty)} \big(\Psi_{\mathfrak{d}}(\mathbf{s}; g_\infty) | \varphi^H(g_\infty)\big) |\omega_{H\backslash G}|_{F_\infty} = \frac{(\vartheta(\mathfrak{d}) | \varphi^H(e))}{\prod_{v \in \Sigma_\infty} (\nu_v^2 - s_v^2)}.$$

Plugging this into the last formula in (6.3) and then applying [26, Lemma 9.5], we have

$$(-1)^{d_F} \mathrm{vol}\big(H(\mathbb{A}_F^\infty) \cap \mathcal{U}; |\omega_H|_{\mathbb{A}^\infty}\big)^{-1} \langle \mathbf{\Phi} | \varphi \rangle$$
$$= \Big(\frac{1}{2\pi i}\Big)^{d_F} \int_{\mathbb{L}(\sigma)} \frac{\alpha(\mathbf{s})}{\prod_{v \in \Sigma_\infty} (\nu_v^2 - s_v^2)} \mathrm{d}\mu_\infty(\mathbf{s}) \big(\vartheta(\mathfrak{d}) | \varphi^H(e)\big)$$
$$= (-1)^{d_F} \Big\{ \prod_{v \in \Sigma_\infty} \alpha_v(\nu_v) \Big\} \big(\vartheta(\mathfrak{d}) | \varphi^H(e)\big).$$

By definition,

$$\varphi^H(e) = \int_{H(\mathbb{A}_F^\infty)\backslash G(\mathbb{A}_F^\infty)} \bar{\phi}^H(g_{\mathrm{fin}}) \Big\{ \int_{H(F)\backslash H(\mathbb{A}_F)} \varphi(h g_{\mathrm{fin}}) |\omega_H|_{\mathbb{A}} \Big\} |\omega_{H\backslash G}|_{\mathbb{A}^\infty}$$
$$= \mathrm{vol}\big(H(\mathbb{A}_F^\infty) \cap \mathcal{U}; |\omega_H|_{\mathbb{A}^\infty}\big)^{-1}$$
$$\times \int_{G(\mathbb{A}_F^\infty)} \bar{\phi}(g_{\mathrm{fin}}) \Big\{ \int_{H(F)\backslash H(\mathbb{A}_F)} \varphi(h g_{\mathrm{fin}}) |\omega_H|_{\mathbb{A}} \Big\} |\omega_{H\backslash G}|_{\mathbb{A}^\infty}$$
$$= \mathrm{vol}\big(H(\mathbb{A}_F^\infty) \cap \mathcal{U}; |\omega_H|_{\mathbb{A}^\infty}\big)^{-1} \mathcal{P}_H(\varphi * \bar{\phi}). \qquad \square$$

Let $\pi \in \Pi_{\mathrm{dis}}(G)$. For $v \in \Sigma_\infty$, let $\nu_v(\pi)$ be a complex number such that the eigenvalue of $\mathcal{C}_{G(F_v)}$ on π is $2^{-1}\{(2\nu_v(\pi))^2 - (m-1)^2\}$; to specify $\nu_v(\pi)$ uniquely, we always impose the condition $\mathrm{Re}(\nu_v(\pi))>0$ or $\mathrm{Re}(\nu_v(\pi))=0$, $\mathrm{Im}(\nu_v(\pi))\geq 0$ on $\nu_v(\pi)$. The vector $\nu_\infty(\pi)=\{\nu_v(\pi)\}_{v\in\Sigma_\infty}$ is called the *Archimedean spectral parameter* of π. Set

$$Q_\infty(\pi)=\prod_{v\in\Sigma_\infty}\bigl(1+|\nu_v(\pi)|^2\bigr).$$

We already introduced the similar quantity $Q_\infty(\eta|\cdot|_F^\nu)$ for $\eta\in Y_F$ and $\nu\in\mathbb{C}$ in Section 5.5.1.

LEMMA 6.5

For any $R>0$, there exists $N>0$ such that

$$\|\varphi(g)\|\ll Q_\infty(\pi)^N a(g)^{-R},\qquad g\in\mathfrak{S}_G,\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi),\pi\in\Pi_{\mathrm{cus}}(G). \tag{6.4}$$

There exists $N_1>0$ such that, for any $\epsilon>0$,

$$\begin{aligned}&\|E(f^{(\nu)};g)\|\ll_\epsilon Q_\infty(\eta|\cdot|_F^\nu)^{N_1}a(g)^{(m-1)/2+\epsilon},\\ &\qquad\qquad g\in\mathfrak{S}_G,f\in\mathcal{B}_{\mathfrak{d}}(\eta),\eta\in Y_F,\nu\in i\mathbb{R}.\end{aligned} \tag{6.5}$$

Proof

The first estimate, as well as the second one with a possibly larger exponent of $a(g)$, follows from the argument of [7, Section 5.3] (see also [27, Section 15.1]). Since the exponent of $a(g)$ in (6.5) is crucial to our purposes, we reproduce the argument closely following [7] but giving a necessary modification. Let $\phi\in C_c^0(G(\mathbb{A}_F))$. Then from [7, p. 636],[1] there exist constants $c_1,c_2>0$ such that

$$E\bigl(I(\eta|\cdot|_F^\nu;\phi)f^{(\nu)},g\bigr)=\int_{G(F)\backslash G(\mathbb{A}_F)}K_\phi(g,x)\Lambda^T E(f^{(\nu)},x)|\omega_G|_{\mathbb{A}}$$

for all $g\in\mathfrak{S}_G$ and all $T>c_1\log\|g\|_G+c_2$, where $K_\phi(g,x)=\sum_{\gamma\in G(F)}\phi(g^{-1}\gamma x)$. Let $\mathcal{V}=\mathrm{supp}(\phi)$. From Lemma 6.2, there is a constant $c_3>0$ such that

$$|K_\phi(g,x)|\leq c_3a(g)^{m-1}\mathbb{1}_{G(F)g\mathcal{V}}(g),\quad g\in\mathfrak{S}_G,x\in G(\mathbb{A}_F).$$

By Lemma 6.2(2), we have a constant $c_4>0$ such that $\|x\|_G\geq c_4a(g)^{1/2}$ for all $x\in\mathfrak{S}_G$ such that $\mathbb{1}_{G(F)g\mathcal{V}}(x)\neq 0$. Taking the integral in $x\in\mathfrak{S}_G$ and by Lemma 6.2 (3) (adapted to G), we have

$$\begin{aligned}\Bigl(\int_{G(F)\backslash G(\mathbb{A}_F)}|K_\phi(g,x)|^2|\omega_G|_{\mathbb{A}}\Bigr)^{1/2}&\leq c_5a(g)^{m-1}\Bigl(\int_{\substack{x\in\mathfrak{S}_G\\ \|x\|_G\geq c_4a(g)^{1/2}}}|\omega_G|_{\mathbb{A}}\Bigr)^{1/2}\\ &\leq c_6a(g)^{m-1}\bigl\{a(g)^{-(m-1)}\bigr\}^{1/2}=c_6a(g)^{(m-1)/2}\end{aligned}$$

with some constant $c_5,c_6>0$. Thus, by the Cauchy–Schwarz inequality

$$\bigl|E\bigl(I(\eta|\cdot|_F^\nu;\phi)f^{(\nu)},g\bigr)\bigr|\leq c_6a(g)^{(m-1)/2}\|\Lambda^T E(f^{(\nu)})\|_2 \tag{6.6}$$

[1] In [7, p. 636] (line 5 from the bottom), $T>c_1\|g\|+c_2$ can be replaced with $T>c_1\log\|g\|+c_2$.

for all $g\in\mathfrak{S}_G$ and all $T\geq c_1\log\|g\|_G+c_2$. Note that $E(f^{(\nu)})$ is a cuspidal Eisenstein series since $M_1\cong G_1$ is F-anisotropic. From the Maass–Selberg relation (see [7, (11)], we have a constant $c_7>0$ (independent of ν, η, and $T>1$) such that

$$\big\|\Lambda^T E(f^{(\nu)})\big\|_2\leq c_7T^{1/2}\big(1+\big|\big(m_G(\bar\eta,-\nu)m'_G(\eta,\nu)f|f\big)_{\eta,\mathfrak{d}}\big|+\big\|m_G(\eta,\nu)f\big\|_{\eta,\mathfrak{d}}\big),$$

where $m_G(\eta,\nu):I(\eta|\cdot|_F^{\nu})\to I(\bar\eta|\cdot|_F^{-\nu})$ is the global intertwining operator defined by (5.4). This, combined with Lemma 5.2, gives us constants c_8 and $N>0$ independent of T such that

$$\big\|\Lambda^T E(f^{(\nu)})\big\|_2\leq c_8T^{1/2}Q_\infty\big(\eta|\cdot|_F^{\nu}\big)^N,\quad \nu\in i\mathbb{R},\eta\in Y_F. \tag{6.7}$$

From (6.6) and (6.7),

$$\big|E\big(I\big(\eta|\cdot|_F^{\nu};\phi\big)f^{(\nu)},g\big)\big|\leq c_9a(g)^{(m-1)/2}T^{1/2}Q_\infty\big(\eta|\cdot|_F^{\nu}\big)^N,\quad \nu\in i\mathbb{R},\eta\in Y_F,$$

for all $g\in\mathfrak{S}_G$ and all $T>c_1\log\|g\|_G+c_2$. By letting $T=2(c_1\log\|g\|_G+c_2)$, we obtain

$$\big|E\big(I\big(\eta|\cdot|_F^{\nu};\phi\big)f^{(\nu)},g\big)\big|\ll a(g)^{(m-1)/2+\epsilon}Q_\infty\big(\eta|\cdot|_F^{\nu}\big)$$

with the implied constant independent of $\nu\in i\mathbb{R}$, $\eta\in Y_F$, and $f\in V(\eta)$. The remaining part of the proof is the same as [7, Section 5.3]. □

Let $\mathcal{V}=\prod_{v\in\Sigma_\infty}\mathcal{V}_v$ be an open compact subgroup of $G(\mathbb{A}_F^\infty)\cap\mathcal{U}$ such that ϕ is right $\mathcal{V}$-invariant. Let $\Pi_{\rm dis}(G)^{\mathcal{V}}$ be the set of all those $\pi\in\Pi_{\rm dis}(G)$ such that $\pi^{\mathcal{V}}\neq\{0\}$, and set $\Pi_{\rm cus}(G)^{\mathcal{V}}=\Pi_{\rm cus}(G)\cap\Pi_{\rm dis}(G)^{\mathcal{V}}$. For each $\pi\in\Pi_{\rm dis}(G)^{\mathcal{V}}$, we always choose our $\mathcal{B}_{\mathfrak{d}}(\pi)$ in such a way that any $\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)$ belongs to $\pi[\mathfrak{d}]^{\mathcal{V}}$ or the orthogonal complement of $\pi[\mathfrak{d}]^{\mathcal{V}}$ in $\pi[\mathfrak{d}]$. Similarly, for each $\eta\in Y_F$, we always require that $\mathcal{B}_{\mathfrak{d}}(\eta)$ is a disjoint union of $\mathcal{B}_{\mathfrak{d}}(\eta)\cap\hat{\mathcal{H}}_{\mathfrak{d}}(\eta)^{\mathcal{V}}$ and $\mathcal{B}_{\mathfrak{d}}(\eta)\cap(\hat{\mathcal{H}}_{\mathfrak{d}}(\eta)^{\mathcal{V}})^{\perp}$.

LEMMA 6.6

Let U be a compact subset of $G(\mathbb{A}_F)$. There exists N_1 such that if $R>N_1$, then the series

$$\mathfrak{C}_R^{\mathcal{V}}(g)=\sum_{\pi\in\Pi_{\rm cus}(G)^{\mathcal{V}}}\ \sum_{\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}}Q_\infty(\pi)^{-R}\big\|\varphi(g)\big\|$$

and the series-integral

$$\mathfrak{E}_R^{\mathcal{V}}(g)=\sum_{\eta\in Y_F}\ \sum_{f\in\mathcal{B}_{\mathfrak{d}}(\eta)^{\mathcal{V}}}\int_{i\mathbb{R}}Q_\infty\big(\eta|\cdot|_F^{\nu}\big)^{-R}\big\|E(f^{(\nu)};g)\big\||\mathrm{d}\nu|$$

converges uniformly on $g\in U$.

