Classifying spaces of degenerating mixed Hodge structures, IV: The fundamental diagram

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Abstract We complete the construction of the fundamental diagram of various partial compactifications of the moduli spaces of mixed Hodge structures with polarized graded quotients. The diagram includes the space of nilpotent orbits, the space of SL(2)-orbits, and the space of Borel–Serre orbits. We give amplifications of this fundamental diagram and amplify the relations of these spaces. We describe how this work is useful in understanding asymptotic behaviors of Beilinson regulators and of local height pairings in degeneration. We discuss mild degenerations in which regulators converge.

Contents

0. Introduction

0.1. The fundamental diagram and its amplification

0.1.1

Let D be the period domain which classifies mixed Hodge structures with polarized graded quotients with respect to the weight filtration (see [11], [22]), with fixed Hodge numbers of graded quotients. In Parts I–III [15, I–III] of this series

Kyoto Journal of Mathematics, Vol. 58, No. 2 (2018), 289–426
First published online December 19, 2017.
DOI 10.1215/21562261-2017-0024, © 2018 by Kyoto University
Received February 2, 2016. Accepted September 23, 2016.
2010 Mathematics Subject Classification: Primary 14C30; Secondary 14D07, 32G20.

of articles, we constructed extended period domains in the diagram

$$\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \overset{\eta}{\underset{\subset}{\to}} & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
\Gamma \backslash D_{\Sigma,\mathrm{val}} & \leftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \overset{\psi}{\to} & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\
\downarrow & & \downarrow & & & & \\
\Gamma \backslash D_{\Sigma} & \leftarrow & D^{\sharp}_{\Sigma} & & & &
\end{array}$$

which we call the fundamental diagram, as the mixed Hodge versions of the extended period domains in [17] for the pure case. We have constructed the maps in the diagram except the map η. In this Part IV of our series of articles, we define the injective map η. There is a big issue concerning this map η, which did not appear in the pure case, as we explain below soon. In this Part IV, we amplify this fundamental diagram as in Sections 0.1.4 and 0.2.3 below, and we remedy the issue as a result of the amplification.

0.1.2

The spaces in the fundamental diagram in Section 0.1.1 are topological spaces, the right six spaces have D as dense open sets, and the left two spaces have the quotient $\Gamma \backslash D$ of D by a discrete group Γ as dense open subsets. These left two spaces have sheaves of holomorphic functions extending that of $\Gamma \backslash D$ (though these spaces need not be complex analytic spaces) and have log structures, and the right four spaces have sheaves of real analytic functions extending that of D (though these spaces need not be real analytic spaces) and have log structures. The maps in the fundamental diagram except η respect these structures.

Among these eight spaces, the main spaces are the three spaces $\Gamma \backslash D_{\Sigma}$ (the space of nilpotent orbits), $D_{\mathrm{SL}(2)}$ (the space of SL(2)-orbits), and D_{BS} (the space of Borel–Serre orbits). We defined and studied D_{BS} in [15, I], $D_{\mathrm{SL}(2)}$ in [15, II], and $\Gamma \backslash D_{\Sigma}$ in [15, III]. The other five spaces appear to help the connection of these three spaces.

The map ψ in the center of the fundamental diagram connects the four spaces in the world of nilpotent orbits on the left with the world of SL(2)-orbits. We call ψ the *Cattani–Kaplan–Schmid (CKS) map*, for it is obtained in the pure case by using the work of Cattani, Kaplan, and Schmid [9] on the relation between nilpotent orbits and SL(2)-orbits.

However, to connect the world of SL(2)-orbits and the world of Borel–Serre orbits on the right, the map η has the following defect. Though the map η is a natural map and is continuous in the pure case (see [17]), a big issue is that, in the mixed case, the map η is not necessarily continuous (see Section 3.5).

0.1.3

To remedy this issue and to amplify the connections of the spaces in the fundamental diagram, we will introduce new spaces

$$D^{\star}_{\mathrm{SL}(2)} \text{ and } D^{\diamond}_{\mathrm{SL}(2)} \text{ in the world of SL(2)-orbits (see Section 2) and}$$

$D^\sharp_{\Sigma,[\mathrm{val}]}$ and $D^\sharp_{\Sigma,[:]}$ in the world of nilpotent orbits (see Section 4).

These spaces are topological spaces, and the first two have sheaves of real analytic functions and log structures. They have the following special properties.

The space $D^\star_{\mathrm{SL}(2)}$ has better relations to Borel–Serre orbits than $D_{\mathrm{SL}(2)}$ (see Section 3.4), and this space remedies the above issue. The spirit of the definition of $D^\star_{\mathrm{SL}(2)}$ (see Section 2) is near that of D_{BS}.

As is shown in Section 5, the space $D^\diamond_{\mathrm{SL}(2)}$ has better relations to nilpotent orbits of mild degeneration (see Section 0.2 for the meaning of mildness) than $D_{\mathrm{SL}(2)}$, though among $D_{\mathrm{SL}(2)}$, $D^\star_{\mathrm{SL}(2)}$, and $D^\diamond_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2)}$ is the best for the relation with general nilpotent orbits. In the pure case, we have

$$D_{\mathrm{SL}(2)} = D^\star_{\mathrm{SL}(2)} = D^\diamond_{\mathrm{SL}(2)}.$$

The space $D^\sharp_{\Sigma,[\mathrm{val}]}$ has a nice relation to $D_{\mathrm{SL}(2),\mathrm{val}}$ (see Section 4), which $D^\sharp_{\Sigma,\mathrm{val}}$ does not have. The space $D^\sharp_{\Sigma,[:]}$ is a quotient of $D^\sharp_{\Sigma,[\mathrm{val}]}$ and also a quotient of $D^\sharp_{\Sigma,\mathrm{val}}$ and has a nice relation to $D_{\mathrm{SL}(2)}$ (see Section 4).

The symbols $\star$ and $\diamond$ are used to express that the spaces are shiny like stars and diamonds in the relations to Borel–Serre orbits and nilpotent orbits, respectively. The symbol [:] is used because $D^\sharp_{\Sigma,[:]}$ is regarded as a space of ratios. The symbol [val], is similar to [:], is used because $D^\sharp_{\Sigma,[\mathrm{val}]}$ is the valuative space associated to $D^\sharp_{\Sigma,[:]}$ for a certain log structure.

Actually, as is explained in [15, II], $D_{\mathrm{SL}(2)}$ has two structures $D^I_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ of a topological space with sheaves of real analytic functions and log structures. Everything in this introduction is true for $D^{II}_{\mathrm{SL}(2)}$.

0.1.4

By using the above spaces, we have the following amplified fundamental diagram and supplemental amplifications in Sections 0.1.5 and 0.2.3, which connect the "three worlds" better.

$$\begin{array}{ccccccccc}
 & & & & & & D^\star_{\mathrm{SL}(2),\mathrm{val}} & \underset{\subset}{\overset{\eta^\star}{\to}} & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & D^\sharp_{\Sigma,[\mathrm{val}]} & \overset{\psi}{\to} & D_{\mathrm{SL}(2),\mathrm{val}} & & D_{\mathrm{BS}} \\
 & & & & \downarrow & & \downarrow & & \\
\Gamma\backslash D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \to & D^\sharp_{\Sigma,[:]} & \overset{\psi}{\to} & D_{\mathrm{SL}(2)} & & \\
\downarrow & & & & \downarrow & & & & \\
\Gamma\backslash D_\Sigma & & \leftarrow & & D^\sharp_\Sigma & & & &
\end{array}$$

This diagram is commutative, and the maps respect the structures of the spaces. As indicated in this diagram, the valuative space $D^\star_{\mathrm{SL}(2),\mathrm{val}}$ associated to $D^\star_{\mathrm{SL}(2)}$ has an injective morphism $\eta^\star : D^\star_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ (see Theorem 3.4.4), which is an improved version of η, and a proper surjective morphism $D^\star_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2),\mathrm{val}}$ (see Theorem 2.5.5). Here morphism means a morphism of topological spaces endowed with sheaves of real analytic functions and with log structures.

As also indicated in the diagram, the CKS map $\psi : D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ factors as $D^{\sharp}_{\Sigma,\mathrm{val}} \to D^{\sharp}_{\Sigma,[:]} \to D_{\mathrm{SL}(2)}$, and we have a continuous map $\psi : D^{\sharp}_{\Sigma,[\mathrm{val}]} \to D_{\mathrm{SL}(2),\mathrm{val}}$ (see Theorem 4.5.2).

0.1.5
In the case in which Σ is the fan Ξ of all rational nilpotent cones of rank at most 1, we have

$$D^{\sharp}_{\Xi,[\mathrm{val}]} = D^{\sharp}_{\Xi,[:]} = D^{\sharp}_{\Xi,\mathrm{val}} = D^{\sharp}_{\Xi}.$$

Furthermore, in this case, we have a CKS map $\psi : D^{\sharp}_{\Xi} \to D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$, and hence, the three worlds are connected directly by (see Theorem 6.2.2)

$$D^{\sharp}_{\Xi} \xrightarrow{\psi} D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \underset{\subset}{\xrightarrow{\eta^{\star}}} D_{\mathrm{BS},\mathrm{val}}.$$

0.1.6
As is described above, the spaces $D_{\mathrm{SL}(2)}$, $D^{\star}_{\mathrm{SL}(2)}$, and D_{BS} are related via their associated valuative spaces $D_{\mathrm{SL}(2),\mathrm{val}}$, $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$, and $D_{\mathrm{BS},\mathrm{val}}$. The associated valuative space is a kind of a projective limit of blowups. In Section 2, we will construct also spaces $D^{\star,+}_{\mathrm{SL}(2)}$, $D^{\star,-}_{\mathrm{SL}(2)}$, and $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ which are related to $D^{\star}_{\mathrm{SL}(2)}$ via kinds of blowups and blowdowns and which work as bridges between $D_{\mathrm{SL}(2)}$, $D^{\star}_{\mathrm{SL}(2)}$, and D_{BS} before going to the valuative spaces (see Section 2).

0.1.7
A nilpotent orbit appears as the limit of a variation of mixed Hodge structure in degeneration. SL(2)-orbits are simpler objects, and Borel–Serre orbits are further simpler. The theory of SL(2)-orbits (see [21], [9] for the pure case and [19], [14] for the mixed case) tells us that, roughly speaking, an SL(2)-orbit is associated to a nilpotent orbit, and we can read real analytic behaviors of the degeneration better by looking at the simpler object SL(2)-orbit. The map ψ gives the SL(2)-orbit associated to a nilpotent orbit.

We hope that the above extended period domains and their relations are useful in the study of degeneration of mixed Hodge structures. Actually, as illustrated in Section 0.3 below and in Section 7, our theory has an application to the study (see [5]) of asymptotic behaviors of degenerations of Beilinson regulators and local height pairings. In these subjects, the asymptotic behaviors are understood by degeneration of mixed Hodge structures.

0.2. Mild degenerations

0.2.1
We will define the subsets

$$D^{\mathrm{mild}}_{\Sigma} \subset D_{\Sigma}, \qquad D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \subset D^{\star}_{\mathrm{SL}(2)}, \qquad D^{\mathrm{mild}}_{\mathrm{BS}} \subset D_{\mathrm{BS}}$$

of elements with mild degenerations. Any element of $D^{\diamond}_{\mathrm{SL}(2)}$ is regarded as having mild degeneration.

0.2.2

Let $D_{\Sigma}^{\mathrm{mild}}$ be the subset of D_{Σ} consisting of all points p satisfying the following condition: for any element N of the monodromy cone associated to p, there is a splitting of W which is compatible with N. (The splitting can depend on N and need not have any relation with the Hodge filtration.)

Denote the subset $D_{\mathrm{BS}}^{(A)}$ of D_{BS} [15, I, Section 8.1] by $D_{\mathrm{BS}}^{\mathrm{mild}}$. Let $D_{\mathrm{BS,val}}^{\mathrm{mild}} \subset D_{\mathrm{BS,val}}$ be the inverse image of $D_{\mathrm{BS}}^{\mathrm{mild}}$. There is also a subset $D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}}$ of $D_{\mathrm{SL}(2)}^{\star}$ consisting of A-orbits (see Section 2) whose inverse image $D_{\mathrm{SL}(2),\mathrm{val}}^{\star,\mathrm{mild}}$ in $D_{\mathrm{SL}(2),\mathrm{val}}^{\star}$ coincides with the inverse image of $D_{\mathrm{BS,val}}^{\mathrm{mild}}$ under $\eta^{\star}$. We also have the mild parts of $D_{\Sigma,[:]}^{\sharp}$ and $D_{\Sigma,[\mathrm{val}]}^{\sharp}$; that is, let $D_{\Sigma,[:]}^{\sharp,\mathrm{mild}}$ and $D_{\Sigma,[\mathrm{val}]}^{\sharp,\mathrm{mild}}$ be the inverse images of $\Gamma \backslash D_{\Sigma}^{\mathrm{mild}}$ in $D_{\Sigma,[:]}^{\sharp}$ and in $D_{\Sigma,[\mathrm{val}]}^{\sharp}$, respectively. All these mild parts $\Gamma \backslash D_{\Sigma}^{\mathrm{mild}}$, $D_{\mathrm{BS}}^{\mathrm{mild}}$, $\ldots$, and so on are open sets of $\Gamma \backslash D_{\Sigma}$, D_{BS}, $\ldots$, and so on, respectively.

0.2.3

For mild degenerations, we can replace the upper right part of the amplified fundamental diagram by the following commutative diagram (maps respect structures of the spaces) which contains the space $D_{\mathrm{SL}(2)}^{\diamond}$ and its associated valuative space $D_{\mathrm{SL}(2),\mathrm{val}}^{\diamond}$ (see Theorem 5.1.10).

$$
\begin{array}{ccccccc}
D_{\Sigma,[\mathrm{val}]}^{\sharp,\mathrm{mild}} & \stackrel{\psi}{\to} & D_{\mathrm{SL}(2),\mathrm{val}}^{\diamond} & \to & D_{\mathrm{SL}(2),\mathrm{val}}^{\star,\mathrm{mild}} & \underset{\subset}{\stackrel{\eta^{\star}}{\to}} & D_{\mathrm{BS,val}}^{\mathrm{mild}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D_{\Sigma,[:]}^{\sharp,\mathrm{mild}} & \stackrel{\psi}{\to} & D_{\mathrm{SL}(2)}^{\diamond} & \to & D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}} & & D_{\mathrm{BS}}^{\mathrm{mild}} \\
 & & & & \downarrow & & \\
 & & & & D_{\mathrm{SL}(2)} & &
\end{array}
$$

0.2.4

In the applications of our work as in Section 7, the following part of the fundamental diagrams in Sections 0.1 and 0.2 becomes important.

$$
\begin{array}{ccc}
D_{\Sigma,[:]}^{\sharp,\mathrm{mild}} & \to & D_{\mathrm{SL}(2)}^{\diamond} \\
\cap & & \downarrow \\
D_{\Sigma,[:]}^{\sharp} & \to & D_{\mathrm{SL}(2)}
\end{array}
$$

Via this diagram, we can understand degeneration of mixed Hodge structure in the space $D_{\mathrm{SL}(2)}$ and understand mild degeneration better in $D_{\mathrm{SL}(2)}^{\diamond}$. The right vertical arrow is usually not injective, and hence, $D_{\mathrm{SL}(2)}^{\diamond}$ can give information about mild degeneration which is lost in $D_{\mathrm{SL}(2)}$. This is explained in Section 0.3 below and explained more precisely in Section 7.

0.3. Relations with regulators and local height pairings

We illustrate the relations of this work to the work [5].

0.3.1
Let S be a smooth curve over $\mathbf{C}$, and let $f : X \to S$ be a proper surjective morphism from a smooth algebraic variety X. Let $0 \in S$ be a point, and assume that $X \smallsetminus X_0 \to S \smallsetminus \{0\}$ is smooth and X is of semistable reduction at $0 \in S$.

For $Z \in K_n(X \smallsetminus X_0)$ $(n \geq 1)$, the asymptotic behavior of the regulator of the restriction $Z(t) \in K_n(X_t)$ of Z to X_t $(t \in S \smallsetminus \{0\},\ t \to 0)$ is studied in [5] by using our theory of degeneration of mixed Hodge structure. For each $r \geq 0$, Z defines a variation of mixed Hodge structure H_Z on $S \smallsetminus \{0\}$ with an exact sequence $0 \to H^m(X/S)(r) \to H_Z \to \mathbf{Z} \to 0$, where $m = 2r - n - 1$, $H^m(X/S)$ is the mth direct image $R^m f_* \mathbf{Z}$ on $S \smallsetminus \{0\}$ with Hodge filtration, and (r) is the Tate twist. The (rth) regulator of $Z(t)$ is determined by the fiber $H_Z(t)$ of H_Z at t.

0.3.2
We describe how our theory is related to this subject. Our description in the rest of Section 0.3 is rough and imprecise. More precise matters are described in Section 7.2, and details are given in [5].

We have the period map

$$\bigl(S \smallsetminus \{0\}\bigr) \times K_n(X \smallsetminus X_0) \to \Gamma \backslash D, \quad (t, Z) \mapsto \mathrm{class}\bigl(H_Z(t)\bigr).$$

By [15, III], this extends to

$$S \times K_n(X \smallsetminus X_0) \to \Gamma \backslash D_\Xi, \qquad S^{\log} \times K_n(X \smallsetminus X_0) \to \Gamma \backslash D^\sharp_\Xi,$$

where $S^{\log}$ is the space associated to S defined in [13]. If Z comes from $K_n(X)$, then H_Z has mild degeneration at $0 \in S$ (see Proposition 7.2.3). The diagram in Section 0.2.4 produces the following commutative diagram.

$$\begin{array}{ccccc} S^{\log} \times K_n(X) & \to & \Gamma \backslash D^{\sharp,\mathrm{mild}}_\Xi & \to & \Gamma \backslash D^\diamond_{\mathrm{SL}(2)} \\ \downarrow & & \cap & & \downarrow \\ S^{\log} \times K_n(X \smallsetminus X_0) & \to & \Gamma \backslash D^\sharp_\Xi & \to & \Gamma \backslash D_{\mathrm{SL}(2)} \end{array}$$

0.3.3
We can prove that, for $Z \in K_n(X)$, the regulator of $Z(t)$ converges when $t \to 0$ (see Theorem 7.2.4). In fact, this is a consequence of the fact that the period map $S \smallsetminus \{0\} \to \Gamma \backslash D$, $t \mapsto \mathrm{class}(H_Z(t))$ induced by Z extends to a continuous map $S^{\log} \to \Gamma \backslash D^\diamond_{\mathrm{SL}(2)}$ as indicated by the upper row of the above diagram. We recover the limit of the regulator of $Z(t)$ for $t \to 0$ from the image of a point b of $S^{\log}$ over 0 in $\Gamma \backslash D^\diamond_{\mathrm{SL}(2)}$. On the other hand, for $Z \in K_n(X \smallsetminus X_0)$, which need not come from $K_n(X)$, the regulator of $Z(t)$ need not converge when $t \to 0$, and the image of b in $\Gamma \backslash D_{\mathrm{SL}(2)}$ tells us how rapidly it diverges. When Z comes from $K_n(X)$, the image of b in $\Gamma \backslash D_{\mathrm{SL}(2)}$ has less information than the image of b in $\Gamma \backslash D^\diamond_{\mathrm{SL}(2)}$ and cannot tell us the limit of the regulator of $Z(t)$.

0.3.4
We have a similar story for the asymptotic behavior of the local height pairing (at the Archimedean place). This is introduced in Section 7.4.

0.4. Organization of this article

The organization of this article is as follows. Section 1 consists of preparation. In Section 2, we consider the new space $D^{\star}_{\mathrm{SL}(2)}$ of SL(2)-orbits. In Section 3, we consider the spaces $D_{\mathrm{SL}(2),\mathrm{val}}$ and $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ of valuative SL(2)-orbits and the space $D_{\mathrm{BS},\mathrm{val}}$ of valuative Borel–Serre orbits. In Section 4, we consider the new spaces $D^{\sharp}_{\Sigma,[\mathrm{val}]}$ and $D^{\sharp}_{\Sigma,[:]}$ in the world of nilpotent orbits and improve CKS maps by using these spaces. In Section 5, we consider the new space $D^{\diamond}_{\mathrm{SL}(2)}$ of SL(2)-orbits and construct mild CKS maps. In Section 6, we give complementary results on properties of extended period domains, on relations of nilpotent orbits, SL(2)-orbits, and Borel–Serre orbits, and on extended period maps. In Section 7, we illustrate the relations to the work [5] and give examples.

In Appendix, we give corrections to [17] and supplements to [15, III]. Sections A.1 and A.3 in this appendix are directly related to Section 5.5 of this article.

1. Preliminaries

1.1. The setting

We recall the basic setting and the notation used throughout this series of articles.

1.1.1
We fix $\Lambda = (H_0, W, (\langle\cdot,\cdot\rangle_w)_w, (h^{p,q})_{p,q})$, where H_0 is a finitely generated free $\mathbf{Z}$-module, W is a finite increasing rational filtration on $H_{0,\mathbf{R}} = \mathbf{R} \otimes H_0$, $\langle\cdot,\cdot\rangle_w$ for each $w \in \mathbf{Z}$ is a rational nondegenerate $\mathbf{R}$-bilinear form $\mathrm{gr}^W_w \times \mathrm{gr}^W_w \to \mathbf{R}$ which is symmetric if w is even and is antisymmetric if w is odd, and $h^{p,q}$ is a nonnegative integer given for each $(p,q) \in \mathbf{Z}^2$, satisfying the following conditions (1)–(3):

(1) $\sum_{p,q} h^{p,q} = \mathrm{rank}_{\mathbf{Z}}(H_0)$,
(2) $\sum_{p+q=w} h^{p,q} = \dim_{\mathbf{R}}(\mathrm{gr}^W_w)$ for any $w \in \mathbf{Z}$,
(3) $h^{p,q} = h^{q,p}$ for any (p,q).

1.1.2
Let D be the classifying space of gradedly polarized mixed Hodge structures in [22] associated to the data fixed in Section 1.1.1. As a set, D consists of all increasing filtrations F on $H_{0,\mathbf{C}} = \mathbf{C} \otimes H_0$ such that $(H_0, W, (\langle\cdot,\cdot\rangle_w)_w, F)$ is a gradedly polarized mixed Hodge structure with $\dim_{\mathbf{C}} F^p(\mathrm{gr}^W_{p+q})/F^{p+1}(\mathrm{gr}^W_{p+q}) = h^{p,q}$ for all p, q.

The space D is an open subset of a simpler complex analytic manifold $\check{D}$ [15, I, Section 1.5], which is defined by replacing the condition of positivity for $\langle\cdot,\cdot\rangle_w$ in the definition of D with the weaker condition that $F^p(\mathrm{gr}^W_w)$ is the exact annihilator of $F^{w-p+1}(\mathrm{gr}^W_w)$ for $\langle\cdot,\cdot\rangle_w$.

1.1.3

For $A=\mathbf{Z},\mathbf{Q},\mathbf{R},\mathbf{C}$, let G_A be the group of the A-automorphisms of $(H_{0,A},W)$ whose gr_w^W's are compatible with $\langle\cdot,\cdot\rangle_w$ for all w. Here $H_{0,A}=A\otimes H_0$. Then $G_{\mathbf{C}}$ (resp., $G_{\mathbf{R}}$) acts on $\check{D}$ (resp., D). For $A=\mathbf{Q},\mathbf{R},\mathbf{C}$, let $\mathfrak{g}_A$ be the associated Lie algebra of G_A.

Let $G_{A,u}=\{\gamma\in G_A \mid \mathrm{gr}^W(g)=1\}$, $\mathfrak{g}_{A,u}=\{N\in\mathfrak{g}_A \mid \mathrm{gr}^W(N)=0\}$. Then $G_A/G_{A,u}$ is isomorphic to $G_A(\mathrm{gr}^W):=\prod_w G_A(\mathrm{gr}_w^W)$ and $\mathfrak{g}_A/\mathfrak{g}_{A,u}$ is isomorphic to $\mathfrak{g}_A(\mathrm{gr}^W):=\prod_w \mathfrak{g}_A(\mathrm{gr}_w^W)$, where $G_A(\mathrm{gr}_w^W)$ (resp., $\mathfrak{g}_A(\mathrm{gr}_w^W)$) is "the G_A (resp., $\mathfrak{g}_A$) for gr_w^W."

1.1.4

For each $w\in\mathbf{Z}$, let $D(\mathrm{gr}_w^W)$ be the D for the graded quotient $((H_0\cap W_w)/(H_0\cap W_{w-1}),\langle\cdot,\cdot\rangle_w,(h^{p,q})_{p+q=w})$. Let $D(\mathrm{gr}^W)=\prod_{w\in\mathbf{Z}}D(\mathrm{gr}_w^W)$. Then the canonical morphism

$$D\to D(\mathrm{gr}^W),\qquad F\mapsto F(\mathrm{gr}^W):=\big(F(\mathrm{gr}_w^W)\big)_{w\in\mathbf{Z}}$$

is surjective.

1.1.5

Let W' be a finite increasing filtration on $H_{0,\mathbf{R}}$. A *splitting* of W' is an isomorphism

$$s\colon\ \mathrm{gr}^{W'}:=\bigoplus_w \mathrm{gr}_w^{W'}\overset{\simeq}{\to}H_{0,\mathbf{R}}$$

of $\mathbf{R}$-vector spaces such that, for any $w\in\mathbf{Z}$ and $v\in\mathrm{gr}_w^{W'}$, $s(v)\in W'_w$ and $v=(s(v)\bmod W'_{w-1})$.

Let $\mathrm{spl}(W')$ be the set of all splittings of W'. Consider the case $W'=W$. Then $\mathrm{spl}(W)$ is regarded as a $G_{\mathbf{R},u}$-torsor.

Let $D_{\mathrm{spl}}:=\{s(F)\mid s\in\mathrm{spl}(W),F\in D(\mathrm{gr}^W)\}\subset D$ be the subset of $\mathbf{R}$-*split* elements. Here $s(F)^p:=s(\bigoplus_w F^p_{(w)})$ for $F=(F_{(w)})_w\in D(\mathrm{gr}^W)$. Then, D_{spl} is a real analytic closed submanifold of D, and we have a real analytic isomorphism $\mathrm{spl}(W)\times D(\mathrm{gr}^W)\overset{\sim}{\to}D_{\mathrm{spl}}$, $(s,F)\mapsto s(F)$. Let $D_{\mathrm{nspl}}:=D\smallsetminus D_{\mathrm{spl}}$.

1.2. Canonical splitting of the weight filtration and the invariant δ of nonsplitting

1.2.1

We review the fact that the weight filtration of an $\mathbf{R}$-mixed Hodge structure has a canonical splitting over $\mathbf{R}$ (which does not split the Hodge filtration except the case of an $\mathbf{R}$-split mixed Hodge structure) and the fact that there is an important map δ which tells us how the $\mathbf{R}$-mixed Hodge structure is far from $\mathbf{R}$-split. We review that we have an isomorphism of real analytic manifolds [15, II, Proposition 1.2.5]

$$\begin{aligned}&D\overset{\cong}{\to}\big\{(F,s,\delta)\in D(\mathrm{gr}^W)\times\mathrm{spl}(W)\times\mathcal{L}\mid\delta\in\mathcal{L}(F)\big\},\\&x\mapsto\big(x(\mathrm{gr}^W),\mathrm{spl}_W(x),\delta_W(x)\big)\end{aligned}$$

by using the canonical splitting $\mathrm{spl}_W(x)$ of W associated to x and the invariant $\delta_W(x)$ of nonsplitting associated to x. $\mathcal{L}$ and $\mathcal{L}(F)$ are explained in Section 1.2.2, $\delta_W(x)$ is explained in Section 1.2.3, and $\mathrm{spl}_W(x)$ is explained in Section 1.2.5 below.

1.2.2

Let $\mathcal{L} = W_{-2}\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)$ be the set of all $\mathbf{R}$-linear maps $\delta : \mathrm{gr}^W \to \mathrm{gr}^W$ such that $\delta(\mathrm{gr}^W_w) \subset \bigoplus_{w' \le w-2} \mathrm{gr}^W_{w'}$ for all $w \in \mathbf{Z}$ [15, II, Section 1.2.1]. This is a finite-dimensional weighted $\mathbf{R}$-vector space.

For $F \in D(\mathrm{gr}^W)$, let $\mathcal{L}(F)$ be the weighted subspace of $\mathcal{L}$ consisting of all elements whose (p,q)-Hodge components for F are 0 unless $p < 0$ and $q < 0$. That is, $\mathcal{L}(F)$ is the set of all $\delta \in \mathcal{L}$ such that $\delta(H_F^{p,q}) \subset \bigoplus_{p'<p, q'<q} H_F^{p',q'}$ for all $p, q \in \mathbf{Z}$. Here $H_F^{p,q}$ denotes the (p,q)-Hodge component of $F(\mathrm{gr}^W_{p+q})$ (see [15, II, Section 1.2.1]).

1.2.3

We explain $\delta_W(x) \in \mathcal{L}(x(\mathrm{gr}^W))$. For $x \in D$, there is a unique pair of $s' \in \mathrm{spl}(W)$ and $\delta \in \mathcal{L}(x(\mathrm{gr}^W))$ such that (see [9, Proposition 2.20])

$$x = s'\big(\exp(i\delta)x(\mathrm{gr}^W)\big).$$

We write $\delta_W(x)$ (or $\delta(x)$) for this δ.

1.2.4

Roughly speaking, $\delta_W(x)$ is the invariant of the mixed Hodge structure x which measures how x is far from D_{spl} in D. We have $\delta_W(x) = 0$ if and only if $x \in D_{\mathrm{spl}}$ (Section 1.1.5).

This $\delta_W(x)$ plays important roles in our series of articles. It is related to the regulator in number theory and in arithmetic geometry as is discussed in [5] and in Section 7 of this article. Hence, we call $\delta_W(x)$ the regulator of the mixed Hodge structure x.

1.2.5

We explain $\mathrm{spl}_W(x) \in \mathrm{spl}(W)$. Let $x \in D$, and let $s' \in \mathrm{spl}(W)$ and δ be as in Section 1.2.3. Then the *canonical splitting* $s = \mathrm{spl}_W(x)$ of W associated to x is defined by

$$s = s' \exp(\zeta),$$

where $\zeta = \zeta(x(\mathrm{gr}^W), \delta)$ is a certain element of $\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)$ determined by $x(\mathrm{gr}^W)$ and $\delta = \delta_W(x)$ roughly as in the following way.

Let $\delta_{p,q}$ $(p, q \in \mathbf{Z})$ be the (p,q)-Hodge component of δ with respect to $x(\mathrm{gr}^W)$. Then the (p,q)-Hodge component $\zeta_{p,q}$ of $\zeta = \zeta(x(\mathrm{gr}^W), \delta)$ with respect to $x(\mathrm{gr}^W)$ is given as a certain universal Lie polynomial of $\delta_{p',q'}$ $(p', q' \in \mathbf{Z}, p' \le -1, q' \le -1)$ (see [9, Lemma 6.60] and [14, Section 1, Appendix] for more explanations).

For $x \in D$, $x_{\mathrm{spl}} := s(x(\mathrm{gr}^W)) \in D_{\mathrm{spl}}$ with $s = \mathrm{spl}_W(x)$ is called the *associated* $\mathbf{R}$*-split mixed Hodge structure*. We have $x \in D_{\mathrm{spl}}$ if and only if $x = x_{\mathrm{spl}}$.

1.2.6
We have the following action of the group $\prod_{w \in \mathbf{Z}} \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}_w^W)$ on D, which we call the *lifted action*. For $a = (a_w)_w \in \prod_{w \in \mathbf{Z}} \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}_w^W)$, a sends $x \in D$ to $x' \in D$, which is characterized by $x'(\mathrm{gr}_w^W) = a_w x(\mathrm{gr}_w^W)$, $\mathrm{spl}_W(x') = \mathrm{spl}_W(x)$, and $\delta_W(x') = \mathrm{Ad}(a)\delta_W(x)$. In other words, a sends the Hodge filtration $F \in D$ to the Hodge filtration $s_F a(s_F^{-1}(F))$, where $s_F := \mathrm{spl}_W(F)$ and $s_F^{-1}(F)$ denotes the filtration on $\mathrm{gr}_{\mathbf{C}}^W = \prod_w \mathrm{gr}_{w,\mathbf{C}}^W$ induced by F via $s_F^{-1} : H_{0,\mathbf{C}} \xrightarrow{\cong} \mathrm{gr}_{\mathbf{C}}^W$. This lifted action will be used in Section 2.

1.3. Spaces with real analytic structures and with fs log structures with sign

This is essentially a review of [15, II, Section 3.1].

1.3.1
Endow $\mathbf{R}^n$ $(n \geq 0)$ with the sheaf $\mathcal{O}_{\mathbf{R}^n}$ of real analytic functions. Let $\mathcal{B}'_{\mathbf{R}}$ be the category of locally ringed spaces S over $\mathbf{R}$ satisfying the following condition (i) locally on S.

(i) There are $n \geq 0$ and a morphism $\iota : S \to \mathbf{R}^n$ of locally ringed spaces over $\mathbf{R}$ such that ι is injective, the topology of S coincides with the topology induced from that of $\mathbf{R}^n$, and the map $\iota^{-1}(\mathcal{O}_{\mathbf{R}^n}) \to \mathcal{O}_S$ is surjective.

For an object S of $\mathcal{B}'_{\mathbf{R}}$, we often call the structural sheaf $\mathcal{O}_S$ the sheaf of real analytic functions on S (though S need not be a real analytic space).

Let $\mathcal{C}_{\mathbf{R}}$ be the category of locally ringed spaces S over $\mathbf{R}$ satisfying the following condition (ii).

(ii) For any open set U of S and for any $n \geq 0$, the canonical map $\mathrm{Mor}(U, \mathbf{R}^n) \to \mathcal{O}_S(U)^n$ is bijective.

1.3.2
We have

$$\mathcal{B}'_{\mathbf{R}} \subset \mathcal{C}_{\mathbf{R}}.$$

For the proof, see [15, II, Lemma 3.1.2].

1.3.3
For a topological field K and for a locally ringed space S over K, the following three conditions (i)–(iii) are equivalent.

(i) For any $s \in S$, the map $K \to \mathcal{O}_{S,s}/m_s$ (m_s denotes the maximal ideal of $\mathcal{O}_{S,s}$) is an isomorphism. Furthermore, for any open set U of S and for any $f \in \mathcal{O}_S(U)$, the map $U \to K$, $s \mapsto f(s)$ is continuous. Here $f(s)$ denotes the image of f in $\mathcal{O}_{S,s}/m_s = K$.

(ii) Let $\mathcal{O}'_S$ be the sheaf on S of all K-valued continuous functions. Then there is a homomorphism $\mathcal{O}_S \to \mathcal{O}'_S$ of sheaves of rings over K.

(iii) Let S' be the topological space S endowed with the sheaf of all K-valued continuous functions. Then there is a morphism of locally ringed spaces $S' \to S$ over K whose underlying map $S' \to S$ is the identity map.

If these equivalent conditions are satisfied, then there is only one homomorphism $\mathcal{O}_S \to \mathcal{O}'_S$ of sheaves of rings over K, and there is only one morphism $S' \to S$ of locally ringed spaces over K lying over the identity map of S. These can be proved easily.

1.3.4
Note that objects of $\mathcal{C}_{\mathbf{R}}$ satisfy the equivalent conditions in Section 1.3.3 with $K = \mathbf{R}$.

1.3.5
Let S be a locally ringed space over $\mathbf{R}$ satisfying the equivalent conditions in Section 1.3.3 with $K = \mathbf{R}$. By a *log structure with sign* on S, we mean a log structure M on S endowed with a submonoid sheaf $M_{>0}$ of M satisfying the following (i) and (ii).

(i) $M_{>0} \supset \mathcal{O}^\times_{S,>0}$. Here $\mathcal{O}^\times_{S,>0}$ denotes the subgroup sheaf of $\mathcal{O}^\times_S$ consisting of all local sections whose values are greater than 0.

(ii) The map $M_{>0} \times \{\pm 1\} \to M$, $(f, \varepsilon) \mapsto \varepsilon f$ is an isomorphism of sheaves. Here we regard $\{\pm 1\} \subset \mathcal{O}^\times_S \subset M$.

Note that the map $\mathcal{O}^\times_{S,>0} \times \{\pm 1\} \to \mathcal{O}^\times_S$, $(f, \varepsilon) \mapsto \varepsilon f$ is an isomorphism. Indeed, if $f \in \mathcal{O}^\times_S$ has value greater than 0 (resp., less than 0) at $s \in S$, then f (resp., $-f$) belongs to $\mathcal{O}^\times_{S,>0}$ on some open neighborhood of s. Hence, this map is surjective. The injectivity is clear.

1.3.6
In [15, II, Section 3.1], we defined the notion of log structure with sign in a more restrictive situation where S is an object of $\mathcal{C}_{\mathbf{R}}$ requiring that M be integral (i.e., the canonical map $M \to M^{\mathrm{gp}}$ is injective), and the presentation of the definition there was more complicated. So here we are improving the generality and the presentation of the definition. (But in this article, we do not need this generalization.) If M is integral, then the present definition is equivalent to the definition in [15, II, Definition 3.1.5], which uses a subgroup sheaf $M^{\mathrm{gp}}_{>0}$. The relation with the present definition is that $M^{\mathrm{gp}}_{>0}$ in [15, II, Definition 3.1.5] is obtained from $M_{>0}$ in the present definition as $M^{\mathrm{gp}}_{>0} = (M_{>0})^{\mathrm{gp}}$, and $M_{>0}$ here is obtained from $M^{\mathrm{gp}}_{>0}$ there as $M_{>0} = M \cap M^{\mathrm{gp}}_{>0}$. To prove the equivalence, the nontrivial point is to show that

(1) $M_{>0} \cap \mathcal{O}^\times_S = \mathcal{O}^\times_{S,>0}$

for a log structure with sign in the present sense. We prove (1). If $f \in M_{>0} \cap \mathcal{O}^\times_S$ has a value less than 0 at $s \in S$, then $-f$ belongs to $\mathcal{O}^\times_{S,>0} \subset M_{>0}$ on some open neighborhood of s, and this contradicts condition (ii) in Section 1.3.5. Hence, $f \in \mathcal{O}_{S,>0}^{\times}$.

Note that (1) implies condition (3) in [15, II, Definition 3.1.5] on M; that is, the values of $f \in M_{>0}$ are at least 0. (The values of f mean the values of the image of f in $\mathcal{O}_S$.) Indeed, for $s \in S$, if the image of f in M_s belongs to $\mathcal{O}_{S,s}^{\times}$, then it belongs to $\mathcal{O}_{S,>0,s}^{\times}$ by the above (1), and hence, f has value greater than 0 at s. If the image of f in M_s does not belong to $\mathcal{O}_{S,s}^{\times}$, then f has value 0 at s.

1.3.7

Let $\mathcal{B}'_{\mathbf{R}}(\log)$ be the category of objects of $\mathcal{B}'_{\mathbf{R}}$ (see Section 1.3.1) endowed with a finitely generated and saturated (fs) log structure with sign. Let $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$ be the category of objects of $\mathcal{C}_{\mathbf{R}}$ endowed with a saturated log structure with sign.

Here a log structure M on a locally ringed space S is said to be *saturated* if all stalks of M are saturated in the following sense. We say a commutative monoid $\mathcal{S}$ is saturated if it is integral (i.e., the canonical map $\mathcal{S} \to \mathcal{S}^{\mathrm{gp}}$ is injective) and if, for any $a \in \mathcal{S}^{\mathrm{gp}}$ such that $a^n \in \mathcal{S} \subset \mathcal{S}^{\mathrm{gp}}$ for some integer $n \geq 1$, we have $a \in \mathcal{S}$.

We have

$$\mathcal{B}'_{\mathbf{R}}(\log) \subset \mathcal{C}_{\mathbf{R}}(\mathrm{sat}).$$

1.3.8. Examples

(1) *The object* $\mathbf{R}^n_{\geq 0}$ *of* $\mathcal{B}'_{\mathbf{R}}(\log)$. The sheaf $\mathcal{O}$ of real analytic functions is the inverse image of the sheaf of real analytic functions on $\mathbf{R}^n$. The log structure M with sign is as follows. M (resp., $M_{>0}$) is the multiplicative submonoid sheaf of $\mathcal{O}$ generated by $\mathcal{O}^{\times}$ (resp., $\mathcal{O}^{\times}_{>0}$) and the coordinate functions $t_1, \ldots, t_n$.

(2) *A real analytic manifold with corners* [6, Appendix] is regarded as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. The log structure with sign is given as follows. Let S be a real analytic manifold with corners, and let $\mathcal{O}$ be the sheaf of real analytic functions. If S is an open set of $\mathbf{R}^n_{\geq 0}$ (endowed with the sheaf of real analytic functions), then the log structure with sign $(M, M_{>0})$ is defined as the inverse image of that of $\mathbf{R}^n_{\geq 0}$. In this situation, the canonical map $M \to \mathcal{O}$ is injective, and hence, M and $M_{>0}$ are regarded as subsheaves of $\mathcal{O}$. In general, S is locally isomorphic to an open set of $\mathbf{R}^n_{\geq 0}$, and the log structure with sign on S induced from such an isomorphism is independent of the choice of the isomorphism (M and $M_{>0}$ are independent of the choice as subsheaves of $\mathcal{O}$). By this, we have that a real analytic manifold with corners equals an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ which is locally isomorphic to an open subobject of $\mathbf{R}^n_{\geq 0}$ $(n \geq 0)$.

(3) *The real toric variety* $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ *for an fs monoid* $\mathcal{S}$. (Here $\mathbf{R}^{\mathrm{mult}}_{\geq 0}$ is the set $\mathbf{R}_{\geq 0}$ regarded as a multiplicative monoid.) This is also an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. (The above (1) is the case $\mathcal{S} = \mathbf{N}^n$ of this (3).)

The sheaf $\mathcal{O}$ of real analytic functions is defined as follows. Take a surjective homomorphism $\mathbf{N}^n \to \mathcal{S}$ of monoids for some $n \geq 0$. It gives an embedding $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \subset \mathbf{R}^n$. We say that an $\mathbf{R}$-valued function on an open set of

$\mathrm{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0}^{\mathrm{mult}})$ is real analytic if it is locally a restriction of a *real analytic* function on an open set of $\mathbf{R}^n$. This defines $\mathcal{O}$, and it is independent of the choice of the surjective homomorphism $\mathbf{N}^n \to \mathcal{S}$.

The log structure M is the one associated to the canonical embedding $\mathcal{S} \to \mathcal{O}$. $M_{>0}$ is the submonoid sheaf of M generated by $\mathcal{O}_{>0}^{\times}$ and the image of $\mathcal{S}$.

(4) *The compactified vector space.* Let V be a finite-dimensional graded $\mathbf{R}$-vector space $V = \bigoplus_{w\in\mathbf{Z}, w\leq -1} V_w$ of weight at most -1. Then we have a real analytic manifold with corners $\bar{V}$ (see [15, I, Section 7]). It is covered by two open sets V and $\bar{V} \smallsetminus \{0\}$. Here V has the usual sheaf of real analytic functions and the trivial log structure, and $\bar{V} \smallsetminus \{0\}$ is described as follows. For $a \in \mathbf{R}_{>0}$ and $v \in V$, let $a \circ v = \sum_w a^w v_w \in V$, where v_w denotes the component of v of weight w. By choosing a real analytic closed submanifold $V^{(1)}$ of $V \smallsetminus \{0\}$ such that $\mathbf{R}_{>0} \times V^{(1)} \to V \smallsetminus \{0\}$, $(a, v) \mapsto a \circ v$ is an isomorphism of real analytic manifolds, we have an isomorphism of real analytic manifolds with corners

$$\mathbf{R}_{\geq 0} \times V^{(1)} \cong \bar{V} \smallsetminus \{0\}$$

extending the above isomorphism. We will denote this extended isomorphism as $(a, v) \mapsto a \circ v$.

For example, in the cases $V = \mathcal{L}$ and $V = \mathcal{L}(F)$ (see Section 1.2.2), we have the compactified vector spaces $\bar{\mathcal{L}}$ and $\bar{\mathcal{L}}(F)$, respectively. We can identify $\bar{\mathcal{L}}(F)$ with the closure of $\mathcal{L}(F)$ in $\bar{\mathcal{L}}$.

PROPOSITION 1.3.9

Let $\mathcal{S}$ be an fs monoid, and consider the real toric variety $T := \mathrm{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0}^{\mathrm{mult}})$. Then if S is an object of $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$, we have a natural bijection between the set of all morphisms $S \to T$ in $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$ and the set of all homomorphisms $\mathcal{S} \to \Gamma(S, M_{S,>0})$.

Proof

Since $\mathcal{S} \subset \Gamma(T, M_{T,>0})$, a morphism $S \to T$ induces $\mathcal{S} \to \Gamma(S, M_{S,>0})$. It is easy to see that this correspondence is bijective. $\square$

1.3.10

If M is an fs log structure with sign, then locally we have a chart $\mathcal{S} \to M$ whose image is contained in $M_{>0}$. (Here $\mathcal{S}$ is an fs monoid.) In fact, if $\mathcal{S} \to M$ is a chart, the composition $\mathcal{S} \to M \cong M_{>0} \times \{\pm 1\} \to M_{>0} \subset M$ is also a chart. We will call such a chart $\mathcal{S} \to M_{>0}$ a *positive chart.*

PROPOSITION 1.3.11

(1) *The category $\mathcal{B}'_{\mathbf{R}}(\log)$ has fiber products.*
(2) *A fiber product in $\mathcal{B}'_{\mathbf{R}}(\log)$ is a fiber product in $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$.*

We have already proved (1) in [15, II, Proposition 3.1.7]. We give here a proof which proves both (1) and (2).

Proof
For a diagram $S_1 \to S_0 \leftarrow S_2$ in $\mathcal{B}'_{\mathbf{R}}(\log)$, locally on S_0, S_1, S_2, we can find fs monoids $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ with homomorphisms $\mathcal{S}_1 \leftarrow \mathcal{S}_0 \to \mathcal{S}_2$ and a morphism $\iota_j : S_j \to T_j := \mathrm{Hom}(\mathcal{S}_j, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ of $\mathcal{B}'_{\mathbf{R}}(\log)$ for each $j = 0, 1, 2$, satisfying the following conditions (i) and (ii).

(i) The diagram

$$\begin{array}{ccccc} S_1 & \to & S_0 & \leftarrow & S_2 \\ \downarrow & & \downarrow & & \downarrow \\ T_1 & \to & T_0 & \leftarrow & T_2 \end{array}$$

is commutative.

(ii) For each $j = 0, 1, 2$, the underlying map $S_j \to T_j$ of ι_j is injective, the topology and the log structure of S_j with sign are induced from those of T_j, and the homomorphism $\iota_j^{-1}(\mathcal{O}_{T_j}) \to \mathcal{O}_{S_j}$ is surjective.

This is proved by using positive charts (see Section 1.3.10) on S_0, S_1, S_2 which are compatible.

To prove Proposition 1.3.11, it is sufficient to prove that, in this situation, we have the fiber product S_3 of $S_1 \to S_0 \leftarrow S_2$ in $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$ which belongs to $\mathcal{B}'_{\mathbf{R}}(\log)$. Let $\mathcal{S}_3$ be the pushout of the diagram $\mathcal{S}_1 \leftarrow \mathcal{S}_0 \to \mathcal{S}_2$ in the category of fs monoids. This $\mathcal{S}_3$ is obtained from the pushout $\mathcal{S}'_3$ of $\mathcal{S}_1 \leftarrow \mathcal{S}_0 \to \mathcal{S}_2$ in the category of commutative monoids as follows. $\mathcal{S}_3$ is the submonoid of $(\mathcal{S}'_3)^{\mathrm{gp}}$ consisting of all elements a such that, for some integer $n \geq 1$, a^n belongs to the submonoid of $(\mathcal{S}'_3)^{\mathrm{gp}}$ generated by the images of $\mathcal{S}_1$ and $\mathcal{S}_2$. Let T_3 be the real toric variety $\mathrm{Hom}(\mathcal{S}_3, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$, let S'_3 be the fiber product of $S_1 \to S_0 \leftarrow S_2$ in the category of topological spaces, and let T'_3 be the fiber product of $T_1 \to T_0 \leftarrow T_2$ which is identified with $\mathrm{Hom}(\mathcal{S}'_3, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ as a topological space. As a topological space, we define S_3 as the fiber product of $S'_3 \to T'_3 \leftarrow T_3$. Let $\iota_3 : S_3 \to T_3$ be the canonical injection. We define the structure sheaf $\mathcal{O}_{S_3}$ on S_3 as follows. For $j = 0, 1, 2$, let I_j be the kernel of $\iota_j^{-1}(\mathcal{O}_{T_j}) \to \mathcal{O}_{S_j}$. Let I_3 be the ideal of $\iota_3^{-1}(\mathcal{O}_{T_3})$ generated by the images of I_1 and I_2. Define $\mathcal{O}_{S_3} = \iota_3^{-1}(\mathcal{O}_{T_3})/I_3$. Define the log structure with sign on S_3 as the inverse image of that of T_3. Then S_3 is clearly an object of $\mathcal{B}'_{\mathbf{R}}(\log)$.

We prove that S_3 is the fiber product of $S_1 \to S_0 \leftarrow S_2$ in $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$. By Proposition 1.3.9, for an object X of $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$ and $j = 0, 1, 2, 3$, a morphism $X \to S_j$ corresponds in a one-to-one manner to a homomorphism $\mathcal{S}_j \to \Gamma(X, M_{X,>0})$ such that the associated morphism $X \to T_j$ has the following two properties (1) and (2).

(1) The image of the set X in T_j is contained in S_j.
(2) The image of I_j in $\mathcal{O}_X$ is 0.

Since $\Gamma(X, M_{X,>0})$ is a saturated monoid, a homomorphism $\mathcal{S}'_3 \to \Gamma(X, M_{X,>0})$ and a homomorphism $\mathcal{S}_3 \to \Gamma(X, M_{X,>0})$ correspond in a one-to-one manner. These prove that S_3 is the fiber product of $S_1 \to S_0 \leftarrow S_2$ in $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$. $\square$

1.3.12

The proof of Proposition 1.3.11 shows that the underlying topological space of a fiber product in $\mathcal{B}'_{\mathbf{R}}(\log)$ need not be the fiber product of the underlying topological spaces. We consider this point.

We call a homomorphism $\mathcal{S}_0 \to \mathcal{S}_1$ of saturated commutative monoids (see Section 1.3.7) *universally saturated* if, for any saturated commutative monoid $\mathcal{S}_2$ and any homomorphism $\mathcal{S}_0 \to \mathcal{S}_2$, the pushout of $\mathcal{S}_1 \leftarrow \mathcal{S}_0 \to \mathcal{S}_2$ in the category of commutative monoids is saturated.

For a morphism $S_1 \to S_0$ of $\mathcal{B}'_{\mathbf{R}}(\log)$, we say f is *universally saturated* if, for any $s_1 \in S_1$ with image s_0 in S_0, the homomorphism $M_{S_0,s_0} \to M_{S_1,s_1}$ is universally saturated. (The last condition is equivalent to the condition that the homomorphism $(M_{S_0}/\mathcal{O}^{\times}_{S_0})_{s_0} \to (M_{S_1}/\mathcal{O}^{\times}_{S_1})_{s_1}$ is universally saturated.)

The following can be proved easily. Let $f : S_1 \to S_0$ be a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$. Let the triple of homomorphisms $\mathcal{S}_j \to M_{S_j}$ $(j = 0, 1)$ and $h : \mathcal{S}_0 \to \mathcal{S}_1$ be a chart of f. If h is universally saturated, then f is universally saturated. Conversely, if f is universally saturated, then locally on S_0 and S_1, there are positive charts (see Section 1.3.10) and a homomorphism h of charts as above such that h is universally saturated.

LEMMA 1.3.13

Let $S_1 \to S_0$ be a universally saturated morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$, let $S_2 \to S_0$ be a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$, and let S_3 be the fiber product of $S_1 \to S_0 \leftarrow S_2$ in the category $\mathcal{B}'_{\mathbf{R}}(\log)$. Then the underlying topological space of S_3 is the fiber product of the underlying topological spaces of S_j $(j = 0, 1, 2)$.

This follows from the proof of Proposition 1.3.11.

PROPOSITION 1.3.14

(1) *For $r \geq 1$, the homomorphism $\mathbf{N} \to \mathbf{N}^r$, $m \mapsto (m, m, \ldots, m)$ is universally saturated.*

(2) *For any saturated commutative monoid $\mathcal{S}$, the homomorphisms $\{1\} \to \mathcal{S}$ and $\mathcal{S} \to \{1\}$ are universally saturated.*

(3) *Let $\mathcal{S}_j$ $(j = 0, 1, 2)$ be saturated commutative monoids, let $\mathcal{S}_0 \to \mathcal{S}_1$ be a universally saturated homomorphism, let $\mathcal{S}_0 \to \mathcal{S}_2$ be a homomorphism, and let $\mathcal{S}_3$ be the pushout of $\mathcal{S}_1 \leftarrow \mathcal{S}_0 \to \mathcal{S}_2$ in the category of commutative monoids. Then the homomorphism $\mathcal{S}_2 \to \mathcal{S}_3$ is universally saturated.*

(4) *Let $\mathcal{S}_j \to \mathcal{S}'_j$ $(j = 1, \ldots, n)$ be universally saturated homomorphisms of saturated commutative monoids. Then the homomorphism $\prod_{j=1}^n \mathcal{S}_j \to \prod_{j=1}^n \mathcal{S}'_j$ is universally saturated.*

(5) *A homomorphism $\mathcal{S} \to \mathcal{S}'$ of saturated commutative monoids is universally saturated if and only if the induced homomorphism $\mathcal{S}/\mathcal{S}^{\times} \to \mathcal{S}'/(\mathcal{S}')^{\times}$ is universally saturated.*

(6) *For a saturated commutative monoid $\mathcal{S}$ and for $a \in \mathcal{S}$, the canonical homomorphism $\mathcal{S} \to \mathcal{S}[1/a]$ is universally saturated. Here $\mathcal{S}[1/a]$ denotes the submonoid $\{xa^{-n} \mid x \in \mathcal{S}, n \geq 0\}$ of $\mathcal{S}^{\mathrm{gp}}$.*

Proof
The proofs of (1), (2), (5), and (6) are easy, and (3) is evident. We can prove (4) by induction on n as follows. We may assume $n \geq 2$. Then the homomorphism between products in (4) is the composition $(\prod_{j=1}^{n-1} \mathcal{S}_j) \times \mathcal{S}_n \to (\prod_{j=1}^{n-1} \mathcal{S}'_j) \times \mathcal{S}_n \to (\prod_{j=1}^{n-1} \mathcal{S}'_j) \times \mathcal{S}'_n$ in which the first homomorphism is universally saturated by induction on n and by (3) and the second homomorphism is universally saturated by (3). □

COROLLARY 1.3.15
For a diagram $S_1 \to S_0 \leftarrow S_2$ in $\mathcal{B}'_{\mathbf{R}}(\log)$, the underlying topological space of the fiber product is the fiber product of the underlying topological spaces in the following cases (i) and (ii).

(i) *The case where at least one of $S_1 \to S_0$ and $S_2 \to S_0$ is strict. Here for a morphism $f : X \to Y$ of locally ringed spaces with log structures, we say f is strict if the log structure of X coincides with the inverse image of the log structure of Y via f.*

(ii) *The case where the log structure of S_0 is trivial.*

The following will be used many times in this article.

1.3.16
Let X be an object of $\mathcal{B}'_{\mathbf{R}}(\log)$, and let Y be a subset of X. Assume that the following condition (C) is satisfied.

(C) The homomorphism from $\mathcal{O}_X$ to the sheaf of $\mathbf{R}$-valued continuous functions on X is injective.

Then we have a structure on Y as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$, which also satisfies (C), as follows. The topology of Y is the one as a subspace of X. $\mathcal{O}_Y$ is the sheaf of $\mathbf{R}$-valued functions on Y which are locally restrictions of functions in $\mathcal{O}_X$. The log structure with sign is the pullback of that of X.

For an object S of $\mathcal{B}'_{\mathbf{R}}(\log)$ which satisfies (C), the map $\mathrm{Mor}(S, Y) \to \mathrm{Mor}(S, X)$ is injective and the image coincides with $\{f \in \mathrm{Mor}(S, X) \mid f(S) \subset Y\}$.

1.4. Review of toric geometry

We recall toric varieties over a field and the real toric varieties associated to fans by comparing them.

1.4.1
Let L be a finitely generated free Abelian group, and let $N := \mathrm{Hom}(L, \mathbf{Z})$. We will denote the group law of L multiplicatively and that of N additively.

For a rational finitely generated sharp cone σ in $N_{\mathbf{R}}$, define an fs monoid $\mathcal{S}(\sigma)$ by

$$\mathcal{S}(\sigma) := \bigl\{ l \in L \mid l(\sigma) \geq 0 \bigr\}.$$

For a rational fan Σ in $N_{\mathbf{R}}$, we have a toric variety $\mathrm{toric}_k(\Sigma)$ over a field k associated to Σ, which is an fs log scheme over k, and a real toric variety $|\mathrm{toric}|(\Sigma)$, which is an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. We review these.

1.4.2
The toric variety $\mathrm{toric}_k(\Sigma)$ over k is described as

$$\mathrm{toric}_k(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathrm{Spec}\bigl(k\bigl[\mathcal{S}(\sigma)\bigr]\bigr) \quad \text{(an open covering)},$$

where $k[\mathcal{S}(\sigma)]$ denotes the semigroup algebra of $\mathcal{S}(\sigma)$ over k and $\mathrm{Spec}(k[\mathcal{S}(\sigma)])$ is endowed with the standard log structure. It represents the contravariant functor from the category (fs/k) of fs log schemes over k to the category of sets, which sends S to the set of all homomorphisms $h : L \to M_S^{\mathrm{gp}}$ satisfying the following condition.

(C) Let $s \in S$. Then there exists $\sigma \in \Sigma$ such that, for any homomorphism $a : (M_S/\mathcal{O}_S^\times)_{\bar{s}} \to \mathbf{N}$, the homomorphism $a \circ h : L \to \mathbf{Q}$ belongs to σ. Here $\bar{s}$ is a geometric point over s.

Note that this condition is equivalent to the following condition.

(C′) Étale locally on S, there is $\sigma \in \Sigma$ such that $h(\mathcal{S}(\sigma)) \subset M_S$.

The set $\mathrm{toric}_k(\Sigma)(k)$ of all k-rational points of $\mathrm{toric}_k(\Sigma)$ is identified with the set of pairs (σ, h) consisting of $\sigma \in \Sigma$ and a homomorphism $h : \mathcal{S}(\sigma)^\times \to k^\times$. The point corresponding to this pair is the element of $\mathrm{Spec}(k[\mathcal{S}(\sigma)])(k) = \mathrm{Hom}(\mathcal{S}(\sigma), k)$ which sends $a \in \mathcal{S}(\sigma)^\times$ to $h(a)$ and sends $a \in \mathcal{S}(\sigma) \smallsetminus \mathcal{S}(\sigma)^\times$ to 0.

1.4.3
The real toric variety $|\mathrm{toric}|(\Sigma)$ is described as

$$|\mathrm{toric}|(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathrm{Hom}\bigl(\mathcal{S}(\sigma), \mathbf{R}_{\geq 0}^{\mathrm{mult}}\bigr) \quad \text{(an open covering)},$$

where $\mathrm{Hom}(\mathcal{S}(\sigma), \mathbf{R}_{\geq 0}^{\mathrm{mult}})$ is regarded as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ as in Section 1.3.8(3). It represents the contravariant functor from $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$ to the category of sets, which sends S to the set of all homomorphisms $h : L \to M_{S,>0}^{\mathrm{gp}}$ satisfying the following condition.

(C) Let $s \in S$. Then there exists $\sigma \in \Sigma$ such that, for any homomorphism $a : (M_S/\mathcal{O}_S^\times)_{\bar{s}} \to \mathbf{N}$, the homomorphism $a \circ h : L \to \mathbf{Q}$ belongs to σ.

Note that this condition is equivalent to the following condition.

(C′) Locally on S, there is $\sigma \in \Sigma$ such that $h(\mathcal{S}(\sigma)) \subset M_{S,>0}$.

The set $|\mathrm{toric}|(\Sigma)$ is identified with the set of pairs (σ, h) consisting of $\sigma \in \Sigma$ and a homomorphism $h : \mathcal{S}(\sigma)^{\times} \to \mathbf{R}_{>0}$. The point corresponding to this pair is the element of $\mathrm{Hom}(\mathcal{S}(\sigma), \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ which sends $a \in \mathcal{S}(\sigma)^{\times}$ to $h(a)$ and sends $a \in \mathcal{S}(\sigma) \smallsetminus \mathcal{S}(\sigma)^{\times}$ to 0. By this understanding, we can regard $|\mathrm{toric}|(\Sigma)$ as a closed subset of $\mathrm{toric}_{\mathbf{R}}(\Sigma)(\mathbf{R})$.

1.4.4

The set $|\mathrm{toric}|(\Sigma)$ is also identified with the set of all pairs (σ, Z) consisting of $\sigma \in \Sigma$ and a subset Z of $\mathrm{Hom}(L, \mathbf{R}^{\mathrm{mult}}_{>0})$ which is a $\mathrm{Hom}(L/\mathcal{S}(\sigma)^{\times}, \mathbf{R}^{\mathrm{mult}}_{>0})$-orbit. In fact, (σ, Z) corresponds to (σ, h) in Section 1.4.3, where h is the restriction of any element of Z to $\mathcal{S}(\sigma)^{\times}$.

1.4.5

If Σ is finite and Σ' is a rational finite subdivision of Σ, we have a proper surjective morphism $\mathrm{toric}_k(\Sigma') \to \mathrm{toric}_k(\Sigma)$. In the case $k = \mathbf{R}$, this induces a morphism $|\mathrm{toric}|(\Sigma') \to |\mathrm{toric}|(\Sigma)$ which is proper and surjective.

1.4.6

A morphism $S' \to S$ in the category (fs/k) (resp., $\mathcal{B}'_{\mathbf{R}}(\log)$) is called a *log modification* if, locally on S, there are a homomorphism $\mathcal{S} \to M_S$ (resp., $\mathcal{S} \to M_{S,>0}$) with $\mathcal{S}$ a sharp fs monoid and a rational finite subdivision Σ' of the fan Σ of all faces of the cone $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{add}}_{\geq 0}) \subset \mathrm{Hom}(\mathcal{S}^{\mathrm{gp}}, \mathbf{R}^{\mathrm{add}})$ such that S' is isomorphic over S to $S \times_{\mathrm{toric}_k(\Sigma)} \mathrm{toric}_k(\Sigma')$ (resp., $S \times_{|\mathrm{toric}|(\Sigma)} |\mathrm{toric}|(\Sigma')$). The underlying map of topological spaces of a log modification is proper and surjective.

1.4.7

We introduce a functor $[\Sigma]$ associated to a fan Σ, and consider its relation to log modification. Let L and N be as in Section 1.4.1. For a rational fan Σ in $N_{\mathbf{R}}$, let $[\Sigma]$ be the contravariant functor from (fs/k) (resp., $\mathcal{B}'_{\mathbf{R}}(\log)$) to the category of sets which sends S to the set of all homomorphisms $h : L \to M^{\mathrm{gp}}_S/\mathcal{O}^{\times}_S$ satisfying the condition (C) in Section 1.4.2 (resp., Section 1.4.3). In the present situation, (C) is equivalent to (C′) with M_S (resp., $M_{S,>0}$) replaced by $M_S/\mathcal{O}^{\times}_S$.

Let S be an object of (fs/k) (resp., $\mathcal{B}'_{\mathbf{R}}(\log)$), and assume that we are given $h \in [\Sigma](S)$. This induces a continuous map $S \to \Sigma$ which sends $s \in S$ to the unique cone $\sigma \in \Sigma$ such that $\mathcal{S}(\sigma) \subset L$ coincides with the inverse image of $(M_S/\mathcal{O}^{\times}_S)_s$ under $L \to (M^{\mathrm{gp}}_S/\mathcal{O}^{\times}_S)_s$.

Assume Σ is finite, and let Σ' be a rational finite subdivision of Σ. Then we have a morphism of functors $[\Sigma'] \to [\Sigma]$. The contravariant functor $\mathrm{Mor}(\cdot, S) \times_{[\Sigma]} [\Sigma']$ from (fs/k) (resp., $\mathcal{B}'_{\mathbf{R}}(\log)$) to the category of sets is represented by a log modification $S' \to S$. In fact, locally on S, $h : L \to M^{\mathrm{gp}}_S/\mathcal{O}^{\times}_S$ lifts to a morphism

$S \to \mathrm{toric}_k(\Sigma)$ (see Section 1.4.2) (resp., $S \to |\mathrm{toric}|(\Sigma)$ (see Section 1.4.3)), and this functor is represented by $S \times_{\mathrm{toric}_k(\Sigma)} \mathrm{toric}_k(\Sigma')$ (resp., $S \times_{|\mathrm{toric}|(\Sigma)} |\mathrm{toric}|(\Sigma')$).

1.4.8

This section will be used in Sections 2.4–2.6. Let $\mathcal{S}_1$ be an fs monoid, let $T := \mathrm{Hom}(\mathcal{S}_1, \mathbf{R}^{\mathrm{mult}}_{>0})$, and let Z be a T-torsor. The purpose of this section is to introduce an object $\bar{Z}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$ and to give a set-theoretical description (1) below of a log modification of $\bar{Z}$.

Let $\bar{T} := \mathrm{Hom}(\mathcal{S}_1, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \supset T$, and let $\bar{Z} := Z \times^T \bar{T}$. We regard $\bar{Z}$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ as follows. Take $\mathbf{r} \in Z$. Then we have the bijection $T \to Z$, $t \mapsto t\mathbf{r}$, and this induces a bijection $\bar{T} \to \bar{Z}$. Via the last bijection from the real toric variety $\bar{T}$, we obtain a structure of $\bar{Z}$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. This structure is independent of the choice of $\mathbf{r}$.

We prepare notation. For $s \in \bar{Z}$, we define a subgroup $T(s)$ of T and a $T(s)$-orbit $Z(s)$ inside Z as follows. In the case $Z = T$ and hence $\bar{Z} = \bar{T}$, s is a homomorphism $\mathcal{S}_1 \to \mathbf{R}^{\mathrm{mult}}_{\geq 0}$. In this case, let $T(s)$ be the subgroup of $T = \mathrm{Hom}(\mathcal{S}_1, \mathbf{R}^{\mathrm{mult}}_{>0})$ consisting of all elements which kill $s^{-1}(\mathbf{R}_{>0}) \subset \mathcal{S}_1$, and let $Z(s) \subset T$ be the set of all elements of $\mathcal{S}_1 \to \mathbf{R}^{\mathrm{mult}}_{>0}$ whose restriction to $s^{-1}(\mathbf{R}_{>0})$ coincides with the homomorphism induced by s. Then $Z(s)$ is a $T(s)$-orbit. In general, take $\mathbf{r} \in Z$, consider the induced isomorphism $\bar{Z} \cong \bar{T}$, let t be the image of s in $\bar{T}$, let $T(s) := T(t)$, and let $Z(s)$ be the $T(s)$-orbit in Z corresponding to the $T(t)$-orbit $Z(t)$ in T via the isomorphism $Z \cong T$. Then $T(s)$ and $Z(s)$ are independent of the choice of $\mathbf{r}$.

Consider L and N in Section 1.4.1, let σ be a rational finitely generated sharp cone in $N_{\mathbf{R}}$, and let Σ be the fan of all faces of σ. Assume that we are given a homomorphism $\mathcal{S}(\sigma) \to \mathcal{S}_1$. Then we have a morphism of functors $\mathrm{Mor}(\cdot, \bar{Z}) \to [\Sigma]$, where $[\Sigma]$ is as in Section 1.4.7. This morphism is obtained as follows. The homomorphism $\mathcal{S}(\sigma) \to \mathcal{S}_1$ induces $\mathrm{Mor}(\cdot, \bar{T}) \to [\Sigma]$. Take $\mathbf{r} \in Z$. Then $\mathbf{r}$ gives an isomorphism $\bar{Z} \cong \bar{T}$ and, hence, the composite morphism $\mathrm{Mor}(\cdot, \bar{Z}) \cong \mathrm{Mor}(\cdot, \bar{T}) \to [\Sigma]$. This composite morphism is independent of the choice of $\mathbf{r}$.

Assume further that the homomorphism $\mathcal{S}(\sigma) \to \mathcal{S}_1$ is universally saturated (Section 1.3.12). Let Σ' be a rational finite subdivision of Σ, and let E be the log modification of $\bar{Z}$ which represents the fiber product $\mathrm{Mor}(\cdot, \bar{Z}) \times_{[\Sigma]} [\Sigma']$ (Section 1.4.7). We give a description of E as a set.

For $s \in \bar{Z}$ and for $\sigma' \in \Sigma'$ such that the image τ of s in Σ coincides with the image of σ' in Σ, let $T(s, \sigma')$ be the subgroup of $T(s)$ consisting of all elements whose image in $\mathrm{Hom}(L/\mathcal{S}(\tau)^{\times}, \mathbf{R}^{\mathrm{mult}}_{>0})$ is contained in its subgroup $\mathrm{Hom}(L/\mathcal{S}(\sigma')^{\times}, \mathbf{R}^{\mathrm{mult}}_{>0})$. Then we have the following.

(1) There is a canonical bijection between E and the set of all triples (s, σ', Z'), where $s \in \bar{Z}$, σ' is an element of Σ' whose image in Σ coincides with the image of s in Σ, and Z' is a $T(s, \sigma')$-orbit in $Z(s)$.

In fact, if $Z = T$, then $E = \bar{T} \times_{|\mathrm{toric}|(\Sigma)} |\mathrm{toric}|(\Sigma')$, and hence, the bijection is given by Section 1.4.4. In general, for $\mathbf{r} \in Z$, if t denotes the image of s under the isomorphism $\bar{Z} \cong \bar{T}$, we have $T(s, \sigma') = T(t, \sigma')$, the isomorphism $Z \cong T$ sends a

$T(t,\sigma')$-orbit in T to a $T(s,\sigma')$-orbit in Z, and the induced composite bijection from the set of triples (s,σ',Z') to E is independent of the choice of $\mathbf{r}$.

2. The new space $D^{\star}_{\mathrm{SL}(2)}$ of $\mathrm{SL}(2)$-orbits

In [15, II], we defined and studied the space $D_{\mathrm{SL}(2)}$ of SL(2)-orbits. Here we introduce a variant $D^{\star}_{\mathrm{SL}(2)}$. It is an object of the category $\mathcal{B}'_{\mathbf{R}}(\log)$ (see Section 1.3.7).

Recall that D_{BS} is an object of $\mathcal{B}'_{\mathbf{R}}(\log)$, $D_{\mathrm{SL}(2)}$ has two structures $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$, and the identity map of $D_{\mathrm{SL}(2)}$ gives a morphism $D^{I}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$. We will relate the three spaces $D^{\star}_{\mathrm{SL}(2)}$, $D^{II}_{\mathrm{SL}(2)}$, and D_{BS} in the following way. These three spaces are not connected directly, but as we will see in this section, they are connected as in the diagram

$$\begin{array}{ccccccc}
D^{\star,+}_{\mathrm{SL}(2)} & \to & D^{\star}_{\mathrm{SL}(2)} & \to & D^{\star,-}_{\mathrm{SL}(2)} & \leftarrow & D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} \\
\downarrow & & & & & & \downarrow \\
D^{II}_{\mathrm{SL}(2)} & & & & & & D_{\mathrm{BS}}
\end{array}$$

in $\mathcal{B}'_{\mathbf{R}}(\log)$ in which the horizontal arrows are log modifications (Section 1.4.6) and the left vertical arrow is proper surjective.

As will be seen in Section 3, this diagram will induce morphisms

$$D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D^{II}_{\mathrm{SL}(2),\mathrm{val}}, \qquad D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$$

of associated valuative spaces, which appeared in Section 0, since log modifications induce isomorphisms of the associated valuative spaces $D^{\star,+}_{\mathrm{SL}(2),\mathrm{val}} \overset{\cong}{\to} D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \overset{\cong}{\to} D^{\star,-}_{\mathrm{SL}(2),\mathrm{val}} \overset{\cong}{\leftarrow} D^{\star,\mathrm{BS}}_{\mathrm{SL}(2),\mathrm{val}}$. In the pure case, the arrows in $D_{\mathrm{SL}(2)} \leftarrow D^{\star,+}_{\mathrm{SL}(2)} \to D^{\star}_{\mathrm{SL}(2)} \to D^{\star,-}_{\mathrm{SL}(2)}$ are isomorphisms.

In Section 2.1, we review SL(2)-orbits in the pure situation. In Section 2.2, we continue reviews of [15, II]. In Section 2.3, we define the spaces $D^{\star}_{\mathrm{SL}(2)}$ and $D^{\star,-}_{\mathrm{SL}(2)}$. After preparations in Section 2.4, we connect $D^{\star}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ in Section 2.5 by introducing the space $D^{\star,+}_{\mathrm{SL}(2)}$, and we connect $D^{\star,-}_{\mathrm{SL}(2)}$ and D_{BS} in Section 2.6 by introducing the space $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$. In Section 2.7, we show that our spaces of SL(2)-orbits belong to a full subcategory $\mathcal{B}'_{\mathbf{R}}(\log)^{+}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$ consisting of nice objects.

2.1. Review of $\mathrm{SL}(2)$-orbits in the pure case

Let the setting be as in Section 1.1, and assume that we are in the pure situation of weight w.

2.1.1

In this pure case, an SL(2)-*orbit in n variables* means a pair (ρ,φ), where ρ is a homomorphism $\mathrm{SL}(2,\mathbf{C})^n \to G(\mathbf{C})$ of algebraic groups defined over $\mathbf{R}$ and φ is a holomorphic map $\mathbf{P}^1(\mathbf{C})^n \to \check{D}$, satisfying

$$\varphi(gz) = \rho(g)\varphi(z) \quad \text{for } g \in \mathrm{SL}(2,\mathbf{C})^n \text{ and } z \in \mathbf{P}^1(\mathbf{C})^n,$$

$$\varphi(\mathfrak{h}^n) \subset D \quad \big(\mathfrak{h} \text{ is the upper half-plane } \{x+iy \mid x,y \in \mathbf{R}, y>0\}\big),$$

$$\rho_*\big(\mathrm{fil}_z^p\big(\mathfrak{sl}(2,\mathbf{C})^n\big)\big) \subset \mathrm{fil}_{\varphi(z)}^p(\mathfrak{g}_{\mathbf{C}}) \quad \big(z \in \mathbf{P}^1(\mathbf{C})^n, p \in \mathbf{Z}\big).$$

Here ρ_* denotes the homomorphism $\mathfrak{sl}(2,\mathbf{C})^n \to \mathfrak{g}_{\mathbf{C}}$ of Lie algebras induced by ρ, and $\mathrm{fil}_z^\bullet$ and $\mathrm{fil}_{\varphi(z)}^\bullet$ are filtrations given by z and $\varphi(z)$, respectively (see [15, II, Section 2.1.2]).

2.1.2

Let (ρ,φ) be an SL(2)-orbit in n variables. Define the *associated homomorphisms* $\tau,\tau^\star : \mathbf{G}_{m,\mathbf{R}}^n \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}})$ of algebraic groups as

$$\tau^\star(t) = \rho(g_1,\dots,g_n), \quad \text{where } t = (t_j)_{1\le j\le n} \qquad \text{and}$$
$$g_j = \begin{pmatrix} 1/\prod_{k=j}^n t_k & 0 \\ 0 & \prod_{k=j}^n t_k \end{pmatrix},$$
$$\tau(t) = \Big(\prod_{j=1}^n t_j\Big)^w \cdot \tau^\star(t).$$

The image of the homomorphism $\tau^\star$ is contained in $G_{\mathbf{R}}$.

For $1 \le j \le n$, we define the increasing filtration $W^{(j)}$ on $H_{0,\mathbf{R}}$ as follows. We have $H_{0,\mathbf{R}} = \bigoplus_{1\le j\le n, k\in\mathbf{Z}} H_{0,\mathbf{R}}(j,k)$, where $H_{0,\mathbf{R}}(j,k)$ is the part of $H_{0,\mathbf{R}}$ on which the action τ of $\mathbf{G}_{m,\mathbf{R}}^n$ is given by $(t_\ell)_{1\le \ell\le n} \mapsto t_j^k$. Define $W^{(j)}$ by $W_k^{(j)} = \bigoplus_{k'\le k} H_{0,\mathbf{R}}(j,k')$. We call $W^{(j)}$ $(1 \le j \le n)$ the *associated weight filtrations.*

2.1.3

Let (ρ,φ) be an SL(2)-orbit in n variables.

For $1 \le j \le n$, the following conditions (i)–(iii) are equivalent.

(i) The jth component $\mathrm{SL}(2,\mathbf{C}) \to G(\mathbf{C})$ of ρ is trivial.

(ii) φ factors through the projection $\mathbf{P}^1(\mathbf{C})^n \to \mathbf{P}^1(\mathbf{C})^{n-1}$ which removes the jth component.

(iii) Either $j \ge 2$ and $W^{(j)} = W^{(j-1)}$, or $j = 1$ and $W^{(1)} = W$ (i.e., $W_w^{(1)} = H_{0,\mathbf{R}}$ and $W_{w-1}^{(1)} = 0$).

2.1.4

We consider the following equivalence relation on SL(2)-orbits. We say an SL(2)-orbit in n variables (ρ,φ) is nondegenerate if there is no j $(1 \le j \le n)$ which satisfies the equivalent conditions in Section 2.1.3. For a nondegenerate SL(2)-orbit (ρ,φ) in n variables and for a nondegenerate SL(2)-orbit (ρ',φ') in n' variables, (ρ,φ) and (ρ',φ') are equivalent if and only if $n = n'$ and there is $t \in \mathbf{R}_{>0}^n$ such that

$$\rho'(g) = \tau^\star(t)\rho(g)\tau^\star(t)^{-1}, \qquad \varphi'(z) = \tau^\star(t)\varphi(z)$$

for any $g \in \mathrm{SL}_2(\mathbf{C})^n$ and $z \in \mathbf{P}^1(\mathbf{C})^n$. Here $\tau^\star$ is the homomorphism associated to (ρ,φ) in Section 2.1.2. We have the same equivalence relation when we replace $\tau^\star(t)$ in the above by $\tau(t)$ in Section 2.1.2 associated to (ρ,φ).

Any SL(2)-orbit uniquely factors through a nondegenerate SL(2)-orbit, called the *associated nondegenerate* SL(2)-*orbit*, which is described as below. Two SL(2)-orbits are equivalent if and only if their associated nondegenerate SL(2)-orbits are equivalent in the above sense.

For an SL(2)-orbit (ρ,φ) in n variables, the associated nondegenerate SL(2)-orbit (ρ',φ') is as follows. Let $J=\{a(1),\dots,a(r)\}$ $(a(1)<\cdots<a(r))$ be the set of j $(1\le j\le n)$ such that the jth component of ρ is nontrivial. Then (ρ',φ') is the SL(2)-orbit in r variables defined by

$$\rho(g_1,\dots,g_n)=\rho'(g_{a(1)},\dots,g_{a(r)}),\qquad \varphi(z_1,\dots,z_n)=\varphi'(z_{a(1)},\dots,z_{a(r)}).$$

This number r is called the rank of the (equivalence class of the) SL(2)-orbit (ρ,φ).

2.1.5

The set $D_{\mathrm{SL}(2)}$ is defined as the set of all equivalence classes of SL(2)-orbits (ρ,φ) such that all members of the set of weight filtrations associated to (ρ,φ) (see Section 2.1.2) are rational (i.e., defined already on $H_{0,\mathbf{Q}}$).

D is embedded in $D_{\mathrm{SL}(2)}$ as the set of classes of SL(2)-orbits of rank 0.

2.1.6

Let $p\in D_{\mathrm{SL}(2)}$. We define objects

$$\tau_p^\star,\tau_p,Z(p),\mathcal{W}(p)$$

associated to p. Let n be the rank of p. Let (ρ,φ) be a nondegenerate SL(2)-orbit which represents p. The homomorphism $\tau^\star$ (resp., τ) (see Section 2.1.2) associated to (ρ,φ) depends only on the class p (it does not depend on the choice of (ρ,φ)). We denote it as $\tau_p^\star$ (resp., τ_p).

The subset

$$\{\varphi\big((iy_j)_{1\le j\le n}\big)\mid y_j\in\mathbf{R}_{>0}(1\le j\le n)\}=\tau^\star(\mathbf{R}^n_{>0})\varphi(\mathbf{i})=\tau(\mathbf{R}^n_{>0})\varphi(\mathbf{i})\subset D$$

$(\mathbf{i}:=(i,\dots,i)\in\mathfrak{h}^n)$ depends only on the class p. We denote it as $Z(p)$ and call it the *torus orbit associated to* p.

The family $\{W^{(j)}\mid 1\le j\le n\}$ of weight filtrations associated to (ρ,φ) (see Section 2.1.2) depends only on the class p. Let $\mathcal{W}(p)=\{W^{(j)}\mid 1\le j\le n\}$, and call it the set of weight filtrations associated to p. It consists of n elements (see [15, II, Proposition 2.1.13]).

2.1.7

$D_{\mathrm{SL}(2)}$ has a structure as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. For this, see [15, II, Section 3.2]. A basic property of the topology of $D_{\mathrm{SL}(2)}$ is that if $p\in D_{\mathrm{SL}(2)}$ is the class of an SL(2)-orbit (ρ,φ), then p is the limit of $\varphi(iy_1,\dots,iy_n)\in D$, where $y_j\in\mathbf{R}_{>0}$ and $y_j/y_{j+1}\to\infty$ $(1\le j\le n$, y_n denotes 1).

2.2. Reviews of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ and $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$

We now consider the mixed Hodge situation. We review the spaces $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ and $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ considered in [15, II] and prepare notation which we will use later. Actually there was an error concerning the definition of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ in [15, II]. We correct it in Remark 2.2.3.

2.2.1

Let

$$D_{\mathrm{SL}(2)}(\mathrm{gr}^W) := \prod_{w\in\mathbf{Z}} D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w),$$

where $D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)$ denotes the space $D_{\mathrm{SL}(2)}$ (see Section 2.1) for the graded quotient gr^W_w.

2.2.2

The set $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ is defined as follows (cf. [15, II, Section 3.5.1]). By an SL(2)-*orbit on* gr^W *of rank* n, we mean a family $(\rho_w, \varphi_w)_{w\in\mathbf{Z}}$ of SL(2)-orbits (ρ_w, φ_w) on gr^W_w in n variables in the sense of Section 2.1.1 satisfying the following condition (1).

(1) For each $1 \le j \le n$, there is a $w \in \mathbf{Z}$ such that the jth component of ρ_w is nontrivial.

The equivalence relation is defined as follows. For an SL(2)-orbit $(\rho_w, \varphi_w)_w$ on gr^W of rank n, the homomorphisms $\tau, \tau^\star : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ associated to the SL(2)-orbit (ρ_w, φ_w) in n variables of weight w for $w \in \mathbf{Z}$ (see Section 2.1.2) define homomorphisms

$$\tau, \tau^\star : \mathbf{G}^n_{m,\mathbf{R}} \to \prod_{w\in\mathbf{Z}} \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$$

of algebraic groups, respectively.

An SL(2)-orbit $(\rho_w, \varphi_w)_w$ on gr^W of rank n and an SL(2)-orbit $(\rho'_w, \varphi'_w)_w$ on gr^W of rank n' are *equivalent* if and only if $n' = n$ and $(\rho'_w(g))_w = \tau^\star(t)(\rho_w(g))_w \tau^\star(t)^{-1}$, $(\varphi'_w(z))_w = \tau^\star(t)(\varphi_w(z))_w$ for some $t \in \mathbf{R}^n_{>0}$. (We have the same equivalence relation when we replace $\tau^\star$ here by τ.)

The set $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ is defined as the set of all equivalence classes of SL(2)-orbits $(\rho_w, \varphi_w)_w$ on gr^W such that the weight filtrations on gr^W_w associated to (ρ_w, φ_w) are rational (i.e., defined over $\mathbf{Q}$) for any $w \in \mathbf{Z}$.

REMARK 2.2.3

In the definition of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ in [15, II, Section 3.5.1], we forgot to put the condition of the rationality of the associated weight filtrations. This error does not affect the rest of [15, II].

2.2.4

We have the embedding

$$D(\mathrm{gr}^W) \hookrightarrow D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$$

by identifying $D(\mathrm{gr}^W)$ with the set of SL(2)-orbits on gr^W of rank 0. We have a map

$$D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W), \quad p \mapsto \big(p(\mathrm{gr}^W_w)\big)_w,$$

which sends the class p of $(\rho_w, \varphi_w)_w$ to (the class $p(\mathrm{gr}^W_w)$ of $(\rho_w, \varphi_w))_w$.

2.2.5

For $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$, we define a finite set $\overline{\mathcal{W}}(p)$ of increasing filtrations on $\mathrm{gr}^W = \prod_w \mathrm{gr}^W_w$ as follows. Let $(\rho_w, \varphi_w)_w$ be an SL(2)-orbit on gr^W in n variables which represents p, let $W^{(w,j)}$ $(w \in \mathbf{Z}, 1 \le j \le n)$ be the jth weight filtration on gr^W_w associated to the SL(2)-orbit $(\rho_w, \varphi_w)_w$ on gr^W_w in n variables, and let $W^{(j)} = \bigoplus_w W^{(w,j)}$. Let $\overline{\mathcal{W}}(p) := \{W^{(j)} \mid 1 \le j \le n\}$. Then $\overline{\mathcal{W}}(p)$ is independent of the choice of the representative $(\rho_w, \varphi_w)_w$ of p.

By an *admissible set of weight filtrations on* gr^W [15, II, Section 3.2.2], we mean a set of increasing filtrations on gr^W which coincides with the set $\overline{\mathcal{W}}(p)$ of weight filtrations associated to some point p of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$. An admissible set Φ of weight filtrations on gr^W has a natural structure of a totally ordered set (given by the *variance* of $W'(\mathrm{gr}^W)$ for $W' \in \Phi$; see [15, II, Section 2.1.11, Proposition 2.1.13]). For any $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ of rank n such that $\Phi = \overline{\mathcal{W}}(p)$, if $(W^{(j)})_{1 \le j \le n}$ denotes the family of weight filtrations associated to p, then $W^{(j)} \le W^{(k)}$ for this order if and only if $j \le k$. By using this ordering, we will identify Φ with the totally ordered set $\{1, \dots, n\}$. By this, we will identify $\mathbf{G}_m^{\Phi}$, $\mathbf{Z}^{\Phi}$, and so on with $\mathbf{G}_m^n$, $\mathbf{Z}^n$, and so on.

Let $\overline{\mathcal{W}}$ be the set of all admissible sets of weight filtrations on gr^W. Let $\mathcal{W}(\mathrm{gr}^W_w)$ be the set of all admissible sets of weight filtrations on gr^W_w; that is, $\mathcal{W}(\mathrm{gr}^W_w) = \{\mathcal{W}(p) \mid p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)\}$ (see Section 2.1.6). We have a map

$$\overline{\mathcal{W}} \to \prod_w \mathcal{W}(\mathrm{gr}^W_w), \qquad \Phi \mapsto \big(\Phi(w)\big)_w,$$

where $\Phi(w) := \{W'(\mathrm{gr}^W_w) \mid W' \in \Phi, W'(\mathrm{gr}^W_w) \ne W(\mathrm{gr}^W_w)\}$. This map sends $\overline{\mathcal{W}}(p)$ for $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ to $(\mathcal{W}(p(\mathrm{gr}^W_w)))_w$.

2.2.6

For $\Phi \in \overline{\mathcal{W}}$ and $Q = (Q(w))_w \in \prod_w \mathcal{W}(\mathrm{gr}^W_w)$ such that $\Phi(w) \subset Q(w)$ for any $w \in \mathbf{Z}$, let

$$\mathbf{G}_m^{\Phi} \to \prod_{w \in \mathbf{Z}} \mathbf{G}_m^{Q(w)}$$

be the homomorphism which sends $(t_{W'})_{W' \in \Phi}$ to $(t'_{w,j})_{w \in \mathbf{Z}, j \in Q(w)}$, where $t'_{w,j}$ is the product of $t_{W'}$ for all elements W' of Φ such that $W'(\mathrm{gr}^W_w) = j$. If $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ and $p' = (p(\mathrm{gr}^W_w))_w \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, then for $\Phi = \overline{\mathcal{W}}(p)$ and $Q(w) = \mathcal{W}(p(\mathrm{gr}^W_w))$ $(w \in \mathbf{Z})$, $\tau_p^{\star}$ coincides with the composition $\mathbf{G}_m^{\Phi} \to \prod_w \mathbf{G}_m^{Q(w)} \to G_{\mathbf{R}}(\mathrm{gr}^W)$, where the first arrow is as above and the second arrow is $\tau_{p'}^{\star}$.

2.2.7
Let $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ (resp., $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$). We define objects

$$S_p, X(S_p)^+, A_p, B_p, \bar{A}_p, \bar{B}_p, \tau_p^\star, \tau_p, \tilde{\tau}_p^\star, \tilde{\tau}_p, Z(p)$$

associated to p. Let

$$S_p = \mathbf{G}_m^{\overline{\mathcal{W}}(p)} \quad \Bigl(\text{resp., } \prod_w \mathbf{G}_m^{\mathcal{W}(p_w)}\Bigr).$$

Then the character group $X(S_p)$ of S_p is identified with $\prod_w \mathbf{Z}^{\overline{\mathcal{W}}(p)}$ (resp., $\prod_w \mathbf{Z}^{\mathcal{W}(p_w)}$). We define the submonoid $X(S_p)^+$ of $X(S_p)$ as the part corresponding to $\mathbf{N}^{\overline{\mathcal{W}}(p)}$ (resp., $\prod_w \mathbf{N}^{\mathcal{W}(p_w)}$).

Let A_p be the connected component in $S_p(\mathbf{R})$ which contains the unit element. We identify

$$A_p = \mathrm{Hom}\bigl(X(S_p), \mathbf{R}_{>0}^{\mathrm{mult}}\bigr) = \mathbf{R}_{>0}^{\overline{\mathcal{W}}(p)} \quad \Bigl(\text{resp., } \prod_w \mathbf{R}_{>0}^{\mathcal{W}(p_w)}\Bigr).$$

Let

$$\bar{A}_p = \mathrm{Hom}\bigl(X(S_p)^+, \mathbf{R}_{\geq 0}^{\mathrm{mult}}\bigr) = \mathbf{R}_{\geq 0}^{\overline{\mathcal{W}}(p)} \quad \Bigl(\text{resp., } \prod_w \mathbf{R}_{\geq 0}^{\mathcal{W}(p_w)}\Bigr) \supset A_p,$$

$$\bar{B}_p = \mathbf{R}_{\geq 0} \times \bar{A}_p \supset B_p = \mathbf{R}_{>0} \times A_p.$$

We regard $\bar{A}_p$ and $\bar{B}_p$ as real toric varieties (see Section 1.3.8(3)). We define homomorphisms

$$\tau_p, \tau_p^\star : S_p \to \prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}_w^W)$$

of algebraic groups over $\mathbf{R}$ and a subset $Z(p)$ of $D(\mathrm{gr}^W)$.

Assume first $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$. For an SL(2)-orbit $(\rho_w, \varphi_w)_w$ on gr^W of rank n which represents p, the associated homomorphisms $\tau, \tau^\star : S_p = \mathbf{G}_{m,\mathbf{R}}^{\overline{\mathcal{W}}(p)} = \mathbf{G}_m^n \to \prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}_w^W)$ depend only on p. We denote τ as τ_p and $\tau^\star$ as $\tau_p^\star$. The set

$$\begin{aligned} Z(p) &:= \bigl\{\bigl(\varphi_w(iy_1, \dots, iy_n)\bigr)_w \mid y_j \in \mathbf{R}_{>0}(1 \leq j \leq n)\bigr\} \\ &= \bigl\{\tau_p(t)\bigl(\varphi_w(\mathbf{i})\bigr)_w \mid t \in A_p\bigr\} = \bigl\{\tau_p^\star(t)\bigl(\varphi_w(\mathbf{i})\bigr)_w \mid t \in A_p\bigr\} \\ &\subset D(\mathrm{gr}^W) = \prod_{w \in \mathbf{Z}} D(\mathrm{gr}_w^W) \quad \bigl(\text{where } \mathbf{i} = (i, \dots, i) \in \mathfrak{h}^n\bigr) \end{aligned}$$

depends only on p.

Next for $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, define τ_p and $\tau_p^\star$ as $(\tau_{p_w})_w$ and $(\tau_{p_w}^\star)_w$, respectively, and let $Z(p) = \prod_w Z(p_w)$ (see Section 2.1.6). Both for $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ and for $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, we call $Z(p)$ the *torus orbit* of p. It is an A_p-torsor.

We define extended homomorphisms

$$\tilde{\tau}_p, \tilde{\tau}_p^\star : \mathbf{G}_m \times S_p \to \prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}_w^W),$$

for $t_0 \in \mathbf{G}_m$ and $t \in S_p$, by

$$\tilde{\tau}_p(t_0, t) = (t_0^w)_w \tau_p(t) = \tau_p(t)(t_0^w)_w,$$

$$\tilde{\tau}_p^\star(t_0,t) = (t_0^w)_w \tau_p^\star(t) = \tau_p^\star(t)(t_0^w)_w.$$

Here $(t_0^w)_w$ acts on gr_w^W as the multiplication by t_0^w.

2.2.8

Let $\Phi \in \overline{\mathcal{W}}$. By a *splitting of* Φ [15, II, Section 3.2.3], we mean a homomorphism $\alpha = (\alpha_w)_w : \mathbf{G}_m^\Phi \to \prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}_w^W)$ of algebraic groups over $\mathbf{R}$ such that, for any $W' \in \Phi$ and $k \in \mathbf{Z}$, W'_k coincides with the sum of the parts of gr^W of α-weight m for all $m \in \mathbf{Z}^\Phi$ such that $m(W') \le k$.

For a splitting α of Φ, let $\alpha^\star : \mathbf{G}_m^\Phi \to G_{\mathbf{R}}(\mathrm{gr}^W)$ be the homomorphism whose $G_{\mathbf{R}}(\mathrm{gr}_w^W)$-component $\alpha_w^\star$ is $t = (t_j)_{j\in\Phi} \mapsto (\prod_{j\in\Phi} t_j)^{-w} \cdot \alpha_w(t)$. Note that the actions of $\alpha(t)$ and $\alpha^\star(t)$ $(t \in \mathbf{R}_{>0}^\Phi)$ on $D(\mathrm{gr}^W)$ are the same.

A splitting of Φ exists: if $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ and $\Phi = \overline{\mathcal{W}}(p)$, then τ_p is a splitting of Φ. In this case, for $\alpha = \tau_p$, $\alpha^\star$ in the above coincides with $\tau_p^\star$ in Section 2.2.7.

Let $Q = (Q(w))_w \in \prod_w \mathcal{W}(\mathrm{gr}_w^W)$. By a *splitting of* Q, we mean a family $\alpha = (\alpha_w)_w$, where α_w is a splitting of $Q(w)$. Let $\alpha^\star = (\alpha_w^\star)_w$.

2.2.9

Let $\Phi \in \overline{\mathcal{W}}$. By a *distance to* Φ*-boundary* [15, II, Section 3.2.4], we mean a real analytic map $\beta\colon D(\mathrm{gr}^W) \to \mathbf{R}_{>0}^\Phi$ such that $\beta(\alpha(t)x) = t\beta(x)$ $(t \in \mathbf{R}_{>0}^\Phi, x \in D(\mathrm{gr}^W))$ for any splitting α of Φ. (The last condition is equivalent to $\beta(\alpha^\star(t)x) = t\beta(x)$ $(t \in \mathbf{R}_{>0}^\Phi, x \in D(\mathrm{gr}^W)$.) A distance to Φ-boundary exists [15, II, Proposition 3.2.5].

Let $Q = (Q(w))_w \in \prod_w \mathcal{W}(\mathrm{gr}_w^W)$. By a *distance* to Q-boundary, we mean a family $(\beta_w)_{w\in\mathbf{Z}}$, where β_w is a distance to $Q(w)$-boundary for the pure situation gr_w^W.

2.2.10

In [15, II], we endowed $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ and $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ with structures as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$. These spaces satisfy condition (C) in Section 1.3.16, that is, the sheaf of real analytic functions is a subsheaf of the sheaf of all $\mathbf{R}$-valued continuous functions. $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ is just the product of $D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W)$ (see Section 2.1.7) in $\mathcal{B}'_{\mathbf{R}}(\log)$. The canonical map $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ (see Section 2.2.4) is a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$, and it is a log modification (see Section 1.4.6) as is explained in [15, II, Theorem 3.5.9, Section 3.5.10]. We review some properties of these spaces.

2.2.11

Let $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ (resp., $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$), and let $\mathbf{r} \in Z(p)$ (see Section 2.2.7). Then p is the limit of $\tau_p(t)\mathbf{r} = \tau_p^\star(t)\mathbf{r}$, where $t \in A_p$ tends to $0 \in \bar{A}_p$. Here $0 \in \bar{A}_p$ denotes $(0,\dots,0) \in \mathbf{R}_{\ge 0}^\Phi$, where $\Phi = \overline{\mathcal{W}}(p)$ (resp., $\prod_w \mathbf{R}_{\ge 0}^{Q(w)}$, where $Q(w) = \mathcal{W}(p(\mathrm{gr}_w^W)))$ (see Section 2.2.5) in the identifications $\bar{A}_p = \mathbf{R}_{\ge 0}^\Phi$ (resp., $\prod_w \mathbf{R}_{\ge 0}^{Q(w)}$) $\supset A_p = \mathbf{R}_{>0}^\Phi$ (resp., $\prod_w \mathbf{R}_{>0}^{Q(w)}$).

2.2.12
For $\Phi \in \overline{\mathcal{W}}$, let

$$D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) = \{p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \mid \overline{\mathcal{W}}(p) \subset \Phi\}.$$

For $Q = (Q(w))_{w\in\mathbf{Z}} \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$, let

$$D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) = \{p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W) \mid \mathcal{W}(p_w) \subset Q(w) \text{ for all } w \in \mathbf{Z}\}.$$

Then $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi)$ (resp., $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$) is open in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ (resp., $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$). When Φ (resp., Q) moves, these open sets cover $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ (resp., $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$). If $\Phi \in \overline{\mathcal{W}}$, $Q = (Q(w))_w \in \prod_w \mathcal{W}(\mathrm{gr}^W_w)$, and $\Phi(w) \subset Q(w)$ (see Section 2.2.5) for any $w \in \mathbf{Z}$, then the map $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ induces a map $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$.

2.2.13
Let $\Phi \in \overline{\mathcal{W}}$ (resp., $Q = (Q(w))_w \in \prod_w \mathcal{W}(\mathrm{gr}^W_w)$), and let β be a distance to Φ-boundary (resp., Q-boundary). Then the map β extends uniquely to a morphism

$$\beta : D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) \to \mathbf{R}^{\Phi}_{\geq 0} \quad \Big(\text{resp., } D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q) \to \prod_w \mathbf{R}^{Q(w)}_{\geq 0}\Big)$$

of $\mathcal{B}'_{\mathbf{R}}(\log)$. The log structure with sign of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi)$ (resp., $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$) coincides with the inverse image of the canonical log structure with sign of $\mathbf{R}^{\Phi}_{\geq 0}$ (resp., $\prod_w \mathbf{R}^{Q(w)}_{\geq 0}$) (see Section 1.3.8(1)).

For a distance β to Φ-boundary (resp., Q-boundary), each component β_j ($j \in \Phi$) (resp., $\beta_{w,j}$ ($w \in \mathbf{Z}$, $j \in Q(w)$)) of β is a section of the log structure M_S, where $S = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi)$ (resp., $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$). We have a chart $\mathbf{N}^{\Phi} \to M_S$ (resp., $\prod_w \mathbf{N}^{Q(w)} \to M_S$) defined as $m \mapsto \prod_j \beta_j^{m(j)}$ (resp., $m \mapsto \prod_w \beta_{w,j}^{m(w,j)}$). The induced homomorphism from $\mathbf{N}^{\Phi}$ (resp., $\prod_w \mathbf{N}^{Q(w)}$) to $M_S/\mathcal{O}_S^{\times}$ is independent of the choice of β. If $\Phi = \overline{\mathcal{W}}(p)$ (resp., $Q(w) = \mathcal{W}(p_w)$) for $p \in S$, then this induces an isomorphism from $\mathbf{N}^{\Phi}$ (resp., $\prod_w \mathbf{N}^{Q(w)}$) to $(M_S/\mathcal{O}_S^{\times})_p$.

If $\Phi(w) \subset Q(w)$ (cf. Section 2.2.5) for any $w \in \mathbf{Z}$, we have a commutative diagram

$$\begin{array}{ccccc} \prod_w \mathbf{N}^{Q(w)} & \to & M_S/\mathcal{O}_S^{\times} & & \big(S := D_{\mathrm{SL}(2)}(\mathrm{gr}^W)\big) \\ \downarrow & & \downarrow & & \\ \mathbf{N}^{\Phi} & \to & M_{S'}/\mathcal{O}_{S'}^{\times} & & \big(S' := D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}\big) \end{array}$$

where the left vertical arrow is the homomorphism induced from the homomorphism $\mathbf{G}_m^{\Phi} \to \prod_w \mathbf{G}_m^{Q(w)}$ (see Section 2.2.6) on the character groups.

2.2.14
Let $\Phi \in \overline{\mathcal{W}}$ (resp., $Q \in \prod_w \mathcal{W}(\mathrm{gr}^W_w)$), let α be a splitting of Φ (resp., Q), and let β be a distance to Φ-boundary (resp., Q-boundary). Then the map $D(\mathrm{gr}^W) \to D(\mathrm{gr}^W)$, $x \mapsto \alpha(\beta(x))^{-1}x = \alpha^{\star}(\beta(x))^{-1}x$ extends uniquely to a morphism (see [15, II, Proposition 3.2.6])

$$b_{\alpha,\beta} : D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}\big(\text{resp., } D_{\mathrm{SL}(2)}(\mathrm{gr}^W)\big) \to D(\mathrm{gr}^W).$$

2.3. The space $D^\star_{\mathrm{SL}(2)}$

We define the space $D^\star_{\mathrm{SL}(2)}$, comparing it with the space $D^{II}_{\mathrm{SL}(2)}$, which we defined in [15, II]. We also define a related space $D^{\star,-}_{\mathrm{SL}(2)}$.

2.3.1

Let $D^\star_{\mathrm{SL}(2)}$ (resp., $D^{\star,-}_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2)}$) be the set of all pairs (p, Z), where $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ (resp., $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$) and Z is a subset of D satisfying the following two conditions (i) and (ii). Denote $\tau^\star_p$ (resp., $\tau^\star_p$, τ_p) by α_p, and denote $\tilde{\tau}^\star_p$ (resp., $\tilde{\tau}^\star_p$, $\tilde{\tau}_p$) by $\tilde{\alpha}_p$.

(i) Z is either
(i.A) an $\alpha_p(A_p)$-orbit in D or
(i.B) an $\tilde{\alpha}_p(B_p)$-orbit in D_{nspl} (see Section 1.1.5)
for the lifted action (see Section 1.2.6).

(ii) The image of Z in $D(\mathrm{gr}^W)$ coincides with the torus orbit $Z(p)$ (see Section 2.2.7) of p.

We call an element (p, Z) an *A-orbit* if it satisfies (i.A) and a *B-orbit* if it satisfies (i.B). This is similar to the case of D_{BS}, which also consists of A_P-orbits and B_P-orbits for $\mathbf{Q}$-parabolic subgroups P of $G_{\mathbf{R}}(\mathrm{gr}^W)$ (see [15, I, Section 5.1, Definition 5.3]).

2.3.2

We embed D in $D^\star_{\mathrm{SL}(2)}$ (resp., $D^{\star,-}_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2)}$) by $F \mapsto (F(\mathrm{gr}^W), \{F\})$. We have canonical maps

$$D^\star_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim, \qquad D^{\star,-}_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W),$$
$$D_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$$

defined by $(p, Z) \mapsto p$. We have a canonical map

$$D^\star_{\mathrm{SL}(2)} \to D^{\star,-}_{\mathrm{SL}(2)}, \qquad (p, Z) \mapsto (p', Z'), \qquad p' := \big(p(\mathrm{gr}^W_w)\big)_w, \qquad Z' := \tau^\star_{p'}(A_{p'})Z.$$

2.3.3

The style of the definition of the set $D_{\mathrm{SL}(2)}$ in Section 2.3.1 is slightly different from the one in [15, II, Section 2.5]. We explain the relation between the two styles. Let $(p, Z) \in D_{\mathrm{SL}(2)}$ in the present style, and let $(\rho_w, \varphi_w)_w$ be an SL(2)-orbit on gr^W which represents p. If (p, Z) is an A-orbit (see Section 2.3.1), then it is the class of $((\rho_w, \varphi_w)_w, \mathbf{r}) \in \mathcal{D}'_{\mathrm{SL}(2),n}$ in [15, II, Section 2.3.1] with $\mathbf{r} \in Z$. If (p, Z) is a B-orbit (see Section 2.3.1), then it is the class of $((\rho'_w, \varphi'_w)_w, \mathbf{r}) \in \mathcal{D}'_{\mathrm{SL}(2),n+1}$ in [15, II, Section 2.3.1], where $\mathbf{r} \in Z$ and ρ'_w (resp., φ'_w) is the composition $\mathrm{SL}(2, \mathbf{R})^{n+1} \to \mathrm{SL}(2, \mathbf{R})^n \to G_{\mathbf{R}}(\mathrm{gr}^W_w)$ (resp., $\mathbf{P}^1(\mathbf{C})^{n+1} \to \mathbf{P}^1(\mathbf{C})^n \to D(\mathrm{gr}^W_w)$) of the projection to the last n factors and ρ_w (resp., φ_w).