Proof

By Lemma 6.5, we have a constant $C>0$ and $N_0>0$ such that $\|\varphi(g)\|\leq CQ_\infty(\pi)^{N_0}$ for all $\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}$ and $g\in U$. Thus, the series

$$\sum_{\pi\in\Pi_{\rm cus}(G)^{\mathcal{V}}}\ \sum_{\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}}Q_\infty(\pi)^{-R+N_0} \tag{6.8}$$

is a uniform majorant for $\mathfrak{C}_R^{\mathcal{V}}(g)$ on $g \in U$. By Weyl's law for the C^∞-bundle $G(F)\backslash G(\mathbb{A}_F)/\mathcal{V} \times_{\mathcal{U}_\infty,\tau_\mathfrak{d}} W(\mathfrak{d}) \to G(F)\backslash G(\mathbb{A}_F)/\mathcal{U}_\infty\mathcal{V}$ (see [2], [15]), the series (6.8) turns out to be convergent if $R - N_0 > (m-1)d_F$. In the same way, by Lemma 6.5, the series-integral $\mathfrak{E}_R^{\mathcal{V}}(g)$ is majorized uniformly on U by

$$\sum_{\eta \in Y_F} \sum_{f \in \mathcal{B}_\mathfrak{d}(\eta)^{\mathcal{V}}} \int_{i\mathbb{R}} Q_\infty\big(\eta|\cdot|_F^\nu\big)^{-R+N_0} |\mathrm{d}\nu| \tag{6.9}$$

with some constant $N_0 > 0$. Since $f \in \mathcal{H}_\mathfrak{d}(\eta)$ is determined by its restriction to $\mathcal{U}_{\mathrm{fin}} = \mathcal{U} \cap G(\mathbb{A}_F^\infty)$, we see that $\mathcal{B}_\mathfrak{d}(\eta)^{\mathcal{V}}$ is contained in the space of $W(\mathfrak{d})$-valued functions on the finite set $\mathcal{U}_{\mathrm{fin}}/\mathcal{V}$, and hence $\#\mathcal{B}_\mathfrak{d}(\eta)^{\mathcal{V}} \le [\mathcal{U}_{\mathrm{fin}} : \mathcal{V}] \dim W(\mathfrak{d})$. Thus to see the convergence of (6.9), we may ignore the summation over f. There exists an open compact subgroup $\mathcal{V}_1 \subset \mathcal{U}$ such that any function $f : \mathcal{U}_{\mathrm{fin}}/\mathcal{V} \to W(\mathfrak{d})$ is left $\mathcal{V}_1$-invariant. Let $\mathfrak{f}_{\mathcal{V}}$ be an ideal of $\mathfrak{o}_E$ such that, for any $\tau \in \prod_{v \in \Sigma_{\mathrm{fin}}} \{(1 + \mathfrak{f}_{\mathcal{V}}\mathfrak{o}_{E_v}) \cap \mathfrak{o}_{E_v}^\times\}$, the element $[\tau]$ (see Section 2.1 for the definition) is contained in $\mathcal{V}_1$. Let $Y_F(\mathfrak{f}_{\mathcal{V}})$ be the set of $\eta \in Y_F$ such that the conductor of $\eta \circ \mathrm{N}_{E/F}$ divides the ideal $\mathfrak{f}_{\mathcal{V}}$. Since $\mathcal{B}_\mathfrak{d}(\eta) = \varnothing$ unless $\eta \in Y_F(\mathfrak{f}_{\mathcal{V}})$, the series (6.9) is majorized by

$$\sum_{\eta \in Y_F(\mathfrak{f}_{\mathcal{V}})} \int_{i\mathbb{R}} Q_\infty\big(\eta|\cdot|_F^\nu\big)^{-R+N_0} |\mathrm{d}\nu|.$$

There exists a lattice L of $\{b = (b_v) \in i\mathbb{R}^{\Sigma_\infty} \mid \sum_v b_v = 0\}$ such that $Y(\mathfrak{f}_{\mathcal{V}}) = \{\eta \prod_v |\cdot|_{F_v}^{b_v} \mid b \in L, \eta \in Y_F^0(\mathfrak{f}_{\mathcal{V}})\}$, where $Y_F^0(\mathfrak{f}_{\mathcal{V}})$ denotes the finite-order elements in $Y_F(\mathfrak{f}_{\mathcal{V}})$. Since $\#Y_F^0(\mathfrak{f}_{\mathcal{V}}) < \infty$, we are reduced to see the convergence of

$$\sum_{b \in L} \int_{\mathbb{R}} \prod_{v \in \Sigma_\infty} \big(1 + |it + b_v|^2\big)^{-R+N_0} \,\mathrm{d}t.$$

Since $\int_{\mathbb{R}^{d_F}} \prod_{j=1}^{d_F} (1 + |x_j|^2)^{-R} \,\mathrm{d}x < +\infty$ if $R > d_F/2$, we have the desired convergence for any R such that $R > d_F/2 + N_0$. □

PROPOSITION 6.7

The series-integral

$$\begin{aligned} &\sum_{\pi \in \Pi_{\mathrm{dis}}(G)^{\mathcal{V}}} \sum_{\varphi \in \mathcal{B}_\mathfrak{d}(\pi)^{\mathcal{V}}} \alpha\big(\nu_\infty(\pi)\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H(\varphi * \bar{\phi})\big)\varphi(g) \\ &+ \sum_{\eta \in Y_F} \sum_{f \in \mathcal{B}_\mathfrak{d}(\eta)^{\mathcal{V}}} \frac{1}{4\pi i} \int_{i\mathbb{R}} \alpha\big(\nu_\infty(\eta|\cdot|_F^\nu)\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H\big(E\big(I\big(\eta|\cdot|_F^\nu;\bar{\phi}\big)f^{(\nu)}\big)\big)\big) \\ &\times E(f^{(\nu)};g)\,\mathrm{d}\nu \end{aligned} \tag{6.10}$$

converges to the value $\mathbf{\Phi}(g)$ absolutely and locally uniformly in $g \in G(\mathbb{A}_F)$.

Proof

By Lemma 6.3, $\mathbf{\Phi} \in L^{2+\epsilon}(G(F)\backslash G(\mathbb{A}_F))[\mathfrak{d}]^{\mathcal{V}}$ for any $\epsilon \in [0, m-3)$; since $m \ge 4$, such ϵ's are nonempty. We have the spectral expansion of $\mathbf{\Phi}$ of the form

$$\begin{aligned}\mathbf{\Phi}(g) = &\sum_{\pi\in\Pi_{\mathrm{dis}}(G)^{\mathcal{V}}}\sum_{\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}}\langle\mathbf{\Phi}|\varphi\rangle\varphi(g)\\ &+\sum_{(\chi,\sigma)}\sum_{f\in\mathcal{B}_{\mathfrak{d}}(\chi,\sigma)^{\mathcal{V}}}\frac{1}{4\pi i}\int_{i\mathbb{R}}\big\langle\mathbf{\Phi}|E(f^{(-\nu)})\big\rangle E(f^{(\nu)})\,\mathrm{d}\nu,\end{aligned}\tag{6.11}$$

which should be understood in the L^2-sense for a moment. Here (χ,σ) runs over a set of pairs of unitary idèle class characters χ of $E^\times$ and an irreducible closed submodule $\sigma\subset L^2(G_1(F)\backslash G_1(\mathbb{A}_F))$, and $\mathcal{B}_{\mathfrak{d}}(\chi,\sigma)^{\mathcal{V}}$ is an orthonormal basis of $I(\chi,\sigma)[\mathfrak{d}]^{\mathcal{V}}$ (see Section 5.5). Since $\varphi=E(f^{(\nu)})$ with $\nu\in i\mathbb{R}$ satisfies the conditions (i), (ii), and (iii) in Lemma 6.4, $\langle\mathbf{\Phi}|E(f^{(\nu)})\rangle$ is proportional to $(\vartheta(\mathfrak{d})|\mathcal{P}_H(E(f^{(\nu)}*\bar\phi))$. By Theorem 5.6, $\mathcal{P}_H(E(f^{(\nu)}))$ is zero unless $\chi=\eta\circ\mathrm{N}_{E/F}$ with some $\eta\in Y_F$ and σ coincides with the constant functions on $G_1(F)\backslash G_1(\mathbb{A}_F)$. Let $\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)$ with $\pi\in\Pi_{\mathrm{dis}}(G)-\Pi_{\mathrm{cus}}(G)$. From [18, Proposition IV.1.11], φ is a finite linear combination of the residues $\varphi_r=\mathrm{Res}_{s=r}E(f^{(\nu)})$ with $f\in I(\chi,\sigma)[\mathfrak{d}]$ and $r\in(0,\frac{m-1}{2}]$. By applying Lemma 6.4 to φ_r, we see that $\langle\mathbf{\Phi}|\varphi_r\rangle$ is a constant multiple of $\mathcal{P}_H(\varphi_r*\bar\phi)$, which, from Theorem 5.6(2), should be zero unless $\eta\circ\mathrm{N}_{E/F}=1$, σ is trivial, and $r=\frac{m-1}{2}$. Applying Lemma 6.4 to all cuspidal components $\langle\mathbf{\Phi}|\varphi\rangle$, we see that (6.11) is simplified to

$$\begin{aligned}\mathbf{\Phi}(g) = &\sum_{\pi\in\Pi_{\mathrm{dis}}(G)^{\mathcal{V}}}\sum_{\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}}\alpha\big(\nu_\infty(\pi)\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H(\varphi*\bar\phi)\big)\varphi(g)\\ &+\sum_{\eta\in Y_F}\sum_{f\in\mathcal{B}_{\mathfrak{d}}(\eta)^{\mathcal{V}}}\frac{1}{4\pi i}\int_{i\mathbb{R}}\alpha\big(\nu_\infty(\eta|\cdot|_F^{\nu})\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H\big(I\big(\eta|\cdot|_F^{\nu};\bar\phi\big)f^{(\nu)}\big)\big)\\ &\times E(f^{(\nu)})\,\mathrm{d}\nu,\end{aligned}\tag{6.12}$$

where the only noncuspidal $\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}$ contributing to the sum is the constant function, which occurs only when $d_v=0$ for all v. To complete the proof, it suffices to show the uniform convergence on U of the series-integral (6.12). From Lemmas 5.4 and 6.5, we have a constant $N>0$ such that

$$\begin{aligned}\big\|\mathcal{P}_H(\varphi*\bar\phi)\big\|\le&\int_{H(F)\backslash H(\mathbb{A}_F)}\big\|(\varphi*\bar\phi)(h)\big\||\omega_H|_{\mathbb{A}}\\ &\ll Q_\infty(\pi)^N,\quad \pi\in\Pi_{\mathrm{cus}}(G)^{\mathcal{V}},\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}.\end{aligned}\tag{6.13}$$

For an arbitrary large $R>0$, we have $|\alpha(\mathbf{s})|\ll\prod_{v\in\Sigma_\infty}(1+|\mathrm{Im}(s_v)|)^{-R}$ compact uniformly in $\mathrm{Re}(\mathbf{s})$. Hence

$$\begin{aligned}\big\|\alpha\big(\nu_\infty(\pi)\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H(\varphi*\bar\phi)\big)\varphi(g)\big\|\le&\big|\alpha\big(\nu_\infty(\pi)\big)\big|\big\|\mathcal{P}_H(\varphi*\bar\phi)\big\|\big\|\varphi(g)\big\|\\ &\ll Q_\infty\big(\nu_\infty(\pi)\big)^{-R+N}\big\|\varphi(g)\big\|\end{aligned}$$

with the implied constant independent of π and φ. Thus, the discrete part of (6.12) is majorized by the series $\mathfrak{C}^{\mathcal{V}}_{R-N}(g)$, which is uniformly convergent on U by Lemma 6.6. From Lemmas 5.4 and 6.5, we obtain a polynomial bound $\|\mathcal{P}_H(E(I(\eta|\cdot|_F^{\nu};\bar\phi)f^{(\nu)}))\|\ll Q_\infty(\eta|\cdot|_F^{\nu})^N$ uniformly in $f\in\mathcal{B}_{\mathfrak{d}}(\eta)^{\mathcal{V}}$, η, and ν.

Using this estimate and Lemma 6.6, we see the uniform convergence on U of the continuous part of (6.12) in the same way as in the discrete part. □

6.5. Period integral of the relative kernel function

We consider the H-period integral $\mathcal{P}_H(\mathbf{\Phi}|\vartheta(\mathfrak{d}))$ of $g \mapsto (\mathbf{\Phi}(g)|\vartheta(\mathfrak{d}))$.

PROPOSITION 6.8

The period integral $\mathcal{P}_H(\mathbf{\Phi}|\vartheta(\mathfrak{d}))$ *converges absolutely and equals* $(-1)^{d_F} \times \mathrm{vol}(H(\mathbb{A}_F^\infty)\cap\mathcal{U})$ *times*

$$\sum_{\pi\in\Pi_{\mathrm{dis}}(G)^{\mathcal{V}}}\sum_{\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}}\alpha\big(\nu_\infty(\pi)\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H(\varphi*\bar\phi)\big)\big(\mathcal{P}_H(\varphi)|\vartheta(\mathfrak{d})\big)$$
$$+\sum_{\eta\in Y_F}\sum_{f\in\mathcal{B}_{\mathfrak{d}}(\eta)^{\mathcal{V}}}\frac{1}{4\pi i}\int_{i\mathbb{R}}\alpha\big(\nu_\infty(\eta|\cdot|_F^\nu)\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H\big(E\big(I\big(\eta|\cdot|_F^\nu;\bar\phi\big)f^{(\nu)}\big)\big)\big)$$
$$\times\big(\mathcal{P}_H\big(E(f^{(\nu)})\big)|\vartheta(\mathfrak{d})\big)\,\mathrm{d}\nu.$$

Proof

The formula is obtained by the termwise integration of the series-integral (6.10). To justify this process by Fubini's theorem, we only have to ensure the convergence of the following series-integrals:

$$\begin{aligned}\sum_{\pi\in\Pi_{\mathrm{dis}}(G)^{\mathcal{V}}}\sum_{\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}}&\big|\alpha\big(\nu_\infty(\pi)\big)\big|\big|\big(\vartheta(\mathfrak{d})|\mathcal{P}_H(\varphi*\bar\phi)\big)\big|\\ &\times\int_{H(F)\backslash H(\mathbb{A}_F)}\big\|\varphi(h)\big\||\omega_H|_{\mathbb{A}},\end{aligned}\tag{6.14}$$

$$\begin{aligned}\sum_{\eta\in Y_F}\sum_{f\in\mathcal{B}_{\mathfrak{d}}(\eta)^{\mathcal{V}}}\int_{i\mathbb{R}}&\big|\alpha\big(\nu_\infty(\eta|\cdot|_F^\nu)\big)\big|\big|\big(\vartheta(\mathfrak{d})|\mathcal{P}_H\big(E\big(I\big(\eta|\cdot|_F^\nu;\bar\phi\big)f^{(\nu)}\big)\big)\big)\big|\\ &\times\Big\{\int_{H(F)\backslash H(\mathbb{A}_F)}\big\|E(f^{(\nu)})\big\||\omega_H|_{\mathbb{A}}\Big\}|\mathrm{d}\nu|.\end{aligned}\tag{6.15}$$

By (6.13), the series (6.14) is majorized by

$$\sum_{\pi\in\Pi_{\mathrm{dis}}(G)^{\mathcal{V}}}\sum_{\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}}\big|\alpha\big(\nu_\infty(\pi)\big)\big|Q_\infty\big(\nu_\infty(\pi)\big)^{2N},$$

whose convergence can be confirmed in the same way as in the previous proof. The convergence of (6.15) is shown similarly. □

7. The geometric side

Let $\mathbf{\Phi}=\mathbf{\Phi}_{\mathfrak{d}}(\alpha,\phi)$ be the relative kernel function defined in Section 6.3. We compute the period integral $\mathcal{P}_H(\mathbf{\Phi}|\vartheta(\mathfrak{d}))$ in a different way than in Section 6.5.