2.3.4
Let $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ (resp., $D^{\star,-,\mathrm{mild}}_{\mathrm{SL}(2)}$) be the subset of $D^{\star}_{\mathrm{SL}(2)}$ (resp., $D^{\star,-}_{\mathrm{SL}(2)}$) consisting of all A-orbits. (We do not define the mild part of $D_{\mathrm{SL}(2)}$. The part of A-orbits in $D_{\mathrm{SL}(2)}$ does not fit our formulation of the mild part.)

2.3.5
Consider the following three situations (a)–(c).

(a) $\mathfrak{D} = D^{\star}_{\mathrm{SL}(2)}$, $\mathfrak{E} = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$.
(b) $\mathfrak{D} = D^{\star,-}_{\mathrm{SL}(2)}$, $\mathfrak{E} = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$.
(c) $\mathfrak{D} = D_{\mathrm{SL}(2)}$, $\mathfrak{E} = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$.

We endow $\mathfrak{D}$ with a structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ as follows in Sections 2.3.6–2.3.11. In situation (c), this coincides with the structure $D^{II}_{\mathrm{SL}(2)}$ treated in [15, II].

2.3.6
In situations (a) and (c) (resp., situation (b)) in Section 2.3.5, for $\Phi \in \overline{\mathcal{W}}$ (resp., $Q \in \prod_w \mathcal{W}(\mathrm{gr}^W_w))$, let $\mathfrak{D}(\Phi)$ (resp., $\mathfrak{D}(Q)$) be the inverse image of $\mathfrak{E}(\Phi)$ (resp., $\mathfrak{E}(Q)$) (see Section 2.2.12) in $\mathfrak{D}$.

2.3.7
In situations (a)–(c) in Section 2.3.5, for $x = (p, Z) \in \mathfrak{D}$, $\mathrm{spl}_W(\mathbf{r})$ for $\mathbf{r} \in Z$ is independent of the choice of $\mathbf{r}$. We denote this $\mathrm{spl}_W(\mathbf{r})$ ($\mathbf{r} \in Z$) by $\mathrm{spl}_W(x)$.

2.3.8
In situations (a) and (c) (resp., situation (b)) in Section 2.3.5, let $\Phi \in \overline{\mathcal{W}}$ (resp., $Q = (Q(w))_w \in \prod_w \mathcal{W}(\mathrm{gr}^W_w))$, let α be a splitting of Φ (resp., Q) (see Section 2.2.8), and let β be a distance to Φ-boundary (resp., Q-boundary) (see Section 2.2.9).

In situations (a) and (b) (resp., situation (c)), for $x \in D$, let $\delta_{\alpha,\beta}(x) \in \mathcal{L}$ (see Section 1.2.2) be $\mathrm{Ad}(\alpha^{\star}(\beta(p)))^{-1}\delta_W(x)$ (resp., $\mathrm{Ad}(\alpha(\beta(p)))^{-1}\delta_W(x)$), where p denotes the image of x in $D(\mathrm{gr}^W)$ (for $\alpha^{\star}$, see Section 2.2.8). Let $\mathfrak{D}' = \mathfrak{D}(\Phi)$ (resp., $\mathfrak{D}(Q)$). Then, for $x = (p, Z) \in \mathfrak{D}'$ and $\mathbf{r} \in Z$, $\delta_{\alpha,\beta}(\tau^{\star}_p(t)\mathbf{r})$ (resp., $\delta_{\alpha,\beta}(\tau_p(t)\mathbf{r})$) converges in $\bar{\mathcal{L}}$ (see Section 1.3.8(4)) when $t \in A_p$ tends to 0 in $\bar{A}_p$, and the limit depends only on x and is independent of the choice of $\mathbf{r}$. We denote this limit by $\delta_{\alpha,\beta}(x)$. We have $\delta_{\alpha,\beta}(x) \in \bar{\mathcal{L}}(b_{\alpha,\beta}(p))$, where $b_{\alpha,\beta}(p)$ is as in Section 2.2.14.

These $\delta_{\alpha,\beta}(x)$ and $b_{\alpha,\beta}(p)$ ($x = (p, Z)$) are described as follows. In situations (a) and (c) (resp., situation (b)), let α' and $(\alpha^{\star})'$ be the restrictions of α and $\alpha^{\star}$ (see Section 2.2.8) to the subgroup $\mathbf{G}_m^{\overline{\mathcal{W}}(p)}$ (resp., $\prod_w \mathbf{G}_m^{\mathcal{W}(p_w)}$) of $\mathbf{G}_m^{\Phi}$ (resp., $\prod_w \mathbf{G}_m^{Q(w)}$), respectively. Since both α' and τ_p split $\overline{\mathcal{W}}(p)$ (resp., $(\mathcal{W}(p_w))_w$), there is $u \in \prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ such that, for all $W' \in \overline{\mathcal{W}}(p)$ (resp., for all $w \in \mathbf{Z}$ and all $W' \in \mathcal{W}(p_w)$), u preserves W' and induces the identity maps on $\mathrm{gr}^{W'}$ and such that

$$\tau_p(t) = u\alpha'(t)u^{-1}, \qquad \tau_p^\star(t) = u(\alpha^\star)'(t)u^{-1}$$

for any $t \in \mathbf{G}_m^{\overline{\mathcal{W}}(p)}$ (resp., $\prod_w \mathbf{G}_m^{\mathcal{W}(p_w)}$). Take $\mathbf{r} \in Z$, and let $\bar{\mathbf{r}}$ be the image of $\mathbf{r}$ in $Z(p)$ (see Section 2.2.7). Then we have

$$b_{\alpha,\beta}(p) = b_{\alpha,\beta}(u^{-1}\bar{\mathbf{r}}),$$

$$\delta_{\alpha,\beta}(x) = \mathrm{Ad}\big(u\alpha^\star\big(\beta(u^{-1}\bar{\mathbf{r}})\big)\big)^{-1}\delta_W(\mathbf{r}) \quad \big(\text{resp., } \mathrm{Ad}\big(u\alpha\big(\beta(u^{-1}\mathbf{r})\big)\big)^{-1}\delta_W(\mathbf{r})\big).$$

These are shown in [15, II, Section 3.3.9] in situation (c). The proofs for situations (a) and (b) are similar.

PROPOSITION 2.3.9

Consider the three situations in Section 2.3.5. In situations (a) and (c), let $\Phi \in \overline{\mathcal{W}}$, $\mathfrak{D}' = \mathfrak{D}(\Phi)$, and $\mathfrak{E}' = \mathfrak{E}(\Phi)$. In situation (b), let $Q \in \prod_w \mathcal{W}(\mathrm{gr}_w^W)$, $\mathfrak{D}' = \mathfrak{D}(Q)$, and $\mathfrak{E}' = \mathfrak{E}(Q)$. In situations (a) and (c) (resp., situation (b)), fix a splitting α of Φ (resp., Q) and a distance β to Φ-boundary (resp., Q-boundary). Then we have a bijection

$$\nu : \mathfrak{D}' \to \big\{(p, s, \delta) \in \mathfrak{E}' \times \mathrm{spl}(W) \times \bar{\mathcal{L}} \mid \delta \in \bar{\mathcal{L}}\big(b_{\alpha,\beta}(p)\big)\big\}$$

(where $\bar{\mathcal{L}}(\cdot)$ is as in Section 1.3.8(4)) defined as $x \mapsto (p, s, \delta)$, where p is the image of x in $\mathfrak{E}$, $s = \mathrm{spl}_W(x)$ (see Section 2.3.7), and $\delta = \delta_{\alpha,\beta}(x)$ (see Section 2.3.8).

Proof

The inverse map of ν is defined as $(p, s, \delta) \mapsto (p, Z)$, where Z is as follows. Consider situations (a) and (c) (resp., situation (b)). Take $u \in G_{\mathbf{R}}(\mathrm{gr}^W)$ for p as in Section 2.3.8.

In the case $\delta \in \mathcal{L} \subset \bar{\mathcal{L}}$, Z is the subset of D whose image in $D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ under the map in Section 1.2.1 is

$$\big\{\big(\mathbf{r}, s, \mathrm{Ad}\big(u\alpha^\star\big(\beta(u^{-1}\mathbf{r})\big)\big)\delta\big) \mid \mathbf{r} \in Z(p)\big\}$$

$$\big(\text{resp., } \big\{\big(\mathbf{r}, s, \mathrm{Ad}\big(u\alpha\big(\beta(u^{-1}\mathbf{r})\big)\big)\delta\big) \mid \mathbf{r} \in Z(p)\big\}\big).$$

In the case $\delta = 0 \circ \delta' \in \bar{\mathcal{L}} \smallsetminus \mathcal{L}$ with $\delta' \in \mathcal{L} \smallsetminus \{0\}$ (see Section 1.3.8(4)), Z is $\mathbf{R}_{>0} \circ Z'$, where Z' is the above set Z for (p, s, δ'). □

PROPOSITION 2.3.10

Let the three situations be as in Section 2.3.5.

(1) *In situations (a) and (c) (resp., situation (b)), endow $\mathfrak{D}(\Phi)$ (resp., $\mathfrak{D}(Q)$) (see Section 2.3.6) with a structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by using the bijection ν in Proposition 2.3.9. (The target of ν is regarded as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by regarding it as Y in $X = \mathfrak{E}' \times \mathrm{spl}(W) \times \bar{\mathcal{L}}$ in Section 1.3.16.) Then this structure is independent of the choice of (α, β).*

(2) *There is a unique structure on $\mathfrak{D}$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ such that, for any $\Phi \in \overline{\mathcal{W}}$ (resp., $Q \in \prod_w \mathcal{W}(\mathrm{gr}_w^W)$), $\mathfrak{D}' := \mathfrak{D}(\Phi)$ (resp., $\mathfrak{D}' := \mathfrak{D}(Q)$) is an open*

subset and the restriction of this structure to $\mathfrak{D}'$ coincides with the structure given in (1).

Proof
For situation (c), this follows from [15, II, Proposition 3.2.9, Theorem 3.2.10]. The proofs for situations (a) and (b) are similar. □

2.3.11
The structures of $D^{\star}_{\mathrm{SL}(2)}$, $D^{\star,-}_{\mathrm{SL}(2)}$, and $D^{II}_{\mathrm{SL}(2)}$ as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$ are given by situations (a), (b), and (c) in Proposition 2.3.10, respectively. In situations (a)–(c) in Section 2.3.5, the canonical map $\mathfrak{D} \to \mathfrak{E}$ is evidently a morphism of $\mathcal{B}'_{\mathbf{R}}(\log)$.

2.3.12
Via the bijection ν of Proposition 2.3.9, A-orbits in $\mathfrak{D}'$ correspond to elements (p, s, δ) of the target of ν such that $\delta \in \mathcal{L}$. Hence, the subset of $\mathfrak{D}$ consisting of all A-orbits is open in $\mathfrak{D}$. Elements (p, Z) of $\mathfrak{D}'$ such that $Z \subset D_{\mathrm{spl}}$ (see Section 1.1.5) correspond to elements (p, s, δ) of the target of ν such that $\delta = 0$.

2.3.13
Consider situations (a)–(c) in Section 2.3.5. In situation (c), we consider the structure $D^{II}_{\mathrm{SL}(2)}$ of $D_{\mathrm{SL}(2)}$.

In Theorem 2.3.14 below, we extend the result [15, II, Theorem 3.4.4] on the local structure of $D^{II}_{\mathrm{SL}(2)}$ to all situations in Section 2.3.5. This section is a preparation for it.

Let $p \in \mathfrak{E}$. We consider the local structure of $\mathfrak{D}$ around the inverse image of p in $\mathfrak{D}$.

Consider situations (a) and (c) (resp., situation (b)). Let $\Phi := \overline{\mathcal{W}}(p)$ (resp., $Q = (Q(w))_w$ with $Q(w) := \mathcal{W}(p_w)$). Fix $\mathbf{r} \in Z(p)$.

Let $K_{\mathbf{r}}$ be the maximal compact subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ associated to $\mathbf{r}$ [15, II, Section 3.4.1], and let $K'_{\mathbf{r}} \subset K_{\mathbf{r}}$ be the isotropy subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ at $\mathbf{r}$. We use the notation in Section 2.2.7. Let R be an $\mathbf{R}$-subspace of $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ satisfying the following conditions (C1) and (C2).

(C1) $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) = \mathrm{Lie}(\tau^{\star}(A_p)) \oplus R \oplus \mathrm{Lie}(K_{\mathbf{r}})$.

(C2) $R = \sum_{m \in X(S_p)} R \cap ((\mathfrak{g}_{\mathbf{R}})_m + (\mathfrak{g}_{\mathbf{R}})_{-m})$. Here $(\cdot)_m$ denotes the part of weight m for the adjoint action of S_p via $\tau_p^{\star}$. (The definition of the part $(\cdot)_m$ does not change if we replace $\tau_p^{\star}$ by τ_p.)

Let S be an $\mathbf{R}$-subspace of $\mathrm{Lie}(K_{\mathbf{r}})$ such that $\mathrm{Lie}(K_{\mathbf{r}}) = \mathrm{Lie}(K'_{\mathbf{r}}) \oplus S$.

For a subset J of Φ (resp., for $J = (J(w))_{w \in \mathbf{Z}}$, $J(w) \subset Q(w)$), let S_J be the subset of S consisting of all elements k such that $\exp(k)\mathbf{r} \in (K_{\mathbf{r}} \cap G_{\mathbf{R},J}(\mathrm{gr}^W)) \cdot \mathbf{r}$, where $G_{\mathbf{R},J}(\mathrm{gr}^W)$ is the subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ consisting of all $g \in G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $gW' = W'$ for any $W' \in J$ (resp., for any $w \in \mathbf{Z}$ and any $W' \in J(w)$).

We define an object Y of $\mathcal{B}'_{\mathbf{R}}(\log)$ as follows. Let

$$X = \bar{A}_p \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S.$$

Note that $\bar{A}_p$ is $\mathbf{R}^{\Phi}_{\geq 0}$ (resp., $\prod_w \mathbf{R}^{Q(w)}_{\geq 0}$) (Section 2.2.7).

Let Y be the subset of X consisting of all elements (t,f,g,h,k) satisfying the following conditions (i)–(iv). In (ii) and (iv) below, let $J = \{j \in \Phi \mid t_j = 0\}$ (resp., $J = (J(w))_{w \in \mathbf{Z}}$ with $J(w) = \{j \in Q(w) \mid t_{w,j} = 0\}$).

For $\chi \in X(S_p)$, write $\chi = \chi_+ (\chi_-)^{-1}$ with $\chi_+, \chi_- \in X(S_p)^+$, which are defined as follows. In the identification $X(S_p) = \prod_w \mathbf{Z}^{Q(w)}$, if we denote by $m(w,j) \in \mathbf{Z}$ the (w,j)-component of χ for $w \in \mathbf{Z}$ and $j \in Q(w)$, then the (w,j)-component of χ_+ is $\max(m(w,j),0)$ and the (w,j)-component of χ_- is $\max(-m(w,j),0)$.

(i) For any $\chi \in X(S_p)$, $t(\chi_+) g_\chi = t(\chi_-) f_\chi$ and $t(\chi_+) h_\chi = t(\chi_-) g_\chi$. Here f_χ, g_χ, h_χ denote the χ-component for the adjoint action of S_p via $\tau_p^\star$, and $t(\chi_+), t(\chi_-) \in \mathbf{R}_{\geq 0}$ are defined by the understanding $\bar{A}_p = \mathrm{Hom}(X(S_p)^+, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$.

(ii) Let $\chi \in X(S_p)$. If $t(\chi_+) = 0$, then $g_\chi = f_\chi = 0$. If $t(\chi_-) = 0$, then $g_\chi = h_\chi = 0$. In other words, if $m(j) \in \mathbf{Z}$ for $j \in \Phi$ (resp., $m(w,j) \in \mathbf{Z}$ for $w \in \mathbf{Z}$ and $j \in Q(w)$) denotes the j-component (resp., (w,j)-component) of χ in the identification $X(S_p) = \mathbf{Z}^\Phi$ (resp., $\prod_w \mathbf{Z}^{Q(w)}$), then $f_\chi = 0$ unless $m(j) \leq 0$ for any $j \in J$ (resp., unless $m(w,j) \leq 0$ for any $w \in \mathbf{Z}$ and $j \in J(w)$), $g_m = 0$ unless $m(j) = 0$ for any $j \in J$ (resp., unless $m(w,j) = 0$ for any $w \in \mathbf{Z}$ and $j \in J(w)$), and $h_m = 0$ unless $m(j) \geq 0$ for any $j \in J$ (resp., unless $m(w,j) \geq 0$ for any $w \in \mathbf{Z}$ and $j \in J(w)$).

(iii) $g_\chi \in R$ and $f_\chi + h_{\chi^{-1}} \in R$ for any $\chi \in X(S_p)$.

(iv) $k \in S_J$.

Regard X as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ in the natural way, and regard $Y \subset X$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by Section 1.3.16. Let

$$Y_0 = \{(t,f,g,h,k) \in Y \mid t \in A_p\} \subset Y.$$

THEOREM 2.3.14

Consider the three situations in Section 2.3.5. Let the notation be as in Section 2.3.13.

(1) *For a sufficiently small open neighborhood U of $(0,0,0,0,0)$ in Y, there exists a unique open immersion $U \to \mathfrak{E}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$ which sends $(t,f,g,h,k) \in U \cap Y_0$ to*

$$\exp(f)\tau_p^\star(t)\exp(k)\mathbf{r} = \exp(f)\tau_p(t)\exp(k)\mathbf{r}$$

of $D(\mathrm{gr}^W) \subset \mathfrak{E}$. This morphism sends $(0,\dots,0) \in Y$ to p.

(2) *Let $\bar{L} = \bar{\mathcal{L}}(\mathbf{r})$, $L = \mathcal{L}(\mathbf{r})$. Then for a sufficiently small open neighborhood U of $(0,0,0,0,0)$ in Y, there exists a unique open immersion $U \times \mathrm{spl}(W) \times \bar{L} \to \mathfrak{D}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$ having the following property. In situations (a) and (b) (resp., situation (c)), it sends $(t,f,g,h,k,s,\delta) \in Y \times \mathrm{spl}(W) \times L$, where $(t,f,g,h,k) \in U \cap Y_0$, $s \in \mathrm{spl}(W)$, and $\delta \in L$, to the element of D whose image in $D(\mathrm{gr}^W) \times$*

$\mathrm{spl}(W) \times \mathcal{L}$ *under the isomorphism from Section 1.2.1 is*

$$\big(\exp(f)\tau_p^{\star}(t)\exp(k)\mathbf{r}, s, \mathrm{Ad}\big(\exp(f)\tau_p^{\star}(t)\exp(k)\big)\delta\big)$$
$$\big(\textit{resp.}, \big(\exp(f)\tau_p(t)\exp(k)\mathbf{r}, s, \mathrm{Ad}\big(\exp(f)\tau_p(t)\exp(k)\big)\delta\big)\big).$$

(3) *For a sufficiently small open neighborhood U of $(0,0,0,0,0)$ in Y, the diagram*

$$\begin{array}{ccc} U \times \mathrm{spl}(W) \times \bar{L} & \to & \mathfrak{D} \\ \downarrow & & \downarrow \\ U & \to & \mathfrak{E} \end{array}$$

is Cartesian in $\mathcal{B}'_{\mathbf{R}}(\log)$ and in the category of topological spaces.

(4) *In situations (a) and (c) (resp., situation (b)), the image of the map in (1) is contained in $\mathfrak{E}(\Phi)$ (resp., $\mathfrak{E}(Q)$) and the image of the map in (2) is contained in $\mathfrak{D}(\Phi)$ (resp., $\mathfrak{D}(Q)$), where $\Phi = \overline{\mathcal{W}}(p)$ (resp., $Q = (\mathcal{W}(p_w))_w$).*

(5) *The underlying maps of the morphisms in (1) and (2) are described in Section 2.3.15 below.*

Proof
In situation (c), this is given in [15, II, Theorem 3.4.4, Section 3.4.12]. The proofs for situations (a) and (b) are similar. □

2.3.15
The maps in Theorem 2.3.14(1) and 2.3.14(2) are induced from the maps

$$Y \to \mathfrak{E}, \qquad Y \times \mathrm{spl}(W) \times \bar{L} \to \mathfrak{D},$$

respectively, defined as follows.

The first map sends $(t, f, g, h, k) \in Y$ to the following element $p' \in \mathfrak{E}$. Assume we are in situation (a) or (c) (resp., situation (b)). Let $J = \{j \in \Phi \mid t_j = 0\}$ (resp., $J = (J(w))_{w \in \mathbf{Z}}$, where $J(w) = \{j \in Q(w) \mid t_{w,j} = 0\}$). Define $p_J \in \mathfrak{E}$ as follows. Let $n = \sharp(\Phi)$ (resp., $n(w) = \sharp(Q(w))$ for $w \in \mathbf{Z}$). Let (ρ, φ) be the SL(2)-orbit on gr^W which represents p (resp., (ρ_w, φ_w) for $w \in \mathbf{Z}$ be the SL(2)-orbit on gr^W_w in $n(w)$ variables which represents p_w) such that $\mathbf{r} = \varphi(i, \dots, i)$ (resp., $\mathbf{r}_w = \varphi_w(i, \dots, i)$). Write $J = \{j_1, \dots, j_m\}$, $j_1 < \dots < j_m$ (resp., $J(w) = \{j_{w,1}, \dots, j_{w,m(w)}\}$, $j_1 < \dots < j_{m(w)}$). Then p_J is the class of the following SL(2)-orbit (ρ', φ') on gr^W of rank m (resp., the family $(\rho'_w, \varphi'_w)_w$ of SL(2)-orbits in $m(w)$ variables):

$$\rho'(g_1, \dots, g_m) = \rho(g'_1, \dots, g'_n), \varphi'(z_1, \dots, z_m) = \varphi(z'_1, \dots, z'_{n(w)})$$
$$\big(\mathrm{resp.}, \rho'_w(g_1, \dots, g_{m(w)}) = \rho_w(g'_1, \dots, g'_{n(w)}), \varphi'_w(z_1, \dots, z_{m(w)})$$
$$= \varphi_w(z'_1, \dots, z'_{n(w)})\big).$$

Here $g'_j = g_k$ and $z'_j = z_k$, where k is the smallest among integers a such that $1 \le a \le m$ (resp., $1 \le a \le m(w)$) and $j \le j_a$ if such an a exists, and $g'_j = 1$ and $z'_j = i$ if such an a does not exist.

Let A' be the set of all elements t' of A_p such that $t'_j = t_j$ for any $j \in \Phi \smallsetminus J$ (resp., $t'_{w,j} = t_{w,j}$ for any $w \in \mathbf{Z}$ and any $j \in Q(w) \smallsetminus J(w)$). Then

$$p' = \exp(f)\tau_p(t')\exp(k)p_J$$

with $t' \in A'$. This p' is independent of the choice of $t' \in A'$.

Next the second map $Y \times \mathrm{spl}(W) \times \bar{L} \to \mathfrak{D}$ sends $(t, f, g, h, k, s, \delta)$ to $(p', Z) \in \mathfrak{D}$, where p' is as above and $Z \subset D$ is as follows. Consider situations (a) and (b) (resp., situation (c)). If $\delta \in L \subset \bar{L}$, Z is the subset of D whose image under the embedding $D \to D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \bar{\mathcal{L}}$ in Section 1.2.1 is the set of elements

$$\big(\exp(f)\tau_p^\star(t')\exp(k)\mathbf{r}, s, \mathrm{Ad}\big(\exp(f)\tau_p^\star(t')\exp(k)\big)\delta\big)$$
$$\big(\text{resp., } \big(\exp(f)\tau_p(t')\exp(k)\mathbf{r}, s, \mathrm{Ad}\big(\exp(f)\tau_p(t')\exp(k)\big)\delta\big)\big),$$

where t' ranges over all elements of A'. If $\delta \in \bar{L} \smallsetminus L$ and $\delta = 0 \circ \delta^{(1)}$ for $\delta^{(1)} \in L \smallsetminus \{0\}$ (see Section 1.3.8(4)), Z is the subset of D whose image under the embedding $D \to D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \bar{\mathcal{L}}$ in Section 1.2.1 is the set of elements

$$\big(\exp(f)\tau_p^\star(t')\exp(k)\mathbf{r}, s, \mathrm{Ad}\big(\exp(f)\tau_p^\star(t')\exp(k)\big)(c \circ \delta^{(1)})\big)$$
$$\big(\text{resp., } \big(\exp(f)\tau_p(t')\exp(k)\mathbf{r}, s, \mathrm{Ad}\big(\exp(f)\tau_p(t')\exp(k)\big)(c \circ \delta^{(1)})\big)\big),$$

where t' ranges over all elements of A' and c ranges over all elements of $\mathbf{R}_{>0}$.

The part for $D_{\mathrm{SL}(2)}$ of the following proposition is [15, II, Theorem 3.5.15].

PROPOSITION 2.3.16

Consider the situations in Section 2.3.5. Fix any $F \in D(\mathrm{gr}^W)$, and let $\bar{L} = \bar{\mathcal{L}}(F)$ (see Section 1.3.8(4)). Then $\mathfrak{D}$ is an $\bar{L}$-bundle over $\mathfrak{E} \times \mathrm{spl}(W)$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. Consequently, the map $\mathfrak{D} \to \mathfrak{E} \times \mathrm{spl}(W)$ is proper.

Proof

This follows from Theorem 2.3.14. □

Note that $\bar{\mathcal{L}}(F)$ for all $F \in D(\mathrm{gr}^W)$ are isomorphic to each other as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$.

PROPOSITION 2.3.17

The map $D^\star_{\mathrm{SL}(2)} \to D^{\star,-}_{\mathrm{SL}(2)}$ (see Section 2.3.2) is a morphism of $\mathcal{B}'_{\mathbf{R}}(\log)$. The following diagram is Cartesian in $\mathcal{B}'_{\mathbf{R}}(\log)$ and also Cartesian in the category of topological spaces:

$$\begin{array}{ccc} D^\star_{\mathrm{SL}(2)} & \to & D^{\star,-}_{\mathrm{SL}(2)} \\ \downarrow & & \downarrow \\ D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim & \to & D_{\mathrm{SL}(2)}(\mathrm{gr}^W). \end{array}$$

Proof

We deduce this from Theorem 2.3.14. Let $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$, and let p' be the image of p in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$. Take R and S for situation (b) in Section 2.3.5 as in Section 2.3.13 by using p' as p in Section 2.3.13, and write this R as R'. Let C be an $\mathbf{R}$-subspace of $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ such that $\mathrm{Lie}(\tau^\star_{p'}(A_{p'}))$ is the direct sum of $\mathrm{Lie}(\tau^\star_p(A_p))$ and C. Let $R = C \oplus R'$. Then R and S satisfy the conditions on R and S in Section 2.3.13 for situation (a) in Section 2.3.5 and for p. The homomorphism $S_p \to S_{p'}$ (see Section 2.2.6) induces a homomorphism $X(S_{p'})^+ \to X(S_p)^+$ and, hence, a morphism $\bar{A}_p = \mathrm{Hom}(X(S_p)^+, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \to \bar{A}_{p'} = \mathrm{Hom}(X(S_{p'})^+, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$. Let Y be the Y in Section 2.3.13 defined by (p,R,S) for situation (a) in Section 2.3.5, and let Y' be the Y in Section 2.3.13 defined by (p',R',S) for situation (b) in Section 2.3.5.

For $(t,f,g,h,k) \in Y$, since $g \in R = C \oplus R'$, we can write $g = c + g'$ with $c \in C$ and $g' \in R'$ in a unique way, and we have $(t',f',g',h',k) \in Y'$, where t' is the image of t in $\bar{A}_{p'}$, $f' = f - c$, and $h' = h - c$. We have a morphism $Y \to Y'$ which sends $(t,f,g,h,k) \in Y$ to $(t't'',f',g',h',k) \in Y'$, where t'' is the unique element of $A_{p'}$ such that $\tau^\star_{p'}(t'') = \exp(c)$. For a sufficiently small open neighborhood U of $(0,0,0,0,0)$ in Y and for a sufficiently small open neighborhood U' of $(0,0,0,0,0)$ in Y' such that the image of U in Y' is contained in U', we have commutative diagrams

$$\begin{array}{ccc} U & \to & \mathfrak{E} \\ \downarrow & & \downarrow \\ U' & \to & \mathfrak{E}' \end{array} \qquad \begin{array}{ccc} U \times \mathrm{spl}(W) \times \bar{L} & \to & \mathfrak{D} \\ \downarrow & & \downarrow \\ U' \times \mathrm{spl}(W) \times \bar{L} & \to & \mathfrak{D}' \end{array}$$

where $\mathfrak{E} = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$, $\mathfrak{E}' = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, $\mathfrak{D} = D^\star_{\mathrm{SL}(2)}$, and $\mathfrak{D}' = D^{\star,-}_{\mathrm{SL}(2)}$. This reduces Proposition 2.3.17 to Theorem 2.3.14. $\square$

2.4. Basic facts on $\mathrm{SL}(2)$-orbits and Borel–Serre orbits

This section is a preparation for the rest of Section 2. In Sections 2.4.1–2.4.3, we review the space D_{BS} defined and studied in [15, I], and then in Sections 2.4.5–2.4.10 we give some basic facts about the spaces $D^\star_{\mathrm{SL}(2)}$, $D^{\star,-}_{\mathrm{SL}(2)}$, $D^{II}_{\mathrm{SL}(2)}$, and D_{BS}.

2.4.1

We review the definition of the set D_{BS} shortly (see [15, I] for details).

Parabolic subgroups play central roles in the theory of Borel–Serre spaces. Following [6], for a linear algebraic group Z over a field, we call an algebraic subgroup P of Z a *parabolic subgroup* if it is geometrically connected and Z/P is a projective variety.

In our setting, there are bijections

$$\begin{aligned} &\{\mathbf{Q}\text{-parabolic subgroup of } G\} \\ &\quad \leftrightarrow \{\mathbf{Q}\text{-parabolic subgroup of } G(\mathrm{gr}^W)\} \\ &\quad \leftrightarrow \{\text{family } (P_w)_{w\in\mathbf{Z}} \text{ of } \mathbf{Q}\text{-parabolic subgroups } P_w \text{ of } G(\mathrm{gr}^W_w)\}. \end{aligned}$$

The bijection from the last set to the second set is given by $(P_w)_w \mapsto \prod_w P_w$, and the bijection from the second set to the first set is given by taking the inverse image under $G_{\mathbf{R}} \to G_{\mathbf{R}}(\mathrm{gr}^W)$.

Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$. Let P_u be the unipotent radical of P, let S_P be the largest $\mathbf{Q}$-split torus in the center of P/P_u, and let A_P (resp., B_P) be the connected component including 1 of the topological group $S_P(\mathbf{R})$ (resp., $(\mathbf{G}_m \times S_P)(\mathbf{R})$).

For each $p \in D(\mathrm{gr}^W)$, we have a canonical homomorphism $S_P \to P$ of algebraic groups over $\mathbf{R}$ such that the composition $S_P \to P \to P/P_u$ is the identify map, which we call the *Borel–Serre lifting at* p and denote by $t \mapsto t_p$. This t_p is characterized by the following two properties.

(i) The image of t_p in P/P_u coincides with t.

(ii) $\theta_{K_p}(t_p) = t_p^{-1}$, where $\theta_{K_p} : G_{\mathbf{R}}(\mathrm{gr}^W) \to G_{\mathbf{R}}(\mathrm{gr}^W)$ denotes the Cartan involution associated to the maximal compact subgroup K_p (cf. [15, I, 2.1]) of $G_{\mathbf{R}}(\mathrm{gr}^W)$ associated to p.

We have the following action of B_P on D, which we call the Borel–Serre action and denote as $(b, F) \mapsto b \circ F$ $(b \in B_P,\ F \in D)$. For $b = (c, a) \in B_P$ with $c \in \mathbf{R}_{>0}$ and $a \in A_P$, we define $b \circ F := (c^w)_w a_{F(\mathrm{gr}^W)} F$, where $a_{F(\mathrm{gr}^W)}$ is the Borel–Serre lifting of a at $F(\mathrm{gr}^W)$, $(c^w)_w$ is the element of $\prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ which acts on gr^W_w as the multiplication by c^w, and $(c^w)_w a_{F(\mathrm{gr}^W)}$ acts on D by the lifted action from Section 1.2.6. The action of A_P on D and the action of B_P on D_{nspl} are fixed-point-free.

D_{BS} is defined as the set of pairs (P, Z), where P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ and Z is either

(i) an A_P-orbit in D or

(ii) a B_P-orbit in D_{nspl}

for the Borel–Serre action.

In case (i), we call (P, Z) an A_P*-orbit*. In case (ii), we call (P, Z) a B_P*-orbit*.

We denote by $D^{\mathrm{mild}}_{\mathrm{BS}}$ the subset of D_{BS} consisting of A_P-orbits. This subset was written as $D^{(A)}_{\mathrm{BS}}$ in [15, I].

2.4.2

We review the structure of D_{BS} as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. (Actually it is a real analytic manifold with corners.)

For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}(\mathrm{gr}^W)$, let

$$D_{\mathrm{BS}}(P) = \{(Q, Z) \in D_{\mathrm{BS}} \mid Q \supset P\}.$$

Then $D_{\mathrm{BS}}(P)$ forms an open covering of D_{BS} when P varies. D_{BS} is also covered by the open sets $D^{\mathrm{mild}}_{\mathrm{BS}}$ (Section 2.4.1) and $D_{\mathrm{BS,nspl}}$, where $D_{\mathrm{BS,nspl}}$ denotes the subset of D_{BS} consisting of all elements (P, Z) such that $Z \subset D_{\mathrm{nspl}}$. The structures of $D^{\mathrm{mild}}_{\mathrm{BS}}(P) := D_{\mathrm{BS}}(P) \cap D^{\mathrm{mild}}_{\mathrm{BS}}$ and $D_{\mathrm{BS,nspl}}(P) := D_{\mathrm{BS}}(P) \cap D_{\mathrm{BS,nspl}}$ as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$ are described as follows.

Let $X(S_P)$ be the character group of S_P, and let $\Delta(P) \subset X(S_P)$ be the set of simple roots (see [6]). This set $\Delta(P)$ is characterized by the following two properties (i) and (ii).

(i) Let n be the rank of S_P. Then $\Delta(P)$ is of order n and generates $\mathbf{Q} \otimes X(S_P)$ over $\mathbf{Q}$.

(ii) Let $X(S_P)^+$ be the submonoid of $X(S_P)$ generated by $\Delta(P)$. Lift S_P to a subtorus of P. Then $X(S_P)^+$ coincides with the submonoid of $X(S_P)$ generated by χ^{-1}, where χ ranges over all elements of $X(S_P)$ which appear in the adjoint action of S_P on $\mathrm{Lie}(P)$.

Define a real toric variety (see Section 1.3.8(3)) $\bar{A}_P$ and $\bar{B}_P$ as

$$\bar{A}_P := \mathrm{Hom}\big(X(S_P)^+, \mathbf{R}^{\mathrm{mult}}_{\geq 0}\big) = \mathbf{R}^{\Delta(P)}_{\geq 0} \supset A_P = \mathrm{Hom}\big(X(S_P), \mathbf{R}^{\mathrm{mult}}_{>0}\big) = \mathbf{R}^{\Delta(P)}_{>0},$$

$$\bar{B}_P := \mathbf{R}_{\geq 0} \times \bar{A}_P \supset B_P = \mathbf{R}_{>0} \times A_P.$$

For a $\mathbf{Q}$-parabolic subgroup Q of $G_{\mathbf{R}}(\mathrm{gr}^W)$ with $Q \supset P$, there is a canonical injection $\Delta(Q) \to \Delta(P)$, and $Q \mapsto \Delta(Q) \subset \Delta(P)$ is a bijection from the set of all $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $Q \supset P$ to the set of all subsets of $\Delta(P)$. This is explained as follows.

For such Q, we have $Q_u \subset P_u$, the composition $S_Q \to Q/Q_u \to Q/P_u$ is injective, and the image of this composite map is contained in $S_P \subset P/P_u \subset Q/P_u$. Hence, A_Q is regarded as a subgroup of A_P. There is a unique injection $\Delta(Q) \to \Delta(P)$ such that the composition $\mathbf{R}^{\Delta(Q)}_{>0} \cong A_Q \subset A_P = \mathbf{R}^{\Delta(P)}_{>0}$ coincides with the map $f \mapsto g$, where $g(j) = f(j)$ for $j \in \Delta(Q)$ and $g(j) = 1$ for $j \in \Delta(P) \smallsetminus \Delta(Q)$.

We have bijections

$$D^{\mathrm{mild}}_{\mathrm{BS}}(P) \cong D \times^{A_P} \bar{A}_P, \qquad D_{\mathrm{BS,nspl}}(P) \cong D_{\mathrm{nspl}} \times^{B_P} \bar{B}_P,$$

which send the element (Q, Z) of $D^{\mathrm{mild}}_{\mathrm{BS}}(P)$ (resp., $D_{\mathrm{BS,nspl}}(P)$) to the class of (z, h) (resp., $(z, \tilde{h})$), where $z \in Z$ and $h \in \bar{A}_P = \mathbf{R}^{\Delta(P)}_{\geq 0}$ (resp., $\tilde{h} = (0, h) \in \bar{B}_P = \mathbf{R}_{\geq 0} \times \mathbf{R}^{\Delta(P)}_{\geq 0}$) is defined by

$$h(j) = 0 \quad \text{for } j \in \Delta(Q) \subset \Delta(P), \qquad h(j) = 1 \quad \text{for } j \in \Delta(P) \smallsetminus \Delta(Q).$$

The right-hand sides of these bijections are regarded as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$ [15, I, Section 8] as is explained below, and the left-hand sides have the structures as objects of $\mathcal{B}'_{\mathbf{R}}(\log)$ for which these bijections are isomorphisms of $\mathcal{B}'_{\mathbf{R}}(\log)$.

There is a closed real analytic submanifold $D^{(1,A)}$ (resp., $D^{(1,B)}$) of D (resp., D_{nspl}) such that we have an isomorphism $A_P \times D^{(1,A)} \xrightarrow{\cong} D$ (resp., $B_P \times D^{(1,B)} \xrightarrow{\cong} D_{\mathrm{nspl}}$), $(a, F) \mapsto a \circ F$ of real analytic manifolds. This induces a bijection $\bar{A}_P \times D^{(1,A)} \to D \times^{A_P} \bar{A}_P$ (resp., $\bar{B}_P \times D^{(1,B)} \to D_{\mathrm{nspl}} \times^{B_P} \bar{B}_P$), and by this, $D \times^{A_P} \bar{A}_P$ (resp., $D_{\mathrm{nspl}} \times^{B_P} \bar{B}_P$) has a structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$. This structure is independent of the choice of $D^{(1,A)}$ (resp., $D^{(1,B)}_{\mathrm{nspl}}$).

2.4.3

The definition of the set D_{BS} can be rewritten in a style which is similar to the definitions of the spaces of SL(2)-orbits in Section 2.3. Let $D_{\mathrm{BS}}(\mathrm{gr}^W) =$

$\prod_{w\in\mathbf{Z}} D_{\mathrm{BS}}(\mathrm{gr}_w^W)$, where $D_{\mathrm{BS}}(\mathrm{gr}_w^W)$ is the space D_{BS} for the graded quotient gr_w^W. For $p=(P_w,Z_w)_{w\in\mathbf{Z}}\in D_{\mathrm{BS}}(\mathrm{gr}^W)$, we denote $\prod_{w\in\mathbf{Z}} Z_w\subset D(\mathrm{gr}^W)$ as $Z(p)$. We call $Z(p)$ the *torus orbit* of p, and we call $\prod_{w\in\mathbf{Z}} P_w\subset G_{\mathbf{R}}(\mathrm{gr}^W)$ the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ associated to p. Then, D_{BS} is understood as the set of pairs (p,Z), where $p\in D_{\mathrm{BS}}(\mathrm{gr}^W)$ and Z is a subset of D satisfying the following conditions (i) and (ii).

(i) Z is either:
(i.A) an A_P-orbit in D for the Borel–Serre action, or
(i.B) a B_P-orbit in D_{nspl} for the Borel–Serre action.
Here P is the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ associated to p.
(ii) The image of Z in $D(\mathrm{gr}^W)$ coincides with the torus orbit $Z(p)$ of p.

2.4.4
In the rest of Section 2.4, we consider situations (a)–(c) in Section 2.3.5 and also situation

(d) $\mathfrak{D}=D_{\mathrm{BS}}$, $\mathfrak{E}=D_{\mathrm{BS}}(\mathrm{gr}^W)$.

2.4.5
For $x\in\mathfrak{D}$, we define objects

$$S_x, X(S_x)^+, T(x), \bar{T}(x), Z(x), \bar{Z}(x)$$

associated to x.

In situations (a)–(c), write $x=(p,Z)$ ($p\in\mathfrak{E}$, $Z\subset D$). In situation (d), write $x=(P,Z)$.

In situations (a)–(c), let $S_x=S_p$ if x is an A-orbit, and let $S_x=\mathbf{G}_m\times S_p$ if x is a B-orbit (see Sections 2.2.7, 2.3.1). In situation (d), let $S_x=S_P$ if x is an A_P-orbit, and let $S_x=\mathbf{G}_m\times S_P$ if x is a B_P-orbit (see Section 2.4.1).

We define a submonoid $X(S_x)^+$ of the character group $X(S_x)$ of S_x as follows. In situations (a)–(c), let $X(S_x)^+:=X(S_p)^+$ if x is an A-orbit (see Section 2.2.7), and let $X(S_x)^+:=\mathbf{N}\times X(S_p)^+\subset\mathbf{Z}\times X(S_p)=X(S_x)$ if x is a B-orbit. In situation (d), let $X(S_x)^+:=X(S_P)^+$ if x is an A_P-orbit, and let $X(S_x)^+:=\mathbf{N}\times X(S_P)^+\subset\mathbf{Z}\times X(S_P)=X(S_x)$ if x is a B_P-orbit, where $X(S_P)^+$ is as in Section 2.4.2.

Let $T(x)$ be the connected component of $S_x(\mathbf{R})$ containing the unit element. Let

$$\bar{T}(x):=\mathrm{Hom}\big(X(S_x)^+,\mathbf{R}^{\mathrm{mult}}_{\geq 0}\big)\supset T(x)=\mathrm{Hom}\big(X(S_x)^+,\mathbf{R}^{\mathrm{mult}}_{>0}\big).$$

We regard $\bar{T}(x)$ as a real toric variety.

Define $Z(x):=Z$. We call $Z(x)$ the torus orbit associated to x.

$T(x)$ acts on $Z(x)$, and $Z(x)$ is a $T(x)$-torsor. Let $\bar{Z}(x):=Z(x)\times^{T(x)}\bar{T}(x)$. Then $\bar{Z}(x)$ has the unique structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ such that, for any $\mathbf{r}\in Z(x)$, the bijection $\bar{T}(x)\to\bar{Z}(x)$ induced from the bijection $T(x)\to Z(x)$, $t\mapsto t\mathbf{r}$ becomes an isomorphism in $\mathcal{B}'_{\mathbf{R}}(\log)$. We call $\bar{Z}(x)$ the extended torus orbit associated to x. In Section 2.4.8 below, we will embed $\bar{Z}(x)$ in $\mathfrak{D}$ satisfying $x \in \bar{Z}(x)$.

2.4.6

This section is a preparation for Proposition 2.4.7. Consider the three situations in Section 2.3.5.

In situations (a) and (b) (resp., situation (c)), we have a global section $\beta_0^\star$ (resp., β_0) of $M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times$ defined as follows. In situations (a) and (c) (resp., situation (b)), let $\Phi \in \overline{\mathcal{W}}$ (resp., $Q = (Q(w))_w \in \prod_w \mathcal{W}(\mathrm{gr}_w^W)$), let $\mathfrak{D}' = \mathfrak{D}(\Phi)$ (resp., $\mathfrak{D}' = \mathfrak{D}(Q)$), let α be a splitting of Φ (resp., Q), and let β be a distance to Φ-boundary (resp., Q-boundary). Fix a real analytic closed submanifold $\mathcal{L}^{(1)}$ of $\mathcal{L} \smallsetminus \{0\}$ such that $\mathbf{R}_{>0} \times \mathcal{L}^{(1)} \to \mathcal{L} \smallsetminus \{0\}$, $(a, \delta) \mapsto a \circ \delta$ is an isomorphism of real analytic manifolds, and let $\mathbf{R}_{\geq 0} \times \mathcal{L}^{(1)} \xrightarrow{\cong} \bar{\mathcal{L}} \smallsetminus \{0\}$ be the induced isomorphism in $\mathcal{B}'_{\mathbf{R}}(\log)$.

Let $\mathfrak{D}'_{\mathrm{nspl}}$ be the open subset of $\mathfrak{D}'$ defined by $\delta \neq 0$ via the bijection ν in Proposition 2.3.9 associated to (α, β). Then in situations (a) and (b) (resp., situation (c)), we have the composite morphism $\mathfrak{D}'_{\mathrm{nspl}} \to \bar{\mathcal{L}} \smallsetminus \{0\} \cong \mathbf{R}_{\geq 0} \times \mathcal{L}^{(1)} \to \mathbf{R}_{\geq 0}$, where the first arrow is ν. We denote this composite morphism $\mathcal{D}_{\mathrm{nspl}}(\Phi) \to \mathbf{R}_{\geq 0}$ by $\beta_0^\star$ (resp., β_0). Then as is easily seen, this $\beta_0^\star$ (resp., β_0) belongs to $M_{\mathfrak{D}'_{\mathrm{nspl}}}$, the class of $\beta_0^\star$ (resp., β_0) in $M_{\mathfrak{D}'_{\mathrm{nspl}}}/\mathcal{O}_{\mathcal{D}'_{\mathrm{nspl}}}^\times$ is independent of the choices of α, β, and $\mathcal{L}^{(1)}$, this class extends uniquely to a section of $M_{\mathfrak{D}'}/\mathcal{O}_{\mathfrak{D}'}^\times$ which is trivial on the part of A-orbits of $\mathfrak{D}$, and this local section of $M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times$ on $\mathfrak{D}' = \mathfrak{D}(\Phi)$ (resp., $\mathfrak{D}' = \mathfrak{D}(Q)$) extends, when Φ (resp., Q) moves, to a global section $\beta_0^\star$ (resp., β_0) of $M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times$ on $\mathfrak{D}$ uniquely.

PROPOSITION 2.4.7

Consider the four situations in Section 2.4.4. For $x \in \mathfrak{D}$, we have a canonical isomorphism

$$(M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times)_x \cong X(S_x)^+.$$

Proof

We first consider situations (a)–(c). Write $x = (p, Z)$. As in Section 2.2.13, we have a canonical isomorphism $(M_{\mathfrak{E}}/\mathcal{O}_{\mathfrak{E}}^\times)_p \cong X(S_p)^+$. In the case when x is an A-orbit, we have $(M_{\mathfrak{E}}/\mathcal{O}_{\mathfrak{E}}^\times)_p \xrightarrow{\cong} (M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times)_x$. If x is a B-orbit, we have $\mathbf{N} \times (M_{\mathfrak{E}}/\mathcal{O}_{\mathfrak{E}}^\times)_p \xrightarrow{\cong} (M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times)_x$, where $1 \in \mathbf{N}$ is sent to $\beta_0^\star$ in situations (a) and (b) and to β_0 in situation (c).

We next consider situation (d). Write $x = (P, Z)$. Assume first that x is an A_P-orbit. Consider the composite morphism $S := D_{\mathrm{BS}}^{\mathrm{mild}}(P) \cong \bar{A}_P \times D^{(1,A)} \to \bar{A}_P = \mathbf{R}_{\geq 0}^{\Delta(P)}$, where the first isomorphism is as in Section 2.4.2. For $j \in \Delta(P)$, let $\beta_j : S \to \mathbf{R}_{\geq 0}$ be the j-component of this composite morphism. Then β_j is a section of M_S, and the class of β_j in $M_S/\mathcal{O}_S^\times$ is independent of the choice of $D^{(1,A)}$ in Section 2.4.2. We have a canonical isomorphism $X(S_x)^+ = \mathbf{N}^{\Delta(P)} \xrightarrow{\cong} (M_S/\mathcal{O}_S^\times)_x$ which sends $m \in \mathbf{N}^{\Delta(P)}$ to the class of $\prod_{j \in \Delta(P)} \beta_j^{m(j)}$. Assume next that x is a B_P-orbit. Consider the composite morphism $S := D_{\mathrm{BS,nspl}}(P) \cong \bar{B}_P \times D^{(1,B)} \to$

$\bar{B}_P = \mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}^{\Delta(P)}$, where the first isomorphism is as in Section 2.4.2. Let $\beta_0^{\mathrm{BS}} : S \to \mathbf{R}_{\geq 0}$ be the first component of this composite morphism, and for $j \in \Delta(P)$, let $\beta_j : S \to \mathbf{R}_{\geq 0}$ be the j-component of this composite morphism. Then β_0^{BS} and β_j $(j \in \Delta(P))$ are sections of M_S, and their classes in $M_S/\mathcal{O}_S^\times$ are independent of the choice of $D^{(1,B)}$ in Section 2.4.2. We have an isomorphism $X(S_x)^+ \cong \mathbf{N} \times \mathbf{N}^{\Delta(P)} \to (M_S/\mathcal{O}_S^\times)_x$ which sends $(m_0, (m(j))_{j\in\Delta(P)}) \in \mathbf{N} \times \mathbf{N}^{\Delta(P)}$ to the class of $(\beta_0^{\mathrm{BS}})^{m_0} \cdot \prod_{j\in\Delta(P)} \beta_j^{m(j)}$. □

2.4.8

Let situations (a)–(d) be as in Section 2.4.4. Let $x \in \mathfrak{D}$. The inclusion map $Z(x) \to D$ extends uniquely to a morphism

$$\bar{Z}(x) \to \mathfrak{D}$$

of $\mathcal{B}'_{\mathbf{R}}(\log)$. This morphism is described as follows.

Assume first that we are in one of the situations (a)–(c). Write $x = (p, Z)$, and fix $\mathbf{r} \in Z(p)$. Consider the morphism $Y \times \mathrm{spl}(W) \times \bar{L} \to \mathfrak{D}$ in Section 2.3.15 defined for $(p, \mathbf{r}, R, S)$ by fixing R and S (see Section 2.3.13). Then the morphism $\bar{Z}(x) \to \mathfrak{D}$ is the composite morphism $\bar{Z}(x) \to Y \times \mathrm{spl}(W) \times \bar{L} \to \mathfrak{D}$, where the first morphism is as follows. Let F be an element of $Z(x)$ whose image under the embedding $D \to D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ is $(\mathbf{r}, s, \delta)$. Let $t \in \bar{A}_p$. Then the first morphism sends $(F, t) \in \bar{Z}(x) = Z(x) \times^{T(x)} \bar{T}(x)$ to $(t, 0, 0, 0, 0, s, \delta) \in Y \times \mathrm{spl}(W) \times \bar{L}$, and if x is a B-orbit, then for $(c, t) \in \bar{B}_p$ $(c \in \mathbf{R}_{\geq 0})$, the first morphism sends $(F, (c, t)) \in \bar{Z}(x)$ to $(t, 0, 0, 0, 0, s, c \circ \delta) \in Y \times \mathrm{spl}(W) \times \bar{L}$.

Next assume that we are in situation (d). Write $x = (P, Z)$. If x is an A_P-orbit, then this morphism $\bar{Z}(x) \to \mathfrak{D}$ is the composition $\bar{Z}(x) = Z \times^{A_P} \bar{A}_P \subset D \times^{A_P} \bar{A}_P \cong D_{\mathrm{BS}}^{\mathrm{mild}}(P)$. If x is a B_P-orbit, then this morphism is the composition $\bar{Z}(x) = Z \times^{B_P} \bar{B}_P \subset D_{\mathrm{nspl}} \times^{B_P} \bar{B}_P \cong D_{\mathrm{BS,nspl}}(P)$.

This morphism $\bar{Z}(x) \to \mathfrak{D}$ is injective and strict (Corollary 1.3.15) and sends $0 \in \bar{Z}(x)$ to x. Here 0 denotes the class of $(\mathbf{r}, 0)$, where $\mathbf{r} \in Z(x)$ and $0 \in \bar{T}(x)$ is the homomorphism $(M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times)_x \to \mathbf{R}_{\geq 0}^{\mathrm{mult}}$ which sends any nontrivial element of $(M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times)_x$ to 0. (Then $0 \in \bar{Z}(x)$ is independent of the choice of $\mathbf{r}$.) We will identify $\bar{Z}(x)$ with its image in $\mathfrak{D}$, which coincides with the closure of $Z(x)$ in $\mathfrak{D}$.

2.4.9

Consider situations (a)–(d) as in Section 2.4.4. In Lemma 2.4.10 below, we give descriptions of log modifications of $\mathfrak{D}$ as sets by using the extended torus orbit $\bar{Z}(x) \subset \mathfrak{D}$ associated to $x \in \mathfrak{D}$ (Section 2.4.8), which we will use in Sections 2.5 and 2.6. Let U be an open set of $\mathfrak{D}$. Let L and N be as in Section 1.4.1, let Σ be a finite rational fan in $N_{\mathbf{R}}$, and let Σ' be a rational finite subdivision of Σ.

Let $\mathrm{Mor}(\cdot, U) \to [\Sigma]$ be a morphism of functors (see Section 1.4.7) such that, for any $x \in U$, if σ denotes the image of x in Σ (see Section 1.4.7), then the homomorphism $\mathcal{S}(\sigma) \to (M_U/\mathcal{O}_U^\times)_x$ is universally saturated. For $x \in U$ and $\sigma' \in \Sigma'$ whose images in Σ coincide, we define a subgroup $T(x, \sigma')$ of $T(x) = \mathrm{Hom}((M_U^{\mathrm{gp}}/\mathcal{O}_U^\times)_x,$

$\mathbf{R}_{>0}^{\mathrm{mult}}$) as follows. Let σ be the image in Σ. Then the homomorphism $L \to (M_U^{\mathrm{gp}}/\mathcal{O}_U^\times)_x$ factors through $L/\mathcal{S}(\sigma)^\times$. $T(x,\sigma')$ is the inverse image of $\mathrm{Hom}(L/\mathcal{S}(\sigma')^\times, \mathbf{R}_{>0}^{\mathrm{mult}}) \subset \mathrm{Hom}(L/\mathcal{S}(\sigma)^\times, \mathbf{R}_{>0}^{\mathrm{mult}})$ in $T(x)$. Let $U' \to U$ be the log modification which represents the functor $\mathrm{Mor}(\cdot, U) \times_{[\Sigma]} [\Sigma']$ (Section 1.4.7).

LEMMA 2.4.10

Let the notation and the assumptions be as in Section 2.4.9. There exists a canonical bijection between U' and the set of all triples (x, σ', Z'), where $x \in U$, σ' is an element of Σ' whose image in Σ coincides with the image of x in Σ, and Z' is a $T(x,\sigma')$-orbit in $Z(x)$.

Proof

Let $x \in U$, and let U'' be the fiber product of $\bar{Z}(x) \to U \leftarrow U'$. Then the fiber on x of $U' \to U$ coincides with the fiber on x of $U'' \to \bar{Z}(x)$. Since U'' represents the functor $\mathrm{Mor}(\cdot, U) \times_{[\Sigma]} [\Sigma']$, this lemma follows from Section 1.4.8. □

2.5. Relations with $D_{\mathrm{SL}(2)}$ and $D^\star_{\mathrm{SL}(2)}$

We connect the spaces $D^\star_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ by introducing a new space $D^{\star,+}_{\mathrm{SL}(2)}$ of SL(2)-orbits.

2.5.1

We define a log modification (see Section 1.4.6)

$$D^{\star,+}_{\mathrm{SL}(2)} \to D^\star_{\mathrm{SL}(2)}.$$

On $\mathfrak{D} := D^\star_{\mathrm{SL}(2)}$, there is a unique section β_{tot} of $M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times$ such that, for any $\Phi \in \overline{\mathcal{W}}$, the restriction of β_{tot} to $\mathfrak{D}(\Phi)$ coincides with the image of the product $\prod_{j\in\Phi} \beta_j$ in $M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times$, where $\beta = (\beta_j)_{j\in\Phi}$ is a distance to Φ-boundary. Let $\beta_0^\star$ be the section of $M_{\mathfrak{D}}/\mathcal{O}_{\mathfrak{D}}^\times$ defined in Section 2.4.6. Consider the homomorphism $\mathbf{N}^2 \to M_S/\mathcal{O}_S^\times$, $(a,b) \mapsto \beta_{\mathrm{tot}}^a (\beta_0^\star)^b$.

Take $L = \mathbf{Z}^2$ in Section 1.4.1, and let Σ be the fan of all faces of the cone $\mathbf{R}^2_{\geq 0} \subset N_{\mathbf{R}}^2 = \mathbf{R}^2$, so we have a morphism $\mathrm{Mor}(\cdot, \mathfrak{D}) \to [\Sigma]$. Let Σ' be the rational finite subdivision of Σ consisting of the cones

$$\sigma_1 := \{(x,y) \in \mathbf{R}^2_{\geq 0} \mid x \geq y\}, \qquad \sigma_2 := \{(x,y) \in \mathbf{R}^2_{\geq 0} \mid x \leq y\}$$

and their faces. Let $D^{\star,+}_{\mathrm{SL}(2)}$ be the log modification of $\mathfrak{D}$ which represents the fiber product $\mathrm{Mor}(\cdot, \mathfrak{D}) \times_{[\Sigma]} [\Sigma']$ (see Section 1.4.7).

$D^{\star,+}_{\mathrm{SL}(2)}$ is covered by the open sets $D^{\star,+}_{\mathrm{SL}(2)}(\sigma_j)$ for $j = 1, 2$ corresponding to the cone σ_j, which represents $\mathrm{Mor}(\cdot, \mathfrak{D}) \times_{[\Sigma]} [\mathrm{face}(\sigma_j)]$, where $\mathrm{face}(\sigma_j)$ denotes the fan of all faces of σ_j. On the open set $U = D^{\star,+}_{\mathrm{SL}(2)}(\sigma_1)$ (resp., $U = D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2)$), the pullback of $\beta_{\mathrm{tot}}/\beta_0^\star$ (resp., $\beta_0^\star/\beta_{\mathrm{tot}}$) in $M_U^{\mathrm{gp}}/\mathcal{O}_U^\times$ belongs to $M_U/\mathcal{O}_U^\times$.

2.5.2
Since the restriction of $\beta_0^\star$ to $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ is trivial, the canonical morphism $D^{\star,+}_{\mathrm{SL}(2)} \to D^\star_{\mathrm{SL}(2)}$ is an isomorphism over $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$, and hence, $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ is embedded in $D^{\star,+}_{\mathrm{SL}(2)}$ as an open set. Via this, $D \subset D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ is embedded in $D^{\star,+}_{\mathrm{SL}(2)}$ as an open set.

2.5.3
We describe $D^{\star,+}_{\mathrm{SL}(2)}$ as a set. We have

$$\Sigma = \{\tau_{1,2}, \tau_1, \tau_2, \tau_0\}, \qquad \Sigma' = \{\sigma_1, \sigma_2, \sigma_0, \tau_1, \tau_2, \tau_0\},$$

where

$$\tau_{1,2} := \mathbf{R}^2_{\geq 0}, \qquad \tau_1 := \mathbf{R}_{\geq 0} \times \{0\}, \qquad \tau_2 := \{0\} \times \mathbf{R}_{\geq 0}, \qquad \tau_0 := \big\{(0,0)\big\},$$
$$\sigma_0 := \big\{(x,x) \mid x \in \mathbf{R}_{\geq 0}\big\}.$$

So, Σ is the set of all faces of $\tau_{1,2}$, and $\mathrm{face}(\sigma_j) = \{\sigma_j, \tau_j, \sigma_0, \tau_0\}$ for $j = 1, 2$. The image of $x = (p, Z) \in D^\star_{\mathrm{SL}(2)}$ in Σ is τ_0 if and only if $x \in D$, τ_1 if and only if $x \in D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \smallsetminus D$, τ_2 if and only if x is a B-orbit and $p \in D(\mathrm{gr}^W)$, and $\tau_{1,2}$ if and only if x is a B-orbit and $p \notin D(\mathrm{gr}^W)$.

We apply Lemma 2.4.10 to describe the log modification $D^{\star,+}_{\mathrm{SL}(2)}$ of $D^\star_{\mathrm{SL}(2)}$ as a set. For this, we show that the homomorphism $\mathbf{N}^2 \to (M_{\mathfrak{D}}/\mathcal{O}^\times_{\mathfrak{D}})_x$ ($\mathfrak{D} := D^\star_{\mathrm{SL}(2)}$), given by $(\beta_{\mathrm{tot}}, \beta_0^\star)$ in Section 2.5.1, is universally saturated for any $x \in \mathfrak{D}$. If the image of x in Σ is τ_0 or τ_1 (resp., τ_2 or $\tau_{1,2}$), then this homomorphism has the shape $\mathbf{N}^2 \to \mathbf{N}^r$, $(a,b) \mapsto (b, \dots, b)$ (resp., $\mathbf{N}^2 \to \mathbf{N} \times \mathbf{N}^r$, $(a,b) \mapsto (a, b, \dots, b)$) for some integer $r \geq 0$ and, hence, is universally saturated by Proposition 1.3.14.

By Lemma 2.4.10, we have the following list of points of $D^{\star,+}_{\mathrm{SL}(2)}$.

(1) $(x, \tau_j, Z(x))$ ($x \in D^\star_{\mathrm{SL}(2)}$ and the image of x in Σ is τ_j). (Here $j = 0, 1, 2$.)
(2) $(x, \sigma_j, Z(x))$ ($x \in D^\star_{\mathrm{SL}(2)}$ and the image of x in Σ is $\tau_{1,2}$). (Here $j = 1, 2$.)
(3) (x, σ_0, Z') ($x = (p, Z) \in D^\star_{\mathrm{SL}(2)}$, the image of x in Σ is $\tau_{1,2}$, and Z' is a $\tau_p(A_p)$-orbit in $Z(x)$).

Actually, in (3), what Lemma 2.4.10 directly tells is that a $\tau_p^\star(T(x,\sigma_0))$-orbit Z' in the $\tilde{\tau}_p^\star(B_p)$-orbit $Z(x)$ appears instead of a $\tau_p(A_p)$-orbit in $Z(x)$. But $\tau_p^\star(T(x,\sigma_0)) = \tau_p(A_p)$ inside $\tilde{\tau}_p^\star(B_p)$.

2.5.4
We have a map $D^{\star,+}_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}$ defined as follows.

(1) (x, τ_j, Z) ($x = (p, Z)$ with image τ_j in Σ for $j = 0, 2$) and (x, σ_2, Z) ($x = (p, Z)$ with image $\tau_{1,2}$ in Σ) are sent to $(p, Z) \in D_{\mathrm{SL}(2)}$.
(2) (x, τ_1, Z) ($x = (p, Z)$ with image τ_1 in Σ) and (x, σ_1, Z) ($x = (p, Z)$ with image $\tau_{1,2}$ in Σ) are sent to $(p, Z_{\mathrm{spl}}) \in D_{\mathrm{SL}(2)}$. Here $Z_{\mathrm{spl}} = \{F_{\mathrm{spl}} \mid F \in Z\}$, where F_{spl} is as in Section 1.2.5.
(3) (x, σ_0, Z') ($x = (p, Z)$ with image $\tau_{1,2}$ in Σ and Z' is a $\tau_p(A_p)$-orbit inside Z) is sent to $(p, Z') \in D_{\mathrm{SL}(2)}$.

THEOREM 2.5.5

(1) *The identity map of* D *extends uniquely to a morphism* $D^{\star,+}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ *in* $\mathcal{B}'_{\mathbf{R}}(\log)$*. Its underlying map of sets is the map in Section 2.5.4. This map is proper and surjective.*

(2) *Let* U *be the open set* $D^{II}_{\mathrm{SL}(2),\mathrm{nspl}} \cup D$ *of* $D^{II}_{\mathrm{SL}(2)}$*. Then the inverse image of* U *in* $D^{\star,+}_{\mathrm{SL}(2)}$ *coincides with the open set* $D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2)$*, and the induced morphism* $D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2) \to U$ *of* $\mathcal{B}'_{\mathbf{R}}(\log)$ *is an isomorphism.*

Proof

We prove (1). It is sufficient to prove that the map in Section 2.5.4 is a morphism $D^{\star,+}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$. For an admissible set of weight filtrations Φ on gr^W, let $D^{\star,+}_{\mathrm{SL}(2)}(\Phi) \subset D^{\star,+}_{\mathrm{SL}(2)}$ be the inverse image of $D^{\star}_{\mathrm{SL}(2)}(\Phi) \subset D^{\star}_{\mathrm{SL}(2)}$. It is sufficient to prove that the induced map $D^{\star,+}_{\mathrm{SL}(2)}(\Phi) \to D^{II}_{\mathrm{SL}(2)}(\Phi)$ is a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$.

Let $D^{\star,+}_{\mathrm{SL}(2),\mathrm{nspl}} \subset D^{\star,+}_{\mathrm{SL}(2)}$ be the inverse image of the open set $D^{\star}_{\mathrm{SL}(2),\mathrm{nspl}}$ of $D^{\star}_{\mathrm{SL}(2)}$. Then $D^{\star,+}_{\mathrm{SL}(2)}(\sigma_1)$ is the union of the two open sets $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ (which is embedded in $D^{\star,+}_{\mathrm{SL}(2)}$) and $D^{\star,+}_{\mathrm{SL}(2),\mathrm{nspl}} \cap D^{\star,+}_{\mathrm{SL}(2)}(\sigma_1)$, and $D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2)$ is contained in $D^{\star,+}_{\mathrm{SL}(2),\mathrm{nspl}}$.

Take a splitting α of Φ and a distance β to Φ-boundary. First, the induced map $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}(\Phi) \to D^{II}_{\mathrm{SL}(2)}(\Phi)$ is a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$, because this map is embedded in a commutative diagram

$$\begin{array}{ccc} D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}(\Phi) & \hookrightarrow & D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) \times \mathrm{spl}(W) \times \mathcal{L} \\ \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \hookrightarrow & D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) \times \mathrm{spl}(W) \times \bar{\mathcal{L}} \end{array}$$

where the horizontal arrows are the maps ν in Proposition 2.3.9 associated to (α,β) and the right vertical arrow is the morphism $(p,s,\delta) \mapsto (p,s,\sum_{w\le -2}(\prod_{j\in\Phi}\beta_j(p))^{-w}\delta_w)$, and because the structure of $D^{II}_{\mathrm{SL}(2)}(\Phi)$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ is induced from that of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathrm{spl}(W) \times \bar{\mathcal{L}}$ in the sense of Section 1.3.16.

Next we consider the induced map $D^{\star,+}_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi) \to D^{II}_{\mathrm{SL}(2)}(\Phi)$. Take a closed real analytic subset $\mathcal{L}^{(1)}$ of $\mathcal{L} \smallsetminus \{0\}$ such that $\mathbf{R}_{>0} \times \mathcal{L}^{(1)} \to \mathcal{L} \smallsetminus \{0\}$, $(a,\delta) \mapsto a\circ\delta$ is an isomorphism, and consider the induced isomorphism $\mathbf{R}_{\ge 0} \times \mathcal{L}^{(1)} \xrightarrow{\cong} \bar{\mathcal{L}} \smallsetminus \{0\}$. Let $\beta^{\star}_0 : D^{\star}_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi) \to \mathbf{R}_{\ge 0}$ be the composition $D^{\star}_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi) \to \bar{\mathcal{L}} \smallsetminus \{0\} \cong \mathbf{R}_{\ge 0} \times \mathcal{L}^{(1)} \to \mathbf{R}_{\ge 0}$, where the first arrow is induced by the map ν in Proposition 2.3.9 associated to (α,β). For $j = 1,2$, let $U_j := D^{\star,+}_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi) \cap D^{\star,+}_{\mathrm{SL}(2)}(\sigma_j)$. When we regard β_j $(j \in \Phi)$ and $\beta^{\star}_0$ as sections of M_{U_j}, then in $M^{\mathrm{gp}}_{U_j}$, $(\prod_{j\in\Phi}\beta_j)/\beta^{\star}_0$ belongs to M_{U_1} and $\beta^{\star}_0/\prod_{j\in\Phi}\beta_j$ belongs to M_{U_2}. Furthermore, $\beta^{\star}_0/\prod_{j\in\Phi}\beta_j$ on U_2 is the pullback of the section β_0 of the log structure of $D_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi)$ which is defined as the composition $D_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi) \to \bar{\mathcal{L}} \smallsetminus \{0\} \cong \mathbf{R}_{\ge 0} \times \mathcal{L}^{(1)} \to \mathbf{R}_{\ge 0}$, where the first arrow is induced by ν of Proposition 2.3.9 associated to (α,β).

The induced maps $U_j \to D^{II}_{\mathrm{SL}(2)}(\Phi)$ for $j = 1, 2$ are morphisms, because they are embedded in the commutative diagrams

$$\begin{array}{ccc} U_1 & \subsetneq\!\!\to & D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) \times \mathrm{spl}(W) \times (\mathbf{R}_{\geq 0} \times \mathcal{L}^{(1)}) \times \mathbf{R}_{\geq 0} \\ \downarrow & & \downarrow \\ D^{II}_{\mathrm{SL}(2)}(\Phi) & \subsetneq\!\!\to & D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) \times \mathrm{spl}(W) \times \bar{\mathcal{L}}, \end{array}$$

$$\begin{array}{ccc} U_2 & \subsetneq\!\!\to & D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) \times \mathrm{spl}(W) \times \mathcal{L}^{(1)} \times \mathbf{R}_{\geq 0} \\ \downarrow & & \| \\ D^{II}_{\mathrm{SL}(2),\mathrm{nspl}}(\Phi) & \subsetneq\!\!\to & D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi) \times \mathrm{spl}(W) \times \mathcal{L}^{(1)} \times \mathbf{R}_{\geq 0}. \end{array}$$

Here in both diagrams, the lower horizontal arrows are induced by ν in Proposition 2.3.9 associated to (α, β) and the isomorphism $\bar{\mathcal{L}} \smallsetminus \{0\} \cong \mathcal{L}^{(1)} \times \mathbf{R}_{\geq 0}$. In the first diagram, the part $U_1 \to \mathbf{R}_{\geq 0} \times \mathcal{L}^{(1)}$ in the upper row is the composition $U_1 \to D^{\star}_{\mathrm{SL}(2),\mathrm{nspl}} \to \bar{\mathcal{L}} \smallsetminus \{0\} \cong \mathbf{R}_{\geq 0} \times \mathcal{L}^{(1)}$, the map from U_1 to the last $\mathbf{R}_{\geq 0}$ in the upper row is $(\prod_{j\in\Phi} \beta_j)/\beta^{\star}_0$, and the right vertical arrow is $(p, s, t, \delta, t') \mapsto (p, s, \sum_{w\leq -2}(tt')^{-w}\delta_w)$. In the second diagram, the part $U_2 \to \mathcal{L}^{(1)}$ in the upper row is the composition $U_2 \to D^{\star}_{\mathrm{SL}(2),\mathrm{nspl}} \to \bar{\mathcal{L}} \smallsetminus \{0\} \cong \mathbf{R}_{\geq 0} \times \mathcal{L}^{(1)} \to \mathcal{L}^{(1)}$, the map $U_2 \to \mathbf{R}_{\geq 0}$ in the upper row is $\beta^{\star}_0/\prod_{j\in\Phi} \beta_j$, and the right vertical arrow is the identity map.

The surjectivity of $D^{\star,+}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ is easily seen. The map is proper, because $D^{\star,+}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ are proper over $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathrm{spl}(W)$. This completes the proof of (1).

We prove (2). It is easy to check that the inverse image of U in $D^{\star,+}_{\mathrm{SL}(2)}$ is $D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2)$ and that the map $D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2) \to U$ is bijective. Hence, for the proof of (2), it is sufficient to prove that the converse map $D^{II}_{\mathrm{SL}(2),\mathrm{nspl}} \to D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2)$ is a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)$. This is a morphism as is seen from the last commutative diagram above. (In the upper row of this diagram, the structure of the space of U_2 as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ is induced from that of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathrm{spl}(W) \times \mathcal{L}^{(1)} \times \mathbf{R}_{\geq 0}$ in the sense of Section 1.3.16.) □

2.5.6

In Proposition 2.5.7, we consider when the identity map of D extends to an isomorphism $D^{\star}_{\mathrm{SL}(2)} \cong D^{II}_{\mathrm{SL}(2)}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$.

Let $\lambda : D_{\mathrm{SL}(2)} \to D^{\star}_{\mathrm{SL}(2)}$ be the map which coincides on $D_{\mathrm{SL}(2),\mathrm{nspl}} \cup D$ with the composition of morphisms $D^{II}_{\mathrm{SL}(2),\mathrm{nspl}} \cup D \cong D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2) \to D^{\star}_{\mathrm{SL}(2)}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$ and which coincides on $D_{\mathrm{SL}(2),\mathrm{spl}} := \{(p, Z) \in D_{\mathrm{SL}(2)} \mid Z \subset D_{\mathrm{spl}}\}$ with the composition of two isomorphisms $D_{\mathrm{SL}(2),\mathrm{spl}} \cong D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathrm{spl}(W) \cong D^{\star}_{\mathrm{SL}(2),\mathrm{spl}} := \{(p, Z) \in D^{\star}_{\mathrm{SL}(2)} \mid Z \subset D_{\mathrm{spl}}\}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$.

PROPOSITION 2.5.7

The following conditions (i)–(vii) are equivalent.

(i) *Either $D = D_{\mathrm{spl}}$ or $D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = D(\mathrm{gr}^W)$.*

(ii) *The identity map of D extends to an isomorphism $D^{II}_{\mathrm{SL}(2)} \cong D^{\star}_{\mathrm{SL}(2)}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$.*

(iii) *The identity map of D extends to a homeomorphism $D^{II}_{\mathrm{SL}(2)} \cong D^{\star}_{\mathrm{SL}(2)}$.*

(iv) *The map $\lambda : D^{I}_{\mathrm{SL}(2)} \to D^{\star}_{\mathrm{SL}(2)}$ (see Section 2.5.6) is continuous.*

(v) *The identity map of D extends to a continuous map $D^{\star}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$.*

(vi) *The map $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}$ is injective.*

(vii) *The map $D_{\mathrm{SL}(2),\mathrm{nspl}} \to D^{\star}_{\mathrm{SL}(2)}$ is injective.*

Proof

(i) $\Rightarrow$ (ii). If $\mathcal{L}(F) = 0$ (see Section 1.2.2) for any $F \in D(\mathrm{gr}^W)$, then the isomorphism in Section 1.2.1 extends to isomorphisms from $D^{II}_{\mathrm{SL}(2)}$ and $D^{\star}_{\mathrm{SL}(2)}$ onto $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathrm{spl}(W)$ in $\mathcal{B}'_{\mathbf{R}}(\log)$. If $D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = D(\mathrm{gr}^W)$, then the isomorphism in Section 1.2.1 extends to isomorphisms from $D^{II}_{\mathrm{SL}(2)}$ and $D^{\star}_{\mathrm{SL}(2)}$ onto $\{(F, s, \delta) \in D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \bar{\mathcal{L}} \mid \delta \in \bar{\mathcal{L}}(F)\}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$.

(ii) $\Rightarrow$ (iii). This is clear.

(iii) $\Rightarrow$ (iv), (v), and (vi). This is clear.

(v) $\Rightarrow$ (vii). If (v) is satisfied, then the composition $D^{II}_{\mathrm{SL}(2),\mathrm{nspl}} \to D^{\star}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ will be the inclusion map.

We prove (iv) $\Rightarrow$ (i), (vi) $\Rightarrow$ (i), and (vii) $\Rightarrow$ (i).

In the rest of this proof, assume $D \neq D_{\mathrm{spl}}$ and $D_{\mathrm{SL}(2)}(\mathrm{gr}^W) \neq D(\mathrm{gr}^W)$. That is, assume (i) does not hold. Then there is $x = (p, Z) \in D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ with p of rank 1 such that $Z \subset D_{\mathrm{nspl}}$. Let $x_{\mathrm{spl}} := (p, Z_{\mathrm{spl}}) \in D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$. We have $x \neq x_{\mathrm{spl}}$.

We prove (iv) $\Rightarrow$ (i). Take $\mathbf{r} \in Z$. Then when $t \in \mathbf{R}_{>0}$ tends to 0, $\tau^{\star}_p(t)\mathbf{r}$ converges to x and $\tau^{\star}_p(t)\mathbf{r}_{\mathrm{spl}}$ converges to x_{spl} in $D^{\star}_{\mathrm{SL}(2)}$.

CLAIM

Let $y = (p, Z_{\mathrm{spl}}) \in D_{\mathrm{SL}(2)}$. Then when $t \in \mathbf{R}_{>0}$ converges to 0, $\tau^{\star}_p(t)\mathbf{r}$ and $\tau^{\star}_p(t)\mathbf{r}_{\mathrm{spl}}$ converge to y in $D^{I}_{\mathrm{SL}(2)}$.

We prove the claim. Let $s := \mathrm{spl}_W(p) = \mathrm{spl}_W(\mathbf{r})$. By [15, II, Proposition 3.2.12], it is sufficient to prove that, when $t \in \mathbf{R}_{>0}$ tends to 0, $(s\tau_p(t)s^{-1})^{-1}(s\tau^{\star}_p(t)s^{-1})\mathbf{r}$ and $(s\tau_p(t)s^{-1})^{-1}(s\tau^{\star}_p(t)s^{-1})\mathbf{r}_{\mathrm{spl}}$ converge to $\mathbf{r}_{\mathrm{spl}}$. The former is equal to $s(t^{-w})_w s^{-1}\mathbf{r}$ and hence converges to $\mathbf{r}_{\mathrm{spl}}$. Here $(t^{-w})_w$ denotes the linear automorphism of $\mathrm{gr}^W = \prod_w \mathrm{gr}^W_w$ that acts on gr^W_w as the multiplication by t^{-w}. The latter is equal to $\mathbf{r}_{\mathrm{spl}}$. This proves the claim.

By the claim, if the continuous map $D^{I}_{\mathrm{SL}(2)} \to D^{\star}_{\mathrm{SL}(2)}$ exists, it should send y to x and also to $x_{\mathrm{spl}} \neq x$, which is a contradiction.

We prove (vi) $\Rightarrow$ (i). The elements x and x_{spl} of $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ have the same image $(p, Z_{\mathrm{spl}}) \in D^{II}_{\mathrm{SL}(2)}$. Hence, the map $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}$ is not injective.

We prove (vii) $\Rightarrow$ (i). Take $\mathbf{r} \in Z$. Take $a \in \mathbf{R}_{>0} \smallsetminus \{1\}$, and let $\mathbf{r}' = a \circ \mathbf{r}$. Then the elements of $D_{\mathrm{SL}(2),\mathrm{nspl}}$ of the forms $(p, \tau_p(\mathbf{R}_{>0})\mathbf{r})$ and $(p, \tau_p(\mathbf{R}_{>0})\mathbf{r}')$

for the lifted action (see Section 1.2.6) are different, but they have the same image $(p, \mathbf{R}_{>0} \circ Z)$ in $D^{\star}_{\mathrm{SL}(2)}$. Hence, the map $D_{\mathrm{SL}(2),\mathrm{nspl}} \to D^{\star}_{\mathrm{SL}(2)}$ is not injective. $\square$

2.6. Relations with D_{BS} and $D^{\star}_{\mathrm{SL}(2)}$

We connect the spaces $D^{\star,-}_{\mathrm{SL}(2)}$ and D_{BS} by introducing a new space $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ of SL(2)-orbits.

2.6.1

For $Q = (Q(w))_w \in \prod_{w\in\mathbf{Z}} \mathcal{W}(\mathrm{gr}^W_w)$, let

$$G_{\mathbf{R}}(\mathrm{gr}^W)_Q := \prod_w G_{\mathbf{R}}(\mathrm{gr}^W_w)_{Q(w)}, \quad \text{where}$$

$$G_{\mathbf{R}}(\mathrm{gr}^W_w)_{Q(w)} := \{g \in G_{\mathbf{R}}(\mathrm{gr}^W_w) \mid gW' = W' \text{ for all } W' \in Q(w)\}.$$

Let $G_{\mathbf{R}}(\mathrm{gr}^W)_{Q,u}$ be the unipotent radical of $G_{\mathbf{R}}(\mathrm{gr}^W)_Q$.

2.6.2

Let $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$. We define a set $\mathcal{P}(p)$ of $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}(\mathrm{gr}^W)$. Let $X(S_p)$ be the character group of the torus S_p (see Section 2.2.7) associated to p. For $\chi \in X(S_p)$, let

$$\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi = \{v \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \mid \mathrm{Ad}(\tau^{\star}_p(t))v = \chi(t)v \text{ for all } t \in S_p\}.$$

Let $\mathcal{P}(p)$ be the set of all $\mathbf{Q}$-parabolic subgroups P of $G_{\mathbf{R}}(\mathrm{gr}^W)$ satisfying the following conditions (i) and (ii).

- (i) $P \supset G_{\mathbf{R}}(\mathrm{gr}^W)_Q$ and $P_u \supset G_{\mathbf{R}}(\mathrm{gr}^W)_{Q,u}$, where $Q = (\mathcal{W}(p_w))_w$.
- (ii) There is a subset I of $X(S_p)$ such that $\mathrm{Lie}(P) = \sum_{\chi\in I} \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi$.

2.6.3

We define $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ as a set. $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ is the set of all triples (p, P, Z), where $p \in D^{\star,-}_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, $P \in \mathcal{P}(p)$, and $Z \subset D$, satisfying the following conditions (i) and (ii). Let $A_{p,P} \subset A_p$ be the inverse image of $A_P \subset P/P_u$ under the composite map $A_p \to G_{\mathbf{R}}(\mathrm{gr}^W)_Q/G_{\mathbf{R}}(\mathrm{gr}^W)_{Q,u} \to P/P_u$. Let $B_{p,P} = \mathbf{R}_{>0} \times A_{p,P} \subset B_p$.

- (i) Z is either an $\tau^{\star}_p(A_{p,P})$-orbit in D or a $\tilde{\tau}^{\star}_p(B_{p,P})$-orbit in D_{nspl}.
- (ii) The image of Z in $D(\mathrm{gr}^W)$ is contained in the torus orbit $Z(p)$.

For $w \in \mathbf{Z}$, we denote by $D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)^{\mathrm{BS}}$ the set $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ for gr^W_w. Let $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}} := \prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)^{\mathrm{BS}}$. We have an evident map $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} \to D^{\star}_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}}$.

PROPOSITION 2.6.4

(1) *We have a canonical map*

$$D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} \to D^{\star,-}_{\mathrm{SL}(2)}, \quad (p, P, Z) \mapsto (p, \tau^{\star}_p(A_p)Z).$$

(2) *We have a canonical map*

$$D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}, \quad (p, P, Z) \mapsto (P, A_P \circ Z).$$

Here $\circ$ *denotes the Borel–Serre action with respect to* P.

Proof
It is clear that (1) holds.

We prove (2). It is sufficient to prove that, for $\mathbf{r} \in Z$ and $t \in A_{p,P}$, we have $\tau_p^\star(t)\mathbf{r} = (\tau_p^\star(t) \bmod P_u) \circ \mathbf{r}$. This follows from $\theta_{K_\mathbf{r}}(\tau_p^\star(t)) = \tau_p^\star(t)^{-1}$ (see [16, Lemma 3.8]), where $\theta_{K_\mathbf{r}}$ denotes the Cartan involution $G_\mathbf{R}(\mathrm{gr}^W) \to G_\mathbf{R}(\mathrm{gr}^W)$ associated to the maximal compact subgroup $K_\mathbf{r}$ of $G_\mathbf{R}(\mathrm{gr}^W)$. □

We give $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ a structure of an object of $\mathcal{B}'_\mathbf{R}(\log)$.

The following Lemma 2.6.5 and Sections 2.6.6–2.6.12 are preparations.

LEMMA 2.6.5
Let L *and* N *be as in Section 1.4.1. Let* R *be a finite subset of* L *such that* $R^{-1} = R$ *and such that the* $\mathbf{Q}$*-vector space* $\mathbf{Q} \otimes L$ *is generated by* R.

(1) *Let* σ *be a rational finitely generated sharp cone in* $N_\mathbf{R}$*, and let* $\mathcal{S}(\sigma) = \{l \in L \mid h(l) \geq 0$ *for all* $h \in \sigma\}$ *be the corresponding fs submonoid of* L *such that* $\mathcal{S}(\sigma)^{\mathrm{gp}} = L$*. Then* σ *satisfies the following condition (i) if and only if* $\mathcal{S}(\sigma)$ *satisfies the following conditions (ii.1) and (ii.2).*

(i) *There exists a subset* R' *of* R *such that* $R = R' \cup (R')^{-1}$ *and such that*

$$\sigma = \bigl\{h \in N_\mathbf{R} \mid h(l) \geq 0 \text{ for all } l \in R'\bigr\}.$$

(ii.1) $R \subset \mathcal{S}(\sigma) \cup \mathcal{S}(\sigma)^{-1}$.

(ii.2) *For any* $l \in \mathcal{S}(\sigma)$*, there is an integer* $n \geq 1$ *such that* l^n *belongs to the submonoid of* L *generated by* $\mathcal{S}(\sigma) \cap R$.

(2) *The set of all* σ *satisfying condition (i) in (1) is a rational fan whose support is the whole* $N_\mathbf{R}$.

(3) *Assume that we are given a subset* R^+ *of* R *which generates* $\mathbf{Q} \otimes L$ *over* $\mathbf{Q}$*. Let* $\nu := \{h \in N_\mathbf{R} \mid h(R^+) \subset \mathbf{R}_{\geq 0}\}$*. Then the* σ*'s as above such that* $\sigma \subset \nu$ *form a rational finite subdivision of* ν.

Proof
The proof of (1) is straightforward.

We prove (2). Let I be the set of all cones σ satisfying condition (i) in (1). We first prove that I is a fan.

We prove that if $\sigma_j \in I$ $(j = 1, 2)$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1. Let $R'_j \subset R$, and assume $R = R'_j \cup (R'_j)^{-1}$ and $\sigma_j = \{h \in N_\mathbf{R} \mid h(l) \geq 0$ for all $l \in R'_j\}$. Let $R' = R'_1 \cup R'_2$. Then $\sigma_1 \cap \sigma_2 = \{h \in N_\mathbf{R} \mid h(l) \geq 0$ for all $l \in R'\}$. Since $R' \smallsetminus R'_1 \subset (R'_1)^{-1}$, $\sigma_1 \cap \sigma_2$ is a face of σ_1.

We prove that if $\sigma \in I$, then any face τ of σ belongs to I. Since τ is a face of σ, we have $\mathcal{S}(\tau) = \mathcal{S}(\sigma)[b^{-1}] = \{ab^{-n} \mid a \in \mathcal{S}(\sigma), n \geq 0\}$ for some $b \in \mathcal{S}(\sigma)$. By condition (ii.2) in (1) for $\mathcal{S}(\sigma)$, there exist $n \geq 1$, $a_1, \dots, a_r \in \mathcal{S}(\sigma) \cap R$, and $m(j) \geq 1$ $(1 \leq j \leq r)$ such that $b^n = \prod_{j=1}^r a_j^{m(j)}$. We have $\mathcal{S}(\tau) = \mathcal{S}(\sigma)[1/\prod_{j=1}^r a_j]$. For the set $R' \subset R$ such that $R = R' \cup (R')^{-1}$ and $\sigma = \{h \in N_{\mathbf{R}} \mid h(l) \geq 0$ for all $l \in R'\}$, we have $\tau = \{h \in N_{\mathbf{R}} \mid h(l) \geq 0$ for all $l \in R' \cup \{a_1^{-1}, \dots, a_r^{-1}\}\}$. Hence, $\tau \in I$.

These prove that I is a fan. We show that $\bigcup_{\sigma \in I} \sigma = N_{\mathbf{R}}$. Let $h \in N_{\mathbf{R}}$. Let $R' = \{l \in R \mid h(l) \geq 0\}$. Then $R = R' \cup (R')^{-1}$. For $\sigma := \{h' \mid h'(l) \geq 0$ for all $l \in R'\} \in I$, we have $h \in \sigma$. This completes the proof of (2).

We prove (3). By (2), we have $\nu = \bigcup_{\sigma \in I} (\sigma \cap \nu)$. It is sufficient to prove that $\sigma \cap \nu \in I$ for any $\sigma \in I$. For $R' \subset R$ such that $R = R' \cup (R')^{-1}$ and $\sigma = \{h \in N_{\mathbf{R}} \mid h(l) \geq 0$ for all $l \in R'\}$, we have $\sigma \cap \nu = \{h \in N_{\mathbf{R}} \mid h(l) \geq 0$ for all $l \in R' \cup R^+\} \in I$. □

2.6.6

Let $Q = (Q(w))_w \in \prod_w \mathcal{W}(\mathrm{gr}_w^W)$. Let L be the character group of $\prod_{w \in \mathbf{Z}} \mathbf{G}_m^{Q(w)}$, and let $N = \mathrm{Hom}(L, \mathbf{Z})$. We have the situation of Section 1.4.1. As in Section 1.4.1, we denote the group law of L multiplicatively, though L is identified with $\prod_w \mathbf{Z}^{Q(w)}$.

Let $\mathcal{P}(Q)$ be the set of all $\mathbf{Q}$-parabolic subgroups P of $G_{\mathbf{R}}(\mathrm{gr}^W)$ satisfying the following conditions (i) and (ii).

(i) $P \supset G_{\mathbf{R}}(\mathrm{gr}^W)_Q$.

(ii) Take a splitting $\alpha = (\alpha_w)_w$ of Q. For $\chi \in L$, let $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi$ be the part of $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ on which the adjoint action of $\prod_w \mathbf{G}_m^{Q(w)}$ via $\alpha^\star$ is given by χ. Then there is a subset I of L such that $\mathrm{Lie}(P) = \sum_{\chi \in I} \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi$.

Under condition (i), condition (ii) is independent of the choice of α. This is because if α' is another splitting of Q, then $\alpha'(t) = g\alpha(t)g^{-1}$ for some $g \in G_{\mathbf{R}}(\mathrm{gr}^W)_Q \subset P$.

2.6.7

Let the notation be as in Section 2.6.6. Taking a splitting α of Q, define a subset

$$R(Q) = \{\chi \in L \mid \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi \neq 0\},$$

where $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi$ is defined with respect to α. This set is independent of the choice of α, because all splittings of Q are conjugate by elements of $G_{\mathbf{R}}(\mathrm{gr}^W)_Q$.

Let $L^+ = \prod_w \mathbf{N}^{Q(w)} \subset \prod_w \mathbf{Z}^{Q(w)} = L$. We will apply Lemma 2.6.5 by taking $R(Q)$ and $R(Q) \cap L^+$ as R and R^+, respectively. We show that R^+ generates the $\mathbf{Q}$-vector space $\mathbf{Q} \otimes L$, as is assumed in Lemma 2.6.5(3). Let $w \in \mathbf{Z}$, and take $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W)$ such that $Q(w) = \mathcal{W}(p_w)$. Let n be the rank of p, take a representative of p, let $N_1, \dots, N_n \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}_w^W)$ be the monodromy logarithms of the representative, and identify $Q(w)$ with $\{1, \dots, n\}$ (see Section 2.2.5). Then $\mathrm{Ad}(\tau_p^\star(t))N_j = t_j^{-2} N_j$. Hence, $R(Q(w))^+$ generates the $\mathbf{Q}$-vector space $\mathbf{Q}^{Q(w)}$. Hence, $R(Q)^+$ generates the $\mathbf{Q}$-vector space $\mathbf{Q} \otimes L = \prod_w \mathbf{Q}^{Q(w)}$.

Let $\mathcal{P}'(Q)$ be the set of all rational finitely generated sharp cones σ in $N_{\mathbf{R}}$ satisfying the following conditions (i) and (ii).

(i) There is a subset R' of $R(Q)$ such that $R(Q)=R'\cup(R')^{-1}$ and such that $\sigma=\{h\in N_{\mathbf{R}}\mid h(\chi)\geq 0$ for all $\chi\in R'\}$.

(ii) $\sigma\subset\prod_w \mathbf{R}_{\geq 0}^{Q(w)}$ in $N_{\mathbf{R}}=\prod_w \mathbf{R}^{Q(w)}$.

That is, $\mathcal{P}'(Q)$ is the set of σ considered in Lemma 2.6.5(3). Hence, $\mathcal{P}'(Q)$ is a rational fan in $N_{\mathbf{R}}$ and is a rational finite subdivision of the cone $\prod_w \mathbf{R}_{\geq 0}^{Q(w)}\subset N_{\mathbf{R}}$.

2.6.8

Let the notation be as in Sections 2.6.6 and 2.6.7. We have $\mathcal{P}(Q)=\prod_w \mathcal{P}(Q(w))$, where the element $(P_w)_w$ of the left-hand side corresponds to the element $\prod_w P_w$ of the right-hand side. We have $R(Q)=\prod_w R(Q(w))$ in $X(\prod_w \mathbf{G}_m^{Q(w)})=\prod_w X(\mathbf{G}_m^{Q(w)})$. We have $\mathcal{P}'(Q)=\prod_w \mathcal{P}'(Q(w))$, where the element $(\sigma_w)_w$ of the left-hand side corresponds to the element $\prod_w \sigma_w$ of the right-hand side.

PROPOSITION 2.6.9

Let the notation be as in Sections 2.6.6 and 2.6.7. For $P\in\mathcal{P}(Q)$, let

$$\sigma_P=\left\{h\in N_{\mathbf{R}}\mid h(\chi)\geq 0 \text{ for all } \chi\in R(Q) \text{ such that } \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_{\chi^{-1}}\subset \mathrm{Lie}(P)\right\}.$$

Then $\sigma_P\in\mathcal{P}'(Q)$, and we have a bijection

$$\mathcal{P}(Q)\to\mathcal{P}'(Q),\quad P\mapsto\sigma_P.$$

Proof

By Section 2.6.8 and by the fact $\sigma_P=\prod_w \sigma_{P_w}$, we can (and do) assume that we are in the pure situation of weight w. We denote $Q(w)$ by Q.

Take $p\in D_{\mathrm{SL}(2)}$ such that $\mathcal{W}(p)=Q$, and take τ_p as a splitting α of Q. Let $n=\sharp(Q)$ be the rank of p. Let $N_1,\dots,N_n$ be the monodromy logarithms of p. We identify Q with $\{1,\dots,n\}$.

We prove that $\sigma_P\in\mathcal{P}'(Q)$ for $P\in\mathcal{P}(Q)$. Let $R'=\{\chi\in L\mid \mathrm{Lie}(P)\cap \mathrm{Lie}(G_{\mathbf{R}})_{\chi^{-1}}\neq 0\}$. Since $\alpha^\star(\prod_w \mathbf{G}_m^{Q(w)})\subset P$ and since P is parabolic, we have $R(Q)=R'\cup(R')^{-1}$. By property (ii) of P in Section 2.6.6, we have $\mathrm{Lie}(G_{\mathbf{R}})_{\chi^{-1}}\subset \mathrm{Lie}(P)$ for $\chi\in R'$. Hence, $\sigma_P=\{h\in N_{\mathbf{R}}\mid h(\chi)\geq 0$ for all $\chi\in R'\}$. It remains to prove that $\sigma_P\subset\mathbf{R}_{\geq 0}^Q$ in $N_{\mathbf{R}}=\mathbf{R}^Q$. Since $N_j\in\mathrm{Lie}(P)$ and $\mathrm{Ad}(\tau_p^\star(t))(N_j)=t_j^{-2}N_j$ $(1\leq j\leq n)$, for any $\chi\in L^+$, χ^2 is contained in the submonoid of L generated by R'. This proves that $h(\chi)\geq 0$ for any $h\in\sigma_P$ and $\chi\in L^+$. This implies $\sigma_P\subset\mathbf{R}_{\geq 0}^Q$. Thus, we have a map $\mathcal{P}(Q)\to\mathcal{P}'(Q)$.

Next we define a map $\mathcal{P}'(Q)\to\mathcal{P}(Q)$. Let $\sigma\in\mathcal{P}'(Q)$, and let $\mathcal{S}(\sigma)\subset L$ be the corresponding fs submonoid of L. For $\chi\in L$, let $V[\chi]\subset H_{0,\mathbf{R}}$ be the sum of the χ'-components $(H_{0,\mathbf{R}})_{\chi'}$ of $H_{0,\mathbf{R}}$ for all $\chi'\in L$ such that $\chi(\chi')^{-1}\in\mathcal{S}(\sigma)$. For $\chi,\chi'\in L$, we have $V[\chi]\supset V[\chi']$ if and only if $\chi(\chi')^{-1}\in\mathcal{S}(\sigma)$. Let P be the algebraic subgroup of $G_{\mathbf{R}}$ consisting of all elements which preserve $V[\chi]$ for all $\chi \in L$. We prove $P \in \mathcal{P}(Q)$.

Since $L = \mathcal{S}(\sigma) \cup \mathcal{S}(\sigma)^{-1}$ (see Lemma 2.6.5), we have either $V[\chi] \supset V[\chi']$ or $V[\chi] \subset V[\chi']$. As in [16, Section 2.7], this totally ordered property of the set $\{V[\chi] \mid \chi \in L\}$ shows that P is a parabolic subgroup of $G_{\mathbf{R}}$. We show that P is defined over $\mathbf{Q}$. For $\chi \in L$, let $U[\chi] = \sum_{\chi'} (H_{0,\mathbf{R}})_{\chi'}$, where χ' ranges over all elements of L such that $\chi(\chi')^{-1} \in L^+$. Then $U[\chi] = \bigcap_{W' \in Q} W'_{m(W')}$, where $m(W') \in \mathbf{Z}$ is the W'-component of $\chi \in L = \mathbf{Z}^Q$. Since the W''s are rational, $U[\chi]$ is rational. Since $V[\chi]$ is the sum of $U[\chi']$ for all χ' such that $\chi(\chi')^{-1} \in \mathcal{S}(\sigma)$, $V[\chi]$ is also rational. Hence, P is rational. Properties (i) and (ii) of P in Section 2.6.6 are checked easily.

As is easily seen, the maps $\mathcal{P}(Q) \to \mathcal{P}'(Q)$ and $\mathcal{P}'(Q) \to \mathcal{P}(Q)$ are the inverses of each other. $\square$

2.6.10

Let the notation be as in Proposition 2.6.9. Via the bijection in Proposition 2.6.9, we identify the fan $\mathcal{P}'(Q)$ with the set $\mathcal{P}(Q)$ of $\mathbf{Q}$-parabolic subgroups of $G_{\mathbf{R}}(\mathrm{gr}^W)$.

Let Σ be the fan of all faces of the cone $\nu := \prod_w \mathbf{R}_{\geq 0}^{Q(w)} \subset N_{\mathbf{R}}$. By the canonical homomorphism $\mathcal{S}(\nu) = L^+ = \prod_w \mathbf{N}^{Q(w)} \to M_{\mathfrak{E}'}/\mathcal{O}_{\mathfrak{E}'}^{\times}$, where $\mathfrak{E}' = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$ (see Proposition 2.4.7), we have a morphism $\mathrm{Mor}(\cdot, D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)) \to [\Sigma]$. Consider the diagrams

$$\mathrm{Mor}(\cdot, D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)) \to [\Sigma] \leftarrow [\mathcal{P}(Q)], \qquad D_{\mathrm{SL}(2)}(\mathrm{gr}^W) \to \Sigma \leftarrow \mathcal{P}(Q).$$

LEMMA 2.6.11

Let $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$. Then $\mathcal{P}(p) \subset \mathcal{P}(Q)$. For $P \in \mathcal{P}(Q)$, $P \in \mathcal{P}(p)$ if and only if the image v of P in Σ coincides with the image of p in Σ.

Proof

It is clear that $\mathcal{P}(p) \subset \mathcal{P}(Q)$. To prove the rest, we may assume $Q(w) = \mathcal{W}(p_w)$ ($w \in \mathbf{Z}$). It is sufficient to prove that, in this case, for $P \in \mathcal{P}(Q)$, $P_u \supset G_{\mathbf{R}}(\mathrm{gr}^W)_{Q,u}$ if and only if the image of P under the map $\mathcal{P}(Q) \to \Sigma$ coincides with the face ν of ν. Let $\sigma \in \mathcal{P}'(Q)$ be the cone in $N_{\mathbf{R}}$ corresponding to P, and let $\mathcal{S} := \mathcal{S}(\sigma)$ be the corresponding fs monoid in L. Then the image of σ in Σ is ν if and only if $\mathcal{S}^{\times} \cap L^+ = \{1\}$. By the proof of Proposition 2.6.9, we have

$$\mathrm{Lie}(P_u) = \sum_{\chi \in \mathcal{S} \smallsetminus \mathcal{S}^{\times}} \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_{\chi^{-1}}, \qquad \mathrm{Lie}\bigl(G_{\mathbf{R}}(\mathrm{gr}^W)_{Q,u}\bigr) = \sum_{\chi \in L^+ \smallsetminus \{1\}} \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_{\chi^{-1}}.$$

Hence, $P_u \supset G_{\mathbf{R}}(\mathrm{gr}^W)_{Q,u}$ if $L^+ \smallsetminus \{1\} \subset \mathcal{S} \smallsetminus \mathcal{S}^{\times}$, that is, if $L^+ \cap \mathcal{S}^{\times} = \{1\}$. Let $w \in \mathbf{Z}$, and let $N_1, \ldots, N_n \in \mathrm{Lie}(G_{\mathbf{R}}(\mathrm{gr}_w^W)_{Q(w),u})$ ($n = \sharp(Q(w))$) be the monodromy logarithms of p_w. If $P_u \supset G_{\mathbf{R}}(\mathrm{gr}^W)_{Q,u}$, then $N_j \in \mathrm{Lie}(P_u)$. Since $\mathrm{Ad}(\tau_p^{\star}(t))N_j = t_j^{-2} N_j$ ($1 \leq j \leq n$), this proves $L^+ \smallsetminus \{1\} \subset \mathcal{S} \smallsetminus \mathcal{S}^{\times}$. $\square$

2.6.12
Let the notation be as in Section 2.6.10. We show that the object of $\mathcal{B}'_{\mathbf{R}}(\log)$ which represents the fiber product of $\mathrm{Mor}(\cdot, D^{\star,-}_{\mathrm{SL}(2)}(Q)) \to [\Sigma] \leftarrow [\mathcal{P}(Q)]$ is identified, as a set, with the inverse image of $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(Q)$ of $\mathfrak{D}' := D^{\star,-}_{\mathrm{SL}(2)}(Q) \subset \mathfrak{D} := D^{\star,-}_{\mathrm{SL}(2)}$ in $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ (see Proposition 2.6.4). By Lemmas 2.4.10 and 2.6.11, a point of this fiber product is identified with a triple (x, P, Z), where $x \in \mathfrak{D}'$, $P \in \mathcal{P}(p)$, $Z \subset D$, satisfying the following condition (i). Let $\mathcal{S} = \mathcal{S}(\sigma)$ be the fs submonoid of L corresponding to the cone $\sigma \in \mathcal{P}'(Q)$ which corresponds to P. Write $x = (p, Z') \in \mathfrak{D}'$, and define a subgroup $T(x, P)$ of $T(x) = \mathrm{Hom}((M^{\mathrm{gp}}_{\mathfrak{D}}/\mathcal{O}^{\times}_{\mathfrak{D}})_x, \mathbf{R}^{\mathrm{mult}}_{>0})$ as follows. If x is an A-orbit, let $T(x, P) = \mathrm{Hom}(L/\mathcal{S}^{\times}, \mathbf{R}^{\mathrm{mult}}_{>0}) \subset \mathrm{Hom}(L, \mathbf{R}^{\mathrm{mult}}_{>0}) = A_p = T(x)$. If x is a B-orbit, let $T(x, P) = \mathbf{R}_{>0} \times \mathrm{Hom}(L/\mathcal{S}^{\times}, \mathbf{R}^{\mathrm{mult}}_{>0}) \subset \mathbf{R}_{>0} \times \mathrm{Hom}(L, \mathbf{R}^{\mathrm{mult}}_{>0}) = B_p = T(x)$.

(i) Z is a $T(x, P)$-orbit in Z'.

We prove this by showing the following claim.

CLAIM
$T(x, P) = A_{p,P}$ if x is an A-orbit and $T(x, P) = B_{p,P}$ if x is a B-orbit (see Section 2.6.3).

Let $S_{p,P} = \mathrm{Hom}(L/\mathcal{S}^{\times}, \mathbf{G}_m) \subset \mathrm{Hom}(L, \mathbf{G}_m) = S_p$. Then $S_{p,P}$ coincides with the part of S_p consisting of all elements whose adjoint action on $\mathrm{Lie}(P/P_u)$ is trivial. That is, $S_{p,P}$ is the inverse image in S_p of the center of P/P_u. Since $S_{p,P}$ is $\mathbf{Q}$-split, the image of $S_{p.P}$ in P/P_u is contained in S_P. This proves that $A_{p,P}$ coincides with the connected component of $S_{p,P}(\mathbf{R})$ containing the unit element. This proves the above claim.