7.1. The double coset space

For $\gamma \in G(F)$, set

$$b(\gamma) = \mathbf{h}(\gamma^{-1}\ell, \ell), \qquad \ell^\gamma = \gamma^{-1}\ell - b(\gamma)\ell, \qquad \Delta_\gamma = \mathbf{h}[\ell^\gamma].$$

A simple computation shows that ℓ^γ is orthogonal to ℓ, and $\Delta_\gamma = 1 - \mathrm{N}_{E/F} b(\gamma)$.

LEMMA 7.1

There exists a well-defined bijection

$$\mathrm{N}^\flat : H(F)\backslash\big(G(F) - H(F)\big)/H(F) \longrightarrow \mathrm{N}_{E/F}(E)$$

such that $\mathrm{N}^\flat(H(F)\gamma H(F)) = \mathrm{N}_{E/F} b(\gamma)$.

Proof

If $h, h_1 \in H(F)$, then there exist elements $c, c_1 \in E^1$ such that $h^{-1}\ell = c\ell$ and $h_1\ell = c_1\ell$; thus $\mathrm{N}_{E/F}(\mathbf{h}(hgh_1\ell, \ell)) = \mathrm{N}_{E/F}(\mathbf{h}(gh_1\ell, h^{-1}\ell)) = \mathrm{N}_{E/F}(c_1\bar{c}\mathbf{h}(g\ell, \ell)) = \mathrm{N}_{E/F}(\mathbf{h}(g\ell, \ell))$. This shows that the map $\mathrm{N}^\flat$ is well defined.

Let us show the injectivity of $\mathrm{N}^\flat$. Suppose that $\mathrm{N}^\flat(H(F)gH(F)) = \mathrm{N}^\flat(H(F)g_1H(F))$ with $g, g_1 \in G(F) - H(F)$. Then $\mathrm{N}_{E/F}(\mathbf{h}(g^{-1}\ell, \ell)) = \mathrm{N}_{E/F}(\mathbf{h}(g_1^{-1}\ell, \ell))$; equivalently, $\mathbf{h}(g^{-1}\ell, \ell) = c\mathbf{h}(g_1^{-1}\ell, \ell)$ with some $c \in E^1$. If we set $y = \mathbf{h}(g^{-1}\ell, \ell)$, $\xi_0 = g^{-1}\ell - y\ell$, and $\xi_0' = cg_1^{-1}\ell - y\ell$, then we can easily confirm that ξ_0 and ξ_0' belong to $\ell^\perp$. The condition $g, g_1 \notin H(F)$ implies that both ξ_0 and ξ_0' are nonzero. A computation shows the identity $\mathbf{h}[\xi_0] = 1 - \mathrm{N}_{E/F}\mathbf{h}(g^{-1}\ell, \ell)$. Hence $\mathrm{N}^\flat(H(F)gH(F)) = \mathrm{N}^\flat(H(F)g_1H(F))$ means the equality $\mathbf{h}[\xi_0] = \mathbf{h}[\xi_0']$. Since $\mathbf{h}$ restricted to $\ell^\perp$ is nondegenerate, we apply Witt's theorem to have an isometry h of $\ell^\perp$ such that $h\xi_0 = \xi_0'$. Extend h to an element of $G(F)$ by setting $h(\ell) = \ell$. Then $h(g^{-1}\ell) = h(\xi_0 + y\ell) = \xi_0' + y\ell = cg_1^{-1}\ell$; equivalently, $g_1hg^{-1}\ell = c\ell$. This means that $g_1hg^{-1} \in H(F)$. Since $h \in H(F)$, we obtain $H(F)g_1H(F) = H(F)gH(F)$, as desired.

Let $a \in E$. By the Hasse–Minkowski theorem applied to the quadratic space $(\mathbf{h}, \ell^\perp)$ (see the first sentence of Section 5.6), we can find a vector $\xi_a \in \ell^\perp$ such that $\mathbf{h}[\xi_a] = 1 - \mathrm{N}_{E/F}(a)$. Since the vector $\xi = a\ell + \xi_a$ satisfies $\mathbf{h}[\xi] = 1$, by Witt's theorem, we have $\xi = g^{-1}\ell$ with some $g \in G(F)$. Noting that $\xi_a \neq 0$, we obtain $g \in G(F) - H(F)$, and by a computation $\mathrm{N}^\flat(H(F)gH(F)) = \mathrm{N}_{E/F}(a)$ also. This completes the proof. □

We fix a complete set of representatives $X_{G,H}(F)$ in $G(F) - H(F)$ of the double coset space $H(F)\backslash(G(F) - H(F))/H(F)$ once and for all. An element $\gamma \in X_{G,H}(F)$ is said to be *regular* or *unipotent* according to whether $\mathrm{N}^\flat(\gamma) \neq 1$ or $\mathrm{N}^\flat(\gamma) = 1$, respectively. We set

$$X^{\mathrm{r}}_{G,H}(F) = \big\{\gamma \in X_{G,H}(F) \mid \mathrm{N}^\flat(\gamma) \in F - \{1\}\big\},$$

$$X^{\mathrm{u}}_{G,H}(F) = \big\{\gamma \in X_{G,H}(F) \mid \mathrm{N}^\flat(\gamma) = 1\big\}.$$

By Lemma 7.1, the set $X^{\mathrm{u}}_{G,H}(F)$ is a singleton. Indeed, we may take $X^{\mathrm{u}}_{G,H}(F) = \{\gamma_{\mathrm{u}}\}$, where $\gamma_{\mathrm{u}} \in G(F)$ is an element such that $\gamma_{\mathrm{u}}\ell = \ell + e_H$ with e_H being an F-isotropic vector in $\ell^{\perp}$.

7.2. The orbital integrals

Let $\gamma \in X_{G,H}(F)$, and set $H_\gamma = \gamma^{-1}H\gamma \cap H$. The unitary group $U(\mathbf{h}|\ell^{\perp})$ is identified with H_0, and the stabilizer $H_0(\ell^\gamma)$ in H_0 of ℓ^γ coincides with $H_0 \cap H_\gamma$. By the H_0-isomorphism

$$H_0(\ell^\gamma)\backslash H_0 \ni H_0(\ell^\gamma)h \longrightarrow h^{-1}\ell^\gamma \in \mathbf{\Sigma}'(\Delta_\gamma),$$

we transport the gauge-form $|\omega_{\mathbf{\Sigma}'(\Delta_\gamma)}|$ on $\mathbf{\Sigma}'(\Delta_\gamma) = \{\xi \in \ell^\perp - \{0\} \mid \mathbf{h}[\xi] = \Delta_\gamma\}$ to $H_0(\ell^\gamma)\backslash H_0$. When $\mathrm{N}^\flat(\gamma) \neq 0$, we have a gauge-form $\omega_{H_\gamma\backslash H}$ on $H_\gamma\backslash H \cong H_0(\ell^\gamma)\backslash H_0$ (cf. [27, Lemma 4.7]). When $\mathrm{N}^\flat(\gamma) = 0$, it is easy to see that $H_\gamma\backslash H \cong (T \times H_0(\ell^\perp))\backslash H_0$ with $T = E^1$. We fix an H-invariant gauge-form $\omega_{H_\gamma\backslash H}$ on $H_\gamma\backslash H$ so that $\omega_{H_\gamma\backslash H}$, ω_T, and $\omega_{H_0(\ell^\perp)\backslash H_0}$ match together algebraically.

LEMMA 7.2

We have $\mathrm{vol}(H) = \{2L(1,\varepsilon_{E/F})\}^2$. *For* $\gamma \in X_{G,H}(F)$, *the volume* $\mathrm{vol}(H_\gamma)$ *equals* $\{2L(1,\varepsilon_{E/F})\}^2$ *or* $\{2L(1,\varepsilon_{E/F})\}^3$ *according to whether* $\mathrm{N}^\flat(\gamma) \neq 0$ *or* $\mathrm{N}(\gamma)^\flat = 0$.

Proof

Since H_γ is isomorphic to $U(m-2) \times U(1)^2$ or $U(m-2) \times U(1)$ according to whether $\mathrm{N}^\flat(\gamma)$ is zero or not, from Section 5.3, the lemma is immediate. □

For any function $\Phi = \Phi(\mathfrak{d}, \alpha, \phi)$ constructed in Section 6.2, define

$$\mathbb{J}_{\mathfrak{d}}(\gamma; \Phi) = \int_{H_\gamma(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \big(\Phi(\gamma h)|\vartheta(\mathfrak{d})\big)|\omega_{H_\gamma\backslash H}|_{\mathbb{A}}. \tag{7.1}$$

LEMMA 7.3

We have

$$\sum_{\gamma \in X_{G,H}(F)} \int_{H(F)\backslash H(\mathbb{A}_F)} \big\|\Phi(\gamma h)\big\| |\omega_{H_\gamma\backslash H}|_{\mathbb{A}} < +\infty. \tag{7.2}$$

In particular, the integral (7.1) converges absolutely.

Proof

From Lemma 6.1 and the proof of Lemma 6.3, we have the estimate $\sum_{\gamma \in H(F)\backslash G(F)} \|\Phi(\gamma g)\| \ll a(g)$ on any Siegel domain $\mathfrak{S}_G \subset G(\mathbb{A}_F)$, which, combined with Lemma 5.4, shows the convergence of the double integral

$$\int_{H(F)\backslash H(\mathbb{A}_F)} \sum_{\gamma \in H(F)\backslash G(F)} \big\|\Phi(\gamma h)\big\| |\omega_H|_{\mathbb{A}}. \tag{7.3}$$

By the unfolding, we see that this equals

$$\sum_{\gamma\in H(F)\backslash G(F)/H(F)} \mathrm{vol}(H_\gamma)\int_{H_\gamma(\mathbb{A})\backslash H(\mathbb{A}_F)} \big\|\Phi(\gamma h)\big\||\omega_{H_\gamma\backslash H}|_{\mathbb{A}},$$

which, from Lemma 7.2, turns out to be a majorant of (7.2). □

7.3. The relative trace formula

From Proposition 6.8 and Lemma 7.3, we obtain the following.

THEOREM 7.4

Let $\mathcal{V}$ be an open compact subgroup of $G(\mathbb{A}_F^\infty)\cap\mathcal{U}$. For any $\alpha\in\mathcal{A}_\infty$ and a decomposable $\phi\in\mathcal{S}(G(\mathbb{A}_F^\infty))^{\mathcal{V}}$ (see Section 6.2), we have the identity $\mathbb{I}_{\mathfrak{d}}(\alpha,\phi)=\mathbb{J}_{\mathfrak{d}}(\alpha,\phi)$, where

$$\begin{aligned}\mathbb{I}_{\mathfrak{d}}(\alpha,\phi) = {} & \sum_{\pi\in\Pi_{\mathrm{dis}}(G)^{\mathcal{V}}}\sum_{\varphi\in\mathcal{B}_{\mathfrak{d}}(\pi)^{\mathcal{V}}} \alpha\big(\nu_\infty(\pi)\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H(\varphi*\bar\phi)\big)\big(\mathcal{P}_H(\varphi)|\vartheta(\mathfrak{d})\big)\\ & + \sum_{\eta\in Y_F}\sum_{f\in\mathcal{B}_{\mathfrak{d}}(\eta)^{\mathcal{V}}}\frac{1}{4\pi i}\\ & \times\int_{i\mathbb{R}}\alpha\big(\nu_\infty(\eta|\cdot|_F^\nu)\big)\big(\vartheta(\mathfrak{d})|\mathcal{P}_H\big(E\big(I(\eta|\cdot|_F^\nu;\bar\phi)f^{(\nu)}\big)\big)\big)\\ & \times\big(\mathcal{P}_H\big(E(f^{(\nu)})\big)|\vartheta(\mathfrak{d})\big)\,\mathrm{d}\nu\end{aligned}$$

and

$$\begin{aligned}\mathbb{J}_{\mathfrak{d}}(\alpha,\phi) = (-1)^{d_F}\mathrm{vol}\big(H(\mathbb{A}_F^\infty)\cap\mathcal{U}\big)\Big\{&\mathrm{vol}(H)\big(\Phi_{\mathfrak{d}}(\alpha,\phi;e)|\vartheta(\mathfrak{d})\big)\\ & + \sum_{\gamma\in X_{G,H}(F)}\mathrm{vol}(H_\gamma)\mathbb{J}_{\mathfrak{d}}\big(\gamma;\Phi_{\mathfrak{d}}(\alpha,\phi)\big)\Big\}.\end{aligned}$$

All the series and integrals are absolutely convergent.