Since $Z' = \tau^{\star}_p(A_p)Z$, a triple (x, P, Z) as above corresponds to a point (p, P, Z) of $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(Q)$ (see Section 2.6.3) in a one-to-one manner.

2.6.13
For $Q \in \prod_w \mathcal{W}(\mathrm{gr}^W_w)$, we define the structure of $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(Q)$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by identifying it as a log modification of $D^{\star,-}_{\mathrm{SL}(2)}(Q)$ by Section 2.6.12. When Q moves, these structures on $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(Q)$ glue globally to a structure of $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$.

For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}(\mathrm{gr}^W)$, let

$$D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(P) = \{(p, P', Z) \in D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} \mid P' \supset P\}.$$

Then $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(P)$ is an open set of $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$, and when P moves, we have a covering of $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ by these open sets.

PROPOSITION 2.6.14

The diagram

$$\begin{array}{ccc} D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} & \to & D^{\star,-}_{\mathrm{SL}(2)} \\ \downarrow & & \downarrow \\ D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}} & \to & D_{\mathrm{SL}(2)}(\mathrm{gr}^W) \end{array}$$

is Cartesian in $\mathcal{B}'_{\mathbf{R}}(\log)$ *and also in the category of topological spaces.*

Proof

This is because $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(Q)$ represents the fiber product of $\mathrm{Mor}(\cdot, D^{\star,-}_{\mathrm{SL}(2)})(Q) \to [\Sigma] \leftarrow [\mathcal{P}(Q)]$ and $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}}(Q)$ represents the fiber product of $\mathrm{Mor}(\cdot, D_{\mathrm{SL}(2)}(\mathrm{gr}^W)) \to [\Sigma] \leftarrow [\mathcal{P}(Q)]$. □

PROPOSITION 2.6.15

Let $F \in D(\mathrm{gr}^W)$, $\bar{L} = \bar{\mathcal{L}}(F)$. *Then* $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ *is an* $\bar{L}$*-bundle over* $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}} \times \mathrm{spl}(W)$.

Proof

This follows from Proposition 2.6.14 and the corresponding result for $D^{\star,-}_{\mathrm{SL}(2)}$. □

2.6.16

For $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ and $P \in \mathcal{P}(p)$, let $S_{p,P} \subset S_p$ be the torus defined in Section 2.6.12, let $X(S_{p,P})$ be the character group of $S_{p,P}$, and let $X(S_{p,P})^+ = \mathcal{S}/\mathcal{S}^\times$, where $\mathcal{S} := \mathcal{S}(\sigma_P)$ (see Lemma 2.6.5) with σ_P the cone corresponding to P (see Proposition 2.6.9). Define a real toric variety $\bar{A}_{p,P}$ by

$$\bar{A}_{p,P} := \mathrm{Hom}\big(X(S_{p,P})^+, \mathbf{R}^{\mathrm{mult}}_{\geq 0}\big) \supset A_{p,P} = \mathrm{Hom}\big(X(S_{p,P}), \mathbf{R}^{\mathrm{mult}}_{>0}\big).$$

We have a canonical morphism

$$\bar{A}_{p,P} \to \bar{A}_p$$

induced from the homomorphism $X(S_p)^+ \to X(S_{p,P})^+$ which is induced by the inclusion map $S_{p,P} \to S_p$.

LEMMA 2.6.17

(1) *The homomorphism* $X(S_P) \to X(S_{p,P})$ *induced by* $S_{p,P} \to S_P$ *(see Section 2.6.12) sends* $X(S_P)^+$ *to* $X(S_{p,P})^+$.

(2) *The map* $A_{p,P} \to A_P$ *extends uniquely to a morphism* $\bar{A}_{p,P} \to \bar{A}_P$ *in* $\mathcal{B}'_{\mathbf{R}}(\log)$.

Proof

We prove (1). As a monoid, $X(S_P)^+$ is generated by $\Delta(P)$ (see Section 2.4.2). For $\chi \in \Delta(P)$, χ^{-1} appears in $\mathrm{Lie}(P)$. Hence, the image of χ^{-1} in $X(S_{p,P})$ appears in $\mathrm{Lie}(P)$. Hence, the image of χ in $X(S_{p,P})$ belongs to $X(S_{p,P})^+$.

Then (2) follows from (1). In fact, the homomorphism $X(S_P)^+ \to X(S_{p,P})^+$ in (1) induces the morphism $\mathrm{Hom}(X(S_P)^+, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \to \mathrm{Hom}(X(S_{p,P})^+, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$. □

2.6.18

In Theorem 2.6.19, we will consider the local structure of $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$, comparing it with the local structure of D_{BS}. Here we give preparations. We consider the following two situations (bd) and (d).

(bd) $\mathfrak{D} = D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ and $\mathfrak{E} = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}}$.

(d) $\mathfrak{D} = D_{\mathrm{BS}}$ and $\mathfrak{E} = D_{\mathrm{BS}}(\mathrm{gr}^W)$.

Fix $p \in \mathfrak{E}$ and $\mathbf{r} \in Z(p)$ (see Section 2.2.7). In situation (bd) (resp., (d)), fix $P \in \mathcal{P}(p)$ (resp., fix a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $p \in \mathfrak{E}(P)$). Let R be an $\mathbf{R}$-subspace of $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ satisfying the following conditions (C1) and (C2).

(C1) $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) = \mathrm{Lie}(\tau^\star(A_{p,P})) \oplus R \oplus \mathrm{Lie}(K_{\mathbf{r}})$ (resp., $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) = \mathrm{Lie}((A_P)_{\mathbf{r}}) \oplus R \oplus \mathrm{Lie}(K_{\mathbf{r}})$, where $(A_P)_{\mathbf{r}}$ denotes the Borel–Serre lifting from Section 2.4.1 of A_P at $\mathbf{r}$).

(C2) $R \subset \mathrm{Lie}(P)$.

These conditions on R are similar to those in Section 2.3.13. Like in Section 2.3.13, let S be an $\mathbf{R}$-subspace of $\mathrm{Lie}(K_{\mathbf{r}})$ such that $\mathrm{Lie}(K_{\mathbf{r}}) = \mathrm{Lie}(K'_{\mathbf{r}}) \oplus S$.

We define an object Y of $\mathcal{B}'_{\mathbf{R}}(\log)$ as follows. Let

$$Y = \bar{A}_P \times R \times S \quad \text{in situation (d).}$$

In situation (bd), we define Y as follows. Let

$$X = \bar{A}_{p,P} \times R \times S.$$

Let Y be the subset of X consisting of all elements (t, f, k) satisfying the following conditions (i) and (ii).

(i) If $\chi \in X(S_p)$ and $t(\chi_+) = 0$, then $f_\chi = 0$. In other words, if $m(w, j)$ denotes the (w, j)-component of $\chi \in X(S_p) = \prod_w \mathbf{Z}^{Q(w)}$, then $f_\chi = 0$ unless $m(w, j) \leq 0$ for any $w \in \mathbf{Z}$ and $j \in J(w)$. Here χ_+, f_χ, and $J(w)$ are as in Section 2.3.13.

(ii) $k \in S_J$. Here S_J is as in Section 2.3.13.

Regard X as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ in the natural way, and regard $Y \subset X$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by Section 1.3.16. In both situations (bd) and (d), let

$$Y_0 = \{(t, f, k) \in Y \mid t \in A_{p.P}\} \subset Y.$$

THEOREM 2.6.19

Let the notation be as in Section 2.6.18. Consider situation (bd) where $\mathfrak{D} = D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ and $\mathfrak{E} = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}}$ (resp., (d) where $\mathfrak{D} = D_{\mathrm{BS}}$ and $\mathfrak{E} = D_{\mathrm{BS}}(\mathrm{gr}^W)$).

(1) *For a sufficiently small open neighborhood U of $(0,0,0)$ in Y, there exists a unique open immersion $U \to \mathfrak{E}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$ which sends $(t,f,k) \in U \cap Y_0$ to the element*

$$\exp(f)\tau_p^\star(t)\exp(k)\mathbf{r} \quad \bigl(\textit{resp.}, \ t \circ \exp(f)\exp(k)\mathbf{r}\bigr)$$

of $D(\mathrm{gr}^W) \subset \mathfrak{E}$.

(2) *Let $\bar{L} = \bar{\mathcal{L}}(\mathbf{r})$ and $L = \mathcal{L}(\mathbf{r})$. Then for a sufficiently small open neighborhood U of $(0,0,0)$ in Y, there exists a unique open immersion $U \times \mathrm{spl}(W) \times \bar{L} \to \mathfrak{D}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$ having the following property. It sends $(t,f,k,s,\delta) \in Y \times \mathrm{spl}(W) \times L$, where $(t,f,k) \in U \cap Y_0$, $s \in \mathrm{spl}(W)$, and $\delta \in L$, to the element of D (resp., to the element $t \circ x$, where x is the element of D) whose image in $D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ under the isomorphism from Section 1.2.1 is*

$$\bigl(\exp(f)\tau_p^\star(t)\exp(k)\mathbf{r}, s, \mathrm{Ad}\bigl(\exp(f)\tau_p^\star(t)\exp(k)\bigr)\delta\bigr)$$

$$\bigl(\textit{resp.}, \ \bigl(\exp(f)\exp(k)\mathbf{r}, s, \mathrm{Ad}\bigl(\exp(f)\exp(k)\bigr)\delta\bigr)\bigr).$$

(3) *For a sufficiently small open neighborhood U of $(0,0,0)$ in Y, the diagram*

$$\begin{array}{ccc} U \times \mathrm{spl}(W) \times \bar{L} & \to & \mathfrak{D} \\ \downarrow & & \downarrow \\ U & \to & \mathfrak{E} \end{array}$$

is Cartesian in $\mathcal{B}'_{\mathbf{R}}(\log)$ and in the category of topological spaces.

(4) *The image of the map in (1) is contained in $\mathfrak{E}(Q) \cap \mathfrak{E}(P)$ with $Q = (\mathcal{W}(p_w))_w$ (resp., in $\mathfrak{E}(P)$), and the image of the map in (2) is contained in $\mathfrak{D}(Q) \cap \mathfrak{D}(P)$ (resp., in $\mathfrak{D}(P)$).*

(5) *The underlying maps of the morphisms in (1) and (2) are described as in Section 2.6.20 below.*

2.6.20

The maps in Theorem 2.6.19(1) and 2.6.19(2) are induced from the maps

$$Y \to \mathfrak{E}, \qquad Y \times \mathrm{spl}(W) \times \bar{L} \to \mathfrak{D},$$

respectively, defined as follows.

We first consider situation (bd) in Section 2.6.18. Let A' be the subset of $A_{p,P} = \mathrm{Hom}(X(S_{p,P}), \mathbf{R}^{\mathrm{mult}}_{>0})$ consisting of all elements whose restriction to $t^{-1}(\mathbf{R}_{>0}) \cap X(S_{p,P})^+$ coincides with the restriction of $t : X(S_{p,P})^+ \to \mathbf{R}_{\geq 0}$, where t ranges over $\bar{A}_{p,P}$. Let $J = (J(w))_w$ for t be as in Section 2.6.18, and let $p_J \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ be as in Section 2.3.15 for J. Then the first map $Y \to \mathfrak{E}$ sends (t,f,k) to

$$p' := \exp(f)\tau_p^\star(t')\exp(k)p_J, \quad \text{where } t' \in A'.$$

The second map $Y \times \mathrm{spl}(W) \times \bar{L} \to \mathfrak{D}$ sends (t,f,k,s,δ) to the following element (p', P', Z) of $\mathfrak{D} = D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ (see Section 2.6.3), where P' and Z are as follows. Let $\bar{A}_{p,P} \to \bar{A}_P = \mathbf{R}^{\Delta(P)}_{\geq 0}$ be the morphism in Lemma 2.6.17. Let $I = \{j \in \Delta(P) \mid t_j = 0\}$, where t_j denotes the j-component of the image of t in $\mathbf{R}^{\Delta(P)}_{\geq 0}$. Then P'

is the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ such that $P' \supset P$ that corresponds to the subset I of $\Delta(P)$ (see Section 2.4.2).

If $\delta \in L$, then Z is the subset of D whose image under the embedding $D \to D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ is the set

$$\big\{\big(\exp(f)\tau_p^\star(t')\exp(k)\mathbf{r}, s, \mathrm{Ad}\big(\exp(f)\tau_p^\star(t')\exp(k)\big)\delta\big) \mid t' \in A'\big\}.$$

If $\delta = 0 \circ \delta^{(1)} \in \bar{L} \smallsetminus L$ ($\delta^{(1)} \in L \smallsetminus \{0\}$) (see Section 1.3.8(4)), then Z is the subset of D whose image under the embedding $D \to D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ is the set

$$\big\{(\exp(f)\tau_p^\star(t')\exp(k)\mathbf{r}, s, \mathrm{Ad}\big(\exp(f)\tau_p^\star(t')\exp(k)\big)(c \circ \delta^{(1)}) \mid t' \in A', c \in \mathbf{R}_{>0}\big\}.$$

Next consider situation (d) in Section 2.6.18. In this situation, the first map sends (t, f, k) to $t \circ \exp(f)\exp(k)\mathbf{r}$. The second map sends (t, f, k, s, δ) with $\delta \in L$ to the element $t \circ x$ and sends $(t, f, k, s, 0 \circ \delta)$ with $\delta \in L \smallsetminus \{0\}$ to the element $(0, t) \circ x$, where x is the element of D whose image in $D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ (see Section 1.2.1) is $(\exp(f)\exp(k)\mathbf{r}, s, \mathrm{Ad}(\exp(f)\exp(k))\delta)$. Here we denote by $(t, x) \mapsto t \circ x$ the morphisms $\bar{A}_P \times D(\mathrm{gr}^W) \to D_{\mathrm{BS}}(\mathrm{gr}^W)$, $\bar{A}_P \times D \to D_{\mathrm{BS}}$, and $\bar{B}_P \times D_{\mathrm{nspl}} \to D_{\mathrm{BS}}$, which extend the morphisms $A_P \times D(\mathrm{gr}^W) \to D_{\mathrm{BS}}(\mathrm{gr}^W)$, $A_P \times D \to D_{\mathrm{BS}}$, and $B_P \times D \to D_{\mathrm{BS}}$, defined by $(t, x) \mapsto t \circ x$, respectively.

2.6.21

We prove Theorem 2.6.19. The theorem is clear in situation (d) in Section 2.6.18.

We consider situation (bd) in Section 2.6.18. We reduce the theorem in this situation to Theorem 2.3.14.

It is easily seen that the validity of the theorem does not depend on the choices of R and S. We take any S satisfying the condition in Section 2.3.13 and, hence, the condition in Section 2.6.18. We choose R in the following way.

Let $Q = (Q(w))_w$, where $Q(w) = \mathcal{W}(\mathrm{gr}_w^W)$. Take a splitting α of Q, and let $R(Q)$ be as in Section 2.6.7. Let $\sigma_P \in \mathcal{P}'(Q)$ be the cone corresponding to $P \in \mathcal{P}(Q)$ (see Proposition 2.6.9), and let $\mathcal{S} := \mathcal{S}(\sigma_P)$ be the corresponding fs submonoid of $X(S_p)$. Note that $R(Q) \subset \mathcal{S} \cup \mathcal{S}^{-1}$ (see Lemma 2.6.5).

Choose a subset I_1 of $R(Q) \cap \mathcal{S} \cap \mathcal{S}^{-1}$ such that $R(Q) \cap \mathcal{S} \cap \mathcal{S}^{-1}$ is the disjoint union of $\{1\}$, I_1, and I_1^{-1}. Let $I_2 := R(Q) \cap \mathcal{S}^{-1} \smallsetminus R(Q) \cap \mathcal{S} \cap \mathcal{S}^{-1}$. Hence, $R(Q)$ is the disjoint union of $\{1\}$, I_1, I_1^{-1}, I_2, and I_2^{-1}. Choose an $\mathbf{R}$-subspace C of $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ such that the subspace $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_1 = \{v \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \mid \mathrm{Ad}(\tau_p^\star(t))v = v \text{ for all } t \in A_p\}$ of $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ coincides with the direct sum of $\mathrm{Lie}(\tau_p^\star(A_p))$, C, and $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_1 \cap \mathrm{Lie}(K_{\mathbf{r}})$. Let

$$R' = C \oplus \Big(\bigoplus_{\chi \in I_1 \cup I_2} \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi\Big).$$

Then $R' \subset \mathrm{Lie}(P)$, and R' satisfies conditions (C1) and (C2) on R of Section 2.3.13. (Here we used the fact that the Cartan involution $\theta_{K_{\mathbf{r}}}$ associated to the maximal compact subgroup $K_{\mathbf{r}}$ of $G_{\mathbf{R}}(\mathrm{gr}^W)$ sends $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi$ to $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_{\chi^{-1}}$ for any $\chi \in X(S_p)$, and $\mathrm{Lie}(K_{\mathbf{r}})$ coincides with $\{v \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \mid \theta_{K_{\mathbf{r}}}(v) = v\}$.)

Take an $\mathbf{R}$-subspace C' of $\mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$ such that $\mathrm{Lie}(\tau_p^\star(A_p)) = \mathrm{Lie}(\tau_p^\star(A_{p,P})) \oplus C'$. We take $R := C' \oplus R'$ as R of Section 2.6.18.

Define X and Y of Section 2.6.18 by using these R and S. Denote by X' and Y', respectively, the X and Y of Section 2.3.13 defined by taking R' and S as R and S of Section 2.3.13.

As in Section 2.6.10, let Σ be the fan of all faces of the cone $\mathrm{Hom}(X(S_p)^+, \mathbf{R}^{\mathrm{add}}_{\geq 0}) = \prod_w \mathbf{R}^{Q(w)}_{\geq 0}$. Let Σ' be the fan of all faces of the cone σ_P.

Then $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(Q) \cap D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}(P)$ represents the fiber product of $\mathrm{Mor}(\cdot, D^\star_{\mathrm{SL}(2)}(Q)) \to [\Sigma] \leftarrow [\Sigma']$. On the other hand, the fiber product of $\mathrm{Mor}(\cdot, X') \to [\Sigma] \leftarrow [\Sigma']$ is represented by $X'' := \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W) \times S$, and the fiber product of $\mathrm{Mor}(\cdot, Y') \to [\Sigma] \leftarrow [\Sigma']$ is represented by the inverse image Y'' of Y' in X'' under the canonical map $X'' \to X'$, where Y'' is endowed with the structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by using the embedding $Y'' \to X''$ (see Section 1.3.16). We identify X with $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \times R' \times S$ via the isomorphism $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \cong \bar{A}_{p,P} \times C'$.

To reduce Theorem 2.6.19 to Theorem 2.3.14, it is sufficient to prove the following (*).

(*) If $(t, f, g, h, k) \in Y''$, then $(t, f, k) \in Y$ in $X = \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \times R' \times S$. We have an isomorphism

$$Y'' \overset{\cong}{\to} Y, \quad (t, f, g, h, k) \mapsto (t, f, k)$$

in $\mathcal{B}'_{\mathbf{R}}(\log)$.

Before the proof of (*), we note the following (1) and (2).

(1) Let $(t, f, g, h, k) \in X''$ ($t \in \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$, $f, g, h \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)$, $k \in S$). Then (t, f, g, h, k) belongs to Y'' if and only if the conditions (i)–(iv) in Section 2.3.13, among which (iii) and (iv) are modified as follows, are satisfied. We replace R in (iii) in Section 2.3.13 by R'. In (iv) in Section 2.3.13, we define $J = (J(w))_{w \in \mathbf{Z}}$, where $J(w) = \{j \in Q(w) \mid t_{w,j} = 0\}$. Here $t_{w,j} \in \mathbf{R}_{\geq 0}$ denotes the (w, j)-component of the image of t in $\bar{A}_p$. Then $k \in S_J$.

(2) Let $(t, f, k) \in X$ ($t \in \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$, $f \in R'$, $k \in S$). Then (t, f, k) belongs to Y if and only if the following conditions (2.i) and (2.ii) are satisfied.

(2.i) Let $\chi \in X(S_p)$. If $t(\chi_+) = 0$, then $f_\chi = 0$.

(2.ii) The same as the form of (iv) in the above (1).

Now we prove the assertion (*). Let $(t, f, g, h, k) \in Y''$. We first prove that $(t, f, k) \in Y$. To show this, it is sufficient to prove $f \in R'$. Let $\chi \in R(Q)$. If $t(\chi_-) \neq 0$, since $t(\chi_+) g_\chi = t(\chi_-) f_\chi$ and $g_\chi \in R'$, we have $f_\chi = t(\chi_-)^{-1} t(\chi_+) g_\chi \in R'$. Assume $t(\chi_-) = 0$. If $\chi \in \mathcal{S}$, then $t(\chi_+) = t(\chi_-) t(\chi) = 0$. Hence, $f_\chi = 0$. If $\chi \notin \mathcal{S}$, then $\chi \in \mathcal{S}^{-1}$, and hence, $f_\chi \in \mathfrak{g}_{\mathbf{R}}(\mathrm{gr}^W)_\chi \subset R'$.

We next prove that $Y'' \to Y$ is an isomorphism. For this, we define a morphism $Y \to X''$ of the converse direction by $(t, f, k) \mapsto (t, f, g, h, k)$ with $g = \sum_{\chi \in \mathcal{S}^{-1}} t(\chi^{-1}) f_\chi$ and $h = \sum_{\chi \in \mathcal{S}^{-1}} t(\chi^{-1})^2 f_\chi$. We show that the image of this morphism is contained in Y''. Let $\chi \in R(Q)$. We prove $t(\chi_+) g_\chi = t(\chi_-) f_\chi$ and $t(\chi_+) h_\chi = t(\chi_-) g_\chi$. If $\chi \in \mathcal{S}^{-1}$, we have $t(\chi_+) g_\chi = t(\chi_+) t(\chi^{-1}) f_\chi = t(\chi_-) f_\chi$ and $t(\chi_+) h_\chi = t(\chi_+) t(\chi^{-1})^2 f_\chi = t(\chi_-) g_\chi$. If $\chi \notin \mathcal{S}^{-1}$, we have $f_\chi = 0$ by the definition of R', and hence, $g_\chi = h_\chi = 0$ by the definitions of g and h. If $t(\chi_+) = 0$, then $f_\chi = 0$ and hence $g_\chi = 0$. We prove that if $t(\chi_-) = 0$, then $g_\chi = h_\chi = 0$. In the case $t(\chi_+) = 0$, then $f_\chi = 0$ and hence $g_\chi = h_\chi = 0$. In the case $\chi \in \mathcal{S}$, we have $t(\chi_+) = t(\chi_-)t(\chi) = 0$. In the case $t(\chi_+) \neq 0$ and $\chi \notin \mathcal{S}$, we have $\chi \in \mathcal{S}^{-1}$ and $t(\chi_-) = t(\chi_+)t(\chi^{-1})$ and hence $t(\chi^{-1}) = 0$. Hence, $g_\chi = t(\chi^{-1})^2 f_\chi = 0$ and $h_\chi = 0$ similarly. We prove $g_\chi, h_\chi + f_{\chi^{-1}} \in R'$. If $\chi \in \mathcal{S}^{-1}$, then $g_\chi = t(\chi)^{-1} f_\chi \in R'$ and $h_\chi = t(\chi^{-1})^2 \in R'$ and hence $h_\chi + f_{\chi^{-1}} \in R'$. If $\chi \notin \mathcal{S}^{-1}$, then $g_\chi = h_\chi = 0$ and hence $h_\chi + f_{\chi^{-1}} = f_{\chi^{-1}} \in R'$.

Thus, we have a morphism $Y \to Y''$. It is clear that the composition $Y \to Y'' \to Y$ is the identity morphism. We prove that the composition $Y'' \to Y \to Y''$ is also the identify morphism. Let $(t, f, g, h, k) \in Y''$, and let (t, f, g', h', k) be the image of $(t, f, k) \in Y$ under $Y \to Y''$. We prove $g'_\chi = g_\chi$ and $h'_\chi = h_\chi$ for any $\chi \in R(Q)$. Assume first $\chi \in \mathcal{S}^{-1}$. If $t(\chi_+) \neq 0$, then $g_\chi = t(\chi_+)^{-1} t(\chi_-) f_\chi = t(\chi^{-1}) f_\chi = g'_\chi$, and we have similarly $h_\chi = h'_\chi$. If $t(\chi_+) = 0$, then $t(\chi_-) = t(\chi_+)t(\chi^{-1}) = 0$, and hence $f_\chi = g_\chi = h_\chi = 0$, and we have $g'_\chi = 0$ and $h'_\chi = 0$ by $f_\chi = 0$. Next assume $\chi \notin \mathcal{S}^{-1}$. Then by the definition of R', we have $a_\chi = 0$ for any $a \in R'$. Since $f_\chi, g_\chi, h_\chi + f_{\chi^{-1}} \in R'$, we have $f_\chi = g_\chi = h_\chi = 0$, and we have $g'_\chi = h'_\chi = 0$ by $f_\chi = 0$. Theorem 2.6.19 is proved.

THEOREM 2.6.22

(1) *The identity map of D extends uniquely to a morphism $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ in $\mathcal{B}'_{\mathbf{R}}(\log)$. It sends $(p, P, Z) \in D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ to $(P, A_P \circ Z) \in D_{\mathrm{BS}}$.*

(2) *The diagram*

$$\begin{array}{ccc} D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} & \to & D_{\mathrm{BS}} \\ \downarrow & & \downarrow \\ D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\mathrm{BS}} & \to & D_{\mathrm{BS}}(\mathrm{gr}^W) \end{array}$$

is Cartesian in $\mathcal{B}'_{\mathbf{R}}(\log)$ and also in the category of topological spaces.

(3) *The inverse image of $D^{\mathrm{mild}}_{\mathrm{BS}}$ in $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ coincides with $D^{\star,\mathrm{BS},\mathrm{mild}}_{\mathrm{SL}(2)}$.*

Proof

Let $(p, P, Z) \in D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$, and take $\mathbf{r} \in Z$. We compare situations (bd) and (d) in Theorem 2.6.19 by taking p, $\mathbf{r}$ for both situations (bd) and (d), and by taking R and S for these situations as follows. Take R and S for situation (d). Take this S as S for situation (bd). Let C be an $\mathbf{R}$-subspace of $\mathrm{Lie}((A_P)_{\mathbf{r}})$ such that $\mathrm{Lie}((A_P)_{\mathbf{r}}) = \mathrm{Lie}(\tau^\star(A_{p,P})) \oplus C$, and take $C \oplus R$ as the R for situation (bd). Then Theorem 2.6.22(1) and 2.6.22(2) follow from Theorem 2.6.19, Lemma 2.6.17, and the fact that

$$\big(\tau^\star(t) \bmod P_u\big) \circ \exp(f)\exp(k)\mathbf{r} = \exp(f)\tau^\star_p(t)\exp(k)\mathbf{r}.$$

Theorem 2.6.22(3) is clear. □

2.7. The category $\mathcal{B}'_{\mathbf{R}}(\log)^+$

The aim of this section is to define a full subcategory $\mathcal{B}'_{\mathbf{R}}(\log)^+$ of $\mathcal{B}'_{\mathbf{R}}(\log)$, consisting of nice objects, and prove that the spaces of SL(2)-orbits in Section 2 belong to $\mathcal{B}'_{\mathbf{R}}(\log)^+$ (see Theorem 2.7.14). We also discuss full subcategories $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$ and $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$ such that

$$\mathcal{B}'_{\mathbf{R}}(\log) \supset \mathcal{B}'_{\mathbf{R}}(\log)^+ \supset \mathcal{B}'_{\mathbf{R}}(\log)^{[+]} \supset \mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}.$$

2.7.1
We first define a full subcategory $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$. We define standard objects of $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. Take $n \geq 0$, a real analytic manifold A, and a real analytic closed submanifold A_J of A for each subset J of $\{1,\dots,n\}$ satisfying $A_\emptyset = A$, $A_J \subset A_{J'}$ if $J \supset J'$. Define

$$Y = \left\{(t,x) \in \mathbf{R}^n_{\geq 0} \times A \mid x \in A_{J(t)}\right\},$$

where $J(t) = \{j \mid 1 \leq j \leq n, t_j = 0\}$. We regard Y as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by taking $\mathbf{R}^n_{\geq 0} \times A$ as X in Section 1.3.16, where the log structure of X with sign is induced from that of $\mathbf{R}^n_{\geq 0}$ (see Section 1.3.8(1)).

Let $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ be the full subcategory of $\mathcal{B}'_{\mathbf{R}}(\log)$ consisting of all objects which are locally isomorphic to open subobjects of Y as above. Real analytic manifolds with corners belong to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$.

2.7.2
We next define a full subcategory $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$ of $\mathcal{B}'_{\mathbf{R}}(\log)$. We define standard objects of $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$. Take an fs monoid $\mathcal{S}$, a real analytic manifold A, and a real analytic closed submanifold A_I of A for each face I of $\mathcal{S}$ satisfying $A_{\mathcal{S}} = A$, $A_I \subset A_{I'}$ if $I \subset I'$. Define

$$Y = \left\{(t,x) \in \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \times A \mid x \in A_{I(t)}\right\},$$

where $I(t)$ is the face $\{a \in \mathcal{S} \mid t(a) \neq 0\}$ of $\mathcal{S}$. We regard Y as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by taking $\operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \times A$ as X in Section 1.3.16, where the log structure of X with sign is induced from that of $\operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ (see Section 1.3.8(3)).

Let $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$ be the full subcategory of $\mathcal{B}'_{\mathbf{R}}(\log)$ consisting of all objects which are locally isomorphic to open subobjects of Y as above. Since a standard object of $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ is the case $\mathcal{S} = \mathbf{N}^n$ of a standard object of $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$, we have $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]} \supset \mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$.

The following Lemmas 2.7.3 and 2.7.4 are proved easily.

LEMMA 2.7.3
Let S be an object of $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$. Then S belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ if and only if, for any $s \in S$, $(M_S/\mathcal{O}_S^\times)_s$ is isomorphic to $\mathbf{N}^r$ for some $r \geq 0$ (which may depend on s).

LEMMA 2.7.4
Let $S' \to S$ be a log modification in $\mathcal{B}'_{\mathbf{R}}(\log)$. If S belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$, then S' also belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$.

2.7.5
We define a full subcategory $\mathcal{B}'_{\mathbf{R}}(\log)^+$ of $\mathcal{B}'_{\mathbf{R}}(\log)$. Let $\mathcal{B}'_{\mathbf{R}}(\log)^+$ be the full subcategory of $\mathcal{B}'_{\mathbf{R}}(\log)$ consisting of all objects S such that, locally on S, there is a log modification $S' \to S$ such that S' belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. We have clearly $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]} \subset \mathcal{B}'_{\mathbf{R}}(\log)^+$.

LEMMA 2.7.6
Let S be an object of $\mathcal{B}'_{\mathbf{R}}(\log)^+$, and assume that $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ is of rank at most 1 as an Abelian group for any $s \in S$. Then S belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$.

Proof
This is because any log modification $S' \to S$ is an isomorphism. □

PROPOSITION 2.7.7
We have that $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]} \subset \mathcal{B}'_{\mathbf{R}}(\log)^+$.

Proof
Let S be an object of $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$. Locally on S, by the resolution of singularities in toric geometry (see [18, p. 23]), there exists a log modification $S' \to S$ such that, for any $s \in S'$, $(M_{S'}/\mathcal{O}_{S'}^\times)_s \cong \mathbf{N}^r$ for some r. By Lemmas 2.7.3 and 2.7.4, S' belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. □

PROPOSITION 2.7.8
Let $S' \to S$ be a log modification in $\mathcal{B}'_{\mathbf{R}}(\log)$. Then, S belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$ if and only if S' belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$.

Proof
First assume that S belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. We prove that S' belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. We may assume that S belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. Locally on S, there is a log modification $S'' \to S$ which is a composition $S'' \to S' \to S$, where the first arrow is a log modification and the second arrow is the given morphism, such that, for any $s \in S''$, $(M_{S''}/\mathcal{O}_{S''}^\times)_s \cong \mathbf{N}^r$ for some r. By Lemmas 2.7.3 and 2.7.4, S'' belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. Hence, S' belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$.

Next assume that S' belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. We prove that S belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. By the assumption, there are an open covering $(U_\lambda)_\lambda$ of S' and a log modification $V_\lambda \to U_\lambda$ for each λ such that V_λ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. Since $S' \to S$ is proper, locally on S, we can take a finite covering $(U_\lambda)_\lambda$. Hence, locally on S, there is a log modification $S'' \to S$ having the following properties (i)–(iii). (i) $S'' \to S$ is a composition $S'' \to S' \to S$, where the first arrow is a log modification and the second arrow is the given morphism. (ii) For each λ, we have a morphism $U_\lambda \times_{S'} S'' \to V_\lambda$ over U_λ which is a log modification. (iii) For any $s \in S''$, $(M_{S''}/\mathcal{O}_{S''}^\times)_s \cong \mathbf{N}^r$ for some $r \geq 0$. By Lemmas 2.7.3 and 2.7.4, $U_\lambda \times_{S'} S''$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. Since $(U_\lambda \times_{S'} S'')_\lambda$ is an open covering of S'', S'' belongs to $\mathcal{B}'(\log)^{[[+]]}$. Hence, S belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. □

PROPOSITION 2.7.9

The category $\mathcal{B}'_{\mathbf{R}}(\log)^+$ *(resp.,* $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$, $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$*) is stable in* $\mathcal{B}'_{\mathbf{R}}(\log)$ *under taking finite products.*

Proof

This is clear for $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$ and $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. The part for $\mathcal{B}'_{\mathbf{R}}(\log)^+$ follows from the part for $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. □

LEMMA 2.7.10

Let $Y \subset \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \times A$ *be a standard object of* $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$ *in Section 2.7.2, let* S *be an object of* $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$*, and let* $S \to \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ *be a morphism in* $\mathcal{B}'_{\mathbf{R}}(\log)$*. Then the fiber product of* $S \to \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \leftarrow Y$ *in* $\mathcal{B}'_{\mathbf{R}}(\log)$ *belongs to* $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$*.*

Proof

Working locally on S, we may assume that S is an open set of the standard object $\mathbf{R}^n_{\geq 0} \times A'$ in Section 2.7.1 (A' here plays the role of A in Section 2.7.1), and that we have a commutative diagram of functors

$$\begin{array}{ccccc} \operatorname{Mor}(\cdot, S) & & \to & & \operatorname{Mor}(\cdot, \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})) \\ \downarrow & & & & \downarrow \\ \operatorname{Mor}(\cdot, \mathbf{R}^n_{\geq 0}) & \to & [\Sigma'] & \to & [\Sigma] \end{array}$$

where Σ is the fan of all faces of the cone $\operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{add}}_{\geq 0})$ and Σ' is the fan of all faces of the cone $\mathbf{R}^n_{\geq 0} \subset \mathbf{R}^n$. Then the fiber product in the problem coincides with the space

$$\{(t, a, a') \in \mathbf{R}^n_{\geq 0} \times A \times A' \mid a \in A_{I(t)}, a' \in A'_{J(t)}\},$$

where $J(t) = \{j \mid 1 \leq j \leq n, t_j = 0\}$ and $I(t)$ is the face of $\mathcal{S}$ which corresponds to the image of t under $\mathbf{R}^n_{\geq 0} \to \Sigma' \to \Sigma$. □

LEMMA 2.7.11

Let $Y \subset \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \times A$ *be a standard object of* $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$ *in Section 2.7.2, let* S *be an object of* $\mathcal{B}'_{\mathbf{R}}(\log)^+$*, and let* $S \to \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ *be a strict morphism in* $\mathcal{B}'_{\mathbf{R}}(\log)$*. Then the fiber product of* $S \to \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \leftarrow Y$ *in* $\mathcal{B}'_{\mathbf{R}}(\log)$ *belongs to* $\mathcal{B}'_{\mathbf{R}}(\log)^+$*.*

Proof

Since $S \to \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ is strict, working locally on $\operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0})$ and on S, we have a rational finite subdivision Σ' of the cone $\operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{add}}_{\geq 0})$ such that the fiber product S' of $S \to \operatorname{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \leftarrow |\mathrm{toric}|(\Sigma')$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ and such that $\mathcal{S}(\sigma')$ for all $\sigma' \in \Sigma'$ are isomorphic to $\mathbf{N}^r \times \mathbf{Z}^m$ for some r, m. Replacing S by S' and replacing $\mathcal{S}$ by $\mathcal{S}(\sigma')$ ($\sigma' \in \Sigma'$), we are reduced to Lemma 2.7.10. □

PROPOSITION 2.7.12

Let $n \geq 0$, and let V be a finite-dimensional $\mathbf{R}$-vector space endowed with an action of $\mathbf{G}_m^n$. Let Y be the subset of $\mathbf{R}_{\geq 0}^n \times V \times V$ consisting of all elements (t, u, v) satisfying the following conditions (i) and (ii) for any $\chi \in X(\mathbf{G}_m^n)$. In the following, we write $\chi = \chi_+(\chi_-)^{-1}$ as in Section 2.3.13.

(i) $t(\chi_+)v_\chi = t(\chi_-)u_\chi$.
(ii) *If $t(\chi_+) = 0$, then $u_\chi = v_\chi = 0$.*

Endow Y with the structure of an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by the embedding $Y \to \mathbf{R}_{\geq 0}^n \times V \times V$ as in Section 1.3.16. Let S be an object of $\mathcal{B}'_{\mathbf{R}}(\log)^+$, assume that we are given a strict morphism $S \to \mathbf{R}_{\geq 0}^n$, and let E be the fiber product of $S \to \mathbf{R}_{\geq 0}^n \leftarrow Y$ in $\mathcal{B}'_{\mathbf{R}}(\log)$. Then E belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$.

Proof

In Section 1.4.1, we take $L = X(\mathbf{G}_m^n)$. Let $L^+ \subset L$ be the submonoid corresponding to $\mathbf{N}^n$ in the identification $L = \mathbf{Z}^n$. Take a finite subset R of L such that $\{\chi \in L \mid V_\chi \neq 0\} \subset R$, $R = R^{-1}$, and $R^+ := R \cap L^+$ generates L^+ as a monoid. Let Σ be the fan of all faces of the cone $\mathbf{R}_{\geq 0}^n \subset \mathbf{R}^n = N_{\mathbf{R}}$, and let Σ' be the rational finite subdivision of Σ defined in Lemma 2.6.5(3) with respect to R and L^+.

Let Y', S', E' be the fiber products of $Y \to \mathbf{R}_{\geq 0}^n \leftarrow |\mathrm{toric}|(\Sigma')$, $S \to \mathbf{R}_{\geq 0}^n \leftarrow |\mathrm{toric}|(\Sigma')$, $E \to \mathbf{R}_{\geq 0}^n \leftarrow |\mathrm{toric}|(\Sigma')$, respectively. (We identify $\mathbf{R}_{\geq 0}^n$ with $|\mathrm{toric}|(\Sigma)$.) For $\sigma' \in \Sigma'$, let $Y'(\sigma')$, $S'(\sigma')$, $E'(\sigma')$ be the open sets of Y', S', E', respectively, corresponding to σ'. These are the fiber products of $Y \to \mathbf{R}_{\geq 0}^n \leftarrow \mathrm{Hom}(\mathcal{S}(\sigma'), \mathbf{R}_{\geq 0}^{\mathrm{mult}})$, $S \to \mathbf{R}_{\geq 0}^n \leftarrow \mathrm{Hom}(\mathcal{S}(\sigma'), \mathbf{R}_{\geq 0}^{\mathrm{mult}})$, and $E \to \mathbf{R}_{\geq 0}^n \leftarrow \mathrm{Hom}(\mathcal{S}(\sigma'), \mathbf{R}_{\geq 0}^{\mathrm{mult}})$, respectively. In particular, $Y'(\sigma') \subset \mathrm{Hom}(\mathcal{S}(\sigma'), \mathbf{R}_{\geq 0}^{\mathrm{mult}}) \times V \times V$.

We prove that $Y'(\sigma')$ is isomorphic to a standard object of the category $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$ (see Section 2.7.2). Since $R \subset \mathcal{S}(\sigma') \cup \mathcal{S}(\sigma')^{-1}$ (see Lemma 2.6.5), we can take subsets R_1 and R_2 such that R is the disjoint union of R_1 and R_2 and such that $R_1 \subset \mathcal{S}(\sigma')$ and $R_2 \subset \mathcal{S}(\sigma')^{-1}$. Consider the map

$$Y'(\sigma') \to \mathrm{Hom}\big(\mathcal{S}(\sigma'), \mathbf{R}_{\geq 0}^{\mathrm{mult}}\big) \times V, \qquad (t, u, v) \mapsto \Big(t, \sum_{\chi \in A_1} v_\chi + \sum_{\chi \in A_2} u_\chi\Big).$$

This induces an isomorphism

$$(1) \qquad Y'(\sigma') \xrightarrow{\cong} \big\{(t, x) \in \mathrm{Hom}\big(\mathcal{S}(\sigma'), \mathbf{R}_{\geq 0}^{\mathrm{mult}}\big) \times V \mid x \in V_{I(t)}\big\}$$

in $\mathcal{B}'_{\mathbf{R}}(\log)$, where $I(t)$ denotes the face $\{\chi \in \mathcal{S}(\sigma') \mid t(\chi) \neq 0\}$ of $\mathcal{S}(\sigma')$, and for a face I of $\mathcal{S}(\sigma')$, we define

$$V_I = \{x \in V \mid x_\chi = 0 \text{ if } \chi \in L \text{ and } \chi_+ \notin I\}.$$

The inverse map of (1) is given by $(t, x) \mapsto (t, u, v)$, where $u = \sum_{\chi \in R_1} t(\chi) x_\chi + \sum_{\chi \in R_2} x_\chi$ and $v = \sum_{\chi \in R_1} x_\chi + \sum_{\chi \in R_2} t(\chi^{-1}) x_\chi$. We omit more details of the proof of this isomorphism (1), for the argument is straightforward and similar to the proof of $Y'' \cong Y$ in the proof of Theorem 2.6.19 (see Section 2.6.21). Note that the right-hand side of (1) is a standard object of $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$ (see Section 2.7.2).

By Lemma 2.7.11, the fiber product $E'(\sigma')$ of $S'(\sigma') \to \mathrm{Hom}(\mathcal{S}(\sigma'), \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \leftarrow Y'(\sigma')$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. Hence, E' belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. By Proposition 2.7.8, this proves that E belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. □

PROPOSITION 2.7.13

We have that $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[+]}$.

This follows form Theorem 2.6.19 for situation (bd) in Section 2.6.18 and from Proposition 2.7.12.

THEOREM 2.7.14

The spaces $D^{I}_{\mathrm{SL}(2)}$, $D^{II}_{\mathrm{SL}(2)}$, $D^{\star}_{\mathrm{SL}(2)}$, $D^{\star,+}_{\mathrm{SL}(2)}$, $D^{\star,-}_{\mathrm{SL}(2)}$, $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$, and $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ belong to $\mathcal{B}'_{\mathbf{R}}(\log)^+$.

REMARK 2.7.15

We think that Theorem 2.7.14 is a version for the spaces of SL(2)-orbits, treated in Section 2, of the following results (1) and (2) on the spaces of Borel–Serre orbits and of nilpotent orbits.

(1) The space D_{BS} of Borel–Serre orbits is a real analytic manifold with corners (see [15, I]).

(2) For a weak rational fan Σ in $\mathfrak{g}_{\mathbf{R}}$ and for a neat subgroup Γ of $G_{\mathbf{Z}}$ which is strongly compatible with Σ, the space $\Gamma \setminus D_{\Sigma}$ is a log manifold (see [15, III, Theorem 2.5.2]).

These results (1) and (2) tell us that D_{BS} and $\Gamma \setminus D_{\Sigma}$ are beautiful spaces. Theorem 2.7.14 also says that the spaces of SL(2)-orbits are beautiful spaces.

2.7.16

We prove Theorem 2.7.14. Theorem 2.7.14 for $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ follows from Proposition 2.7.13 and Proposition 2.7.7. Theorem 2.7.14 for $D^{\star}_{\mathrm{SL}(2)}$, $D^{\star,+}_{\mathrm{SL}(2)}$, and $D^{\star,-}_{\mathrm{SL}(2)}$ follows from that for $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$ by Proposition 2.7.8. In the pure situation, this implies that $D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$ for any w; hence, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$, and hence, $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$ by Proposition 2.7.8. Theorem 2.7.14 for $D^{II}_{\mathrm{SL}(2)}$ follows from that for $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ by Propositions 2.3.16 and 2.7.9.

Finally we prove that $D^{I}_{\mathrm{SL}(2)}$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$. We apply Proposition 2.7.12. Let $x = (p, Z) \in D_{\mathrm{SL}(2)}$, fix $\mathbf{r} \in Z$, and let $\bar{\mathbf{r}} = \mathbf{r}(\mathrm{gr}^W) \in D(\mathrm{gr}^W)$. Let n be $\mathrm{rank}(p)$ if x is an A-orbit, and let $n = \mathrm{rank}(p) + 1$ if x is a B-orbit. Let $V = \mathrm{Lie}(G_{\mathbf{R},u})$, where $G_{\mathbf{R},u}$ denotes the unipotent radical of $G_{\mathbf{R}}$. We define the action of $\mathbf{G}^n_m$ on V as follows. In the case in which x is an A-orbit (resp., a B-orbit), lift the homomorphism τ_p (resp., $\tilde{\tau}_p$) $: \mathbf{G}^n_m \to G_{\mathbf{R}}(\mathrm{gr}^W)$ (see Section 2.2.7) to $\tau_x : \mathbf{G}_m^n \to G_{\mathbf{R}}$ by using the splitting $\mathrm{spl}_W(\mathbf{r})$ of W. We consider the adjoint action of $\mathbf{G}_m^n$ on $\mathrm{Lie}(G_{\mathbf{R},u})$ via τ_x. Define $Y \subset \mathbf{R}_{\geq 0}^n \times V \times V$ as in Proposition 2.7.12. Then by [15, II, Theorem 3.4.6], in the case where x is an A-orbit (resp., a B-orbit), there are an open neighborhood S of $y := (p, \delta_W(\mathbf{r}))$ in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathcal{L}(\bar{\mathbf{r}})$ (resp., $y := (p, 0 \circ \delta_W(\mathbf{r}))$ in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times (\bar{\mathcal{L}}(\bar{\mathbf{r}}) \smallsetminus \{0\})$), a strict morphism $S \to \mathbf{R}_{\geq 0}^n$ which sends y to $0 = (0, \ldots, 0)$, an open neighborhood U of $(y, 0, 0, 0)$ in the fiber product of $S \to \mathbf{R}_{\geq 0}^n \leftarrow Y$ (here $(0,0,0) \in \mathbf{R}_{\geq 0}^n \times V \times V$), and an open immersion $U \to D_{\mathrm{SL}(2)}^I$ which sends $(y, 0, 0, 0)$ to x. By Proposition 2.7.12, U is an object of $\mathcal{B}'_{\mathbf{R}}(\log)^+$. This shows that $D_{\mathrm{SL}(2)}^I$ is an object of $\mathcal{B}'_{\mathbf{R}}(\log)^+$. Theorem 2.7.14 is proved.

LEMMA 2.7.17

Let $n \geq 0$. Then the part of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ consisting of points of rank at most n is open in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$.

Proof

This part is the union of open sets $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}(\Phi)$, where Φ ranges over all admissible sets of weight filtrations on gr^W associated to points of rank at most n. □

2.7.18

We denote the above part of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ by $(D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim})_{\leq n}$. In the pure situation, this part is written as $D_{\mathrm{SL}(2),\leq n}$.

PROPOSITION 2.7.19

(1) *Let U be the inverse image of $(D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim})_{\leq 1}$ in $D_{\mathrm{SL}(2)}^{\star}$ (resp., $D_{\mathrm{SL}(2)}^{II}$). Then U is an object of $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$.*

(2) *Let U be the inverse image of $\prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W)_{\leq 1}$ in $D_{\mathrm{SL}(2)}^{\star,-}$. Then U is an object of $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$.*

Proof

We prove (1). By Theorem 2.7.14, Section 2.2.13 (which describes the stalks of $M_S/\mathcal{O}_S^{\times}$ for $S = D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$), and Lemma 2.7.6, $(D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim})_{\leq 1}$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. Hence, U belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ by Propositions 2.3.16 and 2.7.9.

We prove (2). Similarly, $D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W)_{\leq 1}$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$, and hence, $\prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}_w^W)_{\leq 1}$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ by Proposition 2.7.9. Hence, U belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ by Propositions 2.3.16 and 2.7.9. □

3. Valuative Borel–Serre orbits and valuative $\mathrm{SL}(2)$-orbits

In this section, we study the spaces $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, and $D_{\mathrm{SL}(2),\mathrm{val}}^{\star}$ and their relations.

3.1. The associated valuative spaces

In Section 3.1,

(1) for an object S of $\mathcal{B}'_{\mathbf{R}}(\log)$, we define a locally ringed space S_{val} over $\mathbf{R}$ with a "valuative log structure with sign," and

(2) more generally, for a field K endowed with a nontrivial absolute value $|\cdot|: K \to \mathbf{R}$ and for a locally ringed space S over K endowed with an fs log structure satisfying the conditions in Section 1.3.3, we construct a topological space S_{val}.

In (2), S_{val} is merely a topological space and does not have more structures as in (1). Note that (1) becomes important in the rest of Section 3, and (2) will become important in Section 4. Furthermore, (1) is shortly explained in [15, II, Section 3.7]. We call S_{val} the *valuative space associated to S*.

3.1.1

Let L be an abelian group whose group law is written multiplicatively. A submonoid V of L is said to be *valuative* if $V \cup V^{-1} = L$. An integral monoid V is said to be *valuative* if it is a valuative submonoid of V^{gp}. For an fs monoid $\mathcal{S}$, let $V(\mathcal{S})$ be the set of all valuative submonoids V of $\mathcal{S}^{\mathrm{gp}}$ such that $V \supset \mathcal{S}$ and $V^\times \cap \mathcal{S} = \mathcal{S}^\times$.

3.1.2

Let K be a field endowed with a nontrivial absolute value $|\cdot|: K \to \mathbf{R}$. Let S be a locally ringed space over K satisfying the equivalent conditions in Section 1.3.3 and endowed with an fs log structure. Let S_{val} be the set of all triples (s, V, h), where $s \in S$, $V \in V((M_S/\mathcal{O}_S^\times)_s)$ (see Section 3.1.1), and by denoting by $\tilde{V}$ the inverse image of V in $M_{S,s}^{\mathrm{gp}}$, h is a homomorphism $(\tilde{V})^\times \to \mathbf{R}_{>0}^{\mathrm{mult}}$ extending $f \mapsto |f(s)|$ on $\mathcal{O}_{S,s}^\times$. Here $\mathbf{R}_{>0}^{\mathrm{mult}}$ denotes the set $\mathbf{R}_{>0}$ regarded as a multiplicative group.

3.1.3

There is a variant, which we denote by $S_{\mathrm{val}(K)}$, of S_{val}. Let $S_{\mathrm{val}(K)}$ be the set of all triples (s, V, h), where s and V are as above but h is a homomorphism $(\tilde{V})^\times \to K^\times$ extending $f \mapsto f(s)$ on $\mathcal{O}_{S,s}^\times$. In [17, Section 3.6], in the case $K = \mathbf{C}$, this space $S_{\mathrm{val},(\mathbf{C})}$ was denoted by S_{val}. But in this article, we consider only S_{val} in the sense of Section 3.1.2 except in the proof of Theorem 6.3.1, and we hope no confusion occurs in this article. If we want to avoid confusion in a forthcoming work, we will denote S_{val} in Section 3.1.2 by $S_{\mathrm{val}(|\cdot|)}$. We will call $S_{\mathrm{val}(K)}$ the valuative space of K-points associated to S and call $S_{\mathrm{val}(|\cdot|)}$ the valuative space of absolute values associated to S.

3.1.4

In the case in which $K = \mathbf{R}$ and M_S is a log structure with sign (as in the case $S \in \mathcal{B}'_{\mathbf{R}}(\log)$) (Section 1.3.5), S_{val} is identified with the set of all triples (s, V, h), where $s \in S$, V is an element of $V((M_S/\mathcal{O}_S^\times)_s)$, and by denoting by $\tilde{V}_{>0}$ the inverse image of V in $M^{\mathrm{gp}}_{S,>0,s}$, h is a homomorphism $(\tilde{V})^\times_{>0} \to \mathbf{R}^{\mathrm{mult}}_{>0}$ extending $f \mapsto f(s)$ on $\mathcal{O}^\times_{S,>0,s}$.

3.1.5

Let S be as in Section 3.1.2. The topology of S_{val} is defined as follows. Let $(s_0, V_0, h_0) \in S_{\mathrm{val}}$. Assume that we are given a chart $\mathcal{S} \to M_S$ near the point $s_0 \in S$. We introduce a fundamental system of neighborhoods of the point $(s_0, V_0, h_0) \in S_{\mathrm{val}}$.

Let U be a neighborhood of s_0 in S, let I be a finite subset of $\mathcal{S}^{\mathrm{gp}}$ such that, for any $f \in I$, the image $\bar{f}_{s_0}$ of f in $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_{s_0}$ is contained in V_0, and let $\varepsilon > 0$. Let $B(U, I, \varepsilon)$ be the set of all points (s, V, h) of S_{val} satisfying the following conditions (i)–(iii).

(i) $s \in U$.
(ii) For any $f \in I$, the image $\bar{f}_s$ of f in $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ belongs to V.
(iii) For any $f \in I$, $|h(f) - h_0(f)| < \varepsilon$. Here we define $h(f)$ (resp., $h_0(f)$) to be 0 unless $\bar{f}_s \in V^\times$ (resp., $\bar{f}_{s_0} \in V_0^\times$).

Define a topology of S_{val} so that the sets $B(U, I, \varepsilon)$, where U, I, and ε vary, form a fundamental system of neighborhoods of the point (s_0, V_0, h_0). This topology is independent of the choice of a chart $\mathcal{S}$ and, hence, is well defined globally.

We now consider the relation of S_{val} and the projective limit of toric varieties for the subdivisions of fans. This will be used to prove properties of S_{val} and to endow S_{val} in the case in which $S \in \mathcal{B}'_{\mathbf{R}}(\log)$ with a structure of a locally ringed space over $\mathbf{R}$ and a log structure with sign.

3.1.6

Let the notation be as in Section 1.4.1. Let V be a valuative submonoid of L. For a submonoid $\mathcal{S}$ of L, we say that V *dominates* $\mathcal{S}$ if $\mathcal{S} \subset V$ and $\mathcal{S}^\times = \mathcal{S} \cap V^\times$. For a rational finitely generated sharp cone σ in $N_{\mathbf{R}}$, we say that V dominates σ if V dominates $\mathcal{S}(\sigma) := \{l \in L \mid l(\sigma) \geq 0\}$.

For a rational fan Σ in $N_{\mathbf{R}}$, V dominates some cone in Σ if and only if $\mathcal{S}(\sigma) \subset V$ for some $\sigma \in \Sigma$. If V dominates a cone in Σ, then such a cone is unique and is the smallest cone $\sigma \in \Sigma$ such that $\mathcal{S}(\sigma) \subset V$.

If Σ' is a rational finite subdivision of Σ, then V dominates some cone in Σ if and only if V dominates some cone in Σ'. In this case, if V dominates $\sigma' \in \Sigma'$, then V dominates the smallest cone $\sigma \in \Sigma$ such that $\sigma' \subset \sigma$.

LEMMA 3.1.7

Let Σ be a finite rational fan in $N_{\mathbf{R}}$. Then we have a bijection from the set of all valuative submonoids V of L, which dominate some cone in Σ, onto the projective limit $\varprojlim \Sigma'$, where Σ' ranges over all finite rational subdivisions of Σ.

This bijection sends V to $(\sigma_{\Sigma'})_{\Sigma'}$, where $\sigma_{\Sigma'}$ denotes the cone in Σ' dominated by V. The inverse map is given by $(\sigma_{\Sigma'})_{\Sigma'} \mapsto \bigcup_{\Sigma'} \mathcal{S}(\sigma_{\Sigma'})$.

Proof
This is straightforward. $\square$

LEMMA 3.1.8
Let Σ be a finite rational fan in $N_{\mathbf{R}}$. Then we have the following bijection from $\varprojlim_{\Sigma'} |\mathrm{toric}|(\Sigma')$, where Σ' ranges over all finite rational subdivisions of Σ, to the set of all pairs (V,h) of a valuative submonoid V of L dominating some cone in Σ and a homomorphism $h\colon V^\times \to \mathbf{R}_{>0}$. If $(x_{\Sigma'})_{\Sigma'}$ is an element of $\varprojlim_{\Sigma'} |\mathrm{toric}|(\Sigma')$ and $(\sigma_{\Sigma'}, h_{\Sigma'})$ (where $\sigma_{\Sigma'} \in \Sigma'$, $h_{\Sigma'} : \mathcal{S}(\sigma_{\Sigma'}) \to \mathbf{R}_{\geq 0}$) is the pair corresponding to $x_{\Sigma'}$ (Section 1.4.1), then the pair (V,h) corresponding to $(x_{\Sigma'})_{\Sigma'}$ is as follows: $V = \bigcup_{\Sigma'} \mathcal{S}(\sigma_{\Sigma'})$, and h is the homomorphism $V^\times \to \mathbf{R}_{>0}$ whose restriction to $\mathcal{S}(\sigma_{\Sigma'})^\times$ is $h_{\Sigma'}$ for any Σ'.

Proof
This can be shown by using Lemma 3.1.7. $\square$

PROPOSITION 3.1.9
Let S be as in Section 3.1.2, and assume that we are given a chart $\mathcal{S} \to M_S$ with $\mathcal{S}$ an fs monoid, let $L = \mathcal{S}^{\mathrm{gp}}$, let $N = \mathrm{Hom}(L, \mathbf{Z})$, and let Σ be the fan in $N_{\mathbf{R}}$ of all faces of the cone $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{add}}_{\geq 0})$. Here $\mathbf{R}^{\mathrm{add}}$ denotes $\mathbf{R}_{\geq 0}$ regarded as an additive monoid. Then we have a Cartesian diagram of topological spaces

$$\begin{array}{ccc} S_{\mathrm{val}} & \to & \varprojlim_{\Sigma'} |\mathrm{toric}|(\Sigma') \\ \downarrow & & \downarrow \\ S & \to & |\mathrm{toric}|(\Sigma) = \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\geq 0}) \end{array}$$

where Σ' ranges over all finite rational subdivisions of Σ, and the lower row sends $s \in S$ to the homomorphism $f \mapsto |f(s)|$ $(f \in \mathcal{S})$.

Proof
For $s \in S$, let $\mathcal{S}(s) = \mathcal{S}(\sigma)$, where σ is the element of Σ such that the image of s in $|\mathrm{toric}|(\Sigma)$ corresponds to a pair (σ, h) for some $h : \mathcal{S}(\sigma)^\times \to \mathbf{R}^{\mathrm{mult}}_{>0}$ (see Section 1.4.1). Then $\mathcal{S}(s)^\times$ coincides with the inverse image of $\mathcal{O}^\times_{S,s}$ under the canonical map $\mathcal{S}^{\mathrm{gp}} \to M^{\mathrm{gp}}_{S,s}$, and $\mathcal{S}(s)$ is generated by $\mathcal{S}$ and $\mathcal{S}(s)^\times$. We have $\mathcal{S}(\sigma)/\mathcal{S}(\sigma)^\times \overset{\cong}{\to} (M_S/\mathcal{O}^\times_S)_s$.

By Lemma 3.1.8, the fiber product $S \times_{|\mathrm{toric}|(\Sigma)} \varprojlim_{\Sigma'} |\mathrm{toric}|(\Sigma')$ is identified with the set of all triples (s, V, h), where $s \in S$, V is a valuative submonoid of $\mathcal{S}^{\mathrm{gp}}$ such that $V \supset \mathcal{S}$ and $V^\times \cap \mathcal{S} = \mathcal{S}(s)^\times$, and h is a homomorphism $V^\times \to \mathbf{R}^{\mathrm{mult}}_{>0}$ whose restriction to $\mathcal{S}(s)^\times$ coincides with the composition $\mathcal{S}(s)^\times \to \mathcal{O}^\times_{S,s} \to \mathbf{R}^{\mathrm{mult}}_{>0}$, where the last map is $f \mapsto |f(s)|$. By the isomorphism $\mathcal{S}(\sigma)/\mathcal{S}(\sigma)^\times \overset{\cong}{\to} (M_S/\mathcal{O}^\times_S)_s$, a valuative submonoid V of $\mathcal{S}^{\mathrm{gp}}$ such that $V \supset \mathcal{S}$ and $V^\times \cap \mathcal{S} = \mathcal{S}(s)^\times$ corresponds bijectively to a valuative submonoid V' of $(M^{\mathrm{gp}}_S/O^\times_S)_s$ containing

$(M_S/\mathcal{O}_S^\times)_s$ and $(V')^\times \cap (M_S/\mathcal{O}_S^\times)_s = \{1\}$. Furthermore, if $\tilde{V}'$ denotes the inverse image of V' in $M_{S,s}^{\mathrm{gp}}$, $(\tilde{V}')^\times$ is the pushout of $V^\times \leftarrow \mathcal{S}(s)^\times \to \mathcal{O}_{S,s}^\times$. Hence, h corresponds to a homomorphism $h' : (\tilde{V}')^\times \to \mathbf{R}_{>0}^{\mathrm{mult}}$ whose restriction to $\mathcal{O}_{S,s}^\times$ coincides with $f \mapsto |f(s)|$. Hence, we have a bijection $(s, V, h) \mapsto (s, V', h')$ from the fiber product to S_{val}. In the converse map $(s, V', h') \mapsto (s, V, h)$, V is the inverse image of V' under the canonical map $\mathcal{S}^{\mathrm{gp}} \to (M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ and h is the homomorphism $V^\times \to \mathbf{R}_{>0}^{\mathrm{mult}}$ induced by h'. By using these explicit constructions of the bijection between S_{val} and the fiber product, it is easy to see that this bijection is a homeomorphism. □

COROLLARY 3.1.10

For S as in Section 3.1.2, the map $S_{\mathrm{val}} \to S$ is proper.

LEMMA 3.1.11

Let S and S' be as in Section 3.1.2, and assume that we are given a strict morphism $S' \to S$ of locally ringed spaces over K with log structures (for the word "strict," see Corollary 1.3.15). Then the canonical map $S'_{\mathrm{val}} \to S' \times_S S_{\mathrm{val}}$ is a homeomorphism.

Proof

For any $s' \in S'$ with image s in S, the canonical map $(M_S/\mathcal{O}_S^\times)_s \to (M_{S'}/\mathcal{O}_{S'}^\times)_{s'}$ is an isomorphism from the assumption. From this, we see that the map $S'_{\mathrm{val}} \to S' \times_S S_{\mathrm{val}}$ is bijective. Since this map is continuous and since both S'_{val} and $S' \times_S S_{\mathrm{val}}$ are proper over S' (see Corollary 3.1.10), this map is a homeomorphism. □

LEMMA 3.1.12

Let S be as in Section 3.1.2, and let $|S|$ be the topological space S with the sheaf of all $\mathbf{R}$-valued continuous functions. Endow $|S|$ with the log structure $M_{|S|}$ associated to the composition $M_S \to \mathcal{O}_S \to \mathcal{O}_{|S|}$, where the second arrow is $f \mapsto |f|$, which we regard as a prelog structure. Here $|f|$ denotes the function $s \mapsto |f(s)|$. Then $M_{|S|}$ is an fs log structure, and we have a canonical homeomorphism $|S|_{\mathrm{val}} \cong S_{\mathrm{val}}$.

Proof

If $\mathcal{S} \to M_S$ is a chart with $\mathcal{S}$ an fs monoid, then the composition $\mathcal{S} \to M_S \to M_{|S|}$ is also a chart. Hence, $M_{|S|}$ is an fs log structure. The canonical map $(M_S/\mathcal{O}_S^\times)_s \to (M_{|S|}/\mathcal{O}_{|S|}^\times)_s$ is an isomorphism for any $s \in S$, and hence, we have a canonical bijection $|S|_{\mathrm{val}} \to S_{\mathrm{val}}$. It is easy to see that this is a homeomorphism. □

3.1.13

Assume now that S is an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ (see Section 1.3.7). We endow S_{val} with a sheaf $\mathcal{O}_{S_{\mathrm{val}}}$ of rings and a log structure $M_{S_{\mathrm{val}}}$ with sign as follows. Locally on S, take a positive chart $\mathcal{S} \to M_{S,>0}$ (see Section 1.3.10), let Σ be the fan of faces of the cone $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{add}}_{\geq 0})$, and for a rational finite subdivision Σ' of Σ, regard $S(\Sigma') := S \times_{|\mathrm{toric}|(\Sigma)} |\mathrm{toric}|(\Sigma')$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by taking the fiber product in $\mathcal{B}'_{\mathbf{R}}(\log)$. Here we use the fact that the underlying topological space of this fiber product is the same as the fiber product of the underlying topological spaces by Corollary 1.3.15(i) and by the fact that $S \to |\mathrm{toric}|(\Sigma)$ is strict.

We define $\mathcal{O}_{S_{\mathrm{val}}}$ (resp., $M_{S_{\mathrm{val}}}$) as the inductive limit of $\mathcal{O}_{S(\Sigma')}$ (resp., $M_{S(\Sigma')}$) by using Proposition 3.1.9. This sheaf of rings and the log structure with sign are independent of the choice of the chart and, hence, are defined globally. In fact, if we have two charts $\mathcal{S} \to M_{S,>0}$ and $\mathcal{S}' \to M_{S,>0}$, then there is a third chart $\mathcal{S}'' \to M_{S,>0}$ with homomorphisms $\mathcal{S} \to \mathcal{S}''$ and $\mathcal{S}' \to \mathcal{S}''$ of charts. It is easy to see that the sheaf of rings and the log structure with sign given by the chart $\mathcal{S}$ (resp., $\mathcal{S}'$) are isomorphic to the ones given by the chart $\mathcal{S}''$, and that the composite isomorphisms between the ones given by the chart $\mathcal{S}$ and the ones given by the chart $\mathcal{S}'$ are independent of the choice of the third chart $\mathcal{S}''$.

We call $\mathcal{O}_{S_{\mathrm{val}}}$ the *sheaf of real analytic functions*.

3.1.14

A log modification $S' \to S$ in $\mathcal{B}'_{\mathbf{R}}(\log)$ induces an isomorphism

$$(S')_{\mathrm{val}} \overset{\cong}{\to} S_{\mathrm{val}}$$

of locally ringed spaces over $\mathbf{R}$ with log structures with sign.

Proof

This is clear. $\square$

3.1.15

For $S \in \mathcal{B}'_{\mathbf{R}}(\log)$ and for $x = (s, V, h) \in S_{\mathrm{val}}$, V is identified with the inverse image of $(M_{S_{\mathrm{val}}}/\mathcal{O}^{\times}_{S_{\mathrm{val}}})_x$ under the canonical map $(M^{\mathrm{gp}}_S/\mathcal{O}^{\times}_S)_s \to (M^{\mathrm{gp}}_{S_{\mathrm{val}}}/\mathcal{O}^{\times}_{S_{\mathrm{val}}})_x$, and $h : \tilde{V}^{\times}_{>0} \to \mathbf{R}^{\mathrm{mult}}_{>0}$ coincides with the composition $\tilde{V}^{\times}_{>0} \to \mathcal{O}_{S_{\mathrm{val}},>0,x} \to \mathbf{R}^{\mathrm{mult}}_{>0}$, where the first arrow is induced from $\tilde{V}_{>0} \subset M_{S,>0,s} \to M_{S_{\mathrm{val}},>0,x}$ and the second arrow is $f \mapsto f(x)$.

3.1.16

Let S be a locally ringed space. Then a log structure M on S is said to be *valuative* if it is integral and satisfies the following condition: for any local section f of M^{gp}, locally we have either $f \in M$ or $f^{-1} \in M$, that is, every stalk of M is valuative. By Section 3.1.15, for $S \in \mathcal{B}'_{\mathbf{R}}(\mathrm{val})$, the log structure of S_{val} is valuative.

3.1.17

Let $\mathcal{S}_1$, T, $\bar{T}$, Z, $\bar{Z}$, $T(s) \subset T$, and $Z(s) \subset Z$ (for $s \in \bar{Z}$) be as in Section 1.4.8. We give a description (1) below of the valuative space $\bar{Z}_{\mathrm{val}}$ associated to $\bar{Z}$ as a set. This will be used in Section 3.3.3.

For a valuative submonoid V of $\mathcal{S}^{\mathrm{gp}}_1$, let $T(V) := \mathrm{Hom}(\mathcal{S}^{\mathrm{gp}}_1/V^{\times}, \mathbf{R}_{>0}) \subset T = \mathrm{Hom}(\mathcal{S}_1, \mathbf{R}^{\mathrm{mult}}_{>0})$. Then we have the following.

(1) $\bar{Z}_{\mathrm{val}}$ is identified with the set of all triples (s,V,Z'), where $s\in\bar{Z}$, V is a valuative submonoid of $\mathcal{S}_1^{\mathrm{gp}}$ such that $V\supset\mathcal{S}_1$ and such that $V^\times\cap\mathcal{S}_1=\mathrm{Ker}(\mathcal{S}_1\to(M_{\bar{Z}}/\mathcal{O}_{\bar{Z}}^\times)_s)$, and Z' is a $T(V)$-orbit in $Z(s)$. (Note that $Z(s)$ is a $T(s)$-torsor and $T(V)\subset T(s)$.)

This is proved as follows. Let $L=\mathcal{S}_1^{\mathrm{gp}}$, and let Σ be the fan of all faces of the cone $\mathrm{Hom}(\mathcal{S}_1,\mathbf{R}^{\mathrm{add}}_{\geq 0})\subset N_{\mathbf{R}}=\mathrm{Hom}(L,\mathbf{R})$. Then by Proposition 3.1.9, $\bar{Z}_{\mathrm{val}}$ is the projective limit of the log modifications of $\bar{Z}$ corresponding to rational finite subdivisions Σ' of Σ. By Lemma 3.1.7, the projective limit of the sets Σ' is identified with the set of valuative cones V as above. Hence, the above description (1) of $\bar{Z}_{\mathrm{val}}$ follows from the descriptions of log modifications of $\bar{Z}$ in Section 1.4.8 as sets by taking the projective limit.

3.2. The category $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$

We define categories $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$ and $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+\subset\mathcal{C}_{\mathbf{R}}(\mathrm{val})$. In Section 3.3, we will see that the valuative spaces associated to the spaces of SL(2)-orbits and the space of Borel–Serre orbits belong to $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$.

3.2.1

Let $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$ be the category of objects of $\mathcal{C}_{\mathbf{R}}$ (see Section 1.3.1) endowed with a valuative log structure (see Section 3.1.16) with sign. We have $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})\supset\mathcal{C}_{\mathbf{R}}(\mathrm{val})$ (see Section 1.3.7).

PROPOSITION 3.2.2

Let S be an object of $\mathcal{B}'_{\mathbf{R}}(\log)^+$. Then S_{val} belongs to $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$.

For the proof, we use the following lemma.

LEMMA 3.2.3

Let $(S_\lambda)_\lambda$ be a directed projective system in $\mathcal{C}_{\mathbf{R}}$, let S be the projective limit of the topological spaces S_λ, and endow S with the inductive limit of the inverse images of $\mathcal{O}_{S_\lambda}$. Assume that there is an open set S' of S satisfying the following conditions (i) and (ii).

(i) *S' belongs to $\mathcal{C}_{\mathbf{R}}$.*
(ii) *For any open set U of S, the map $\mathcal{O}_S(U)\to\mathcal{O}_S(U\cap S')$ is injective.*

Then $S\in\mathcal{C}_{\mathbf{R}}$.

Proof

Let $\mathcal{F}$ be the sheaf on S of morphisms to $\mathbf{R}^n$ of locally ringed spaces over $\mathbf{R}$, where $\mathbf{R}^n$ is endowed with the sheaf of all real analytic functions. We have a morphism $a:\mathcal{F}\to\mathcal{O}_S^n$ by $f\mapsto(f^*(t_j))_{1\leq j\leq n}$, where the t_j's are the standard coordinate functions of $\mathbf{R}^n$. We also have a morphism $b:\mathcal{O}_S^n\to\mathcal{F}$, which comes from the fact that, since S_λ belongs to $\mathcal{C}_{\mathbf{R}}$, $\mathcal{O}_S^n$ is regarded as the inductive limit of the inverse images of sheaves on S_λ of morphisms to $\mathbf{R}^n$. As is easily seen, the composition $ab: \mathcal{O}_S^n \to \mathcal{O}_S^n$ is the identity morphism. We prove that $ba: \mathcal{F} \to \mathcal{F}$ is the identity morphism. Let $f \in \mathcal{F}(U)$ with U an open set of S. It is easy to see that f and $ba(f)$ induce the same underlying continuous maps $U \to \mathbf{R}^n$, which we denote by g. It remains to prove that the homomorphisms $g^{-1}(\mathcal{O}_{\mathbf{R}^n}) \to \mathcal{O}_U$ given by f and $ba(f)$ coincide. Since $\mathcal{O}_S(V) \to \mathcal{O}_S(V \cap S')$ is injective for any open set V of U, it is sufficient to prove that the restrictions of f and $ba(f)$ to $U \cap S'$ coincide. But S' belongs to $\mathcal{C}_{\mathbf{R}}$, and hence, ba gives the identity morphism of $\mathcal{F}|_{S'}$. $\square$

3.2.4

We prove Proposition 3.2.2. By Section 3.1.14, it is sufficient to prove this for objects of $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. As in Section 2.7.1, let $Y \subset \mathbf{R}^n_{\geq 0} \times A$ be a standard object of $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$. It is sufficient to prove that Y_{val} belongs to $\mathcal{C}_{\mathbf{R}}$. Let $L = \mathbf{Z}^n$, let Σ be the set of all faces of the cone $\mathbf{R}^n_{\geq 0} \subset \mathrm{Hom}(L, \mathbf{R}^{\mathrm{add}}) = \mathbf{R}^n$, and for a rational finite subdivision Σ' of Σ, let $Y(\Sigma') = \{(t,x) \in |\mathrm{toric}|(\Sigma') \times A \mid x \in A_{J(t)}\}$, where $J(t) = \{j \mid 1 \leq j \leq n, t_j = 0\}$ with t_j the jth component of the image of t in $|\mathrm{toric}|(\Sigma) = \mathbf{R}^n_{\geq 0}$. We apply Lemma 3.2.3 by taking the projective system $(Y(\Sigma'))_{\Sigma'}$ in $\mathcal{C}_{\mathbf{R}}$ as $(S_\lambda)_\lambda$ and by taking the open set $\mathbf{R}^n_{>0} \times A$ of Y_{val} as S'. Then the projective limit S in Lemma 3.2.3 is Y_{val}. The injectivity of $\mathcal{O}_S(U) \to \mathcal{O}_S(U \cap S')$ for any open set U of Y_{val} is seen easily. Hence, Y_{val} belongs to $\mathcal{C}_{\mathbf{R}}$ by Lemma 3.2.3.

3.2.5

We define a full subcategory $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ of $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$. This is the category of all objects which are locally isomorphic to open subobjects of S_{val} with objects S of $\mathcal{B}'_{\mathbf{R}}(\log)^+$. We can replace $\mathcal{B}'_{\mathbf{R}}(\log)^+$ by $\mathcal{B}'_{\mathbf{R}}(\log)^{[[+]]}$ in this definition to get the same category $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$. Hence, $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ is the category of objects which are locally an open subobject of

$$Y_{\mathrm{val}} = \{(t,x) \in (\mathbf{R}^n_{\geq 0})_{\mathrm{val}} \times A \mid x \in A_{J(t)}\}.$$

Here n, A, $(A_J)_J$, and Y are as in Section 2.7.1, and $J(t) = \{j \mid 1 \leq j \leq n, t_j = 0\}$, where t_j denotes the jth component of the image of t in $\mathbf{R}^n_{\geq 0}$.

PROPOSITION 3.2.6

For any object S of $\mathcal{B}'_{\mathbf{R}}(\log)$ and for any object X of $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$, the canonical map $\mathrm{Mor}(X, S_{\mathrm{val}}) \to \mathrm{Mor}(X, S)$ is bijective. Consequently, if S is an object of $\mathcal{B}'_{\mathbf{R}}(\log)^+$, then S_{val} represents the functor $X \mapsto \mathrm{Mor}(X, S)$ from $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$ to (Sets).

Proof

It is sufficient to prove that, in the situation of Proposition 3.1.9, the canonical map $\mathrm{Mor}(X, S \times_{|\mathrm{toric}|(\Sigma)} |\mathrm{toric}|(\Sigma')) \to \mathrm{Mor}(X, S)$ is bijective. Here $S \times_{|\mathrm{toric}|(\Sigma)} |\mathrm{toric}|(\Sigma')$ denotes the fiber product in $\mathcal{B}'_{\mathbf{R}}(\log)$. By Proposition 1.3.11, it is the fiber product in $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$. Hence, it is sufficient to prove that the map $\mathrm{Mor}(X,$

$|\mathrm{toric}|(\Sigma')) \to \mathrm{Mor}(X, |\mathrm{toric}|(\Sigma))$ is bijective. This last fact is reduced to Proposition 1.3.9. $\square$

PROPOSITION 3.2.7

(1) *The category $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ has finite products. We will denote the product in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ as $X \times_{\mathrm{val}} Y$.*

(2) *A finite product in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ is a finite product in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$.*

(3) *The functor $\mathcal{B}'_{\mathbf{R}}(\log)^+ \to \mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$, $S \mapsto S_{\mathrm{val}}$ preserves finite products.*

(4) *For objects $S_1, \dots, S_n$ of $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$, the product of $S_1, \dots, S_n$ in the category $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$ exists. As a topological space, it is the product of the topological spaces S_j. We will denote this product in $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$ by $S_1 \times_{\mathrm{sat}} \cdots \times_{\mathrm{sat}} S_n$.*

Proof

If Y and Y' are objects of $\mathcal{B}'_{\mathbf{R}}(\log)^+$, then by Proposition 2.7.9, $(Y \times Y')_{\mathrm{val}}$ is an object of $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$, and for any object X of $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$, we have

$$\begin{aligned}\mathrm{Mor}\bigl(X, (Y \times Y')_{\mathrm{val}}\bigr) &= \mathrm{Mor}(X, Y \times Y') = \mathrm{Mor}(X, Y) \times \mathrm{Mor}(X, Y') \\ &= \mathrm{Mor}(X, Y_{\mathrm{val}}) \times \mathrm{Mor}\bigl(X, (Y')_{\mathrm{val}}\bigr),\end{aligned}$$

where the first and the third equalities follow from Proposition 3.2.6, and the second equality follows from Proposition 1.3.11(2). This proves (1), (2), and (3).

We prove (4). Locally on each S_j, we have $S_j = (S'_j)_{\mathrm{val}}$ for an object S'_j of $\mathcal{B}'_{\mathbf{R}}(\log)^+$. Locally in each S'_j, take a chart $\mathcal{S}_j \to M_{S'_j}$, let Σ_j be the fan of all faces of the cone $\mathrm{Hom}(\mathcal{S}_j, \mathbf{R}^{\mathrm{add}}_{\geq 0})$, and consider $S = \varprojlim \prod_{j=1}^n S'_j \times_{|\mathrm{toric}|(\Sigma_j)} |\mathrm{toric}|(\Sigma'_j)$, where Σ'_j ranges over all rational finite subdivisions of Σ_j. Endow S with the inductive limit of the inverse images of $\mathcal{O}$ and the log structures with sign of $\prod_{j=1}^n S'_j \times_{|\mathrm{toric}|(\Sigma_j)} |\mathrm{toric}|(\Sigma'_j)$. Then S belongs to $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$ by Lemma 3.2.3 and is the product of S_j in $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$. This locally constructed S glues to a global S. $\square$

The following lemma will be used in Section 3.4.

LEMMA 3.2.8

Let $n \geq 0$, and let $S_j \to S'_j$ $(1 \leq j \leq n)$ be morphisms in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ having the Kummer property of log structure in the sense (K) below. Let S (resp., S') be the product $S_1 \times_{\mathrm{sat}} \cdots \times_{\mathrm{sat}} S_n$ (resp., $S'_1 \times_{\mathrm{sat}} \cdots \times_{\mathrm{sat}} S'_n$) in the category $\mathcal{C}_{\mathbf{R}}(\mathrm{sat})$, and let $S_{\mathrm{val}} = S_1 \times_{\mathrm{val}} \cdots \times_{\mathrm{val}} S_n$ (resp., $S'_{\mathrm{val}} = S'_1 \times_{\mathrm{val}} \cdots \times_{\mathrm{val}} S'_n$) be the product in the category $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ (see Proposition 3.2.7).

(K) We say that a morphism $X \to Y$ of locally ringed spaces with log structure has the Kummer property of log structure if, for any $x \in X$ and the image y of x in Y, the homomorphism $(M_Y/\mathcal{O}_Y^\times)_y \to (M_X/\mathcal{O}_X^\times)_x$ is injective, and for any $a \in (M_X/\mathcal{O}_X^\times)_x$, there is $m \geq 1$ such that a^m belongs to the image of $(M_Y/\mathcal{O}_Y^\times)_y$.

Then the diagram

$$\begin{array}{ccc} S_{\mathrm{val}} & \to & S'_{\mathrm{val}} \\ \downarrow & & \downarrow \\ S & \to & S' \end{array}$$

is Cartesian in the category of topological spaces.

Proof

The set S_{val} is identified with the set of all triples (s, V, h), where $s \in S$, V is a valuative submonoid of $(M_S^{\mathrm{gp}}/\mathcal{O}_S^\times)_s$ such that $V \supset (M_S/\mathcal{O}_S^\times)_s$ and $V \cap (M_S/\mathcal{O}_S^\times)_s = \{1\}$, and h is a homomorphism $(\tilde{V})^\times \to \mathbf{R}_{>0}$, where $\tilde{V}$ denotes the inverse image of V in $M_{S,>0,s}$ such that the restriction of h to $\mathcal{O}_{S,>0,s}^\times$ coincides with $f \mapsto f(s)$. Furthermore, $(M_S/\mathcal{O}_S^\times)_s \cong \prod_{j=1}^n (M_{S_j}^{\mathrm{gp}}/\mathcal{O}_{S_j}^\times)_{s_j}$, where s_j denotes the image of s in S_j. Similar statements hold for S'. From these, we see that the diagram is Cartesian in the category of sets. Since S_{val} and the fiber product E of $S \to S' \leftarrow S'_{\mathrm{val}}$ in the category of topological spaces are proper over S, we see that the canonical map $S_{\mathrm{val}} \to E$ is a homeomorphism. □

3.3. $D_{\mathrm{BS,val}}$, $D^{I}_{\mathrm{SL}(2),\mathrm{val}}$, $D^{II}_{\mathrm{SL}(2),\mathrm{val}}$, $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$

3.3.1

We define

$$D_{\mathrm{BS,val}}, \qquad D^{I}_{\mathrm{SL}(2),\mathrm{val}}, \qquad D^{II}_{\mathrm{SL}(2),\mathrm{val}}, \qquad D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$$

as the valuative spaces associated to the objects

$$D_{\mathrm{BS}}, \qquad D^{I}_{\mathrm{SL}(2)}, \qquad D^{II}_{\mathrm{SL}(2)}, \qquad D^{\star}_{\mathrm{SL}(2)}$$

of $\mathcal{B}'_{\mathbf{R}}(\log)$ (see Section 3.1.13), respectively. By Theorem 2.7.14, Proposition 3.2.2, and Section 3.2.5, they belong to $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$. We call $D_{\mathrm{BS,val}}$ the space of valuative Borel–Serre orbits, and we call the other spaces $D^{I}_{\mathrm{SL}(2),\mathrm{val}}$ and so on spaces of valuative SL(2)-orbits.

$D^{I}_{\mathrm{SL}(2),\mathrm{val}}$ and $D^{II}_{\mathrm{SL}(2),\mathrm{val}}$ are identified as sets because $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ are identified as sets and the morphism $D^{I}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}$ is strict (see Lemma 3.1.11). They are denoted by $D_{\mathrm{SL}(2),\mathrm{val}}$ when we regard them just as sets.

3.3.2

Since a log modification induces an isomorphism of associated valuative spaces (see Section 3.1.14), we have

$$D^{\star,+}_{\mathrm{SL}(2),\mathrm{val}} \overset{\cong}{\to} D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \overset{\cong}{\to} D^{\star,-}_{\mathrm{SL}(2),\mathrm{val}} \overset{\cong}{\leftarrow} D^{\star,\mathrm{BS}}_{\mathrm{SL}(2),\mathrm{val}}.$$

Hence, the morphisms

$$D^{\star,+}_{\mathrm{SL}(2)} \to D^{II}_{\mathrm{SL}(2)}, \qquad D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$$

(see Sections 2.5, 2.6) induce morphisms

$$D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D^{II}_{\mathrm{SL}(2),\mathrm{val}}, \qquad D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS,val}}.$$

3.3.3
These valuative spaces are described as sets as follows. Let situations (a)–(d) and the notation be as in Sections 2.4.4 and 2.4.5. By Section 3.1.17, we have the following. As a set, $\mathfrak{D}_{\mathrm{val}}$ is identified with the set of all triples (x, V, Z), where $x \in \mathfrak{D}$, V is a valuative submonoid of $(M_{\mathfrak{D}}^{\mathrm{gp}}/\mathcal{O}_{\mathfrak{D}}^{\times})_x = X(S_x)$ such that $X(S_x)^+ \subset V$ and $X(S_x)^+ \cap V^{\times} = \{1\}$, and Z is a $T(V)$-orbit in the $T(x)$-torsor $Z(x)$. Here

$$T(V) := \mathrm{Hom}\big(X(S_x)/V^{\times}, \mathbf{R}_{>0}^{\mathrm{mult}}\big) \subset T(x) = \mathrm{Hom}\big(X(S_x), \mathbf{R}_{>0}^{\mathrm{mult}}\big).$$

3.3.4
Let the notation be as in Section 3.3.3. For a point $z = (x, V, Z) \in \mathfrak{D}_{\mathrm{val}}$, the stalk $(M_{\mathfrak{D}_{\mathrm{val}}}/\mathcal{O}_{\mathfrak{D}_{\mathrm{val}}}^{\times})_z$ is described as follows. In situations (a)–(c), $(M_{\mathfrak{D}_{\mathrm{val}}}/\mathcal{O}_{\mathfrak{D}_{\mathrm{val}}}^{\times})_z = V/V^{\times}$. In situation (d), $(M_{\mathfrak{D}_{\mathrm{val}}}/\mathcal{O}_{\mathfrak{D}_{\mathrm{val}}}^{\times})_z = V'/(V')^{\times}$, where $V' = V \cap (X(S_x)^+)^{\mathrm{gp}}$.

3.3.5
In [16, Definition 2.6] and [17, Definition 5.1.6], which treated the pure case, we defined the set $D_{\mathrm{BS,val}}$ in a different style. Following the style in [16, 2.6] and [17, 5.1.6], we can define $D_{\mathrm{BS,val}}$ also as the set of all triples (T, V, Z), where T is an $\mathbf{R}$-split torus in $G_{\mathbf{R}}(\mathrm{gr}^W)$, V is a valuative submonoid of the character group $X(T)$ of T, and $Z \subset D$, satisfying the following conditions (i)–(iv).

(i) Let $T_{>0}$ be the connected component of $T(\mathbf{R})$ containing the unit element. Then Z is either a $T_{>0}$-orbit for the lifted action in Section 1.2.6 or an $\mathbf{R}_{>0} \times T$-orbit in D_{nspl} for the lifted action. Here $t \in \mathbf{R}_{>0}$ acts on gr^W by the multiplication by t^w on gr_w^W.

(ii) Let $\mathbf{r} \in Z$, let $\bar{\mathbf{r}} := \mathbf{r}(\mathrm{gr}^W) \in D(\mathrm{gr}^W)$, let $K_{\bar{\mathbf{r}}}$ be the maximal compact subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ associated to $\bar{\mathbf{r}}$, and let $\theta_{K_{\bar{\mathbf{r}}}} : G_{\mathbf{R}}(\mathrm{gr}^W) \to G_{\mathbf{R}}(\mathrm{gr}^W)$ be the Cartan involution associated to $K_{\bar{\mathbf{r}}}$. Then $\theta_{K_{\bar{\mathbf{r}}}}(t) = t^{-1}$ for any $t \in T$.

(iii) $V^{\times} = \{1\}$.

(iv) Consider the direct sum decomposition $\mathrm{gr}^W = \bigoplus_{\chi \in X(T)} (\mathrm{gr}^W)_{\chi}$ by the action of T. Then for any $\chi \in X(T)$, the subspace $\bigoplus_{\chi' \in V^{-1}\chi} (\mathrm{gr}^W)_{\chi'}$ is $\mathbf{Q}$-rational.

The relation with the presentation in Section 3.3.3 of $D_{\mathrm{BS,val}}$ is as follows. $(P, V, Z) \in D_{\mathrm{BS,val}}$ in the presentation in Section 3.3.3 corresponds to (T, V', Z) in the above presentation, where $T \subset S_P$ is the annihilator of $V^{\times}$ in S_P and $V' = V/V^{\times} \subset X(T)$. The group $T(V)$ in Section 3.3.3 coincides with $T_{>0}$ in the above (i).

Conversely, for a triple (T, V, Z) here, the corresponding triple in the presentation of $D_{\mathrm{BS,val}}$ in Section 3.3.3 is (P, V', Z), where P is the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ defined as the connected component (as an algebraic group) of the algebraic subgroup of $G_{\mathbf{R}}(\mathrm{gr}^W)$ consisting of all elements which preserve the subspaces $\bigoplus_{\chi' \in V^{-1}\chi} (\mathrm{gr}^W)_{\chi'}$ of gr^W, and V' is the inverse image of V under the homomorphism $X(S_P) \to X(T)$ induced by the canonical homomorphism $T \to S_P$.

3.3.6
We describe the map $D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D^{II}_{\mathrm{SL}(2),\mathrm{val}}$. This map is described as $x=(p,V,Z)\mapsto(p,V',Z')$, using Section 3.3.3 as follows.

(0) On D, this map is the identity map.

(1) For an A-orbit which does not belong to D, $V'=V$ and $Z'=Z_{\mathrm{spl}}$.

(2) Assume that (p,V,Z) is a B-orbit, let n be the rank of p, and identify $X(S_x)=\mathbf{Z}\times X(S_p)$ with $\mathbf{Z}\times\mathbf{Z}^n$. Let $e=(1,-1,\dots,-1)\in\mathbf{Z}\times\mathbf{Z}^n$.

(2.1) Assume $-e\notin V$ (hence $e\in V$). Then $V'=\{a=(a_0,a_1,\dots,a_n)\in\mathbf{Z}^{n+1}\mid a-a_0e\in V\}$, and $Z'=Z$.

(2.2) Assume $e,-e\in V$. Then $V'=\{a\in\mathbf{Z}^n\mid(0,a)\in V\}$, and $Z'=Z$.

(2.3) Assume $e\notin V$ (hence $-e\in V$). Then $V'=\{a\in\mathbf{Z}^n\mid(0,a)\in V\}$, and $Z'=Z_{\mathrm{spl}}$.

3.4. The morphism $\eta^{\star}: D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$

3.4.1
The map $\eta^{\star}: D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is described as follows. This description is similar to the pure case in [16, Theorem 3.11] and [17, Theorem 5.2.11].

The map $\eta^{\star}$ sends $(p,V,Z)\in D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ in the presentation of $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ in Section 3.3.3 to $(T,V',Z)\in D_{\mathrm{BS},\mathrm{val}}$ in the presentation of $D_{\mathrm{BS},\mathrm{val}}$ in Section 3.3.5, where T and V' are as follows. Let $T'\subset S_p$ be the annihilator of $V^{\times}\subset X(S_p)$. Then T is the image of $T'\to G_{\mathbf{R}}(\mathrm{gr}^W)$ under $\tau_p^{\star}$. V' is the inverse image of $V/V^{\times}\subset X(T')$ under the homomorphism $X(T)\to X(T')$ induced by the canonical homomorphism $T'\to T$.

3.4.2
We also have the following description of $\eta^{\star}$ by regarding $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ as $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2),\mathrm{val}}$. Let the notation be as in Section 2.6. By Section 3.1.17, an element of $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2),\mathrm{val}}$ is written as (p,P,V,Z), where $p\in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$, $P\in\mathcal{P}(p)$, V is a valuative submonoid of $X(S_{p,P})$ such that $X(S_{p,P})^+\subset V$ and $X(S_{p,P})^+\cap V^{\times}=\{1\}$, and Z is either a $\tau^{\star}(\mathrm{Hom}(X(S_{p,P})/V^{\times},\mathbf{R}_{>0}))$-orbit in D or a $\tilde{\tau}^{\star}(\mathbf{R}_{>0}\times\mathrm{Hom}(X(S_{p,P})/V^{\times},\mathbf{R}_{>0}))$-orbit in D_{nspl} for the lifted action such that the image of Z in $D(\mathrm{gr}^W)$ is contained in $Z(p)$. The map $\eta^{\star}$ sends $(p,P,V,Z)\in D^{\star,\mathrm{BS}}_{\mathrm{SL}(2),\mathrm{val}}$ to $(P,V',Z)\in D_{\mathrm{BS},\mathrm{val}}$ in the presentation of $D_{\mathrm{BS},\mathrm{val}}$ as a set in Section 3.3.3, where $V'\subset X(S_P)$ is the inverse image of V under the homomorphism $X(S_P)\to X(S_{p,P})$ induced by the canonical homomorphism $S_{p,P}\to S_P$.

LEMMA 3.4.3
The morphism $\eta^{\star}: D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ *has the Kummer property of log structure in the sense of Lemma 3.2.8 (K).*

Proof
Let $x=(p,P,V,Z)\in D^{\star,\mathrm{BS}}_{\mathrm{SL}(2),\mathrm{val}}=D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ (see Section 3.4.2), and let y be the image of x in $D_{\mathrm{BS},\mathrm{val}}$. By Section 3.3.4, in the case of an A-orbit (resp., a B-orbit), V is a valuative submonoid of $X(S_{p,P})$ (resp., $\mathbf{Z}\times X(S_{p,P})$), and the stalk of $M/\mathcal{O}^\times$ of $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ at x is identified with $V/V^\times$. On the other hand, the stalk of $M/\mathcal{O}^\times$ of $D_{\mathrm{BS},\mathrm{val}}$ at y is identified with $V'/(V')^\times$, where in the case of an A-orbit (resp., a B-orbit), V' is the inverse image of V in $(X(S_P)^+)^{\mathrm{gp}}$ (resp., $\mathbf{Z}\times(X(S_P)^+)^{\mathrm{gp}}$) for the canonical map $(X(S_P)^+)^{\mathrm{gp}}\subset X(S_P)\to X(S_{p,P})$. Note that $(X(S_P)^+)^{\mathrm{gp}}$ is of finite index in $X(S_P)$. Furthermore, since the kernel of $S_{p,P}\to S_P$ is finite, the cokernel of $X(S_P)\to X(S_{p,P})$ is finite. Hence, the map $V'/(V')^\times\to V/V^\times$ is injective, and for any element a of $V/V^\times$, there is $m\geq 1$ such that a^m belongs to the image of $V'/(V')^\times$. □

THEOREM 3.4.4
The map $\eta^\star: D^{\star}_{\mathrm{SL}(2),\mathrm{val}}\to D_{\mathrm{BS},\mathrm{val}}$ in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ has the following properties.

(1) *The map $\eta^\star: D^{\star}_{\mathrm{SL}(2),\mathrm{val}}\to D_{\mathrm{BS},\mathrm{val}}$ is injective.*

(2) *Let $Q\in\prod_w\mathcal{W}(\mathrm{gr}^W_w)$, and define the open set $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(Q)$ of $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ as the inverse image of the open set $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)(Q)$ of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)$. Then the topology of $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(Q)$ coincides with the restriction of the topology of $D_{\mathrm{BS},\mathrm{val}}$ through $\eta^\star$.*

(3) *The diagram*

$$\begin{array}{ccc} D^{\star}_{\mathrm{SL}(2),\mathrm{val}} & \overset{\eta^\star}{\to} & D_{\mathrm{BS},\mathrm{val}} \\ \downarrow & & \downarrow \\ \prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)_{\mathrm{val}} & \overset{\eta}{\to} & \prod_w D_{\mathrm{BS}}(\mathrm{gr}^W_w)_{\mathrm{val}} \end{array}$$

is Cartesian in the category of topological spaces.

Proof
We prove (3) first. For each $x=(x_w)_w\in\prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)_{\mathrm{val}}$, for the image $y=(y_w)_w$ of x in $\prod_w D_{\mathrm{BS}}(\mathrm{gr}^W_w)_{\mathrm{val}}$, and for each $w\in\mathbf{Z}$, there is an open neighborhood U_w of x_w and an open neighborhood V_w of y_w having the following properties (i) and (ii).

(i) The image of U_w in $D_{\mathrm{BS}}(\mathrm{gr}^W_w)_{\mathrm{val}}$ is contained in V_w.

(ii) Let U be the inverse image of $\prod_w U_w$ in $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$, and let V be the inverse image of $\prod_w V_w$ in $D_{\mathrm{BS},\mathrm{val}}$. Take any $F\in D(\mathrm{gr}^W)$, and let $\bar{L}=\bar{\mathcal{L}}(F)$. Then we have a commutative diagram

$$\begin{array}{ccc} U & \cong & \prod_{\mathrm{val},w} U_w\times_{\mathrm{val}}\mathrm{spl}(W)\times_{\mathrm{val}}\bar{L} \\ \downarrow & & \downarrow \\ V & \cong & \prod_{\mathrm{val},w} V_w\times_{\mathrm{val}}\mathrm{spl}(W)\times_{\mathrm{val}}\bar{L} \end{array}$$

where $\prod_{\mathrm{val},w}$ is the product in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$, the upper row is an isomorphism over $\prod_{\mathrm{val},w} U_w$, and the lower row is an isomorphism over $\prod_{\mathrm{val},w} V_w$.

By Lemmas 3.2.8 and 3.4.3, the following diagram is Cartesian in the category of topological spaces:

$$\begin{array}{ccc} \prod_{\mathrm{val},w} U_w \times_{\mathrm{val}} \mathrm{spl}(W) \times_{\mathrm{val}} \bar{L} & \to & \prod_{\mathrm{val},w} V_w \times_{\mathrm{val}} \mathrm{spl}(W) \times_{\mathrm{val}} \bar{L} \\ \downarrow & & \downarrow \\ \prod_w U_w \times \mathrm{spl}(W) \times \bar{L} & \to & \prod_w V_w \times \mathrm{spl}(W) \times \bar{L} \end{array}$$

Theorem 3.4.4(3) follows from these two Cartesian diagrams.

Next we prove (1). The injectivity was proved in [16, Theorem 3.11] in the pure case. Hence, the map $\prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)_{\mathrm{val}} \to \prod_w D_{\mathrm{BS}}(\mathrm{gr}^W_w)_{\mathrm{val}}$ is injective. By (3), this proves the injectivity of $D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$.

We prove (2). Assume first that we are in the pure situation of weight w. Let $\mathcal{T}_1$ be the topology of $D_{\mathrm{SL}(2)}$ defined in [15, II], and let $\mathcal{T}_{1,\mathrm{val}}$ be the topology of $D_{\mathrm{SL}(2),\mathrm{val}}$ defined in this article. Let $\mathcal{T}_{2,\mathrm{val}}$ be the topology of $D_{\mathrm{SL}(2),\mathrm{val}}$ which is the weakest topology satisfying the following two conditions (i) and (ii).

(i) For any open set U of $D_{\mathrm{BS},\mathrm{val}}$, the pullback of U in $D_{\mathrm{SL}(2),\mathrm{val}}$ is open.
(ii) For any $Q \in \prod_w \mathcal{W}(\mathrm{gr}^W_w)$, $D_{\mathrm{SL}(2),\mathrm{val}}(Q)$ is open.

Let $\mathcal{T}_2$ be the topology of $D_{\mathrm{SL}(2)}$ as a quotient space of $D_{\mathrm{SL}(2),\mathrm{val}}$ which is endowed with the topology $\mathcal{T}_{2,\mathrm{val}}$. Recall that in [16] and [17], which treated the pure case, the topologies of $D_{\mathrm{SL}(2)}$ and $D_{\mathrm{SL}(2),\mathrm{val}}$ were defined as $\mathcal{T}_2$ and $\mathcal{T}_{2,\mathrm{val}}$, respectively (not as in the present series of articles). The study of $\mathcal{T}_2$ in [17, Section 10] and the study of $\mathcal{T}_1$ in [15, II, Section 3.4] show that $\mathcal{T}_1 = \mathcal{T}_2$. Since the map $\eta^{\star}$ from $D_{\mathrm{SL}(2),\mathrm{val}}$ with $\mathcal{T}_{1,\mathrm{val}}$ to $D_{\mathrm{BS},\mathrm{val}}$ is continuous as we have seen in Section 2.6, we have that $\mathcal{T}_{1,\mathrm{val}} \geq \mathcal{T}_{2,\mathrm{val}}$. Since the map $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$ is proper for $\mathcal{T}_{1,\mathrm{val}}$ (see Corollary 3.1.10) and also for $\mathcal{T}_{2,\mathrm{val}}$ (see [17, Theorem 3.14]), we have $\mathcal{T}_{1,\mathrm{val}} = \mathcal{T}_{2,\mathrm{val}}$.

Thus, we have proved (2) in the pure case. By (3), we have a Cartesian diagram of topological spaces

$$\begin{array}{ccc} D^{\star}_{\mathrm{SL}(2),\mathrm{val}} & \to & D_{\mathrm{BS},\mathrm{val}} \\ \downarrow & & \downarrow \\ (\prod_w D_{\mathrm{SL}(2)}(\mathrm{gr}^W_w)_{\mathrm{val}}) \times \mathrm{spl}(W) & \to & (\prod_w D_{\mathrm{BS}}(\mathrm{gr}^W_w)_{\mathrm{val}}) \times \mathrm{spl}(W) \end{array}$$

The vertical arrows are proper by Proposition 2.3.16 and [15, I, Corollary 8.5]. Hence, (2) is reduced to the pure case. □

3.4.5

As in Theorem 3.4.4(2), the topology of $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(Q)$ coincides with the induced topology from $D_{\mathrm{BS},\mathrm{val}}$. We show an example in which the topology of $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ is not the induced one from $D_{\mathrm{BS},\mathrm{val}}$. This example is pure of weight 3. So $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ is written as $D_{\mathrm{SL}(2),\mathrm{val}}$ below.