8. The germ expansion of local orbital integrals

In this section, we return to the setting of Section 4, keeping all notation and conventions introduced there. For any $\gamma\in G-H$, the local orbital integral is defined by

$$J_F(\gamma;f)=\int_{H_\gamma\backslash H} f(\gamma h)|\omega_{H_\gamma\backslash H}|_F,\quad f\in\mathcal{S}(H\backslash G), \tag{8.1}$$

where H_γ and $\omega_{H_\gamma\backslash H}$ are as in Section 7.2. We let $\mathcal{S}_E(F)$ be the space of all the compactly supported $\mathbb{C}$-valued functions φ on F smooth on $F^\times$ such that $\varphi(x)$ equals a constant multiple of $\chi_E(x)$ in a neighborhood of $x=0$, where $\chi_E(x)=\mathrm{vol}(\mathfrak{o}_F^\times)\{1+\mathrm{ord}_F(x)\}\delta(x\in\mathfrak{o}_F)$ if E is not a field and $\chi_E(x)=\delta(x\in\mathfrak{o}_F)\{1+\varepsilon_{E/F}(x)\}$ if E is a field.

PROPOSITION 8.1

(1) *Let* $f \in \mathcal{S}(H\backslash G)$*. The integral (8.1) converges absolutely. There exists* $\varphi_0, \varphi_1 \in \mathcal{S}_E(F)$ *such that*

$$J_F(\gamma; f) = \varphi_1\big(\mathrm{N}^\flat(\gamma)\big) + C_{\mathbf{h}^\ell}\big|\mathrm{N}^\flat(\gamma) - 1\big|_F^{m-2} \varepsilon_{E/F}^{m-1}\big(\mathrm{N}^\flat(\gamma) - 1\big)\varphi_0\big(\mathrm{N}^\flat(\gamma)\big) \tag{8.2}$$

for all $\gamma \in G - H$ *such that* $\mathrm{N}^\flat(\gamma) \neq 0$*, where* $C_{\mathbf{h}^\ell}$ *is the constant to be defined in Section A.1 For any pair* (φ_0, φ_1) *of elements from* $\mathcal{S}_E(F)$ *satisfying this condition, we have*

$$\varphi_1(1) = J_F(\gamma_{\mathrm{u}}; f), \qquad \varphi_0(1) = f(e).$$

(2) *For any pair* (φ_0, φ_1) *of functions from* $\mathcal{S}_E(F)$*, there exists* $f \in \mathcal{S}(H\backslash G)$ *such that (8.2) holds for all* $\gamma \in G - H$ *with* $\mathrm{N}^\flat(\gamma) \neq 0$*.*

Proof

There exists $\phi \in \mathcal{S}(\mathbf{\Sigma}(1))$ such that $f(g) = \int_{\tau \in E^1} \phi(\tau g^{-1}\ell)|\omega_{E^1}|_F$ for all $g \in G$. Since $\mathbf{\Sigma}(1)$ is a closed subset of V, we can extend ϕ to an element of $\mathcal{S}(V)$ denoted by the same symbol ϕ. Let $\gamma \in G - H$ with $\mathrm{N}^\flat(\gamma) \neq 0$. For $t \in F$, set $\mathbf{\Sigma}'(t) = \ell^\perp \cap \mathbf{\Sigma}(t)$. Then

$$\begin{aligned} J_F(\gamma; f) &= \int_{Z \in \mathbf{\Sigma}'(1 - \mathrm{N}_{E/F} b(\gamma))} \int_{\tau \in E^1} \phi\big(\tau\big(b(\gamma)\ell + Z\big)\big)|\omega_{E^1}|_F |\omega_{\mathbf{\Sigma}'(1 - \mathrm{N}_{E/F} b(\gamma))}|_F \\ &= \int_{\tau \in E^1} J^\ell\big(\tau b(\gamma), \phi\big)|\omega_{E^1}|_F, \end{aligned}$$

where $\phi \mapsto J^\ell(\beta, \phi)$ is the linear functional on $\mathcal{S}(V)$ defined by (A.1). From Lemma A.3, there exists $\tilde{\varphi}_1 \in \mathcal{S}(E)$ such that

$$\begin{aligned} J^\ell(\beta, \phi) &= \tilde{\varphi}_1(\beta) + C_{\mathbf{h}^\ell}\delta(\mathrm{N}_{E/F}\beta - 1 \in \mathfrak{o}_F)|\mathrm{N}_{E/F}\beta - 1|_F^{m-2} \\ &\quad \times \varepsilon_{E/F}^{m-1}\big(\mathrm{N}_{E/F}(\beta) - 1\big)\phi(\beta\ell) \end{aligned} \tag{8.3}$$

for all $\beta \in E - E^1$. Setting $\beta = \tau b(\gamma)$ with $\tau \in E^1$ and taking the τ-integral, we obtain

$$\begin{aligned} J_F(\gamma; f) &= \varphi_1\big(\mathrm{N}_{E/F} b(\gamma)\big) + C_{\mathbf{h}^\ell}\big|\mathrm{N}_{E/F} b(\gamma) - 1\big|_F^{m-2} \\ &\quad \times \varepsilon_{E/F}^{m-1}\big(\mathrm{N}_{E/F} b(\gamma) - 1\big)\varphi_0\big(\mathrm{N}_{E/F} b(\gamma)\big), \end{aligned} \tag{8.4}$$

where φ_0 and φ_1 are functions on F defined by integration on the fibers $E(x) = \mathrm{N}_{E/F}^{-1}(x)$ as

$$\begin{aligned} \varphi_1(x) &= \int_{\tau \in E(x)} \tilde{\varphi}_1(\tau)|\omega_{E(x)}|_{F_v}, \\ \varphi_0(x) &= \delta(x - 1 \in \mathfrak{o}_F) \int_{\tau \in E(x)} \phi(\tau\ell)|\omega_{E(x)}|_F. \end{aligned}$$

From Lemma A.2(2), both φ_0 and φ_1 belong to $\mathcal{S}_E(F)$. For any $\tau \in E^1$, by taking the limit $\beta \to \tau$ in (8.3), we have $\tilde{\varphi}_1(\tau) = J^\ell(\tau,\phi)$. Hence,

$$\varphi_1(1) = \int_{\tau \in E^1} J^\ell(\tau,\phi)|\omega_{E^1}|_F = J_F(\gamma_{\mathrm{u}}; f),$$

$$\varphi_0(1) = \int_{\tau \in E^1} \phi(\tau\ell)|\omega_{E^1}|_F = f(e).$$

This proves the first assertion in (1) of the proposition. To show the second part of (1), we let ξ_0 and ξ_1 be functions from $\mathcal{S}_E(F)$ such that the equation (8.4) with (φ_0,φ_1) replaced with (ξ_0,ξ_1) holds. By taking the difference of the two equations, we obtain

$$\text{(8.5)}\quad \varphi_1(\mathrm{N}_{E/F}b) - \xi_1(\mathrm{N}_{E/F}b) = C_{\mathbf{h}^\ell}\chi_{\mathbf{h}^\ell}(\mathrm{N}_{E/F}b - 1)\big\{\xi_0(\mathrm{N}_{E/F}b) - \varphi_0(\mathrm{N}_{E/F}b)\big\}$$

for all $b = b(\gamma)$ with some $\gamma \in G - H$ satisfying $\mathrm{N}^\flat\gamma \neq 0, 1$. From Lemma 7.1, b can be an arbitrary element in $E^\times - E^1$. Since the function $C_{\mathbf{h}^\ell}\chi_{\mathbf{h}^\ell}$ on F is not smooth at zero, a contradiction arises from (8.5) if $\xi_0(1) \neq \varphi_0(1)$. Hence $\xi_0(1) = \varphi_0(1)$ and $\xi_1(1) = \varphi_1(1)$ is obtained. This proves the second assertion of (1) in the proposition. To show the claim (2), suppose that we are given a pair of functions (φ_0,φ_1) from $\mathcal{S}_E(F)$ satisfying (8.2). By Lemma A.2(2), we can find $\tilde{\varphi}_0, \tilde{\varphi}_1 \in \mathcal{S}(E)$ such that $\int_{E(t)} \tilde{\varphi}_j(\tau)|\omega_{E(t)}|_F = \varphi_j(t)$ for all $t \in F^\times$. By Lemma A.3(2), there exists $\phi \in \mathcal{S}(V)$ such that $J^\ell(\beta,\phi) = \tilde{\varphi}_1(\beta) + C_{\mathbf{h}^\ell}\chi_{\mathbf{h}^\ell}(\mathrm{N}_{E/F}\beta - 1)\tilde{\varphi}_0(\beta)$ for all $\beta \in E - E^1$. From this, we obtain (8.4) by taking the fiber integral over $\beta \in E(\mathrm{N}_{E/F}b(\gamma))$. □

9. Archimedean orbital integrals

In this section, we return to the setting of Section 3, keeping all notation and conventions introduced there; thus $F = \mathbb{R}$, $E = \mathbb{C}$, G denotes $G(\mathbb{R})$, and so on. Fix an integer $d \in \mathbb{N}_0$. Recall the function $\hat{\Psi}_d(\alpha; g)$ defined by (3.11). The aim of this section is to study the integral

$$\text{(9.1)}\qquad J_{\vartheta_d}\big(\gamma; \hat{\Psi}_d(\alpha)\big) = \int_{H_\gamma\backslash H} \big(\vartheta_d|\hat{\Psi}_d(\alpha;\gamma h)\big)|\omega_{H_\gamma\backslash H}|_{\mathbb{R}}$$

for $\gamma \in G - H$, where ϑ_d is the fixed unit vector of $W_d^{\mathcal{U}\cap H}$ implicit in the definition of $\hat{\Psi}_d(\alpha)$ (see Section 3.1). The integral behaves differently depending on the signature of $\mathbf{h}[\ell^\gamma] = 1 - \mathrm{N}_{E/F}b(\gamma)$ (for the definition of ℓ^γ and $b(\gamma)$, we refer the reader to Section 7.1). Indeed, we have the following evaluation.

THEOREM 9.1

The integral (9.1) converges absolutely and

$$J_{\vartheta_d}\big(\gamma; \hat{\Psi}_d(\alpha)\big) = \frac{1}{2\pi i}\int_{(\sigma)} \hat{J}_d(s;\gamma)\alpha(s)s\,\mathrm{d}s,$$

where $\sigma > \frac{m-1}{2}$ *and*

$$\hat{J}_d(s;\gamma) = \frac{-\Gamma(m-1)}{2}\frac{\Gamma(s-\frac{m-3}{2}-d)}{\Gamma(s+\frac{m-1}{2}+d)}\sum_{l=0}^{d}\frac{d!}{(d-l)!}\binom{d}{l}(-1)^l\big(1-\mathbf{h}[\ell^\gamma]\big)^{d-l}$$
$$\times\begin{cases}(-\mathbf{h}[\ell^\gamma])^{-s+(m-3)/2-d+l}\frac{\Gamma(s+\frac{m-1}{2}+d)\Gamma(s-\frac{m-3}{2}+d-l)}{\Gamma(2s+1)} \\ \quad\times{}_2F_1(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}+d-l; 2s+1; (\mathbf{h}[\ell^\gamma])^{-1}) \\ \qquad(\mathbf{h}[\ell^\gamma]<0), \\ (2^{-1})^{\delta(\mathbf{h}[\ell^\gamma]=0)}\Gamma(m+l-2)\frac{\Gamma(s-\frac{m-3}{2}+d-l)}{\Gamma(s+\frac{m-1}{2}-d+l)} \quad (\mathbf{h}[\ell^\gamma]\geq 0).\end{cases}$$

We separate three cases $\mathbf{h}[\ell^\gamma]<0$, $\mathbf{h}[\ell^\gamma]=0$, and $\mathbf{h}[\ell^\gamma]>0$ to prove the theorem.