Let $H_{0,\mathbf{Z}} = H'_{0,\mathbf{Z}} \otimes \mathrm{Sym}^2(H'_{0,\mathbf{Z}})$, where $H'_{0,\mathbf{Z}}$ is a free $\mathbf{Z}$-module of rank 2 with basis e_1, e_2. Hence, $H_{0,\mathbf{Z}}$ is of rank 6. The intersection form $\langle\cdot,\cdot\rangle$ on $H_{0,\mathbf{Z}}$ is $b \otimes \mathrm{Sym}^2(b)$, where b is the antisymmetric bilinear form on $H'_{0,\mathbf{Z}}$ characterized by $\langle e_1, e_2\rangle = -1$. We have the following SL(2)-orbit (ρ, φ) in two variables:

$$\rho(g_1, g_2) = g_1 \otimes \mathrm{Sym}^2(g_2), \qquad \varphi(z_1, z_2) = F(z_1) \otimes \mathrm{Sym}^2 F(z_2)$$

$(g_1, g_2 \in \mathrm{SL}(2),\ z_1, z_2 \in \mathbf{C})$, where $F(z)$ is the decreasing filtration on $H'_{0,\mathbf{C}}$ defined by

$$F(z)^2 = 0 \subset F(z)^1 = \mathbf{C} \cdot (ze_1 + e_2) \subset F(z)^0 = H'_{0,\mathbf{C}}.$$

The associated homomorphism $\tau^\star : \mathbf{G}_m^2 \to G_\mathbf{R}$ is as follows. $\tau^\star(t_1, t_2)$ acts on $e_2 \otimes e_2^2$ by $t_1 t_2^3$, on $e_2 \otimes e_1 e_2$ by $t_1 t_2$, on $e_2 \otimes e_1^2$ by $t_1 t_2^{-1}$, on $e_1 \otimes e_2^2$ by $t_1^{-1} t_2$, on $e_1 \otimes e_1 e_2$ by $t_1^{-1} t_2^{-1}$, and on $e_1 \otimes e_1^2$ by $t_1^{-1} t_2^{-3}$. The associated weight filtrations $W^{(1)}$ and $W^{(2)}$ are as follows:

$$\begin{aligned}
&W_1^{(1)} = 0 \subset W_2^{(1)} = e_1 \otimes \mathrm{Sym}^2 H'_{0,\mathbf{R}} = W_3^{(1)} \subset W_4^{(1)} = H_{0,\mathbf{R}},\\
&W_{-1}^{(2)} = 0 \subset W_0^{(2)} = \mathbf{R} e_1 \otimes e_1^2 = W_1^{(2)} \subset W_2^{(2)}\\
&\qquad = W_1^{(2)} + \mathbf{R} e_1 \otimes e_1 e_2 + \mathbf{R} e_2 \otimes e_1^2 = W_3^{(2)}\\
&\qquad\quad \subset W_4^{(2)} = W_3^{(2)} + \mathbf{R} e_1 \otimes e_2^2 + \mathbf{R} e_2 \otimes e_1 e_2 = W_5^{(2)} \subset W_6^{(2)} = H_{0,\mathbf{R}}.
\end{aligned}$$

Let

$$\Phi = \{W^{(1)}, W^{(2)}\}.$$

We show that $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$ is not open for the topology induced from the topology of $D_{\mathrm{BS},\mathrm{val}}$. Let V be the valuative submonoid of $X(\mathbf{G}_m^2)$ which is, under the identification $X(\mathbf{G}_m^2) = \mathbf{Z}^2$, identified with the set of all $(a, b) \in \mathbf{Z}^2$ satisfying either $(a > 0)$ or $(a = 0$ and $b \geq 0)$. Consider the point $x := (p, V, Z) \in D_{\mathrm{SL}(2),\mathrm{val}}$, where p is the class of this SL(2)-orbit and Z is the torus orbit $\{F(iy_1) \otimes \mathrm{Sym}^2 F(iy_2) \mid y_1, y_2 \in \mathbf{R}_{>0}\}$ of p. The map $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ sends x to $y := (T, V, Z)$ in the presentation in Section 3.3.5 of $D_{\mathrm{BS},\mathrm{val}}$, where T is the image of $\tau^\star = \tau_p^\star : \mathbf{G}_m^2 \to G_\mathbf{R}$ and we regard V as a submonoid of $X(T)$ via the canonical isomorphism $\mathbf{G}_m^2 \cong T$ given by $\tau_p^\star$. In the presentation of $D_{\mathrm{BS},\mathrm{val}}$ in Section 3.3.3, this point y coincides with (P, V', Z), where P and V' are as follows. P is the $\mathbf{Q}$-parabolic subgroup of $G_\mathbf{R}$ consisting of all elements which preserve the following subspaces W'_w $(w \in \mathbf{Z})$:

$$W'_1 = \mathbf{R} e_1 \otimes e_1^2, \qquad W'_2 = W'_1 + \mathbf{R} e_1 \otimes e_1 e_2, \qquad W'_3 = W'_2 + \mathbf{R} e_1 \otimes e_2^2,$$
$$W'_4 = W'_3 + \mathbf{R} e_2 \otimes e_1^2, \qquad W'_5 = W'_4 + \mathbf{R} e_2 \otimes e_1 e_2.$$

We have $S_P \cong \mathbf{G}_m^3$. The inclusion map $T \to P$ induces a canonical homomorphism $T \to S_P$. $V' \subset X(S_P)$ is the inverse image of V under the canonical homomorphism $X(S_P) \to X(T) \cong X(\mathbf{G}_m^2)$.

Let f be the element of $\mathrm{Lie}(P_u)$ which sends $e_2 \otimes e_1^2$ to $e_1 \otimes e_2^2$ and kills $e_1 \otimes \mathrm{Sym}^2 H'_{0,\mathbf{R}}$, $e_2 \otimes e_1 e_2$, and $e_2 \otimes e_2^2$. For $c \in \mathbf{R}$, we have the SL(2)-orbit in two variables $(\rho^{(c)}, \varphi^{(c)})$ defined by

$$\rho^{(c)}(g_1, g_2) = \exp(cf)\rho(g_1, g_2)\exp(-cf), \qquad \varphi^{(c)}(z_1, z_2) = \exp(cf)\varphi(z_1, z_2).$$

The associated weight filtrations of $(\rho^{(c)},\varphi^{(c)})$ are $\exp(cf)W^{(1)}$ and $\exp(cf)W^{(2)}$. Since f respects $W^{(1)}$ but not $W^{(2)}$, $\{\exp(cf)W^{(1)},\exp(cf)W^{(2)}\}=\{W^{(1)}, \exp(cf)W^{(2)}\}$ is not contained in Φ if $c\neq 0$. If $c\in\mathbf{Q}$, then the filtration $\exp(cf)W^{(2)}$ is rational, and hence, $(\rho^{(c)},\varphi^{(c)})$ determines an element $p^{(c)}$ of $D_{\mathrm{SL}(2)}$. Let $x^{(c)}:=(p^{(c)},V,Z^{(c)})\in D_{\mathrm{SL}(2),\mathrm{val}}$, where V is the same as above and $Z^{(c)}$ is the torus orbit of $p^{(c)}$.

Now, when $c\in\mathbf{Q}\smallsetminus\{0\}$ converges to 0 in $\mathbf{R}$, the image $y^{(c)}$ of $x^{(c)}$ in $D_{\mathrm{BS,val}}$ converges to y. This is because P acts on $D_{\mathrm{BS,val}}(P)$ continuously in the natural way, $y\in D_{\mathrm{BS,val}}(P)$, and $y^{(c)}=\exp(cf)y$ for this action of P. Since $y^{(c)}\notin D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$, $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$ is not open for the topology induced by the topology of $D_{\mathrm{BS,val}}$.

The set $\{x^{(c)}\mid c\in\mathbf{Q}\}$ is discrete in $D_{\mathrm{SL}(2),\mathrm{val}}$, though the image $\{y^{(c)}\mid c\in\mathbf{Q}\}$ in $D_{\mathrm{BS,val}}$ has the topology of the subspace $\mathbf{Q}$ of $\mathbf{R}$ via the correspondence $y^{(c)}\leftrightarrow c$. Thus, the topology of $D_{\mathrm{SL}(2),\mathrm{val}}$ is not the induced topology from $D_{\mathrm{BS,val}}$.

3.5. The map $\eta: D_{\mathrm{SL}(2),\mathrm{val}}\to D_{\mathrm{BS,val}}$

3.5.1

We define a canonical map

$$\eta: D_{\mathrm{SL}(2),\mathrm{val}}\to D_{\mathrm{BS,val}}$$

following the method in the pure case (see [17]). But we will see that this map need not be continuous. η is the unique map such that, for any $x\in D_{\mathrm{SL}(2)}$ and any $x\in D_{\mathrm{SL}(2),\mathrm{val}}$ lying over x, the restriction of η to the subset $\bar{Z}(x)_{\mathrm{val}}$ of $D_{\mathrm{SL}(2),\mathrm{val}}$ (note $\tilde{x}\in\bar{Z}(x)_{\mathrm{val}}$) is the unique morphism in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})^+$ whose restriction to $Z(\tilde{x})$ is the inclusion morphism $Z(\tilde{x})\overset{\subseteq}{\to} D\subset D_{\mathrm{BS,val}}$.

The map η coincides with the composition of the two maps $D_{\mathrm{SL}(2),\mathrm{val}}\to D^{\star}_{\mathrm{SL}(2),\mathrm{val}}\xrightarrow{\eta^{\star}_{\mathrm{val}}} D_{\mathrm{BS,val}}$, where the first arrow is the following map λ_{val}. The restriction of λ_{val} to $D_{\mathrm{SL}(2),\mathrm{nspl},\mathrm{val}}$ is the morphism on the associated valuative spaces induced from the morphism $\lambda: D^{II}_{\mathrm{SL}(2),\mathrm{nspl}}\to D^{\star}_{\mathrm{SL}(2)}$ in Section 2.5.6. The restriction of λ_{val} to $D_{\mathrm{SL}(2),\mathrm{spl},\mathrm{val}}$ is the morphism on the associated valuative spaces induced from the isomorphism $\eta: D^{II}_{\mathrm{SL}(2),\mathrm{spl}}\overset{\cong}{\to} D^{\star}_{\mathrm{SL}(2),\mathrm{spl}}$ in Section 2.5.6.

The composition $D_{\mathrm{SL}(2),\mathrm{val}}\xrightarrow{\lambda_{\mathrm{val}}} D^{\star}_{\mathrm{SL}(2),\mathrm{val}}\to D_{\mathrm{SL}(2),\mathrm{val}}$ is the identity map. By Theorem 3.4.4(1), the map $\eta: D_{\mathrm{SL}(2),\mathrm{val}}\to D_{\mathrm{BS,val}}$ is injective.

PROPOSITION 3.5.2

(1) *The restriction of η to the open set $D^{II}_{\mathrm{SL}(2),\mathrm{nspl},\mathrm{val}}\cup D$ of $D^{II}_{\mathrm{SL}(2),\mathrm{val}}$ is a morphism in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$.*

(2) *For any $\Phi\in\overline{\mathcal{W}}$, the topology of $D^{II}_{\mathrm{SL}(2),\mathrm{nspl},\mathrm{val}}(\Phi)\cup D$ coincides with the topology induced from the topology of $D_{\mathrm{BS,val}}$.*

Proof
This restriction of η to $D^{II}_{\mathrm{SL}(2),\mathrm{nspl},\mathrm{val}} \cup D$ is the composition $D^{II}_{\mathrm{SL}(2),\mathrm{nspl},\mathrm{val}} \cup D \to D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \xrightarrow{\eta^{\star}_{\mathrm{val}}} D_{\mathrm{BS},\mathrm{val}}$, where the first arrow is the open immersion induced from the isomorphism $D^{II}_{\mathrm{SL}(2)} \cup D \cong D^{\star,+}_{\mathrm{SL}(2)}(\sigma_2)$ in Theorem 2.5.5(2). This proves (1). By this, (2) follows from Theorem 3.4.4(2). □

PROPOSITION 3.5.3
The equivalent conditions (i)–(vii) of Proposition 2.5.7 are equivalent to each of the following conditions.

(viii) *The identity map of D extends to an isomorphism $D^{II}_{\mathrm{SL}(2),\mathrm{val}} \cong D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ in $\mathcal{C}_{\mathbf{R}}(\mathrm{val})$.*

(ix) *The map $\lambda_{\mathrm{val}} : D^{I}_{\mathrm{SL}(2),\mathrm{val}} \to D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ (Section 3.5.1) is continuous.*

(x) *The map $\eta : D^{II}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is continuous.*

(xi) *The map $\eta : D^{I}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is continuous.*

(xii) *The map $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2),\mathrm{val}}$ is injective.*

Proof
(ii) ⇒ (viii). Take the associated valuative spaces.

The implications (viii) ⇒ (ix) and (viii) ⇒ (xii) are clear.

The implications (viii) ⇒ (x) and (x) ⇒ (xi) are easily seen.

(xi) ⇒ (ix). Use the fact that the topology of $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$ is the restriction of the topology of $D_{\mathrm{BS},\mathrm{val}}$ (see Theorem 3.4.4(2)).

(ix) ⇒ (iv). This is because $D^{I}_{\mathrm{SL}(2),\mathrm{val}} \to D^{I}_{\mathrm{SL}(2)}$ is proper surjective (see Corollary 3.1.10).

(xii) ⇒ (i). The proof of (vi) ⇒ (i) of Proposition 2.5.7 actually proves this. In that proof, assuming (i) does not hold, we used $x = (p, Z) \in D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ with p of rank 1 such that $x \neq x_{\mathrm{spl}}$. Since p is of rank 1, these x and x_{spl} are regarded canonically as elements of $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2),\mathrm{val}}$ whose images in $D_{\mathrm{SL}(2),\mathrm{val}}$ coincide. □

4. New spaces $D^{\sharp}_{\Sigma,[:]}$ and $D^{\sharp}_{\Sigma,[\mathrm{val}]}$ of nilpotent orbits

In this Section 4, we define and consider the new spaces $D^{\sharp}_{\Sigma,[:]}$ and $D^{\sharp}_{\Sigma,[\mathrm{val}]}$ of nilpotent orbits (nilpotent i-orbits, to be precise, in our terminology).

In Sections 4.1–4.3, for a topological space S endowed with an fs log structure on the sheaf of all $\mathbf{R}$-valued continuous functions, we define topological spaces $S_{[:]}$ (the space of ratios, Section 4.2) and $S_{[\mathrm{val}]}$ (see Section 4.3), and we define proper surjective continuous maps $S_{[:]} \to S$, $S_{\mathrm{val}} \to S_{[:]}$, and $S_{[\mathrm{val}]} \to S_{[:]}$, where S_{val} is as in Section 3.1. As will be explained in Section 4.4, in the case $S = D^{\sharp}_{\Sigma}$, we obtain the new spaces of nilpotent i-orbits $D^{\sharp}_{\Sigma,[:]}$ as $S_{[:]}$ and $D^{\sharp}_{\Sigma,[\mathrm{val}]}$ as $S_{[\mathrm{val}]}$, and S_{val} coincides with $D^{\sharp}_{\Sigma,\mathrm{val}}$, which we have already defined in [15, III]. We construct CKS maps $D^\sharp_{\Sigma,[:]} \to D^I_{\mathrm{SL}(2)}$ and $D^\sharp_{\Sigma,[\mathrm{val}]} \to D^I_{\mathrm{SL}(2),\mathrm{val}}$ in Section 4.5. We have already constructed the CKS map $D^\sharp_{\Sigma,\mathrm{val}} \to D^I_{\mathrm{SL}(2)}$ in [15, III].

4.1. The space of ratios in toric geometry

4.1.1

The space of ratios which we consider appears in the following way. Consider $S = \mathrm{Spec}(k[T_1,T_2])$ with k a field. Regarding S as the toric variety associated to the cone $\mathbf{R}^2_{\geq 0} \subset \mathbf{R}^2$, consider the toric varieties over k associated to rational finite subdivisions of the cone $\mathbf{R}^2_{\geq 0}$ (see Section 1.4.1), and let X be the projective limit of these toric varieties regarded as topological spaces with Zariski topology. It is the projective limit obtained by blowing up the origin $s = (0,0) \in S$ first and then continuing to blow up the intersections of irreducible components of the inverse image of $\mathrm{Spec}(k[T_1,T_2]/(T_1T_2)) \subset S$ on the blowup.

Let $X_0 \subset X$ be the inverse image of s, and endow X_0 with the topology as a subspace of X. Then we have the following continuous surjective map from X_0 to the interval $[0,\infty] \supset \mathbf{R}_{>0}$ despite the fact that the Zariski topology and the topology of real numbers are very different in nature. If $x \in X_0$, then the image of x in $[0,\infty]$ is defined as

$$\begin{aligned}&\sup\{a/b \mid (a,b) \in \mathbf{N}^2 \smallsetminus \{(0,0)\}, T_1^b/T_2^a \in \mathcal{O}_{X,x}\}\\ &\quad = \inf\{a/b \mid (a,b) \in \mathbf{N}^2 \smallsetminus \{(0,0)\}, T_2^a/T_1^b \in \mathcal{O}_{X,x}\}.\end{aligned}$$

Here $\mathbf{N} = \mathbf{Z}_{\geq 0}$, and $\mathcal{O}_X$ is the inductive limit of the inverse images on X of the structural sheaves of the blowups. The image of x in $[0,\infty]$ is, roughly speaking, something like the ratio $\log(T_1)/\log(T_2)$ at x.

In the definition below, this $[0,\infty]$ is the *space* $R(\mathbf{N}^2)$ *of ratios* of the fs monoid $\mathbf{N}^2 = (M_S/\mathcal{O}_S^\times)_s$ which is generated by the classes of T_1 and T_2. The above relation with the projective limit of blowups is generalized in Proposition 4.1.11.

4.1.2

In this Section 4.1, the notation $\mathcal{S}$ is used for an fs monoid. We denote the semigroup law of $\mathcal{S}$ multiplicatively unless we assume and state that $\mathcal{S} = \mathbf{N}^n$. So the neutral element of $\mathcal{S}$ is denoted by 1.

4.1.3

For a sharp fs monoid $\mathcal{S}$, let $R(\mathcal{S})$ be the set of all maps $r : (\mathcal{S} \times \mathcal{S}) \smallsetminus \{(1,1)\} \to [0,\infty]$ satisfying the following conditions (i)–(iii).

(i) $r(g,f) = r(f,g)^{-1}$.
(ii) $r(f,g)r(g,h) = r(f,h)$ if $\{r(f,g), r(g,h)\} \neq \{0,\infty\}$.
(iii) $r(fg,h) = r(f,h) + r(g,h)$.

We endow $R(\mathcal{S})$ with the topology of simple convergence. It is a closed subset of the product of copies of the compact set $[0,\infty]$ and hence is compact.

REMARK 4.1.4
From condition (i), we have $r(f,f)=1$. (Conversely, $r(f,f)=1$ and (ii) imply (i).) From this and from $r(1,f)+r(f,f)=r(f,f)$, which comes from (iii), we get

$$r(1,f)=0, \qquad r(f,1)=\infty \quad \text{for any } f\in\mathcal{S}\smallsetminus\{1\}.$$

4.1.5
For example, we have $R(\mathbf{N}^2)\cong[0,\infty]$, where $r\in R(\mathbf{N}^2)$ corresponds to $r(q_1,q_2)\in[0,\infty]$ with q_j the standard jth basis of $\mathbf{N}^2$. A description of $R(\mathbf{N}^n)$ for general n is given in Corollary 4.2.22.

4.1.6
We have a canonical bijection between $R(\mathcal{S})$ and the set $R'(\mathcal{S})$ of all equivalence classes of $((\mathcal{S}^{(j)})_{0\le j\le n},(N_j)_{1\le j\le n})$, where $n\ge 0$, $\mathcal{S}^{(j)}$ is a face of $\mathcal{S}$ such that

$$\mathcal{S}=\mathcal{S}^{(0)}\supsetneq\mathcal{S}^{(1)}\supsetneq\cdots\supsetneq\mathcal{S}^{(n)}=\{1\},$$

and N_j is a homomorphism $\mathcal{S}^{(j-1)}\to\mathbf{R}^{\mathrm{add}}$ such that $N_j(\mathcal{S}^{(j)})=0$ and such that $N_j(\mathcal{S}^{(j-1)}\smallsetminus\mathcal{S}^{(j)})\subset\mathbf{R}_{>0}$. The equivalence relation is given by multiplying each N_j by an element of $\mathbf{R}_{>0}$ (which may depend on j).

We define a map $R(\mathcal{S})\to R'(\mathcal{S})$ as follows. Let $r\in R(\mathcal{S})$. We give the corresponding element of $R'(\mathcal{S})$.

For $f\in\mathcal{S}\smallsetminus\{1\}$, let $\mathcal{S}(r,f)=\{g\in\mathcal{S}\mid r(g,f)\neq\infty\}$. Then the conditions (i)–(iii) on r in Section 4.1.3 show that $\mathcal{S}(r,f)$ is a face of $\mathcal{S}$. For $f,g\in\mathcal{S}$, we have $\mathcal{S}(r,f)\subset\mathcal{S}(r,g)$ if and only if $r(f,g)\neq\infty$, and we have $\mathcal{S}(r,f)\supset\mathcal{S}(r,g)$ if and only if $r(f,g)\neq 0$. Hence, the faces of $\mathcal{S}$ of the form $\mathcal{S}(r,f)$ ($f\in\mathcal{S}\smallsetminus\{1\}$) together with the face $\{1\}$ form a totally ordered set for the inclusion relation. Let $\mathcal{S}=\mathcal{S}^{(0)}\supsetneq\mathcal{S}^{(1)}\supsetneq\cdots\supsetneq\mathcal{S}^{(n)}=\{1\}$ be all the members of this set. Take $q_j\in\mathcal{S}^{(j-1)}\smallsetminus\mathcal{S}^{(j)}$ ($1\le j\le n$). We have a homomorphism $N_j:\mathcal{S}^{(j-1)}\to\mathbf{R}$ defined by $N_j(f)=r(f,q_j)$. This N_j kills $\mathcal{S}^{(j)}$ and $N_j(\mathcal{S}^{(j-1)}\smallsetminus\mathcal{S}^{(j)})\subset\mathbf{R}_{>0}$. If we replace q_j by another element q'_j, then N_j is multiplied by $r(q_j,q'_j)\in\mathbf{R}_{>0}$. Thus, we have the map $R(\mathcal{S})\to R'(\mathcal{S})$, $r\mapsto\operatorname{class}((\mathcal{S}^{(j)})_j,(N_j)_j)$.

Next we define the inverse map $R'(\mathcal{S})\to R(\mathcal{S})$. Let $\operatorname{class}((\mathcal{S}^{(j)})_{0\le j\le n},(N_j)_{1\le j\le n})\in R'(\mathcal{S})$. Let $(f,g)\in(\mathcal{S}\times S)\smallsetminus\{(1,1)\}$. We define $r(f,g)$ as follows. Let j be the largest integer greater than or equal to 0 such that f belongs to $\mathcal{S}^{(j)}$, and let k be that of g.

(1) If $j=k<n$, then $r(f,g)=N_{j+1}(f)/N_{j+1}(g)$.
(2) If $j>k$, then $r(f,g)=\infty$.
(3) If $j<k$, then $r(f,g)=0$.

This gives the map $R'(\mathcal{S})\to R(\mathcal{S})$.

It can be seen easily that the maps $R(\mathcal{S})\to R'(\mathcal{S})$ and $R'(\mathcal{S})\to R(\mathcal{S})$ are inverses of each other.

4.1.7

As in Section 3.1.1, for a sharp fs monoid $\mathcal{S}$, let $V(\mathcal{S})$ be the set of all valuative submonoids V of $\mathcal{S}^{\mathrm{gp}}$ such that $V \supset \mathcal{S}$ and $V^{\times} \cap \mathcal{S} = \{1\}$. We endow $V(\mathcal{S})$ with the following topology. For a finite set I of $\mathcal{S}^{\mathrm{gp}}$, let $U(I) = \{V \in V(\mathcal{S}) \mid I \subset V\}$. Then these $U(I)$ form a basis of open sets of $V(\mathcal{S})$.

4.1.8

We define a map

$$V(\mathcal{S}) \to R(\mathcal{S}), \quad V \mapsto r_V.$$

For $V \in V(\mathcal{S})$, $r_V \in R(\mathcal{S})$ is the map $\mathcal{S} \times \mathcal{S} \smallsetminus \{(1,1)\} \to [0,\infty]$ defined by

$$\begin{aligned} r_V(f,g) &= \sup\{a/b \mid (a,b) \in \mathbf{N}^2 \smallsetminus \{(0,0)\}, f^b/g^a \in V\} \\ &= \inf\{a/b \mid (a,b) \in \mathbf{N}^2 \smallsetminus \{(0,0)\}, g^a/f^b \in V\}, \end{aligned}$$

where $(f,g) \in (\mathcal{S} \times \mathcal{S}) \smallsetminus \{(1,1)\}$ (see Section 4.1.3).

PROPOSITION 4.1.9

The map $V(\mathcal{S}) \to R(\mathcal{S})$ is continuous and surjective.

Proof

We first prove the continuity of $V(\mathcal{S}) \to R(\mathcal{S})$. Let $f, g \in \mathcal{S} \smallsetminus \{1\}$, and assume $r_V(f,g) > a/b$, where $a, b \in \mathbf{N}$ and $b > 0$. We have $f^b/g^a \in V$. If $V' \in V(\mathcal{S})$ and $f^b/g^a \in V'$, we have $r_{V'}(f,g) \geq a/b$. This proves the continuity of $V(\mathcal{S}) \to R(\mathcal{S})$ (see Sections 4.1.7, 4.1.3).

We next prove the surjectivity of $V(\mathcal{S}) \to R(\mathcal{S})$. Let $\mathrm{class}((\mathcal{S}^{(j)})_{0 \leq j \leq n}, (N_j)_{1 \leq j \leq n}) \in R'(\mathcal{S})$ (see Section 4.1.6). Then the corresponding element of $R(\mathcal{S})$ is the image in $R(\mathcal{S})$ of the following element $V \in V(\mathcal{S})$. For $1 \leq j \leq n$, define the $\mathbf{Q}$-vector subspace $Q^{(j)}$ of the $\mathbf{Q}$-vector space $\mathcal{S}_{\mathbf{Q}} = \mathbf{Q} \otimes \mathcal{S}^{\mathrm{gp}}$ by $Q^{(j)} := \mathrm{Ker}(N_j : \mathcal{S}_{\mathbf{Q}}^{(j-1)} \to \mathbf{R})$. Then $Q^{(j)} \supset \mathcal{S}_{\mathbf{Q}}^{(j)}$. Take an isomorphism of $\mathbf{Q}$-vector spaces $\lambda_j : Q^{(j)}/\mathcal{S}_{\mathbf{Q}}^{(j)} \overset{\cong}{\to} \mathbf{Q}^{d(j)}$, where $d(j) := \dim(Q^{(j)}/\mathcal{S}_{\mathbf{Q}}^{(j)})$. Define V by the following. Let $a \in \mathcal{S}^{\mathrm{gp}}$. When there is j such that $1 \leq j \leq n$, $a \in \mathcal{S}_{\mathbf{Q}}^{(j-1)}$, and $a \notin Q^{(j)}$, then $a \in V$ if and only if $N_j(a) > 0$. When there is j such that $a \in Q^{(j)}$ and $a \notin \mathcal{S}_{\mathbf{Q}}^{(j)}$, then $a \in V$ if and only if the first nonzero entry of $\lambda_j(a) \in \mathbf{Q}^{d(j)}$ is greater than 0. □

4.1.10

Let k be a field, let S be the toric variety $\mathrm{Spec}(k[\mathcal{S}])$, and let X be the projective limit as a topological space of the toric varieties over k (with Zariski topology) which correspond to finite rational subdivisions of the cone $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{add}}_{\geq 0})$ (see Section 1.4.1). Let $\mathcal{O}_X$ be the inductive limit of the inverse images on X of the structural sheaves of these toric varieties. Let $X_0 \subset X$ be the inverse image of $s \in S = \mathrm{Spec}(k[\mathcal{S}])$, where s is the k-rational point of S at which all nontrivial elements of $\mathcal{S}$ have value 0. Endow X_0 with the topology as a subspace of X.

We have a continuous map $X_0 \to V(\mathcal{S})$ which sends $x \in X_0$ to $\{f \in \mathcal{S}^{\mathrm{gp}} \mid f \in \mathcal{O}_{X,x}\} \in V(\mathcal{S})$. The induced map $X_0(k) \to V(\mathcal{S})$ is surjective. In fact, for each $V \in V(\mathcal{S})$, the inverse image of V in X_0 under the map $X_0 \to V(\mathcal{S})$ is identified with $\mathrm{Spec}(k[V^\times])$. It has a k-rational point which sends all elements of $V^\times$ to 1. Composing with the map in Section 4.1.8 as

$$X_0(k) \subset X_0 \to V(\mathcal{S}) \to R(\mathcal{S})$$

and using Proposition 4.1.9, we have the following result.

PROPOSITION 4.1.11

(1) *The map $X_0 \to R(\mathcal{S})$ is continuous.*

(2) *The induced map $X_0(k) \to R(\mathcal{S})$ is surjective.*

COROLLARY 4.1.12

If we regard $R(\mathcal{S})$ as a quotient space of $V(\mathcal{S})$ or X_0, then the topology of $R(\mathcal{S})$ coincides with the quotient topology.

This is because $V(\mathcal{S})$ and X_0 are quasicompact and $R(\mathcal{S})$ is Hausdorff. Thus, Zariski topology and the topology of real numbers are well connected here.

4.2. The space $S_{[:]}$ of ratios

4.2.1

For a locally ringed space S endowed with an fs log structure, we define the set $S_{[:]}$ as the set of all pairs (s, r), where $s \in S$ and $r \in R((M_S/\mathcal{O}_S^\times)_s)$. We have the canonical surjection $S_{[:]} \to S$, $(s, r) \mapsto s$.

4.2.2

Let K be a field endowed with a nontrivial absolute value $|\cdot| : K \to \mathbf{R}_{\geq 0}$. Let S be a locally ringed space over K satisfying the equivalent conditions in Section 1.3.3, and assume that we are given an fs log structure on S. We define a natural topology of $S_{[:]}$ for which the projection $S_{[:]} \to S$ is a proper continuous map and which induces on each fiber of this projection the topology of $R((M_S/\mathcal{O}_S^\times)_s)$ defined in Section 4.1.3.

4.2.3

Let K and S be as in Section 4.2.2. To define the topology on $S_{[:]}$, the method is, so to speak, to combine the topology of S and the topologies of $R(\mathcal{S})$ (see Section 4.1) for $\mathcal{S} = (M_S/\mathcal{O}_S^\times)_s$ $(s \in S)$ by using a chart of the log structure.

Assume first that we are given a chart $\mathcal{S} \to M_S$ of the log structure, where $\mathcal{S}$ is an fs monoid. Fix $c \in \mathbf{R}_{>0}$. We have a map

$$S_{[:]} \to [0,\infty]^{\mathcal{S}\times\mathcal{S}}, \quad (s,r) \mapsto r_c,$$

where $r_c : \mathcal{S} \times \mathcal{S} \to [0,\infty]$ is defined by the following (1) and (2). Let $f, g \in \mathcal{S}$.

(1) If the images of f and g in $M_{S,s}$ belong to $\mathcal{O}_{S,s}^{\times}$, then

$$r_c(f,g) = \sup\bigl(c, -\log\bigl(\bigl|f(s)\bigr|\bigr)\bigr)/\sup\bigl(c, -\log\bigl(\bigl|g(s)\bigr|\bigr)\bigr).$$

(2) Otherwise,

$$r_c(f,g) = r(\bar{f}_s, \bar{g}_s),$$

where $\bar{f}_s$ (resp., $\bar{g}_s$) denotes the image of f (resp., g) in $(M_S/\mathcal{O}_S^{\times})_s$.

LEMMA 4.2.4

(1) *The map*

$$S_{[:]} \to S \times [0,\infty]^{\mathcal{S}\times\mathcal{S}}, \quad (s,r) \mapsto (s, r_c)$$

is injective.

(2) *The topology on $S_{[:]}$ induced by the embedding in (1) is independent of the choices of the chart and of the constant $c > 0$.*

Proof

We can see that (1) follows from the fact that the map $\mathcal{S} \to (M_S/\mathcal{O}_S^{\times})_s$ is surjective for any $s \in S$.

We prove (2). If we have two charts $\mathcal{S} \to M_S$ and $\mathcal{S}' \to M_S$, we have locally on S a third chart $\mathcal{S}'' \to M_S$ with homomorphisms of charts $\mathcal{S} \to \mathcal{S}''$ and $\mathcal{S}' \to \mathcal{S}''$. It is clear that if this third chart and two homomorphisms of charts are given and if the constant $c > 0$ is fixed, then the topology given by the chart $\mathcal{S}'' \to M_S$ and c is finer than the topology given by $\mathcal{S} \to M_S$ or $\mathcal{S}' \to M_S$ and c. Hence, it is sufficient to prove that if we have a homomorphism $\mathcal{S}' \to \mathcal{S}$ from a chart $\mathcal{S}' \to M_S$ to a chart $\mathcal{S} \to M_S$, then the topology given by the former chart and the constant $c' > 0$ is finer than the topology given by the latter and $c > 0$. It suffices to prove that, for $f, g \in \mathcal{S}$, the map $(s,r) \mapsto r_c(f,g)$ is continuous for the topology given by $\mathcal{S}' \to M_S$ and c'.

CLAIM 1

Let $f, g \in \mathcal{S}$, let $s \in S$, and assume that the images of f and g in $(M_S/\mathcal{O}_S^{\times})_s$ coincide. Let $c, c' > 0$. Then for some neighborhood U of s in S, we have a continuous map $R_{c,c'}(f,g) : U \to \mathbf{R}_{>0}$ whose value at $s' \in U$ is $\sup(c, -\log(|f(s')|))/\sup(c', -\log(|g(s')|))$ *if the images of f and g in $M_{S,s'}$ belong to $\mathcal{O}_{S,s'}^{\times}$, and is* 1 *otherwise.*

This Claim 1 is proved easily.

We continue the proof of (2). Let $f, g \in \mathcal{S}$. Then locally on S, we have $f', g' \in \mathcal{S}'$ and sections u, v of $\mathcal{O}_S^\times$ such that $f = f'u$ and $g = g'v$ in M_S. We have

$$r_c(f,g) = r_{c'}(f',g')R_{c,c'}(f,f')(s)R_{c',c}(g',g)(s).$$

This proves the desired continuity of $r_c(f,g)$. □

4.2.5

By the independence Lemma 4.2.4(2), we have a canonical topology of $S_{[:]}$ (globally).

4.2.6

Assume that $\mathcal{S}$ is sharp and that, for any $f \in \mathcal{S} \smallsetminus \{1\}$ and any $s \in S$, we have $|f(s)| < 1$. (Note that we have such a chart locally on S.) Let $Y = (\mathcal{S} \times \mathcal{S}) \smallsetminus \{(1,1)\}$. Then we have a slightly different embedding

$$S_{[:]} \to S \times [0,\infty]^Y, \quad (s,r) \mapsto (s, r_*),$$

where $r_* : Y \to [0,\infty]$ is defined as follows. Let $(f,g) \in Y$.

(1) If the images of f and g in $M_{S,s}$ belong to $\mathcal{O}_{S,s}^\times$, then

$$r_*(f,g) = \log(|f(s)|)/\log(|g(s)|).$$

(2) Otherwise,

$$r_*(f,g) = r(\bar{f}_s, \bar{g}_s),$$

where $\bar{f}_s$ (resp., $\bar{g}_s$) denotes the image of f (resp., g) in $(M_S/\mathcal{O}_S^\times)_s$.

LEMMA 4.2.7

Let the assumptions be as in Section 4.2.6.

(1) *The map $S_{[:]} \to S \times [0,\infty]^Y$ is injective.*

(2) *The topology of $S_{[:]}$ induced by this embedding coincides with the topology defined in Section 4.2.5.*

(3) *The image of the embedding (1) consists of all pairs $(s,r) \in S \times [0,\infty]^Y$ such that r satisfies conditions (i)–(iii) in Section 4.1.3 and such that the following conditions (iv) and (v) are satisfied. Let $(f,g) \in Y$.*

(iv) *If the images of f and g in $M_{S,s}$ belong to $\mathcal{O}_{S,s}^\times$, then $r(f,g) = \log(|f(s)|)/\log(|g(s)|)$.*

(v) *Otherwise, $r(f,g)$ depends only on the images of f and g in $(M_S/\mathcal{O}_S^\times)_s$.*

(4) *The image of the embedding in (1) is a closed set of $S \times [0,\infty]^Y$.*

Proof

Statements (1) and (3) follow from the fact that the map $\mathcal{S} \to (M_S/\mathcal{O}_S^\times)_s$ is surjective for any $s \in S$. Additionally, (4) follows from (3).

We prove (2). If $f \in \mathcal{S} \smallsetminus \{1\}$, by the property $|f(s)| < 1$ for any $s \in S$, we see that there is a continuous function $R_c(f) : S \to \mathbf{R}_{>0}$ whose value at $s \in S$ is

$-\sup(c, -\log(|f(s)|))/\log(|f(s)|)$ if the image of f in $M_{S,s}$ belongs to $\mathcal{O}_{S,s}^\times$ and is 1 otherwise. For $f, g \in \mathcal{S} \smallsetminus \{1\}$, we have

$$r_c(f,g) = r_*(f,g)R_c(f)(s)R_c(g)(s)^{-1}.$$

Furthermore, for $f \in \mathcal{S}$, $r_c(1,f)$ is the value of the continuous function $c/\sup(c, -\log(|f(s)|))$ at s, and $r_c(f,1)$ is the value of the continuous function $\sup(c, -\log(|f(s)|))/c$ at s, and for $f \in \mathcal{S} \smallsetminus \{1\}$, we have $r_*(1,f) = 0$ and $r_*(1,f) = \infty$. □

PROPOSITION 4.2.8

The canonical map $S_{[:]} \to S$ is proper.

Proof

Since $[0,\infty]^Y$ is compact, this follows from Lemma 4.2.7(4). □

4.2.9

For each $s \in S$, the topology of $R((M_S/\mathcal{O}_S^\times)_s)$ defined in Section 4.1 coincides with the topology of the fiber $R((M_S/\mathcal{O}_S^\times)_s)$ over s of $S_{[:]} \to S$ as a subspace of $S_{[:]}$.

LEMMA 4.2.10

Let S and S' be as in Section 4.2.2, and assume that we are given a strict morphism $S' \to S$ of locally ringed spaces over K with log structures. (For the word "strict," see Corollary 1.3.15.) Then the canonical map $S'_{[:]} \to S' \times_S S_{[:]}$ is a homeomorphism.

Proof

This is proved in the same way as Lemma 3.1.11. □

4.2.11

We consider $S_{[:]}$ more locally. Assume we are given a chart $\mathcal{S} \to M_S$. Let Φ be a set of faces of $\mathcal{S}$ which is totally ordered for the inclusion relation and which contains $\mathcal{S}$. Let $S_{[:]}(\Phi)$ be the subset of $S_{[:]}$ consisting of all (s,r) such that the inverse images in $\mathcal{S}$ of the faces of $(M_S/\mathcal{O}_S^\times)_s$ associated to r (see Section 4.1.6) belong to Φ. Then $S_{[:]}(\Phi)$ for all Φ forms an open covering of $S_{[:]}$.

4.2.12

Let the notation be as in Section 4.2.11. Assume further that, for any $f \in \mathcal{S} \smallsetminus \{1\}$, we have $|f(s)| < 1$ for any $s \in S$. (Such a chart always exists locally on S.) In Proposition 4.2.14, we give a description of the topological space $S_{[:]}(\Phi)$.

Write $\Phi = \{\mathcal{S}^{(j)} \mid 0 \le j \le n\}$, $\mathcal{S} = \mathcal{S}^{(0)} \supsetneq \mathcal{S}^{(1)} \supsetneq \cdots \supsetneq \mathcal{S}^{(n)}$. For each $1 \le j \le n$, fix $q_j \in \mathcal{S}^{(j-1)} \smallsetminus \mathcal{S}^{(j)}$. Consider the topological subspace

$$P \subset \mathbf{R}^n_{\ge 0} \times \prod_{j=0}^{n} \mathrm{Hom}(\mathcal{S}^{(j)}, \mathbf{R}^{\mathrm{add}})$$

(here the Hom space is endowed with the topology of simple convergence) consisting of elements (t,h) $(t = (t_j)_{1\le j\le n},\ t_j \in \mathbf{R}_{\ge 0},\ h = (h_j)_{0\le j\le n},\ h_j : \mathcal{S}^{(j)} \to \mathbf{R})$ satisfying the following conditions (i)–(iii) for $0 \le j < n$.

(i) $h_j(q_{j+1}) = 1$.
(ii) $h_j(f) = t_{j+1}h_{j+1}(f)$ for any $f \in \mathcal{S}^{(j+1)}$.
(iii) $h_j(\mathcal{S}^{(j)} \smallsetminus \mathcal{S}^{(j+1)}) \subset \mathbf{R}_{>0}$.

LEMMA 4.2.13

We have a unique continuous map $P \to \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})$ which sends (t,h) to the following $a \in \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})$. Let j be the smallest integer such that $0 \le j \le n$ and such that $t_k \ne 0$ if $j < k \le n$. Then

$$a(f) = \exp\Big(-h_j(f) \prod_{k=j+1}^{n} t_k^{-1}\Big) \in \mathbf{R}_{>0} \quad \text{if } f \in \mathcal{S}^{(j)},$$

$$a(f) = 0 \quad \text{if } f \in \mathcal{S} \smallsetminus \mathcal{S}^{(j)}.$$

Proof

The problem is the continuity of the map. This is shown as follows. Let $f \in \mathcal{S}$. It is sufficient to prove that the map $P \to \mathbf{R}_{\ge 0}$, $(t,h) \mapsto a(f)$ $(f \in \mathcal{S})$ (with notation as above) is continuous. Let j be the largest integer such that $0 \le j \le n$ and such that $f \in \mathcal{S}^{(j)}$. Then this map is the composition of the continuous map $P \to \mathbf{R}_{\ge 0}$ which sends $((t_j)_j, (h_j)_j) \in P$ to $\prod_{k=j+1}^{n} t_k \cdot h_j(f)^{-1}$ (note $h_j(f) > 0$) and the continuous map $\mathbf{R}_{\ge 0} \to \mathbf{R}_{\ge 0}$ which sends $t \in \mathbf{R}_{>0}$ to $\exp(-t^{-1})$ and 0 to 0. $\square$

PROPOSITION 4.2.14

Let the notation be as above. We have a Cartesian diagram of topological spaces

$$\begin{array}{ccc} S_{[:]}(\Phi) & \to & P \\ \downarrow & & \downarrow \\ S & \to & \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0}) \end{array}$$

where the lower horizontal arrow sends $s \in S$ to the map $f \mapsto |f(s)|$ $(f \in \mathcal{S})$, the right vertical arrow is as $a \mapsto a(f)$ $(f \in \mathcal{S})$ in Lemma 4.2.13, the left vertical arrow is the canonical one, and the upper horizontal arrow sends $(s,r) \in S_{[:]}(\Phi)$ $(s \in S,\ r \in R((M_S/\mathcal{O}_S^\times)_s))$ to $(s, ((t_j)_j, (h_j)_j))$, where $t_j = \log(|q_{j+1}(s)|)/\log(|q_j(s)|)$ (resp., $t_j = r(q_{j+1}, q_j)$) if $1 \le j < n$ and if $q_j q_{j+1}$ is invertible (resp., not invertible) at s, $t_n = -1/\log(|q_n(s)|)$, $h_j(f) = r(f, q_{j+1})$ for $0 \le j < n$, and $h_n(f) = -\log(|f(s)|)$. (Note that if $(s,r) \in S_{[:]}(\Phi)$ and $f \in \mathcal{S}^{(n)}$, then the image of f in $M_{S,s}$ belongs to $\mathcal{O}_{S,s}^\times$ and hence $|f(s)| \in \mathbf{R}_{>0}$.)

Proof
The converse map is given by $(s,(t,h)) \mapsto (s,r)$, where r is as follows. Let $(a,b) \in (M_S/\mathcal{O}_S^\times)_s \times (M_S/\mathcal{O}_S^\times)_s \smallsetminus \{(1,1)\}$, and take $f,g \in \mathcal{S}$ such that the image of f (resp., g) in $(M_S/\mathcal{O}_S^\times)_s$ is a (resp., b). Take the largest j such that $0 \le j \le n-1$ and $f,g \in \mathcal{S}^{(j)}$. Then $r(f,g) = h_j(f)/h_j(g) \in [0,\infty]$. (Note that at least one of f,g is outside $\mathcal{S}^{(j+1)}$, and hence, at least one of $h_j(f)$ and $h_j(g)$ is nonzero.) It is easy to see that this is the converse map and continuous. $\square$

REMARK 4.2.15
In $(t,h) \in P$ $(t = (t_j)_{1\le j\le n} \in \mathbf{R}^n_{\ge 0})$, t_j for $1 \le j \le n-1$ is determined by h as $t_j = h_{j-1}(q_{j+1})$. t_n is determined by the image a of (t,h) in $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})$ as $t_n = -1/\log(a(q_n))$. These explain the fact that, in the above proof of Proposition 4.2.14, the converse map $(s,(t,h)) \mapsto (s,r)$ is described without using t.

4.2.16
Let $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})_{<1}$ be the open set of $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})$ consisting of all elements h such that $h(f) < 1$ for any $f \in \mathcal{S} \smallsetminus \{1\}$. Then the images of S and P in $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})$, under the maps in Proposition 4.2.14, are contained in $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})_{<1}$. Hence, by Proposition 4.2.14, we have the following.

COROLLARY 4.2.17
In the case $S = \mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})_{<1}$ with the sheaf of all $\mathbf{R}$-valued continuous functions and with the natural log structure, $S_{[:]}(\Phi)$ is identified with P.

4.2.18
We give a comment on this space P. For $1 \le j \le n$, we fixed an element q_j of $\mathcal{S}^{(j-1)} \smallsetminus \mathcal{S}^{(j)}$ (see Proposition 4.2.14). Let $m(j) = \dim_{\mathbf{Q}}(\mathcal{S}^{(j-1)}_{\mathbf{Q}}/\mathcal{S}^{(j)}_{\mathbf{Q}}) - 1$ if $1 \le j \le n$, and let $m(n+1) = \dim_{\mathbf{Q}}(\mathcal{S}^{(n)}_{\mathbf{Q}})$. For $1 \le j \le n+1$, fix elements $q_{j,k}$ $(0 \le k \le m(j))$ of $(\mathcal{S}^{(j-1)})^{\mathrm{gp}}$ satisfying the following conditions (i)–(iii).

(i) $q_{j,0} = q_j$ if $1 \le j \le n$.
(ii) For $1 \le j \le n$, $(q_{j,k} \bmod \mathcal{S}^{(j)}_{\mathbf{Q}})_{0\le k\le m(j)}$ is a $\mathbf{Q}$-basis of $\mathcal{S}^{(j-1)}_{\mathbf{Q}}/\mathcal{S}^{(j)}_{\mathbf{Q}}$.
(iii) $(q_{n+1,k})_{1\le k\le m(n+1)}$ is a $\mathbf{Q}$-basis of $\mathcal{S}^{(n)}_{\mathbf{Q}}$.

PROPOSITION 4.2.19
We have an injective open map

$$P \hookrightarrow \mathbf{R}^n_{\ge 0} \times \prod_{j=1}^{n+1} \mathbf{R}^{m(j)}$$

which sends $(t,h) \in P$ $(t \in \mathbf{R}^n_{\ge 0},\ h \in \prod_{j=0}^{n} \mathrm{Hom}(\mathcal{S}^{(j)}, \mathbf{R}^{\mathrm{add}}))$ to (t,a), where $a = (a_j)_{1\le j\le n+1}$, $a_j = (a_{j,k})_{1\le k\le m(j)}$ with

$$a_{j,k} = h_{j-1}(q_{j,k}) \in \mathbf{R}$$

for $1 \le j \le n+1$. Here we define $h_{j-1}(q_{j,k})$ by using the unique extension of $h_{j-1} : \mathcal{S}^{(j-1)} \to \mathbf{R}$ to a homomorphism $(\mathcal{S}^{(j-1)})^{\mathrm{gp}} \to \mathbf{R}^{\mathrm{add}}$.

The proof is easy.

4.2.20

Consider the case $S = |\Delta|^n$, where $|\Delta| = \{t \in \mathbf{R} \mid 0 \le t < 1\}$ with the sheaf of all $\mathbf{R}$-valued continuous functions and with the fs log structure associated to $\mathbf{N}^n \to \mathcal{O}_S$, $m \mapsto \prod_{j=1}^n q_j^{m(j)}$, where q_j $(1 \le j \le n)$ are the coordinate functions. Let $\mathcal{S}$ be the multiplicative monoid generated by q_j $(1 \le j \le n)$ which is identified with $\mathbf{N}^n$. Then $|\Delta|^n$ is identified with $\mathrm{Hom}(\mathcal{S}, \mathbf{R}^{\mathrm{mult}}_{\ge 0})_{<1}$ in Section 4.2.16.

Let $\Phi = \{\mathcal{S}^{(j)} \mid 0 \le j \le n\}$, where $\mathcal{S}^{(j)}$ is generated by q_k $(j < k \le n)$. Then $S_{[:]}$ is covered by the open sets $S_{[:]}(g(\Phi))$, where g ranges over elements of the permutation group $\mathfrak{S}_n$ acting on $\mathcal{S}$, and g induces a homeomorphism $S_{[:]}(\Phi) \cong S_{[:]}(g(\Phi))$. We describe $S_{[:]}(\Phi)$.

PROPOSITION 4.2.21

Let the notation be as in Section 4.2.20. Then we have a commutative diagram

$$\begin{array}{ccc} S_{[:]}(\Phi) & \cong & \mathbf{R}^n_{\ge 0} \\ \downarrow & & \downarrow \\ S & = & |\Delta|^n \end{array}$$

in which the upper horizontal isomorphism sends $(s, r) \in S_{[:]}(\Phi)$ to $(t_1, \dots, t_n)$, where $t_j = r(q_{j+1}, q_j)$ $(1 \le j \le n-1)$ and $t_n = -1/\log(q_n(s))$, and the right vertical arrow is $(t_j)_{1 \le j \le n} \mapsto (q_j)_{1 \le j \le n}$, where $q_j = \exp(-\prod_{k=j}^n t_k^{-1})$.

Proof

This follows from Corollary 4.2.17. □

COROLLARY 4.2.22

Let the notation be as in Section 4.2.20. Regarding $R(\mathcal{S})$ as the fiber of $S_{[:]} \to S = |\Delta|^n$ over the point $(0, \dots, 0) \in S$, define $R(\mathcal{S})(\Phi) = R(\mathcal{S}) \cap S_{[:]}(\Phi)$. Then we have a homeomorphism

$$R(\mathcal{S})(\Phi) \cong \mathbf{R}^{n-1}_{\ge 0}$$

which sends $r \in R(\mathcal{S})(\Phi)$ to $(t_1, \dots, t_{n-1})$, where $t_j = r(q_{j+1}, q_j)$.

Proof

This follows from Proposition 4.2.21. □

LEMMA 4.2.23

Let S, $|S|$, and $M_{|S|}$ be as in Lemma 3.1.12. Then we have a canonical homeomorphism $S_{[:]} \cong |S|_{[:]}$.

Proof

As in the proof of Lemma 3.1.12, we have a canonical isomorphism $(M_S/\mathcal{O}_S^\times)_s \cong (M_{|S|}/\mathcal{O}_{|S|}^\times)_s$ for each $s \in S$. This gives a canonical bijection between $S_{[:]}$ and $|S|_{[:]}$. By Proposition 4.2.14, they have the same topology. $\square$

4.3. $S_{[:]}$, $S_{[\mathrm{val}]}$, and S_{val}

Let K and S be as in Section 4.2.2. We construct a topological space $S_{[\mathrm{val}]}$ and proper surjective continuous maps

$$S_{\mathrm{val}} \to S_{[:]}, \qquad S_{[\mathrm{val}]} \to S_{[:]}.$$

Here S_{val} is as in Section 3.1.

4.3.1

Let $S_{\mathrm{val}} \to S_{[:]}$ be the map $(s,V,h) \mapsto (s,r_V)$, where $V \mapsto r_V$ is the map $V(\mathcal{S}) \to R(\mathcal{S})$ for $\mathcal{S} = (M_S/\mathcal{O}_S^\times)_s$ (see Sections 3.1.2, 4.1.8).

PROPOSITION 4.3.2

The map $S_{\mathrm{val}} \to S_{[:]}$ is continuous, proper, and surjective.

Proof

The surjectivity follows from the surjectivity in Proposition 4.1.9. Once we prove the continuity, properness follows from the properness of $S_{\mathrm{val}} \to S$ and of $S_{[:]} \to S$. We prove the continuity. Working locally on S, we may and do assume that we have a chart $\mathcal{S} \to M_S$ with $\mathcal{S}$ a sharp fs monoid such that, for any $f \in \mathcal{S} \smallsetminus \{1\}$ and $s \in S$, we have $|f(s)| < 1$.

Fix $(s_0,V_0,h_0) \in S_{\mathrm{val}}$, and let $(s_0,r_0) \in S_{[:]}$ be its image. We show that, when $(s,V,h) \in S_{\mathrm{val}}$ converges to (s_0,V_0,h_0), its image $(s,r) \in S_{[:]}$ converges to (s_0,r_0). Let $f,g \in \mathcal{S} \smallsetminus \{1\}$. It is sufficient to prove that $r_*(f,g) \in [0,\infty]$ (see Section 4.2.6) converges to $(r_0)_*(f,g) \in [0,\infty]$. If at least one of f and g is invertible at s_0 (i.e., if at least one of the images of f and g in M_{S,s_0} belongs to $\mathcal{O}_{S,s_0}^\times$), then the function $(s,r) \mapsto r_*(f,g) \in [0,\infty]$ on $S_{[:]}$ comes from the continuous function $s \mapsto \log(|f(s)|)/\log(|g(s)|) \in [0,\infty]$ on some neighborhood of s_0 in S. Hence, we may assume that both f and g are not invertible at s_0. Assume that $(r_0)_*(f,g) > a/b$, $a,b \in \mathbf{N}$, $b > 0$. It is sufficient to prove that $r_*(f,g) > a/b$ when (s,V,h) is sufficiently near (s_0,V_0,h_0). Let $\varphi = f^b/g^a \in \mathcal{S}^{\mathrm{gp}}$. Since the image $\bar{\varphi}_{s_0}$ of φ in $(M_S/\mathcal{O}_S^\times)_{s_0}$ belongs to V_0, there is a neighborhood U of (s_0,V_0,h_0) in S_{val} such that if $(s,V,h) \in U$, then $\bar{\varphi}_s \in V$. If $(s,V,h) \in U$ and if at least one of f and g are not invertible at s, then $r_*(f,g) = r(\bar{f}_s,\bar{g}_s) \ge a/b$ because $\bar{\varphi}_s \in V$. Consider points $(s,V,h) \in U$ such that both f and g are invertible at s. On U, the function $(s,V,h) \mapsto h(\varphi)$ is continuous. (Here $h(\varphi)$ is defined to be 0 if $\bar{\varphi}_s \notin V^\times$.) We have

$$r_*(f,g) = b^{-1}r_*(f^b,g) = b^{-1}r_*(g^a\varphi,g) = (a/b) + b^{-1}\log\bigl(h(\varphi)\bigr)/\log\bigl(|g(s)|\bigr).$$

When $(s,V,h) \in U$ converges to (s_0,V_0,h_0), $h(\varphi)$ converges to $h_0(\varphi) \in \mathbf{R}$ and $g(s)$ converges to 0. If $h_0(\varphi) = 0$, then when (s,V,h) converges to (s_0,V_0,h_0), we have

$h(\varphi) < 1$ and $|g(s)| < 1$ and, hence, $\log(h(\varphi))/\log(|g(s)|) > 0$. If $h_0(\varphi) > 0$, then when (s, V, h) converges to (s_0, V_0, h_0), $\log(h(\varphi))/\log(|g(s)|)$ converges to 0. $\square$

4.3.3

We next discuss $S_{[\mathrm{val}]}$. To define it, we use the following *new log structure on* $S_{[:]}$, which is endowed with the sheaf $\mathcal{O}_{S_{[:]}}$ of all $\mathbf{R}$-valued continuous functions. (We use the word "new log structure," to distinguish this log structure from the "old" log structure on $S_{[:]}$, which is defined as the inverse image of the log structure of S on the inverse image of $\mathcal{O}_S$ on $S_{[:]}$.)