9.1. The case $\mathbf{h}_v[\ell^\gamma]<0$

Fix a vector $\ell_H\in V$ orthogonal to ℓ and ℓ^- such that $\mathbf{h}[\ell_H]=+1$, and define a 1-parameter subgroup $a_H^{(r)}$ $(r\in\mathbb{R})$ by

$$a_H^{(r)}\ell^- = \cosh r\ell^- + \sinh r\ell_H, \qquad a_H^{(r)}\ell_H = \cosh r\ell_H + \sinh r\ell^-,$$
$$a_H^{(r)}|(\mathbb{C}\ell_H+\mathbb{C}\ell^-)^\perp = \mathrm{id}.$$

We define 1-parameter subgroups $t_H^\pm(\varphi)$ $(\varphi\in\mathbb{R})$ as

$$t_H^+(\varphi)\ell_H = e^{i\varphi}\ell_H, \qquad t_H^+(\varphi)|(\mathbb{C}\ell_H)^\perp=\mathrm{id},$$
$$t_H^-(\varphi)\ell^- = e^{i\varphi}\ell^-, \qquad t_H^-(\varphi)|(\mathbb{C}\ell^-)^\perp=\mathrm{id}.$$

LEMMA 9.2

Suppose $\mathbf{h}[\ell^\gamma]<0$. *The integral (9.1) converges absolutely and*

$$J_{\vartheta_d}\big(\gamma;\hat{\Psi}_d(\alpha)\big) = \frac{1}{2\pi i}\int_{(\sigma)}\hat{J}_d(s;\gamma)\alpha(s)s\,\mathrm{d}s,$$

where

$$\hat{J}_d(s;\gamma) = \frac{2-m}{2}\frac{1}{sc_d(s)}\sum_{l=0}^{d}\binom{d}{l}^2\binom{l+m-3}{l}^{-1}$$
$$\times(-1)^l\big(-\mathbf{h}[\ell^\gamma]\big)^{l+m-2}\big(1-\mathbf{h}[\ell^\gamma]\big)^{d-l}$$
$$\times\int_0^\infty\big(1-\mathbf{h}[\ell^\gamma]\cosh^2 r\big)^{-s-\frac{m-1}{2}-d}$$
$$\times{}_2F_1\Big(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}-d; 2s+1; \frac{1}{1-\mathbf{h}[\ell^\gamma]\cosh^2 r}\Big)$$
$$\times(\sinh r)^{2m+2l-5}\cosh r\,\mathrm{d}r.$$

Proof

Since $\mathrm{N}_{E/F}b(\gamma)>1$, we have $\mathrm{N}_{E/F}b(\gamma)=\cosh^2 t_\gamma$ with some $t_\gamma>0$. Since $\mathrm{N}^\flat(a^{(t)})=\mathrm{N}_{E/F}\mathbf{h}(a^{(t)}\ell,\ell)=\cosh^2 t$ for $t\in\mathbb{R}$, we have $\mathrm{N}^\flat(a^{(t_\gamma)})=\mathrm{N}^\flat(\gamma)$, which implies $\gamma \in Ha^{(t_v)}H$ from Lemma 7.1. We can write $\gamma = h_\gamma a^{(t_\gamma)} h'_\gamma$ with some $h_\gamma, h'_\gamma \in H$. Set $a = a^{(t_\gamma)}$ and $\tilde{H}_{\ell^-} = \{h \in H \mid h\ell^- = c\ell^-, h\ell = c\ell\ (\exists c \in \mathbb{C}^{(1)})\}$. Then $H \cap a^{-1}Ha = \tilde{H}_{\ell^-}$ is confirmed easily. We have an H-isomorphism from $\tilde{H}_{\ell^-}\backslash H$ onto $H_\gamma\backslash H$ induced from $h \mapsto h_\gamma^{-1}h$. Thus, by a change of variables,

$$J_{\vartheta_d}\big(\gamma; \hat{\Psi}_d(\alpha)\big) = \int_{H_\gamma\backslash H} \big(\vartheta_d | \hat{\Psi}_d(\alpha;\gamma h)\big) |\omega_{H_\gamma\backslash H}|_{\mathbb{R}}$$
$$= \big|\mathbf{h}[\ell^\gamma]\big|^{m-2} \int_{\tilde{H}_{\ell^-}\backslash H} \big(\vartheta_d | \hat{\Psi}_d(\alpha; ah)\big)\, \mathrm{d}\mu(h),$$

where $\mathrm{d}\mu$ is the H-invariant measure on $\tilde{H}_{\ell^-}\backslash H$ corresponding to $|\omega_{\mathbf{\Sigma}'(-1)}|_{\mathbb{R}}$ on $\mathbf{\Sigma}'(-1) = \{Z \in \ell^\perp \mid \mathbf{h}[Z] = -1\}$ by the isomorphism $\tilde{H}_{\ell^-}\backslash H \cong \mathbf{\Sigma}'(-1)$ induced from $h \mapsto h^{-1}\ell^-$.

Let $\mathrm{d}k_0$ be the Haar measure on $\mathcal{U}_H$ with total volume 1. The measure $\mathrm{d}\mu$ is decomposed as $\mathrm{d}\mu = \pi|\sqrt{\theta}|^{-1} C_H (\sinh r)^{2m-5} \cosh r \frac{\mathrm{d}\varphi}{2|\sqrt{\theta}|}\, \mathrm{d}r\, \mathrm{d}k_0$ with respect to the Cartan decomposition $\tilde{H}_{\ell^-}\backslash H = \{t_H^-(\varphi) a_H^{(r)} \mid \varphi \in \mathbb{R}\ r \geq 0\}\mathcal{U}_H$, where $C_H = |\sqrt{\theta}|^{2-m} 4\pi^{m-2}\Gamma(m-2)^{-1}$. Noting that $\tau_d(k_0)\vartheta_d = \vartheta_d$, by Lemma 9.3, we have

$$\begin{aligned}
&J_{\vartheta_d}\big(\gamma; \hat{\Psi}_d(\alpha)\big) \\
&= \big|\mathbf{h}[\ell^\gamma]\big|^{m-2} C_H \int_0^{+\infty} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2|\sqrt{\theta}|} \big(\vartheta_d | \hat{\Psi}_d\big(\alpha; a^{(t_\gamma)} t_H^-(\varphi) a_H^{(r)}\big)\big) \\
&\quad \times (\sinh r)^{2m-5} \cosh r\, \mathrm{d}r \\
&= \big|\mathbf{h}[\ell^\gamma]\big|^{m-2} \pi |\sqrt{\theta}|^{-1} C_H \int_0^{+\infty} \big(\vartheta_d | \hat{\Psi}_d(\alpha; a^{(u)})\big) (\cosh^2 u)^{-d} \\
&\quad \times \sum_{l=0}^{d} \binom{d}{l}^2 \binom{l+m-3}{l}^{-1} (-1)^l (\cosh t_\gamma)^{2(d-l)} (\sinh t_\gamma \sinh r)^{2l} (\sinh r)^{2m-5} \\
&\quad \times \cosh r\, \mathrm{d}r \\
&= \frac{|\mathbf{h}[\ell^\gamma]|^{m-2}(2-m)1}{2\pi i} \int_0^{\infty} \Big\{ \int_{(\sigma)} (\cosh^2 u)^{-s-\frac{m-1}{2}-d} \\
&\quad \times {}_2F\Big(s + \frac{m-1}{2} + d, s - \frac{m-3}{2} - d; 2s+1; \frac{1}{\cosh^2 u}\Big) \frac{\alpha(s)}{2sc_d(s)} s\, \mathrm{d}s \Big\} \\
&\quad \times \sum_{l=0}^{d} \binom{d}{l}^2 \binom{l+m-3}{l}^{-1} (-1)^l (\cosh t_\gamma)^{2(d-l)} (\sinh t_\gamma \sinh r)^{2l} (\sinh r)^{2m-5} \\
&\quad \times \cosh r\, \mathrm{d}r.
\end{aligned}$$

Since $\cosh^2 u = 1 + \sinh^2 t_\gamma \cosh^2 r = 1 - \mathbf{h}[\ell^\gamma] \cosh^2 r$, we are done by an order change of integrals. To apply Fubini's theorem, we need to make sure that

$$\int_0^\infty \int_{(\sigma)} \big|\psi_d(s;u)\alpha(s)\big||s\,\mathrm{d}s|(1+\sinh^2 t_\gamma \cosh^2 r)^{-d}(\sinh r)^{2m+2l-5}\cosh r\,\mathrm{d}r$$
$$< +\infty.$$

By Lemma 3.4, the integral is majorized from above by

$$\Big\{\int_{(\sigma)} \big(1+|\mathrm{Im}(s)|\big)^N|\alpha(s)||s\,\mathrm{d}s|\Big\}$$
$$\times \Big\{\int_0^\infty (1+\sinh^2 t_\gamma \cosh^2 r)^{-\sigma-(m-1)/2-d}(\sinh r)^{2m+2l-5}\cosh r\,\mathrm{d}r\Big\},$$

which is convergent if $\sigma > (m-3)-2d+2l$ due to the bound $|\alpha(s)| \ll (1+|\mathrm{Im}(s)|)^{-N-3}$ and $(1+\sinh^2 t_\gamma \cosh^2 r)^{-\sigma-(m-1)/2-d}(\sinh r)^{2m+2l-5}\cosh r \ll e^{-(2\sigma-(m-3)+2d-2l)r}$ for $r \geq 0$. □

Let us compute $\hat{J}_d(s,\gamma)$. It suffices to evaluate the integral

$$f_l(z) = \int_0^\infty (1-z\cosh^2 r)^{-s-(m-1)/2-d}$$
$$\times\, {}_2F_1\Big(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}-d; 2s+1; \frac{1}{1-z\cosh^2 r}\Big)$$
$$\times (\sinh r)^{2m+2l-5}\cosh r\,\mathrm{d}r$$

for $z<0$. From the formula ${}_2F_1(a,b,c;z) = (1-z)^{-a}{}_2F_1(a,c-b;c;\frac{z}{z-1})$ (see [17, p. 47]), we have

$$f_l(z) = \int_0^\infty (-z\cosh^2 r)^{-s-(m-1)/2-d}$$
$$\times\, {}_2F_1\Big(s+\frac{m-1}{2}+d, s+\frac{m-1}{2}+d; 2s+1; (z\cosh^2 r)^{-1}\Big)$$
$$\times (\sinh r)^{2m+2l-5}\cosh r\,\mathrm{d}r.$$

Suppose that $|z|>1$ and $\mathrm{Re}(z)<0$ for a moment. By the power series expansion of ${}_2F_1$ and by the formula $\int_0^\infty (\sinh r)^\alpha(\cosh r)^{-\beta}\,\mathrm{d}r = 2^{-1}B(\frac{1+\alpha}{2},\frac{\beta-\alpha}{2})$ for $\mathrm{Re}(\beta)>\mathrm{Re}(\alpha)$, $\mathrm{Re}(\alpha)>-1$ (see [17, p. 10]), this becomes

$$\frac{\Gamma(2s+1)}{\Gamma(s+\frac{m-1}{2}+d)^2}\sum_{n=0}^\infty \frac{(-1)^n\Gamma(s+\frac{m-1}{2}+d+n)^2}{n!\Gamma(2s+1+n)}$$
$$\times (-z)^{-s-\frac{m-1}{2}-d-n}\int_0^\infty (\cosh r)^{-2s-(m-1)-2d-2n+1}(\sinh r)^{2m+2l-5}\,\mathrm{d}r$$
$$= \frac{\Gamma(2s+1)}{2\Gamma(s+\frac{m-1}{2}+d)^2}\sum_{n=0}^\infty \frac{(-1)^n\Gamma(s+\frac{m-1}{2}+d+n)^2}{n!\Gamma(2s+1+n)}(-z)^{-s-\frac{m-1}{2}-d-n}$$
$$\times \frac{\Gamma(m+l-2)\Gamma(s-\frac{m-3}{2}+d-l+n)}{\Gamma(s+\frac{m-1}{2}+d+n)}$$

$$= \frac{\Gamma(m+l-2)\Gamma(s-\frac{m-3}{2}+d-l)}{2\Gamma(s+\frac{m-1}{2}+d)}(-z)^{-s-\frac{m-1}{2}-d}$$
$$\times {}_2F_1\Big(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}+d-l; 2s+1; z^{-1}\Big).$$

When viewed as a function in z, the last expression is holomorphic on $\mathrm{Re}(z)<0$ and equals $f_l(z)$ when $\mathrm{Re}(z)<0$, $|z|>1$, as seen above. By the absolute convergence of the defining integral, $f_l(z)$ is also a holomorphic function on $\mathrm{Re}(z)<0$. Thus the evaluation of $f_l(z)$ remains valid for all $\mathrm{Re}(z)<0$. This settles the first case of the theorem.

LEMMA 9.3

We have $a^{(t)}t_H^-(\varphi)a_H^{(r)} \in Ha^{(u)}k$ with $u \geq 0$ and $k \in \mathcal{U}$ satisfying

$$\cosh^2 u = 1 + \cosh^2 r \sinh^2 t, \tag{9.2}$$

$$\begin{aligned}(\vartheta_d|\tau_d(k)\vartheta_d) = (\cosh^2 u)^{-d} \sum_{l=0}^{d} \binom{d}{l}^2 \binom{l+m-3}{l}^{-1} (-1)^l \\ \times (\cosh t)^{2(d-l)} (\sinh t \sinh r)^{2l}.\end{aligned} \tag{9.3}$$

Proof

Set $g = a^{(t)}t_H^-(\varphi)a_H^{(r)} = ha^{(u)}k$ with $h \in H$, $k \in \mathcal{U}$. On the one hand, from $g = a^{(t)}t_H^-(\varphi)a_H^{(r)}$, we have $|(g^{-1}\ell|\ell^-)| = |(a_H^{(-r)}(\cosh t\ell - e^{-i\varphi}\sinh t\ell^-)|\ell^-) = |\sinh t \cosh r|$. On the other hand, from $g = ha^{(u)}k$, we compute $|(g^{-1}\ell|\ell^-)| = |(a^{(-u)}\ell|\ell^-)| = |\sinh u|$. Comparing these, we obtain $|\sinh u| = |\sinh t \cosh r|$, which is equivalent to (9.2). To show (9.3), we recall the polynomial realization of W_d. We fix an orthonormal basis $\{\ell_j\}_{j=1}^m$ of V such that $\ell_m = \ell^-$, $\ell_{m-1} = \ell$, and $\ell_{m-2} = \ell_H$, and we let x_j $(1 \leq j \leq m)$ be the complex coordinate functions on V dual to this basis. The group G is embedded into $\mathrm{GL}_m(\mathbb{C})$ by sending an element $g \in G$ to the matrix $\rho(g) = (g_{ij})$ such that $g\ell_i = \sum_j g_{ji}\ell_j$. Then $\rho(gh) = \rho(g)\rho(h)$ $(g, h \in G)$ is confirmed easily. We have

$$\rho(\mathcal{U}) = \{\mathrm{diag}(k_1, k_2) \mid k_1 \in U(m-1), k_2 \in U(1)\},$$
$$\rho(\mathcal{U} \cap H) = \{\mathrm{diag}(u, u_1, u_2) \mid u \in U(m-2), u_1, u_2 \in U(1)\}.$$

The functions x_j $(1 \leq j \leq m)$ together with their complex conjugates $\bar{x}_j$ form a $\mathbb{C}$-basis of $X = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$. We let the group $\mathrm{GL}_m(\mathbb{C})$ act on X $\mathbb{C}$-linearly by the rule $gx_i = \sum_j (g^{-1})_{ij}x_j$ and $g\bar{x}_i = \sum_j (\bar{g}^{-1})_{ij}\bar{x}_j$; the action is extended to the symmetric algebra $\mathsf{S}(X)$ in the natural way. We introduce a $\mathbb{C}$-bilinear pairing on X by defining $\langle x_i, x_j\rangle = \langle \bar{x}_i, \bar{x}_j\rangle = 0$, $\langle x_i, \bar{x}_j\rangle = \delta_{ij}$ and extend it to $\mathsf{S}(X)$ by the rule

$$\langle y_1 \cdots y_p, z_1 \cdots z_q\rangle = \delta_{p,q}\frac{1}{p!}\sum_{\sigma \in S_p}\prod_{j=1}^{p}\langle y_j, z_{\sigma(j)}\rangle$$

for all decomposable elements $y = y_1 \cdots y_p$ and $z = z_1 \cdots z_q$. Set

$$\xi_j = x_j \bar{x}_j \quad (1 \le j \le m).$$

Then we have $\langle \xi_j^d, \xi_k^{d'} \rangle = \delta_{jk}\delta_{d,d'}\binom{2d}{d}^{-1}$. The relation

$$\langle gx, y\rangle = \langle x, g^* y\rangle, \quad g \in \mathrm{GL}_m(\mathbb{C}), x, y \in \mathsf{S}(X), \tag{9.4}$$

is confirmed easily, where $g^* = {}^t\bar{g}$. Let ∂_j and $\bar{\partial}_j$ denote the derivation of $\mathsf{S}(X)$ with respect to the variables x_j and $\bar{x}_j$, respectively. For $d \in \mathbb{N}_0$, let $\mathsf{S}^d(X)$ be the subspace of homogeneous bidegree (d,d) elements of $\mathsf{S}(X)$. Let X_0 be the $\mathbb{C}$-span of $x_j, \bar{x}_j$ $(1 \le j \le m-1)$, and let $\mathsf{S}(X_0)$ be the subalgebra of $\mathsf{S}(X)$ generated by X_0; $\mathsf{S}(X_0)$ is a $\mathcal{U}$-submodule of $\mathsf{S}(X)$. The space of harmonic tensors $W_d = \{x \in \mathsf{S}^d(X_0) \mid \sum_{j=1}^{m-1} \partial_j \bar{\partial}_j x = 0\}$ is a $\mathcal{U}$-invariant subspace of $\mathsf{S}^d(X_0) = \mathsf{S}(X_0) \cap \mathsf{S}^d(X)$, which yields a realization of τ_d inside $\mathsf{S}(X)$. Let $\mathrm{pr}_{\mathcal{U}\cap H} : \mathsf{S}(X_0) \to \mathsf{S}(X_0)^{\mathcal{U}\cap H}$ be the projector defined by $\mathrm{pr}_{\mathcal{U}\cap H}(\xi) = \int_{\mathcal{U}\cap H} \rho(k_0)\xi \, \mathrm{d}k_0$. Set

$$\begin{aligned} w_d &= 2^{-1}\big\{(x_{m-1} + \sqrt{-1}x_{m-2})^d(\bar{x}_{m-1} + \sqrt{-1}\bar{x}_{m-2})^d \\ &\qquad + (x_{m-1} - \sqrt{-1}x_{m-2})^d(\bar{x}_{m-1} - \sqrt{-1}\bar{x}_{m-2})^d\big\}, \\ \vartheta_d &= \mathrm{pr}_{\mathcal{U}\cap H}(w_d), \qquad v_d = \xi_{m-1}^d. \end{aligned}$$

A simple computation reveals that $w_d \in W_d$, and the relation $\bar{w}_d = w_d$ is evident. Thus $\vartheta_d \in W_d$ and $\bar{\vartheta}_d = \vartheta_d$. Since $\tau_d(k_0)$ $(k_0 \in \mathcal{U}\cap H)$ preserves x_{m-1} up to a constant, it is easy to see that $\langle \vartheta_d, v_d\rangle \ne 0$, which guarantees $\vartheta_d \ne 0$. We introduce a Hermitian inner product on W_d by

$$(x|y) = \langle x, \bar{y}\rangle / \langle \vartheta_d, \bar{\vartheta}_d\rangle, \quad x, y \in W_d. \tag{9.5}$$

This is $\mathcal{U}$-invariant and satisfies $(\vartheta_d|\vartheta_d) = 1$. To show the second formula in the lemma, we compute the pairing $\langle \rho(g)^{-1}v_d, \vartheta_d\rangle$ in two different ways. From $g = ha^{(u)}k$, using (9.4), we have

$$\begin{aligned} \big\langle \rho(g)^{-1}v_d, \vartheta_d\big\rangle &= \big\langle \rho(ha^{(u)})^{-1}v_d, \rho(k^{-1})^*\vartheta_d\big\rangle \\ &= \big\langle \rho(a^{(u)})^{-1}v_d, \rho(k^{-1})^*\vartheta_d\big\rangle \quad \text{(since } v_d \text{ is } H\text{-invariant)} \\ &= \big\langle (\cosh u x_{m-1} + \sinh u x_m)^d(\cosh u \bar{x}_{m-1} + \sinh u \bar{x}_m)^d, \\ &\qquad \rho(k^{-1})^*\vartheta_d\big\rangle \\ &= (\cosh^2 u)^d \big\langle v_d, \rho(k^{-1})^*\vartheta_d\big\rangle. \end{aligned} \tag{9.6}$$

Here, the last equality is due to $\langle \mathbb{C}x_m + \mathbb{C}\bar{x}_m, W_d\rangle = \{0\}$. From $g = a^{(t)}t_H^-(\varphi)a_H^{(r)}$, the tensor $\rho(g)^{-1}v_d$ is equal to

$$\begin{aligned} &\big\{\cosh t x_{m-1} + e^{-i\varphi}\sinh t(\cosh r x_m + \sinh r x_{m-2})\big\}^d \\ &\quad \times \big\{\cosh t \bar{x}_{m-1} + e^{i\varphi}\sinh t(\cosh r \bar{x}_m + \sinh r \bar{x}_{m-2})\big\}^d \\ &= \sum_{l=0}^{d} \binom{d}{l}^2 (\cosh t)^{2l}(\sinh t \sinh r)^{2(d-l)}\xi_{m-1}^l \xi_{m-2}^{d-l} + \eta, \end{aligned}$$

with η a linear combination of monomials

$$x_{m-1}^{j}\bar{x}_{m-1}^{k}x_{m-2}^{d-j}\bar{x}_{m-2}^{d-k} \quad \text{with } j \neq k,\ 0 \leq j,k \leq m-1, \tag{9.7}$$

and those having x_m or $\bar{x}_m$ as a factor. The algebraic group H contains an F-torus T such that $t \in T$ acts on ℓ by the scalar $\chi(t) \in E^1$ fixing vectors orthogonal to ℓ. We have $\rho(t)x_j = x_j (j \neq m-1)$ and $\rho(t)x_{m-1} = \chi(t)^{-1}x_{m-1}$ for all $t \in T$. Since $T \subset \mathcal{U} \cap H$ and ϑ_d is $(\mathcal{U} \cap H)$-invariant, we automatically have $\rho(t)\vartheta_d = \vartheta_d$ for all $t \in T$. The elements (9.7) have nontrivial T-weights χ^{j-k}; thus the pairings with ϑ_d are all zero. Those monomials having x_m or $\bar{x}_m$ as a factor are paired with ϑ_d to be 0. As a linear combination of these, we have $\langle \eta, \vartheta_d \rangle = 0$. Hence

$$\langle \rho(g)^{-1}v_d, \vartheta_d \rangle = \sum_{l=0}^{d} \binom{d}{l}^2 (\cosh t)^{2l}(\sinh t \sinh r)^{2(d-l)} \langle \xi_{m-1}^{l}\xi_{m-2}^{d-l}, \vartheta_d \rangle. \tag{9.8}$$

LEMMA 9.4

Set $R_0 = \sum_{j=1}^{m-2} \xi_j$. *Then*

$$\mathrm{pr}_{\mathcal{U}\cap H}(\xi_{m-2}^{l}) = \binom{l+m-3}{l}^{-1} R_0^{l} \quad (l \in \mathbb{N}).$$

Proof

Since $\mathsf{S}^{l}(X_0)^{\mathcal{U}\cap H} = \mathbb{C}R_0^{l}$, we have a constant $c_l \in \mathbb{C}$ such that $\mathrm{pr}_{\mathcal{U}\cap H}(\xi_{m-2}^{l}) = c_l R_0^{l}$. Consider the operator $\Delta_0 = \sum_{j=1}^{m-2} \partial_j \bar{\partial}_j$. The obvious formula $\Delta_0 \xi_{m-2}^{l} = l^2 \xi_{m-2}^{l-1}$ implies that $\Delta_0\, \mathrm{pr}_{\mathcal{U}\cap H}(\xi_{m-2}^{l}) = l^2\, \mathrm{pr}_{\mathcal{U}\cap H}(\xi_{m-2}^{l-1})$ because Δ_0 is $(\mathcal{U} \cap H)$-invariant. The latter formula, combined with the easily confirmed formula $\Delta_0(R_0^{l}) = l(l+m-3)R_0^{l-1}$, yields the recurrence relation $c_l = \frac{l}{n+m-3}c_{l-1}$ $(l > 1)$ and $c_1 = (m-2)^{-1}$, which is uniquely solved by $c_l = \binom{l+m-3}{l}^{-1}$. ☐

Recall the $\mathcal{U}$-decomposition

$$\mathsf{S}^{d}(X_0) = \bigoplus_{l=0}^{d} R^{l} W_{d-l} \quad \text{where } R = R_0 + \xi_{m-1} = \sum_{j=1}^{m-1} \xi_j. \tag{9.9}$$

LEMMA 9.5

There exist constants $c(k,l) \in \mathbb{C}$ $(0 \leq l \leq k)$ *such that* $c(k,0) = \langle \xi_{m-1}^{k}, \xi_{m-1}^{k} \rangle / \langle \vartheta_k, \vartheta_k \rangle$ *and*

$$\xi_{m-1}^{k} = \sum_{l=0}^{k} c(k,l) R^{l} \vartheta_{k-l} \quad (k \in \mathbb{N}).$$

Proof

Since $W_l^{\mathcal{U}\cap H} = \mathbb{C}\vartheta_l$, by taking the $\mathcal{U}\cap H$-invariant part, (9.9) yields $\mathsf{S}^{d}(X_0)^{\mathcal{U}\cap H} = \bigoplus_{l=0}^{d} \mathbb{C}R^{l}\vartheta_{d-l}$. Hence the H-invariant tensor ξ_{m-1}^{k} is written as a linear combination $\sum_{l=0}^{k} c_{kl} R^{l} \vartheta_{k-l}$ $(0 \leq l \leq k)$. To determine the coefficient c_{k0}, we compute the pairing $\langle \xi_{m-1}^{k}, \vartheta_k \rangle$. From the definition, the difference $\vartheta_k - \xi_{m-1}^{k}$ is a linear combination of monomials in x_j $(1 \leq j \leq m-1)$ divisible by some x_j $(j \neq m-1)$.

Hence $\langle \xi_{m-1}^k, \vartheta_k\rangle = \langle \xi_{m-1}^k, \xi_{m-1}^k\rangle$. On the other hand, $\langle \xi_{m-1}^k, \vartheta_k\rangle = c_{k0}\langle \vartheta_k, \vartheta_k\rangle$ because $\langle R^l\vartheta_{k-l}, \vartheta_k\rangle = 0$ $(l > 1)$. Thus $c_{k0} = \langle \xi_{m-1}^k, \xi_{m-1}^k\rangle / \langle \vartheta_k, \vartheta_k\rangle$. □

We compute

$$
\begin{aligned}
&\langle \xi_{m-1}^l \xi_{m-2}^{d-l}, \vartheta_d\rangle \\
&= \langle \xi_{m-1}^l \operatorname{pr}_{\mathcal{U}\cap H}(\xi_{m-2}^{d-l}), \vartheta_d\rangle \\
&= \binom{d-l+m-3}{d-l}^{-1} \langle \xi_{m-1}^l R_0^{d-l}, \vartheta_d\rangle \quad (\text{Lemma } 9.4) \\
&= \binom{d-l+m-3}{d-l}^{-1} \sum_{j=0}^{d-l} \binom{d-l}{j} (-1)^{d-l-j} \langle \xi_{m-1}^{d-j} R^j, \vartheta_d\rangle \\
&\qquad \big(\text{by } R_0^{d-l} = (R-\xi_{m-1})^{d-l}\big) \\
&= \binom{d-l+m-3}{d-l}^{-1} \sum_{j=0}^{d-l}\sum_{k=0}^{d-j} \binom{d-l}{j} (-1)^{d-l-j} c(d-j,k) \langle R^{j+k}\vartheta_{d-(j+k)}, \vartheta_d\rangle \\
&\qquad (\text{Lemma } 9.5) \\
&= \binom{d-l+m-3}{d-l}^{-1} (-1)^{d-l}\langle v_d, v_d\rangle \quad \big(\text{by } c(d,0) = \langle v_d, v_d\rangle / \langle \vartheta_d, \vartheta_d\rangle\big).
\end{aligned}
$$

Combining this with (9.8), we obtain

$$
\begin{aligned}
\big\langle \rho(g)v_d, \vartheta_d\big\rangle = \sum_{l=0}^{d} &\binom{d}{l}^2 \binom{d-l+m-3}{d-l}^{-1} \\
&\times (-1)^{d-l}\langle v_d, v_d\rangle (\cosh t)^{2l} (\sinh t \sinh r)^{2(d-l)},
\end{aligned}
$$

which, together with (9.6), shows the identity

$$
\begin{aligned}
\big\langle v_d, \rho(k^{-1})^*\vartheta_d\big\rangle = (\cosh u)^{-2d} \sum_{l=0}^{d} &\binom{d}{l}^2 \binom{d-l+m-3}{d-l}^{-1} \\
&\times (-1)^{d-l} (\cosh t)^{2l} (\sinh t \sinh r)^{2(d-l)} \langle v_d, v_d\rangle.
\end{aligned} \tag{9.10}
$$

Let $P_d : \mathsf{S}^d(X_0) \to W_d$ be the projector determined by (9.9). Then $0 \neq \langle \vartheta_d, v_d\rangle = \langle \vartheta_d, P_d(v_d)\rangle$ implies that $P_d(v_d) \in W_d$ is nonzero. Since v_d is H-invariant, $P_d(v_d)$ is $(\mathcal{U}\cap H)$-invariant. Thus $P_d(v_d) = c\vartheta_d$ with some $c \neq 0$. As we have seen in the proof of Lemma 9.5, $\langle v_d, \vartheta_d\rangle = \langle v_d, v_d\rangle$. Hence $c = \langle v_d, v_d\rangle / \langle \vartheta_d, \vartheta_d\rangle$. The identity (9.10) holds true after we replace v_d with $P_d(v_d)$ on the left-hand side. With this remark, together with (9.5), the equation (9.10) yields (9.3). This completes the proof of Lemma 9.3. □

9.2. The case $\mathbf{h}[\ell^\gamma] > 0$

Let us define a 1-parameter subgroup $\kappa^{(x)}$ $(x \in \mathbb{R})$ by setting

$$\kappa^{(x)}\ell_H = \cos x\ell_H - \sin x\ell, \qquad \kappa^{(x)}\ell = \cos x\ell + \sin x\ell_H,$$
$$\kappa^{(x)}|(\mathbb{C}\ell + \mathbb{C}\ell_H)^\perp = \mathrm{id}.$$

LEMMA 9.6

Suppose $\mathbf{h}[\ell^\gamma] > 0$. *The integral (9.1) converges absolutely and*

$$J_{\vartheta_d}\big(\gamma; \hat{\Psi}_d(\alpha)\big) = \frac{1}{2\pi i}\int_{(\sigma)} \hat{J}_d(s;\gamma)\alpha(s)s\,\mathrm{d}s,$$

where

$$\begin{aligned}\hat{J}_d(s;\gamma) = {} & \frac{(2-m)|\mathbf{h}[\ell^\gamma]|^{m-2}}{2}\frac{1}{sc_d(s)}\sum_{l=0}^{d}\binom{d}{l}^2\binom{l+m-3}{l}^{-1} \\ & \times (-1)^l\big(-\mathbf{h}[\ell^\gamma]\big)^l\big(1-\mathbf{h}[\ell^\gamma]\big)^{d-l} \\ & \times \int_0^\infty \big(1+\mathbf{h}[\ell^\gamma]\sinh^2 r\big)^{-s-\frac{m-1}{2}-d} \\ & \times {}_2F_1\Big(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}-d; 2s+1; \frac{1}{1+\mathbf{h}[\ell^\gamma]\sinh^2 r}\Big) \\ & \times (\cosh r)^{2m+2l-5}\sinh r\,\mathrm{d}r.\end{aligned}$$

Proof

Since $0 \le \mathrm{N}_{E/F}b(\gamma) < 1$, we have $\mathrm{N}_{E/F}b(\gamma) = \cos^2 x_\gamma$ with some $x_\gamma \in (0, \pi/2]$. Then $\mathbf{h}[\ell^\gamma] = 1 - \cos^2 x_\gamma = \sin^2 x_\gamma$. Since $\mathrm{N}^\flat(\kappa^{(x_\gamma)}) = \mathrm{N}_{E/F}\mathbf{h}(\kappa^{(x_\gamma)}\ell, \ell) = \cos^2 x_\gamma = \mathrm{N}^\flat(\gamma)$, we have $H\gamma H = H\kappa^{(x_\gamma)}H$ from Lemma 7.1. Thus $\gamma = h_\gamma \kappa^{(x_\gamma)} h'_\gamma$ with some $h_\gamma, h'_\gamma \in H$. Set $\kappa = \kappa^{(x_\gamma)}$. Then it turns out that $\kappa^{-1}H\kappa \cap H$ coincides with $\tilde{H}_{\ell_H} = \{h \in H \mid h\ell_H = c\ell_H, h\ell = c\ell \ (\exists c \in \mathbb{C}^{(1)})\}$. We remark that $\tilde{H}_{\ell_H}\backslash H \cong U(m-3,1)\backslash U(m-2,1)$. Let $\mathrm{d}\mu$ be the measure on $\tilde{H}_{\ell_H}\backslash H$ corresponding to $|\omega_{\mathbf{\Sigma}'(1)}|_{\mathbb{R}}$ by the isomorphism $\tilde{H}_{\ell_H}\backslash H \cong \mathbf{\Sigma}'(1) = \{Z \in \ell^\perp \mid \mathbf{h}[Z] = +1\}$ induced by $h \mapsto h^{-1}\ell_H$. Then $|\omega_{H_\gamma\backslash H}|_{\mathbb{R}}$ on $H_\gamma\backslash H$ corresponds to $|\mathbf{h}[\ell]|^{m-2}\,\mathrm{d}\mu$ on $H_{\ell_H}\backslash H$ by the isomorphism $h \mapsto h_\gamma h$. For the pair $(H, \tilde{H}_{\ell_H})$ and the measure $\mathrm{d}\mu$, both the decomposition $H = \tilde{H}_{\ell_H}\{t_H^+(\varphi)a_H^{(r)} \mid \varphi \in \mathbb{R}\ r \ge 0\}\mathcal{U}_H$ and an integration formula similar to the one in Lemma 3.2 hold true. Thus,

$$\begin{aligned}J_{\vartheta_d}\big(\gamma; \hat{\Psi}_d(\alpha)\big) &= \big|\mathbf{h}[\ell^\gamma]\big|^{m-2}\int_{H_{\ell_H}\backslash H}\big(\vartheta_d|\hat{\Psi}_d(\alpha, \kappa h)\big)\,\mathrm{d}\mu(h) \\ &= \big|\mathbf{h}[\ell^\gamma]\big|^{m-2}C_H\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2|\sqrt{\theta}|}\int_0^\infty \big(\vartheta_d|\hat{\Psi}_d\big(\alpha, \kappa t_H^+(\varphi)a_H^{(r)}\big)\big) \\ &\quad \times (\cosh r)^{2m-5}\sinh r\,\mathrm{d}r\end{aligned}$$

with $C_H = |\sqrt{\theta}|^{2-m}4\pi^{m-2}\Gamma(m-2)^{-1}$. The remaining part of the proof is similar to Lemma 9.2 except that we use Lemma 9.7 below instead of Lemma 9.3. □

LEMMA 9.7

We have $\kappa^{(x)}t_H^+(\varphi)a_H^{(r)} \in Ha^{(u)}k$ *with* $u \geq 0$ *and* $k \in \mathcal{U}$ *satisfying*

$$\cosh^2 u = 1 + \sinh^2 r \sin^2 x,$$

$$(\vartheta_d|\tau_d(k)\vartheta_d) = (1+\sin^2 x \sinh^2 r)^{-d}\sum_{l=0}^{d}\binom{d}{l}^2\binom{l+m-3}{l}^{-1} \times (-1)^l(\cos x)^{2(d-l)}(\sin x\cosh r)^{2l}.$$

Proof

This is basically the same as Lemma 9.3. □

Let us compute $\hat{J}_d(s,\gamma)$. It suffices to evaluate the integral

$$f_w^-(z) = \int_0^\infty (1+z\sinh^2 r)^{-s-(m-1)/2-d} \times {}_2F_1\Big(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}-d, 2s+1; \frac{1}{1+z\sinh^2 r}\Big) \times (\cosh r)^{2w-1}\sinh r\, \mathrm{d}r$$

for $z>0$, $w=m+l-2$ with $0 \leq l \leq d$. We first keep the w in the region $\mathrm{Re}(w)<1$ and take the Mellin transform of $z \mapsto f_w^-(z)$:

$$\int_0^\infty f_w^-(z)z^{\lambda-1}\,\mathrm{d}z = \int_0^\infty\int_0^\infty z^{\lambda-1}(1+z\sinh^2 r)^{-s-(m-1)/2-d} \times {}_2F_1\Big(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}-d, 2s+1; \frac{1}{1+z\sinh^2 r}\Big) \times (\cosh r)^{2w-1}\sinh r\,\mathrm{d}r\,\mathrm{d}z.$$

By the variable change $z \mapsto (\sinh^2 r)^{-1}z$, the double integral breaks up into a product of

$$\int_0^\infty (\sinh r)^{-2\lambda+1}(\cosh r)^{2w-1}\,\mathrm{d}r \tag{9.11}$$

and

$$\int_0^\infty z^{\lambda-1}(1+z)^{-s-(m-1)/2-d} \times {}_2F_1\Big(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}-d, 2s+1; \frac{1}{1+z}\Big)\,\mathrm{d}z. \tag{9.12}$$

The first integral is absolutely convergent and is evaluated as $\frac{\Gamma(1-\lambda)\Gamma(\lambda-w)}{2\Gamma(1-w)}$ if $\mathrm{Re}(w)<\mathrm{Re}(\lambda)<1$. By the variable change $y=(1+z)^{-1}$, (9.12) becomes

$$\int_0^1 y^{s+d+(m-1)/2-\lambda-1}(1-y)^{\lambda-1}{}_2F_1\Big(s+\frac{m-1}{2}+d, s-\frac{m-3}{2}-d, 2s+1; y\Big)\,\mathrm{d}y,$$

which is evaluated to be $\frac{\Gamma(2s+1)\Gamma(\lambda)^2\Gamma(s+\frac{m-1}{2}+d-\lambda)}{\Gamma(s+\frac{m-1}{2}+d)^2\Gamma(s-\frac{m-3}{2}-d+\lambda)}$ in the absolute convergence region $\mathrm{Re}(s)+\frac{m-1}{2}+d>\mathrm{Re}(\lambda)>0$. Thus we see that the Mellin transform converges absolutely on $\mathrm{Re}(w)<\mathrm{Re}(\lambda)<1$, $\mathrm{Re}(s)+\frac{m-1}{2}+d>\mathrm{Re}(\lambda)>0$, and

$$\int_0^\infty f_w^-(z)z^{\lambda-1}\,\mathrm{d}z=\frac{\Gamma(1-\lambda)\Gamma(\lambda-w)}{2\Gamma(1-w)}\frac{\Gamma(2s+1)\Gamma(\lambda)^2\Gamma(s+\frac{m-1}{2}+d-\lambda)}{\Gamma(s+\frac{m-1}{2}+d)^2\Gamma(s-\frac{m-3}{2}-d+\lambda)}.$$

By Mellin's inversion formula,

$$f_w^-(z)=\frac{1}{2\pi i}\int_{(\sigma)}\frac{\Gamma(1-\lambda)\Gamma(\lambda-w)}{2\Gamma(1-w)}\frac{\Gamma(2s+1)\Gamma(\lambda)^2\Gamma(s+\frac{m-1}{2}+d-\lambda)}{\Gamma(s+\frac{m-1}{2}+d)^2\Gamma(s-\frac{m-3}{2}-d+\lambda)}z^{-\lambda}\,\mathrm{d}\lambda,$$

where the contour is taken as $\mathrm{Re}(w)<\sigma<1$. The integral on the right-hand side defines a holomorphic function on $\mathrm{Re}(w)<1$. To continue it to a half-plane containing the point $w=m+l-2$, we shift the contour leftward beyond $\lambda=w$, which is the only pole of the integrand swept by the moving contour. By the residue theorem,

$$\begin{aligned}f_w^-(z)&=2^{-1}\frac{\Gamma(2s+1)\Gamma(w)^2\Gamma(s+\frac{m-1}{2}+d-w)}{\Gamma(s+\frac{m-1}{2}+d)^2\Gamma(s-\frac{m-3}{2}-d+w)}z^{-w}\\&\quad+\frac{1}{2\pi i}\int_{(\sigma_1)}\frac{\Gamma(1-\lambda)\Gamma(\lambda-w)}{2\Gamma(1-w)}\frac{\Gamma(2s+1)\Gamma(\lambda)^2\Gamma(s+\frac{m-1}{2}+d-\lambda)}{\Gamma(s+\frac{m-1}{2}+d)^2\Gamma(s-\frac{m-3}{2}-d+\lambda)}\\&\quad\times z^{-\lambda}\,\mathrm{d}\lambda,\end{aligned}$$

where $\sigma_1<\mathrm{Re}(w)$. The second term has a zero at $w=m+l-2$ due to the factor $1/\Gamma(1-w)$. Thus $f_{m+l-2}^-(z)$ is given by the first term substituted with $w=m+l-2$.

9.3. The case $\mathbf{h}[\ell^\gamma]=0$

Since $\ell_H+\ell^-$ is an $\mathbb{R}$-isotropic vector orthogonal to ℓ, by Witt's theorem, we can fix $\delta\in G$ such that $\delta^{-1}\ell=\ell+(\ell_H+\ell^-)$. Set $H_\delta=\delta^{-1}H\delta\cap H$. A computation reveals that $H_\delta=\{h\in H\mid h\ell=c\ell, h(\ell_H+\ell^-)=c(\ell_H+\ell^-)\ (\exists c\in\mathbb{C}^{(1)})\}$. Thus $\{a_H^{(r)}\mid r\in\mathbb{R}\}H_\delta$, up to the center of H, coincides with the $\mathbb{R}$-parabolic subgroup of H stabilizing $\mathbb{C}(\ell_H+\ell^-)$. By the isomorphism $H_\delta\backslash H\cong\mathbf{\Sigma}'(0)$ induced by $h\mapsto h(\ell_H+\ell^-)$, we transport the measure $|\omega_{\mathbf{\Sigma}'(0)}|_{\mathbb{R}}$ to obtain an H-invariant measure $\mathrm{d}\mu$ on $H_\delta\backslash H$. From Lemma 3.1, we have the integral formula

$$\begin{aligned}\int_{H_\delta\backslash H}f(h)\,\mathrm{d}\mu(h)&=\frac{2\pi}{|\sqrt{\theta}|}\times|\sqrt{\theta}|^{2-m}\frac{\pi^{m-2}}{\Gamma(m-2)}\\&\quad\times\int_0^\infty\int_{\mathcal{U}_H}f(a_H^{(r)}k_0)e^{-2(m-2)r}\,\mathrm{d}r\,\mathrm{d}k_0\end{aligned}\tag{9.13}$$

associated to the Iwasawa decomposition $H=H_\delta\{a_H^{(r)}\mid r\in\mathbb{R}\}\mathcal{U}_H$, where $\mathrm{d}k_0$ is the probability Haar measure on $\mathcal{U}_H$.

LEMMA 9.8

Suppose $\mathbf{h}[\ell^\gamma]=0$. *The integral (9.1) converges absolutely and*

$$J_{\vartheta_d}\big(\gamma;\hat{\Psi}_d(\alpha)\big)=\frac{1}{2\pi i}\int_{(\sigma)}\hat{J}_d(s;\gamma)\alpha(s)s\,\mathrm{d}s,$$

where

$$\hat{J}_d(s;\gamma)=\frac{2-m}{4}\frac{1}{sc_d(s)}\sum_{l=0}^{d}\binom{d}{l}^2\binom{l+m-3}{l}^{-1}(-1)^l\int_0^\infty(1+e^{-2r})^{-s-\frac{m-1}{2}-d}$$
$$\times\,{}_2F_1\Big(s+\frac{m-1}{2}+d,s-\frac{m-3}{2}-d;2s+1;\frac{1}{1+e^{-2r}}\Big)e^{-2(m+l-2)r}\,\mathrm{d}r.$$

Proof

Since $\mathbf{h}(\delta^{-1}\ell,\ell)=1$, we have $\gamma\in H\delta H$ from Lemma 7.1. By a change of variables, we have $J_{\vartheta_d}(\gamma;\hat{\Psi}_d(\alpha))=\int_{H_\delta\backslash H}(\vartheta_d|\hat{\Psi}_d(\alpha,\delta h))\,\mathrm{d}\mu(h)$. We complete the proof in the same way as in Lemmas 9.2 and 9.6 by using the formula (9.13) and Lemma 9.9 below. □

LEMMA 9.9

Let $\delta a_H^{(r)}=ha^{(u)}k$ *with* $h\in H(F_v)$, $u\in\mathbb{R}$, *and* $k\in\mathcal{U}$. *Then*

$$\cosh^2u=1+e^{-2r},\qquad(\vartheta_d|\tau_d(k)\vartheta_d)=\sum_{l=0}^{d}\binom{d}{l}^2\binom{l+m-3}{l}^{-1}(-1)^le^{-2lr}.$$

Proof

This is basically the same as in Lemma 9.3. We use the formula $\rho(\delta)x_{m-1}=x_{m-1}+x_{m-2}+x_m$. □

By the variable change $z=e^{-2r}$, the integral $\hat{J}_d(s,\gamma)$ is reduced to the same integral (9.12) with $\lambda=l+m-2$ in the case $\mathbf{h}[\ell^\gamma]>0$.

Appendix

Let F be a field of characteristic 0.

LEMMA A.1

Let $Q=\sum_{j=1}^n a_jx_j^2$ *be a nondegenerate quadratic form over* F *of* n*-variables* $x_1,\dots,x_n$. *For any* $t\in F$, *set*

$$\mathbf{\Sigma}_Q(t)=\big\{x\in F^n-\{0\}\mid Q(x)=t\big\},$$

viewing this as an F*-algebraic variety. Let* $\omega_{F^n}=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$ *be the standard gauge-form on* F^n. *Let* $\phi:F^n-\{0\}\to F$ *be the* F*-morphism defined by* $\phi(x)=Q(x)$.

(1) *There exists a gauge-form* $\omega_{\mathbf{\Sigma}_Q(t)}$ *on* $\mathbf{\Sigma}_Q(t)$ *satisfying the relation* $\omega_{F^n}=\omega_{\mathbf{\Sigma}_Q(t)}\wedge\phi^*(\mathrm{d}t)$, *where* t *is the coordinate function on* F.

(2) *For j, let $U_j = \{x \in \boldsymbol{\Sigma}_Q(t) \mid x_j \neq 0\}$. Then*

$$\omega_{\boldsymbol{\Sigma}_Q(t)}|U_j = \frac{1}{2a_j x_j}\,\mathrm{d}x_1 \wedge \cdots \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_m.$$

A.1

Let F be a local non-Archimedean field of characteristic 0, and let $E = F[\sqrt{\theta}]$ be an étale quadratic algebra over F. Let $\varepsilon_{E/F}$ be the quadratic character of $F^\times$ trivial on $\mathrm{N}_{E/F}(E^\times)$ if E is a field and $\varepsilon_{E/F} = 1$ otherwise. Let V be an m-dimensional E-vector space, and let $\mathbf{h} : V \times V \to E$ be a nondegenerate Hermitian form on V. Set $\boldsymbol{\Sigma}(t) = \boldsymbol{\Sigma}_Q(t)$ for $t \in F$, where Q is the quadratic form $\mathbf{h}[Z]$ on the F-vector space V. For $\phi \in \mathcal{S}(V)$, define

$$\mathcal{M}_f(t) = \int_{\boldsymbol{\Sigma}(t)} f(x)|\omega_{\boldsymbol{\Sigma}(t)}|_F, \quad t \in F.$$

If $m = 1$ and $\mathbf{h}$ is isotropic, then we set $C_{\mathbf{h}} = \mathrm{vol}(\mathfrak{o}_F^\times)$. If $m > 1$, or $m = 1$ and $\mathbf{h}$ is anisotropic, then we set $C_{\mathbf{h}} = |\theta|_F^{-m/2}|2|_F^{-m}\rho(\varepsilon_{E/F}^m, m)^{-1}\gamma(Q)$, where $\rho(\chi, s) = \epsilon(\chi, s, \psi)^{-1}\frac{L(s,\chi)}{L(1-s,\chi)}$ for a quasicharacter χ of $F^\times$ with respect to the additive character ψ of F such that $\psi|\mathfrak{o}_F = 1$ and $\psi|\mathfrak{p}_F^{-1} \neq 1$. The factor $|\theta|_F^{-m/2}|2|_F^{-m}$ comes from the difference between our Haar measure on V and the one used in [21]. Define a function $\chi_{\mathbf{h}} : F^\times \to \mathbb{C}^\times$ by

$$\chi_{\mathbf{h}}(t) = \delta(t \in \mathfrak{o}_F)\begin{cases} \varepsilon_{E/F}(t)^m|t|_F^{m-1} & (m > 1), \\ 1 + \mathrm{ord}_F(t) & (m = 1 \text{ and } \mathbf{h} \text{ is isotropic}), \\ 1 + \varepsilon_{E/F}(t) & (m = 1 \text{ and } \mathbf{h} \text{ is anisotropic}). \end{cases}$$

LEMMA A.2

(1) *For any $f \in \mathcal{S}(V)$, there exists $\varphi \in \mathcal{S}(F)$ such that*

$$\mathcal{M}_f(t) = \varphi(t) + f(0)C_{\mathbf{h}}\chi_{\mathbf{h}}(t), \quad t \in F^\times.$$

(2) *Let $\mathcal{S}_{\mathbf{h}}(F)$ be the space of all the compactly supported functions φ on F smooth on $F^\times$ such that $\varphi(t) = C\chi_{\mathbf{h}}(t)$ in a neighborhood of $t = 0$ with some constant C. Then, $f \mapsto \mathcal{M}_f$ is a linear surjection from $\mathcal{S}(V)$ onto $\mathcal{S}_{\mathbf{h}}(F)$.*

Proof

This follows from [21, Proposition 3.5, Théorème 3.7] applied to our $\mathbf{h}$ viewed as a quadratic form on the $2m$-dimensional F-vector space V. We also note that, since $\dim_F V = 2m$ is even, we apply the first case of [21, Proposition 1.7] to determine the constants $\beta_a(Q)$ occurring in [21, Proposition 3.5]. □

Let $\ell \in V$ be an anisotropic vector with $\mathbf{h}[\ell] = 1$. Set $\mathbf{h}^\ell = \mathbf{h}|\ell^\perp$. For $t \in F$, we set $\boldsymbol{\Sigma}^\ell(t) = \ell^\perp \cap \boldsymbol{\Sigma}(t)$. For any $\beta \in E$, define

$$J^\ell(\beta, f) = \int_{y \in \boldsymbol{\Sigma}^\ell(1-\mathrm{N}_{E/F}\beta)} f(\beta\ell + y)|\omega_{\boldsymbol{\Sigma}^\ell(1-\mathrm{N}_{E/F}\beta)}|_F, \quad f \in \mathcal{S}(V). \tag{A.1}$$

PROPOSITION A.3

(1) *Let $f \in \mathcal{S}(V)$. For any $\beta \in E$, the integral $J^\ell(b, f)$ converges absolutely. There exist $\varphi_0, \varphi_1 \in \mathcal{S}(E)$ such that*

$$J^\ell(\beta, f) = \varphi_1(\beta) + C_{\mathbf{h}^\ell}\chi_{\mathbf{h}^\ell}(1 - \mathrm{N}_{E/F}\beta)\varphi_0(\beta), \quad \beta \in E - E^1. \tag{A.2}$$

(2) *Let $\varphi_0, \varphi_1 \in \mathcal{S}(E)$. There exists $f \in \mathcal{S}(V)$ such that (A.2) holds for all $\beta \in E - E^1$.*

Proof
To prove (1), we may assume that $f(\beta\ell + y) = f_1(\beta)f_2(y)$ with $f_1 \in \mathcal{S}(E)$, $f_2 \in \mathcal{S}(\ell^\perp)$. Then $J^\ell(\beta, f) = f_1(\beta\ell)\mathcal{M}_{f_2}(1 - \mathrm{N}_{E/F}\beta)$. Applying Lemma A.2(1), we have assertion (1). Let us show assertion (2). Let $\varphi_0, \varphi_1 \in \mathcal{S}(E)$. By Lemma A.2(2), we have a function $f_0' \in \mathcal{S}(\ell^\perp)$ such that $\mathcal{M}_{f_0'}(t) = C_{\mathbf{h}^\ell}\chi_{\mathbf{h}^\ell}(t)$ for all $t \in F$. Define $f' \in \mathcal{S}(V)$ by setting $f'(\beta\ell + y) = \varphi_0(b)f_0'(y)$ for $b \in E$ and $y \in \ell^\perp$. Then

$$J^\ell(\beta, f') = \varphi_0(b)\mathcal{M}_{f_0'}(1 - \mathrm{N}_{E/F}\beta) = \varphi_0(\beta)C_{\mathbf{h}^\ell}\chi_{\mathbf{h}^\ell}(1 - \mathrm{N}_{E/F}\beta)$$

for all $\beta \in E - E^1$. Let ϕ be the characteristic function of a compact open neighborhood of $1 - \mathrm{N}_{E/F}(\mathrm{supp}(\varphi_1))$. Then $\phi \in \mathcal{S}(F)$. By Lemma A.2(2), we have $f_0'' \in \mathcal{S}(\ell^\perp)$ such that $\mathcal{M}_{f_0''}(t) = \phi(t)$ for all $t \in F$. Define $f''(\beta\ell + y) = \varphi_1(b)f_0''(y)$. Then

$$J^\ell(\beta, f'') = \varphi_1(b)\mathcal{M}_{f_0''}(1 - \mathrm{N}_{E/F}\beta) = \varphi_1(\beta)\phi(1 - \mathrm{N}_{E/F}\beta) = \varphi_1(\beta)$$

for all $\beta \in E - E^1$. Obviously, the function $f = f' + f''$ has the desired property. □

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