Assume that we are given a chart $\mathcal{S} \to M_S$ with $\mathcal{S}$ a sharp fs monoid such that $|f(s)| < 1$ for any $f \in \mathcal{S} \smallsetminus \{1\}$ and for any $s \in S$. Let $\mathcal{S}^{(j)}$ $(0 \le j \le n)$ be faces of $\mathcal{S}$ such that $\mathcal{S} = \mathcal{S}^{(0)} \supsetneq \mathcal{S}^{(1)} \supsetneq \cdots \supsetneq \mathcal{S}^{(n)}$, and let $\Phi = \{\mathcal{S}^{(j)} \mid 0 \le j \le n\}$. Take $q_j \in \mathcal{S}^{(j-1)} \smallsetminus \mathcal{S}^{(j)}$ for $1 \le j \le n$. Then we define the new log structure on $S_{[:]}(\Phi)$ as the fs log structure associated to

$$\mathbf{N}^n \to \mathcal{O}_{S_{[:]}}, \qquad m \mapsto \Bigl(\prod_{j=1}^{n-1} r(q_{j+1}, q_j)^{m(j)/2}\Bigr) \cdot \bigl(-1/\log(|q_n|)\bigr)^{m(n)/2}.$$

Then it is easy to see that this log structure glues to an fs log structure on $S_{[:]}$ which is independent of any choices. In the identification $S_{[:]} = |S|_{[:]}$ (see Lemma 4.2.23), the new log structure of $S_{[:]}$ and that of $|S|_{[:]}$ coincide.

REMARK 4.3.4

It may seem strange to take the square root $(\cdot)^{m(j)/2}$ in the definition of this log structure. But this becomes important in Section 5 to ensure that the CKS map $D^{\sharp}_{\Sigma,[:]} \to D_{\mathrm{SL}(2)}$ respects (and $D^{\sharp,\mathrm{mild}}_{\Sigma,[:]} \to D^{\diamond}_{\mathrm{SL}(2)}$ which appears later (see Theorem 5.1.10) also respects) the log structures.

4.3.5

Let $S_{[\mathrm{val}]}$ be the valuative space $(S_{[:]})_{\mathrm{val}}$ (Section 3.1) associated to $S_{[:]}$ endowed with this new log structure.

By Section 3.1, the map $S_{[\mathrm{val}]} \to S_{[:]}$ is proper and surjective.

LEMMA 4.3.6

Let S (resp., S') be a topological space endowed with the sheaf of all $\mathbf{R}$-valued continuous functions and with an fs log structure, and let $S' \to S$ be a strict morphism (see Corollary 1.3.15) of locally ringed spaces over $\mathbf{R}$ with log structures. Then the canonical map $S'_{[\mathrm{val}]} \to S' \times_S S_{[\mathrm{val}]}$ is a homeomorphism.

Proof

This is proved in the same way as Lemma 3.1.11. $\square$

PROPOSITION 4.3.7

Assume $K = \mathbf{R}$. There is a unique homeomorphism

$$(|\Delta|^n)_{[\mathrm{val}]} \cong (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$$

in which $(q_j)_{1\leq j\leq n} \in (|\Delta| \smallsetminus \{0\})^n \subset (|\Delta|^n)_{[\mathrm{val}]}$ corresponds to $(-1/\log(q_j))_{1\leq j\leq n} \in \mathbf{R}^n_{>0} \subset (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$.

Proof

This is deduced from Proposition 4.2.21. □

4.3.8. Examples

We compare $S_{[:]}$, S_{val}, and $S_{[\mathrm{val}]}$ in the case $K = \mathbf{R}$ and $S = \mathbf{R}^2_{\geq 0}$ with the standard log structure. The maps from these spaces to S are homeomorphisms outside $(0,0) \in S$. We describe the fibers over $(0,0)$ explicitly.

(1) The fiber of $S_{[:]} \to S$ over $(0,0) \in S$ is canonically homeomorphic to the interval $[0,\infty]$. It consists of points $r(a)$ with $a \in [0,\infty]$. $(q_1,q_2) \in \mathbf{R}^2_{>0}$ converges to $r(a)$ if and only if $q_j \to 0$ and $\log(q_2)/\log(q_1) \to a$.

(2) A difference between the surjection $S_{\mathrm{val}} \to S_{[:]}$ and the surjection $S_{[\mathrm{val}]} \to S_{[:]}$ is that the fiber of the former surjection over $r(a)$ has cardinality greater than 1 if and only if $a \in \mathbf{Q}_{>0}$ and the fiber of the latter surjection over $r(a)$ has cardinality greater than 1 if and only if $a = 0$ or $a = \infty$.

(3) The fiber of $S_{\mathrm{val}} \to S$ over $(0,0) \in S$ consists of points $p(a)$ $(a \in [0,\infty] \smallsetminus \mathbf{Q}_{>0})$ and $p(a,c)$ $(a \in \mathbf{Q}_{>0},\ c \in [0,\infty])$. $(q_1,q_2) \in \mathbf{R}^2_{>0}$ converges to $p(a)$ if and only if $q_j \to 0$ and $\log(q_2)/\log(q_1) \to a$. $(q_1,q_2) \in \mathbf{R}^2_{>0}$ converges to $p(a,c)$ if and only if $q_j \to 0$, $\log(q_2)/\log(q_1) \to a$, and $q_1^a/q_2 \to c$. Under the map $S_{\mathrm{val}} \to S_{[:]}$, $p(a)$ goes to $r(a) \in S_{[:]}$, and $p(a,c)$ goes to $r(a) \in S_{[:]}$.

(4) The fiber of $S_{[\mathrm{val}]}$ over $(0,0) \in S$ consists of points $s(a)$ $(a \in [0,\infty] \smallsetminus \mathbf{Q}_{>0})$ and $s(a,c)$ $(a \in \mathbf{Q}_{>0},\ c \in [0,\infty])$. $(q_1,q_2) \in \mathbf{R}^2_{>0}$ converges to $s(a)$ if and only if $q_j \to 0$ and, for $t_j := -1/\log(q_j)$ (so $t_j \to 0$), $\log(t_2)/\log(t_1) \to a$. $(q_1,q_2) \in \mathbf{R}^2_{>0}$ converges to $s(a,c)$ if and only if $q_j \to 0$ and, for $t_j := -1/\log(q_j)$ (so $t_j \to 0$), $\log(t_2)/\log(t_1) \to a$, and t_1^a/t_2 converges to c. Under the map $S_{[\mathrm{val}]} \to S_{[:]}$, $s(1,c)$ goes to $r(c) \in S_{[:]}$, $s(a)$ with $a < 1$ and $s(a,c)$ with $a < 1$ go to $r(0)$ in $S_{[:]}$, and $s(a)$ with $a > 1$ and $s(a,c)$ with $a > 1$ go to $r(\infty)$ in $S_{[:]}$.

(5) We give some examples of convergences.

(5.1) Fix $c \in \mathbf{R}_{>0}$. If $q \in \mathbf{R}_{>0}$ and $q \to 0$, then $(cq,q) \in \mathbf{R}^2_{>0}$ converges to $r(1)$ in $S_{[:]}$, to $p(1,c)$ in S_{val}, and to $s(1,1)$ in $S_{[\mathrm{val}]}$. Thus, the limit in $S_{[:]}$ and the limit in $S_{[\mathrm{val}]}$ are independent of c, but the limit in S_{val} depends on c.

(5.2) Fix $a \in \mathbf{R}$ such that $0 < a < 1$. If $t \in \mathbf{R}_{>0}$ and $t \to 0$, $(\exp(-1/t), \exp(-1/t^a)) \in \mathbf{R}^2_{>0}$ converges to $r(0)$ in $S_{[:]}$, to $p(0)$ in S_{val}, and to $s(a)$ (resp., $s(a,1)$) in $S_{[\mathrm{val}]}$ if $a \notin \mathbf{Q}$ (resp., $a \in \mathbf{Q}$). Thus, the limit in $S_{[:]}$ and the limit in S_{val} are independent of a, but the limit in $S_{[\mathrm{val}]}$ depends on a.

4.4. The spaces $D^\sharp_{\Sigma,[:]}$ and $D^\sharp_{\Sigma,[\mathrm{val}]}$

4.4.1

Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$ (see [15, III, Section 2.2.3]), and let Γ be a neat subgroup of $G_{\mathbf{Z}}$ which is strongly compatible with Σ. Then we have a space $\Gamma\setminus D_\Sigma$ which is endowed with a sheaf of holomorphic functions and an fs log structure. By taking $K=\mathbf{C}$ in Section 4.2, we have a topological space $(\Gamma\setminus D_\Sigma)_{[:]}$ with a proper surjective map $(\Gamma\setminus D_\Sigma)_{[:]}\to\Gamma\setminus D_\Sigma$.

4.4.2

Let Σ be a weak fan in $\mathfrak{g}_{\mathbf{Q}}$, and let $D^\sharp_\Sigma$ be the topological space defined in [15, III, Section 2.2.5]. We define topological spaces $D^\sharp_{\Sigma,[:]}$ and $D^\sharp_{\Sigma,[\mathrm{val}]}$ and proper surjective maps $D^\sharp_{\Sigma,[:]}\to D^\sharp_\Sigma$, $D^\sharp_{\Sigma,\mathrm{val}}\to D^\sharp_{\Sigma,[:]}$, and $D^\sharp_{\Sigma,[\mathrm{val}]}\to D^\sharp_{\Sigma,[:]}$. Here $D^\sharp_{\Sigma,\mathrm{val}}$ is the topological space defined in [15, III, Section 3.2].

4.4.3

Let $\sigma\in\Sigma$, and consider the open set $D^\sharp_\sigma$ of $D^\sharp_\Sigma$. There is a neat subgroup Γ of $G_{\mathbf{Z}}$ which is strongly compatible with the fan face(σ) of all faces of σ. We define the topological space $D^\sharp_{\sigma,[:]}$ as the fiber product of $D^\sharp_\sigma\to\Gamma\setminus D_\sigma\leftarrow(\Gamma\setminus D_\sigma)_{[:]}$. This is independent of the choice of Γ.

Furthermore, the inverse image of the new log structure of $(\Gamma\setminus D_\sigma)_{[:]}$ on $D^\sharp_{\sigma,[:]}$ (given on the sheaf of all $\mathbf{R}$-valued continuous functions), which we call the *new log structure of* $D^\sharp_{\sigma,[:]}$, is independent of the choice of Γ. These $D^\sharp_{\sigma,[:]}$'s glue to a topological space $D^\sharp_{\Sigma,[:]}$ over $D^\sharp_\Sigma$, and the new log structures of $D^\sharp_{\sigma,[:]}$ glue to an fs log structure on the sheaf of all $\mathbf{R}$-valued functions on $D^\sharp_{\Sigma,[:]}$, which we call the *new log structure*.

We define $D^\sharp_{\Sigma,[\mathrm{val}]}$ as the valuative space associated to $D^\sharp_{\Sigma,[:]}$ with the new log structure. We have canonical proper surjective maps $D^\sharp_{\Sigma,[:]}\to D^\sharp_\Sigma$ and $D^\sharp_{\Sigma,[\mathrm{val}]}\to D^\sharp_{\Sigma,[:]}$.

4.4.4

Before we define the canonical map $D^\sharp_{\Sigma,\mathrm{val}}\to D^\sharp_{\Sigma,[:]}$, we remark that, though we have a canonical new log structure on $D^\sharp_{\Sigma,[:]}$, we do not have a canonical log structure on $D^\sharp_\Sigma$. For $\sigma\in\Sigma$ and for a neat subgroup Γ of $G_{\mathbf{Z}}$ which is strongly compatible with face(σ), the pullback of the log structure of $\Gamma\setminus D_\sigma$ on $D^\sharp_\sigma$ depends on the choice of Γ. Here we endow $D^\sharp_\sigma$ with the sheaf of all $\mathbf{C}$-valued continuous functions.

For example, consider the classical case $H_{0,\mathbf{Z}}=\mathbf{Z}^2$ of pure weight 1 of Hodge type $(1,0)+(0,1)$, in which D is the upper half-plane. For the standard choice of σ and $\Gamma=\left(\begin{smallmatrix}1&\mathbf{Z}\\0&1\end{smallmatrix}\right)$, $\Gamma\setminus D_\sigma$ is isomorphic to the unit disk, and the log structure is generated by the coordinate function q. $D^\sharp_\sigma$ is identified with $\{x+iy\mid x\in\mathbf{R}, 0<y\le\infty\}$, and the canonical projection $D^\sharp_\sigma\to\Gamma\setminus D_\sigma$ is identified with $z\mapsto$

$\exp(2\pi iz)$. We have $D^\sharp_{\sigma,[:]} = D^\sharp_\sigma$, and the new log structure on it is generated by $1/y^{1/2}$ or, equivalently, by $1/(\log|q|)^{1/2}$.

Take $n \geq 2$, and replace Γ by $\Gamma' := \left(\begin{smallmatrix} 1 & n\mathbf{Z} \\ 0 & 1 \end{smallmatrix}\right)$. Then the log structure of $\Gamma' \setminus D_\sigma$ is generated by $q^{1/n}$. Hence, the inverse image on $D^\sharp_\sigma$ of the log structure of $\Gamma \setminus D_\sigma$ and that of $\Gamma' \setminus D_\sigma$ do not coincide. This problem does not happen for the new log structure, because $1/(\log|q^{1/n}|)^{1/2} = n^{1/2}/(\log|q|)^{1/2}$ and $1/(\log|q|)^{1/2}$ generate the same log structure.

4.4.5

Endow $D^\sharp_\Sigma$ with the sheaf of all $\mathbf{C}$-valued continuous functions. For $\sigma \in \Sigma$, take a neat subgroup Γ of $G_\mathbf{Z}$ which is strongly compatible with σ, and consider the inverse image on $D^\sharp_\sigma$ of the log structure of $\Gamma \setminus D_\sigma$. We show that $D^\sharp_{\sigma,\mathrm{val}}$ in [15, III] is identified with the valuative space S_{val} in Section 3.1 associated to $S := D^\sharp_\sigma$ with this log structure (with $K = \mathbf{C}$).

In Section 1.4.1, let $N = \{x \in \sigma_\mathbf{R} \mid \exp(x) \in \Gamma \text{ in } G_\mathbf{R}\}$, let $L = \mathrm{Hom}(N, \mathbf{Z})$, and regard σ as a cone in $N_\mathbf{R} := \mathbf{R} \otimes N$. Let Σ be the fan of all faces of σ, and denote $|\mathrm{toric}|(\Sigma)$ by $|\mathrm{toric}|_\sigma$. Then we have a commutative diagram

$$\begin{array}{ccccc} D^\sharp_{\sigma,\mathrm{val}} & \leftarrow & E^\sharp_{\sigma,\mathrm{val}} & \hookrightarrow & |\mathrm{toric}|_{\sigma,\mathrm{val}} \times \check{D} \\ \downarrow & & \downarrow & & \downarrow \\ D^\sharp_\sigma & \leftarrow & E^\sharp_\sigma & \hookrightarrow & |\mathrm{toric}|_\sigma \times \check{D} \end{array}$$

where the squares are Cartesian, $E^\sharp_\sigma$ is a $\sigma_\mathbf{R}$-torsor over $D^\sharp_\sigma$ for a certain natural action of $\sigma_\mathbf{R}$ on $E^\sharp_\sigma$, $E^\sharp_{\sigma,\mathrm{val}}$ is a $\sigma_\mathbf{R}$-torsor over $D^\sharp_{\sigma,\mathrm{val}}$ for a certain natural action of $\sigma_\mathbf{R}$ on $E^\sharp_{\sigma,\mathrm{val}}$, and the pullback of the log structure of $|D^\sharp_\sigma|$ (see Lemma 3.1.12) on $E^\sharp_\sigma$ coincides with the pullback of the canonical log structure of $|\mathrm{toric}|_\sigma$. In the upper row, the space in the middle and the space on the right are the valuative spaces associated to their lower spaces, respectively. Hence, the valuative space associated to $D^\sharp_\sigma$ coincides with the quotient $D^\sharp_{\sigma,\mathrm{val}}$ of $E^\sharp_{\sigma,\mathrm{val}}$ by $\sigma_\mathbf{R}$, that is, $D^\sharp_{\sigma,\mathrm{val}}$.

Here the problem of the dependence of the log structure of $S = D^\sharp_\sigma$ on Γ (Section 4.4.4) does not matter for the following reason. For another choice Γ' of Γ such that $\Gamma' \subset \Gamma$, the identity map of S is a morphism from S with the log structure given by Γ' to S with the log structure given by Γ, and this morphism has the Kummer property from Lemma 3.2.8 of log structure. Hence, the associated valuative space is independent of the choice of Γ.

4.4.6

By Sections 4.4.5, 3.1, and 4.2, we have a proper surjective map $D^\sharp_{\sigma,\mathrm{val}} \to D^\sharp_{\sigma,[:]}$, and this glues to a proper surjective map $D^\sharp_{\Sigma,\mathrm{val}} \to D^\sharp_{\Sigma,[:]}$.

4.4.7. Examples

We describe the differences of the topologies of $D^\sharp_{\Sigma,[:]}$, $D^\sharp_{\Sigma,\mathrm{val}}$, and $D^\sharp_{\Sigma,[\mathrm{val}]}$.

Let $N_1, N_2 \in \mathfrak{g}_{\mathbf{Q}}$, and assume that $N_1N_2 = N_2N_1$ and that N_1 and N_2 are nilpotent and linearly independent over $\mathbf{Q}$. Let $F \in \check{D}$, and assume that (N_1, N_2, F) generates a nilpotent orbit in the sense of [15, III, Section 2.2.2]. Let Σ be the fan of all faces of the cone $\mathbf{R}_{\geq 0}N_1 + \mathbf{R}_{\geq 0}N_2$. When $y_1, y_2 \in \mathbf{R}$ tend to ∞, $\exp(iy_1N_1 + iy_2N_2)F$ converges in $D^{\sharp}_{\Sigma}$.

(1) Fix a constant $a \in \mathbf{R}$. When $y \to \infty$, $\exp(iyN_1 + i(y+a)N_2)F$ converges in $D^{\sharp}_{\Sigma,\mathrm{val}}$, $D^{\sharp}_{\Sigma,[\mathrm{val}]}$, and $D^{\sharp}_{\Sigma,[:]}$. The limit in $D^{\sharp}_{\Sigma,[\mathrm{val}]}$ is independent of a, and hence, the limit in $D^{\sharp}_{\Sigma,[:]}$ is independent of a, but the limit in $D^{\sharp}_{\Sigma,\mathrm{val}}$ depends on a.

(2) Fix a constant $a \in \mathbf{R}$ such that $0 < a < 1$. Then when $y \to \infty$, $\exp(iyN_1 + iy^aN_2)F$ converges in $D^{\sharp}_{\Sigma,\mathrm{val}}$, $D^{\sharp}_{\Sigma,[\mathrm{val}]}$, and $D^{\sharp}_{\Sigma,[:]}$. The limit in $D^{\sharp}_{\Sigma,\mathrm{val}}$ is independent of a, and hence, the limit in $D^{\sharp}_{\Sigma,[:]}$ is independent of a, but the limit in $D^{\sharp}_{\Sigma,[\mathrm{val}]}$ depends on a.

4.5. CKS maps to $D_{\mathrm{SL}(2)}$ and $D_{\mathrm{SL}(2),\mathrm{val}}$

4.5.1
Recall that, in [15, III, Theorem 3.3.2], we proved that the identity map of D extends uniquely to a continuous map

$$D^{\sharp}_{\Sigma,\mathrm{val}} \to D^{I}_{\mathrm{SL}(2)}.$$

Section 3.3 of [15, III] is devoted to its proof. The corresponding result in the pure case is [17, Theorem 5.4.4], and its full proof is given in [17, Chapter 6].

In this section, we prove the following related Theorems 4.5.2 and 4.5.7.

THEOREM 4.5.2

(1) *The identity map of D extends uniquely to continuous maps*

$$D^{\sharp}_{\Sigma,[:]} \to D^{I}_{\mathrm{SL}(2)}, \qquad D^{\sharp}_{\Sigma,[\mathrm{val}]} \to D^{I}_{\mathrm{SL}(2),\mathrm{val}}.$$

These maps respect the log structures on the sheaves of all $\mathbf{R}$-valued continuous functions. Here we use the new log structures on $D^{\sharp}_{\Sigma,[:]}$ in Section 4.4.3 (cf. Section 4.3.3) and the log structure on $D^{I}_{\mathrm{SL}(2)}$ discussed in Theorem 2.7.14.

(2) *The CKS map $D^{\sharp}_{\Sigma,\mathrm{val}} \to D^{I}_{\mathrm{SL}(2)}$ defined in [15, III, Theorem 3.3.2] coincides with the composition $D^{\sharp}_{\Sigma,\mathrm{val}} \to D^{\sharp}_{\Sigma,[:]} \to D^{I}_{\mathrm{SL}(2)}$.*

4.5.3
Let D_{nilp} be the set of $(N_1, \ldots, N_n, F)$, where $n \geq 0$, $N_j \in \mathfrak{g}_{\mathbf{R}}$, and $F \in \check{D}$, which satisfies the following two conditions.

(i) $(N_1, \ldots, N_n, F)$ generates a nilpotent orbit in the sense of [15, III, Section 2.2.2].

(ii) For any $w \in \mathbf{Z}$ and for any $1 \leq j \leq n$, let $W^{(j)}$ be the relative monodromy filtration of $y_1N_1 + \cdots + y_jN_j$ $(y_1, \ldots, y_j \in \mathbf{R}_{>0})$ relative to W. ($W^{(j)}$ exists and does not depend on the choices of $y_j \in \mathbf{R}_{>0}$ by condition (i).) Then the filtrations $W^{(j)}(\mathrm{gr}^W)$ on gr^W $(1 \leq j \leq n)$ are $\mathbf{Q}$-rational.

4.5.4

We review the map $D_{\mathrm{nilp}} \to D_{\mathrm{SL}(2)}$ which sends $(N_1, \ldots, N_n, F) \in D_{\mathrm{nilp}}$ to the class of the associated SL(2)-orbit [15, II, 2.4]. It is the map which sends $(N_1, \ldots, N_n, F)$ to the limit of $\exp(\sum_{j=1}^{n} iy_j N_j)F$, where $y_j \in \mathbf{R}_{>0}$, $y_j/y_{j+1} \to \infty$ ($1 \le j \le n$, y_{n+1} denotes 1) in $D^I_{\mathrm{SL}(2)}$. This map is also characterized as follows. Recall that an element (p, Z) of $D_{\mathrm{SL}(2)}$ is determined by the following (i) and (ii):

(i) Whether (p, Z) is an A-orbit or a B-orbit.

(ii) $(\Phi, \mathbf{r})$, where Φ is the set of weight filtrations on gr^W associated to p (see [15, II, Proposition 2.5.2(ii)]) and $\mathbf{r}$ is any element of Z.

Let $(p, Z) \in D_{\mathrm{SL}(2)}$ be the image of $(N_1, \ldots, N_n, F)$. Then (p, Z) is a B-orbit if and only if there is j such that $N_j \ne 0$, $N_k = 0$ for $1 \le k < j$, and $\mathrm{gr}^W(N_j) = 0$. Φ is the set of $W^{(j)}(\mathrm{gr}^W)$ for all j such that $\mathrm{gr}^W(N_k) \ne 0$ for some $k \le j$. $\mathbf{r}$ in the above (ii) is given as follows.

Since $(N_1, \ldots, N_n, F)$ generates a nilpotent orbit, $(W^{(n)}, F)$ is a mixed Hodge structure. Let $(W^{(n)}, \hat{F}_{(n)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to it. Then $(N_1, \ldots, N_{n-1}, \exp(iN_n)\hat{F}_{(n)})$ generates a nilpotent orbit, and hence, $(W^{(n-1)}, \exp(iN_n)\hat{F}_{(n)})$ is a mixed Hodge structure. Let $(W^{(n-1)}, \hat{F}_{(n-1)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to it. Then $(N_1, \ldots, N_{n-2}, \exp(iN_{n-1})\hat{F}_{(n-1)}$ generates a nilpotent orbit, and hence, $(W^{(n-2)}, \exp(iN_{n-1})\hat{F}_{(n-1)})$ is a mixed Hodge structure. In this way, we have the $\mathbf{R}$-split mixed Hodge structure $(W^{(j)}, \hat{F}_{(j)})$ for $1 \le j \le n$ by a downward induction on j (see [15, II, Section 2.4.6]). We obtain $\mathbf{r} \in D$ as $\mathbf{r} = \exp(iN_k)\hat{F}_{(k)}$ if k is the minimal j such that $N_j \ne 0$, where in the case $N_j = 0$ for all j, we define $\mathbf{r} = F$.

4.5.5

Let Γ be a neat subgroup of G_Z. Assume (Γ, Σ) is strongly compatible. By Section 4.1.6, $D^\sharp_{\Sigma,[:]}$ is identified with the set of $(\sigma, Z, (\mathcal{S}^{(j)})_{0 \le j \le n}, (N_j)_{1 \le j \le n})$, where $(\sigma, Z) \in D^\sharp_\Sigma$, and if s denotes the image of (σ, Z) in $S := \Gamma \setminus D_\Sigma$, then $\mathcal{S}^{(j)}$ are faces of $(M_S/\mathcal{O}_S^\times)_s$ such that $(M_S/\mathcal{O}_S^\times)_s = \mathcal{S}^{(0)} \supsetneq \mathcal{S}^{(1)} \supsetneq \cdots \supsetneq \mathcal{S}^{(n)} = \{1\}$ and N_j is a homomorphism $\mathcal{S}^{(j-1)} \to \mathbf{R}^{\mathrm{add}}$ such that $N_j(\mathcal{S}^{(j)}) = 0$ and $N_j(\mathcal{S}^{(j-1)} \smallsetminus \mathcal{S}^{(j)}) \subset \mathbf{R}_{>0}$. (Here N_j is considered modulo $\mathbf{R}^{\mathrm{mult}}_{>0}$.)

For $s = \mathrm{class}(\sigma, Z) \in S = \Gamma \setminus D_\Sigma$, $(M_S/\mathcal{O}_S^\times)_s$ is canonically isomorphic to $\mathrm{Hom}(\Gamma(\sigma), \mathbf{N})$. Hence, σ is identified with $\mathrm{Hom}((M_S/\mathcal{O}_S^\times)_s, \mathbf{R}^{\mathrm{add}}_{\ge 0})$, and the face $\mathcal{S}^{(j)}$ of $(M_S/\mathcal{O}_S^\times)_s$ in the above corresponds to a face σ_j of σ consisting of all homomorphisms $(M_S/\mathcal{O}_S^\times)_s \to \mathbf{R}^{\mathrm{add}}_{\ge 0}$ which kills $\mathcal{S}^{(j)}$.

Hence, $D^\sharp_{\Sigma,[:]}$ is identified with the set of $(\sigma, Z, (\sigma_j)_{0 \le j \le n}, (N_j)_{1 \le j \le n})$, where $(\sigma, Z) \in D^\sharp_\Sigma$, the σ_j's are faces of σ such that $0 = \sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n = \sigma$, and N_j is an element of $\sigma_{j,\mathbf{R}}/\sigma_{j-1,\mathbf{R}}$ which belongs to the image of an element of the interior of σ_j.

4.5.6
Let $(\sigma, Z, (\sigma_j)_{0\le j\le n}, (N_j)_{1\le j\le n}) \in D^\sharp_{\Sigma,[:]}$ (see Section 4.5.5), and let $\tilde{N}_j$ be an element of the interior of σ_j whose image in $\sigma_{j,\mathbf{R}}/\sigma_{j-1,\mathbf{R}}$ coincides with N_j. Let $F \in Z$. Then $(\tilde{N}_1, \dots, \tilde{N}_n, F)$ generates a nilpotent orbit, as is easily seen. We will prove the following.

THEOREM 4.5.7
The map $D^\sharp_{\Sigma,[:]} \to D^I_{\mathrm{SL}(2)}$ (see Theorem 4.5.2) sends $(\sigma, Z, (\sigma_j)_j, (N_j)_j)$ to the image of $(\tilde{N}_1, \dots, \tilde{N}_n, F) \in D_{\mathrm{nilp}}$ in $D_{\mathrm{SL}(2)}$ (see Section 4.5.4). Here $F \in Z$ and $\tilde{N}_j$ is any element of the interior of σ_j whose image in $\sigma_{j,\mathbf{R}}/\sigma_{j-1,\mathbf{R}}$ coincides with N_j.

LEMMA 4.5.8
Let $\sigma \subset \mathfrak{g}_{\mathbf{R}}$ be a nilpotent cone, let $F \in \check{D}$, and assume that (σ, F) generates a nilpotent orbit. Let $N \in \sigma_{\mathbf{R}}$, and let $F' = \exp(iN)F$. Let $M(\sigma, W)$ be the relative monodromy filtration of σ with respect to W.

(1) *$\delta(M(\sigma, W), F') = \delta(M(\sigma, W), F) + N$, where the last N denotes the homomorphism $\mathrm{gr}^{M(\sigma,W)} \to \mathrm{gr}^{M(\sigma,W)}$ which is the sum of the maps $\mathrm{gr}^{M(\sigma,W)}_k \to \mathrm{gr}^{M(\sigma,W)}_{k-2}$ $(k \in \mathbf{Z})$ induced by N.*
(2) *$\zeta(M(\sigma, W), F') = \zeta(M(\sigma, W), F)$.*
(3) *$\mathrm{spl}_{M(\sigma,W)}(F') = \mathrm{spl}_{M(\sigma,W)}(F)$.*

Proof
Statement (1) follows from the definition of δ.

By (1), (2) follows from the facts that $\delta(M(\sigma, W), F)$ and N commute, that N is of Hodge type $(-1,-1)$ for $F(\mathrm{gr}^{M(\sigma,W)})$, and that $\zeta_{-1,-1} = 0$ in general.

Then (3) follows from (1) and (2). □

4.5.9
Let $(\sigma, Z, (\sigma_j)_{0\le j\le n}, (N_j)_{1\le j\le n}) \in D^\sharp_{\Sigma,[:]}$, $F \in Z$, and $\tilde{N}_j$ be as in Section 4.5.6. We show that the image of $(\tilde{N}_1, \dots, \tilde{N}_n, F) \in D_{\mathrm{nilp}}$ in $D_{\mathrm{SL}(2)}$ is independent of the choices of $\tilde{N}_j$ and the choice of $F \in Z$.

We prove that the associated element of $D_{\mathrm{SL}(2)}$ does not depend on the choice of $F \in Z$. If $F' \in Z$, then $F' = \exp(iN)F$ for some $N \in \sigma_{\mathbf{R}}$. Hence, by Lemma 4.5.8(3) applied to (σ, F), which generates a nilpotent orbit, $\hat{F}_{(n)}$ is independent of the choice.

We prove that the associated element of $D_{\mathrm{SL}(2)}$ does not depend on the choice of a lifting $\tilde{N}_j$ of N_j. If $\tilde{N}'_j$ is another lifting of N_j, then $\tilde{N}'_j = \tilde{N}_j + R_j$ for some $R_j \in \sigma_{j-1,\mathbf{R}}$. By Lemma 4.5.8(3) applied to $(\sigma_{j-1}, \exp(i\tilde{N}_j)\hat{F}_{(j)})$, which generates a nilpotent orbit, $\hat{F}_{(j-1)}$ is independent of the choice by downward induction on j.

4.5.10
By Section 4.5.9, we have a map $D^\sharp_{\Sigma,[:]} \to D_{\mathrm{SL}(2)}$ which sends $(\sigma, Z, (\sigma_j)_{0\le j\le n}, (N_j)_{1\le j\le n}) \in D^\sharp_{\Sigma,[:]}$ to the image of $(\tilde{N}_1,\dots,\tilde{N}_n, F) \in D_{\mathrm{nilp}}$ in $D_{\mathrm{SL}(2)}$. Comparing the definition of this map and the definition of the map $\psi : D^\sharp_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ (see Section 4.5.1) given in [15, III, Theorem 3.3.1], we see that the composition $D^\sharp_{\Sigma,\mathrm{val}} \to D^\sharp_{\Sigma,[:]} \to D_{\mathrm{SL}(2)}$ coincides with ψ. Theorem 4.5.2(2) is proved.

4.5.11
We complete the proofs of Theorems 4.5.2(1) and 4.5.7. This map $D^\sharp_{\Sigma,[:]} \to D_{\mathrm{SL}(2)}$ in Section 4.5.10 is continuous, because $D^\sharp_{\Sigma,\mathrm{val}} \to D^I_{\mathrm{SL}(2)}$ is continuous and $D^\sharp_{\Sigma,\mathrm{val}} \to D^\sharp_{\Sigma,[:]}$ is proper and surjective.

4.5.12
Endow $D^\sharp_{\Sigma,[:]}$ with the new log structure from Section 4.3.3 on the sheaf of all $\mathbf{R}$-valued continuous functions. Consider the log structure on $D^I_{\mathrm{SL}(2)}$ in Theorem 2.7.14. We show that the continuous map $D^\sharp_{\Sigma,[:]} \to D^I_{\mathrm{SL}(2)}$ respects these log structures. We check this on $E^\sharp_{\sigma,[:]}$. On the toric component of $E^\sharp_{\sigma,[:]}$, the log structure is generated by $t_j := (y_{j+1}/y_j)^{1/2}$ ($1 \le j \le n$, y_{n+1} denotes 1). Let β be a distance to the boundary for Φ, where Φ is the set of $M(\sigma_j, W)$, and let $\tau : \mathbf{G}^n_{m,\mathbf{R}} \to \prod_w \mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}^W_w)$ be the homomorphism whose w-component is the τ (Section 2.1.2) of the SL(2)-orbit in n variables associated to $\mathrm{gr}^W_w(N_1),\dots, \mathrm{gr}^W_w(N_n), F(\mathrm{gr}^W_w))$. Then $\beta(\exp(\sum_{j=1}^n iy_jN_j)F) = tu$, where $u := \beta(\tau(t)^{-1}\exp(\sum_{j=1}^n iy_jN_j)F)$ is invertible in the ring of real analytic functions. Hence, $D^\sharp_{\Sigma,[:]} \to D^I_{\mathrm{SL}(2)}$ respects the log structures.

4.5.13
By Section 4.5.12, the map $D^\sharp_{\Sigma,[:]} \to D^I_{\mathrm{SL}(2)}$ induces the continuous map $D^\sharp_{\Sigma,[\mathrm{val}]} \to D^I_{\mathrm{SL}(2),\mathrm{val}}$ of associated valuative spaces. This proves the second part of Theorem 4.5.2(1).

4.5.14
Consequently, we have an amplified fundamental diagram in Section 0.1.4.

In the pure case,

$$
\begin{array}{ccccccccc}
 & & & & D^\sharp_{\Sigma,[\mathrm{val}]} & \xrightarrow{\psi} & D_{\mathrm{SL}(2),\mathrm{val}} & \xrightarrow{\eta} & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & \downarrow & & D^{\mathrm{BS}}_{\mathrm{SL}(2)} & \to & D_{\mathrm{BS}} \\
 & & & & & & \downarrow & & \\
\Gamma \backslash D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \to & D^\sharp_{\Sigma,[:]} & \xrightarrow{\psi} & D_{\mathrm{SL}(2)} & & \\
\downarrow & & & & \downarrow & & & & \\
\Gamma \backslash D_\Sigma & & \leftarrow & & D^\sharp_\Sigma & & & &
\end{array}
$$

which is commutative and in which the maps respect the structures of the spaces.

5. Mild nilpotent orbits and the space $D^{\diamond}_{\mathrm{SL}(2)}$ of $\mathrm{SL}(2)$-orbits

In this section, we consider the spaces of mild nilpotent orbits and the space $D^{\diamond}_{\mathrm{SL}(2)}$, which is closely related to mild nilpotent orbits. In Section 5.1, we give the main definitions and results of Section 5. In the rest of Section 5, we give the proofs of the results in Section 5.1. These results in Section 5.1 were obtained in our joint efforts with Spencer Bloch.

5.1. Mild nilpotent orbits and the space $D^{\diamond}_{\mathrm{SL}(2)}$

Let $\mathcal{L} = W_{-2}\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)$ be as in Section 1.2.2.

5.1.1

Let $D^{\mathrm{mild}}_{\mathrm{nilp}}$ be the subset of D_{nilp} (see Section 4.5.3) consisting of all elements $(N_1, \dots, N_n, F)$ satisfying the following condition.

For any $y_j \geq 0$ $(1 \leq j \leq n)$, there is a splitting (which may depend on $(y_j)_j$) of W which is compatible with $\sum_{j=1}^n y_j N_j$.

We have the following "SL(2)-orbit theorem for mild degeneration."

THEOREM 5.1.2

Let $(N_1, \dots, N_n, F) \in D^{\mathrm{mild}}_{\mathrm{nilp}}$.

(1) *If* $y_j/y_{j+1} \to \infty$ $(1 \leq j \leq n$, y_{n+1} *denotes* $1)$, *then* $\delta_W(\exp(\sum_{j=1}^n iy_j N_j)F)$ *converges in* $\mathcal{L}$. *Moreover, there are* $a_m \in \mathcal{L}$ *for* $m \in \mathbf{N}^n$ *and* $\varepsilon \in \mathbf{R}_{>0}$ *satisfying the following (i) and (ii).*

(i) $\sum_{m \in \mathbf{N}^n} (\prod_{j=1}^n x_j^{m(j)}) a_m$ *absolutely converges for* $x_j \in \mathbf{R}$, $|X_j| < \varepsilon$ $(1 \leq j \leq n)$.

(ii) *For* $y_j \in \mathbf{R}_{>0}$ $(1 \leq j \leq n)$ *such that* $t_j := (y_{j+1}/y_j)^{1/2} < \varepsilon$ $(1 \leq j \leq n$, y_{n+1} *denotes* $1)$, *we have* $\exp(\sum_{j=1}^n iy_j N_j)F \in D$ *and*

$$\delta_W\Big(\exp\Big(\sum_{j=1}^n iy_j N_j\Big)F\Big) = \sum_{m \in \mathbf{N}^n} \Big(\prod_{j=1}^n t_j^{m(j)}\Big) a_m.$$

(2) *Let* $\tau^\star : \mathbf{G}^n_{m,\mathbf{R}} \to G(\mathrm{gr}^W)$ *be the homomorphism whose* $G_{\mathbf{R}}(\mathrm{gr}^W_w)$ *component is the* $\tau^\star$ *(see Section 2.1.2) of the* $\mathrm{SL}(2)$*-orbit in* n *variables on* gr^W_w *associated to* $(\mathrm{gr}^W_w(N_1), \dots, \mathrm{gr}^W_w(N_n), F(\mathrm{gr}^W_w))$. *Then there are* $a_m \in \mathcal{L}$ *for* $m \in \mathbf{N}^n$ *and* $\varepsilon \in \mathbf{R}_{>0}$ *satisfying the above condition (i) and the modification of the above condition (ii) by replacing* $\delta_W(\exp(\sum_{j=1}^n iy_j N_j)F)$ *with* $\mathrm{Ad}(\tau^\star(t))^{-1}\delta_W(\exp(\sum_{j=1}^n iy_j N_j)F)$, *where* $t = (t_1, \dots, t_n), t_j = (y_{j+1}/y_j)^{1/2}$.

(3) *If* $y_j/y_{j+1} \to \infty$ $(1 \leq j \leq n$, y_{n+1} *denotes* $1)$, *then* $\exp(\sum_{j=1}^n iy_j N_j)F$ *converges in* $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$.

In fact, (3) follows from (2) by the definition of the structure of $D^{\star}_{\mathrm{SL}(2)}$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ given in Section 2.3.11.

5.1.3
By Theorem 5.1.2, we have maps

$$D_{\mathrm{nilp}}^{\mathrm{mild}} \to \mathcal{L}, \qquad D_{\mathrm{nilp}}^{\mathrm{mild}} \to D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}}$$

by taking the limit of the convergence in Theorem 5.1.2.

5.1.4
We define the *mild part* $D_{\Sigma}^{\mathrm{mild}}$ *of the set of nilpotent orbits* D_{Σ} as the part of points (σ, Z) which satisfy the following condition.

(C) For each N in the cone σ, there is a splitting of W (which can depend on N) which is compatible with N.

For the other spaces of nilpotent orbits $D_{\Sigma}^{\sharp}$, $D_{\Sigma,[:]}^{\sharp}$, $D_{\Sigma,[\mathrm{val}]}^{\sharp}$, and so on, we define their mild parts $D_{\Sigma}^{\sharp,\mathrm{mild}}$, $D_{\Sigma,[:]}^{\sharp,\mathrm{mild}}$, $D_{\Sigma,[\mathrm{val}]}^{\sharp,\mathrm{mild}}$, and so on as the inverse images of $D_{\Sigma}^{\mathrm{mild}}$.

5.1.5
In the above definition from Section 5.1.4 of the mildness, the following stronger condition (C′) need not be satisfied.

(C′) There is a splitting of W which is compatible with any element N of the cone σ.

5.1.6
For example, in the case of Example II in [15, I and II] (the case of $0 \to H^1(E)(1) \to * \to \mathbf{Z} \to 0$, where E varies over elliptic curves), we had a nilpotent orbit of rank 2, and that is a mild degeneration in the sense of Section 5.1.4 (i.e., it satisfies (C)), but it does not satisfy (C′).

THEOREM 5.1.7

(1) *There is a unique continuous map* $D_{\Sigma,[:]}^{\sharp,\mathrm{mild}} \to \mathcal{L}$ *which extends the map* $D \to \mathcal{L}$, $x \mapsto \delta_W(x)$.

(2) *There is a unique continuous map* $D_{\Sigma,[:]}^{\sharp,\mathrm{mild}} \to D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}}$ *which extends the identity map of* D.

(3) *The map in (1) (resp., (2)) sends* $(\sigma, Z, (\sigma_j)_j, (N_j)_j) \in D_{\Sigma,[:]}^{\sharp,\mathrm{mild}}$ *(see Section 4.5.5) to the image of* $(\tilde{N}_1, \ldots, \tilde{N}_n, F) \in D_{\mathrm{nilp}}^{\mathrm{mild}}$ *in* $\mathcal{L}$ *(resp.,* $D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}}$*) under the map in Section 5.1.3. Here* $\tilde{N}_j$ *is as in Theorem 4.5.7, and* F *is any element of* Z.

Theorem 5.1.7(1) shows the convergence of Beilinson regulators in a family with mild degeneration (see Section 7.2).

5.1.8

We define a space $D^{\diamond}_{\mathrm{SL}(2)}$. Let $D^{\diamond}_{\mathrm{SL}(2)}$ be the subset of $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \times \mathcal{L}$ consisting of all elements (p, Z, δ) ($(p, Z) \in D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ with $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$ and $Z \subset D$ (see Section 2.3.2), $\delta \in \mathcal{L}$) satisfying the following conditions (i) and (ii).

(i) Let n be the rank of p, and let $\mathbf{0} := (0, \dots, 0) \in \mathbf{Z}^n$. Then δ is of $\mathrm{Ad}(\tau_p^{\star})$-weight at most $\mathbf{0}$.

(ii) For any $F \in Z$, $\delta_W(F)$ coincides with the component of δ of $\mathrm{Ad}(\tau_p^{\star})$-weight $\mathbf{0}$.

We define the structure of $D^{\diamond}_{\mathrm{SL}(2)}$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)$ by regarding $D^{\diamond}_{\mathrm{SL}(2)}$ (resp., $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \times \mathcal{L}$) as Y (resp., X) in Section 1.3.16. We have the evident morphism

$$D^{\diamond}_{\mathrm{SL}(2)} \to D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}, \qquad (p, Z, \delta) \mapsto (p, Z)$$

of $\mathcal{B}'_{\mathbf{R}}(\log)$.

5.1.9

Via the map

$$D \to D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \times \mathcal{L}, F \mapsto \big(F, \delta_W(F)\big),$$

we regard D as a subset of $D^{\diamond}_{\mathrm{SL}(2)}$.

THEOREM 5.1.10

(1) *Let $D^{\mathrm{mild}}_{\mathrm{nilp}} \to D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \times \mathcal{L}$ be the map which sends $(N_1, \dots, N_n, F) \in D^{\mathrm{mild}}_{\mathrm{nilp}}$ to the limit of $(F_y, \delta_W(F_y))$, where $y = (y_j)_{1 \le j \le n} \in \mathbf{R}^n_{>0}$, $F_y := \exp(\sum_{j=1}^n iy_j N_j)F$, and $y_j/y_{j+1} \to \infty$ ($1 \le j \le n$, y_{n+1} denotes 1). Then, the image of this map is contained in $D^{\diamond}_{\mathrm{SL}(2)}$.*

(2) *There is a unique continuous map $D^{\sharp,\mathrm{mild}}_{\Sigma,[:]} \to D^{\diamond}_{\mathrm{SL}(2)}$ which extends the identity map of D.*

(3) *There is a unique continuous map $D^{\sharp,\mathrm{mild}}_{\Sigma,[\mathrm{val}]} \to D^{\diamond}_{\mathrm{SL}(2),\mathrm{val}}$ which extends the identity map of D.*

PROPOSITION 5.1.11

(1) *The map $D^{\diamond}_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathrm{spl}(W) \times \mathcal{L}$ induced by $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathrm{spl}(W)$ is injective, and the image of this map consists of all elements (p, s, δ) satisfying the following conditions (i) and (ii).*

(i) *δ is of $\mathrm{Ad}(\tau_p^{\star})$-weight at most $\mathbf{0}$.*

(ii) *Let $(\rho_w, \varphi_w)_w$ be an $\mathrm{SL}(2)$-orbit on gr^W which represents p. Then the component of δ of $\mathrm{Ad}(\tau_p^{\star})$-weight $\mathbf{0}$ is of Hodge type $(\le -1, \le -1)$ with respect to $(\varphi_w(i, \dots, i))_w$.*

(2) *If $(p, Z, \delta) \in D^{\diamond}_{\mathrm{SL}(2)}$ and if (p, s, δ) is its image in $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim} \times \mathrm{spl}(W) \times \mathcal{L}$, then Z is recovered from (p, s, δ) as follows. Under the embedding $D \to D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ in Section 1.2.1, the image of $Z \subset D$ coincides with*

$(Z(p), s, \delta_{\mathbf{0}}) \subset D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$. *Here* $\delta_{\mathbf{0}}$ *denotes the component of* δ *of* $\mathrm{Ad}(\tau_p^\star)$*-weight* $\mathbf{0}$.

5.1.12
By the *weak topology of* $D^\diamond_{\mathrm{SL}(2)}$, we mean the topology of $D^\diamond_{\mathrm{SL}(2)}$ as a subspace of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \times \mathrm{spl}(W) \times \mathcal{L}$. We denote the topological space $D^\diamond_{\mathrm{SL}(2)}$ endowed with the weak topology by $D^{\diamond,\mathrm{weak}}_{\mathrm{SL}(2)}$. This weak topology is weaker than but need not coincide with the topology defined in Section 5.1.8 (see Section 7.1.7).

REMARK 5.1.13
(1) Unlike other spaces of SL(2)-orbits ($D^\star_{\mathrm{SL}(2)}$, $D^I_{\mathrm{SL}(2)}$, $D^{II}_{\mathrm{SL}(2)}$, and so on), D is not necessarily dense in $D^\diamond_{\mathrm{SL}(2)}$ (even for the weak topology).

(2) The authors believe that $D^\diamond_{\mathrm{SL}(2)}$ belongs to $\mathcal{B}'_{\mathbf{R}}(\log)^+$ and that this can be proved by using the methods in Section 2.7, but they have not yet proved it.

5.1.14
The above results show that we have commutative diagrams

$$\begin{array}{ccccc} D^{\sharp,\mathrm{mild}}_{\Sigma,[:]} & \to & D^\diamond_{\mathrm{SL}(2)} & \to & D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \\ \cap & & & & \downarrow \\ D^\sharp_{\Sigma,[:]} & & \to & & D^{II}_{\mathrm{SL}(2)} \end{array} \qquad \begin{array}{ccccc} D^{\sharp,\mathrm{mild}}_{\Sigma,[\mathrm{val}]} & \to & D^\diamond_{\mathrm{SL}(2),\mathrm{val}} & \to & D^{\star,\mathrm{mild}}_{\mathrm{SL}(2),\mathrm{val}} \\ \cap & & & & \downarrow \\ D^\sharp_{\Sigma,[\mathrm{val}]} & & \to & & D^{II}_{\mathrm{SL}(2),\mathrm{val}} \end{array}$$

The rest of Section 5 is devoted to the proofs of the above results.

5.2. Preparations on pure SL(2)-orbits

We further review pure SL(2)-orbits in one variable.

5.2.1
Assume that we are in the pure case of weight w, and assume that we are given an SL(2)-orbit (ρ, φ) in one variable. Let

$$N, N^+ \in \mathfrak{g}_{\mathbf{R}}$$

be as follows. Let $\rho_* : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}_{\mathbf{R}}$ be the Lie algebra homomorphism induced by ρ. Then N (resp., N^+) is the image of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ (resp., $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$) in $\mathfrak{sl}(2, \mathbf{R})$.

5.2.2
We have a direct sum decomposition

$$H_{0,\mathbf{R}} = \bigoplus_{k,r \geq 0} H_{0,\mathbf{R},(k,r)}$$

defined as follows. Let $Z = \mathrm{Ker}(N : H_{0,\mathbf{R}} \to H_{0,\mathbf{R}})$. Then $Z = \bigoplus_{k \geq 0} Z_{(-k)}$, where $Z_{(-k)}$ is the part of Z of $\tau^\star$-weight $-k$. Let

$$H_{0,\mathbf{R},(k,r)} := (N^+)^r Z_{(-k)}.$$

In particular, $Z_{(-k)} = H_{0,\mathbf{R},(k,0)}$.

5.2.3
We have the following.

(1) Elements of $H_{0,\mathbf{R},(k,r)}$ have $\tau^\star$-weight $2r-k$.

(2) For each $k \geq 0$, $H_{0,\mathbf{R},(k,\bullet)} := \bigoplus_r H_{0,\mathbf{R},(k,r)}$ is stable under the action of $\mathrm{SL}(2,\mathbf{R})$ by ρ. As a representation of $\mathrm{SL}(2,\mathbf{R})$, we have a unique isomorphism

$$H_{0,\mathbf{R},(k,\bullet)} \cong \mathrm{Sym}^k(A) \otimes Z_{(-k)},$$

where $A = \mathbf{R}^2 = \mathbf{R}e_1 \oplus \mathbf{R}e_2$, on which $\mathrm{SL}(2,\mathbf{R})$ acts via the natural action on A and the trivial action on $Z_{(-k)}$, which sends $v \in Z_{(-k)}$ on the left-hand side to $e_1^k \otimes v \in \mathrm{Sym}^k(A) \otimes Z_{(-k)}$ on the right-hand side.

(3) For $e \geq 0$, the kernel of $N^e : H_{0,\mathbf{R}} \to H_{0,\mathbf{R}}$ coincides with the direct sum of $H_{0,\mathbf{R},(k,r)}$ for $k, r \geq 0$ such that $r < e$.

(4) The filtration $\varphi(0)$ is the direct sum of its restrictions $\varphi(0)_{(k,r)}$ to $H_{0,\mathbf{C},(k,r)}$ for all (k,r). $H_{0,\mathbf{R},(k,r)}$ with Hodge filtration $\varphi(0)_{(k,r)}$ on $H_{0,\mathbf{C},(k,r)}$ is an $\mathbf{R}$-Hodge structure of weight $w+2r-k$.

(5) For any $z \in \mathbf{P}^1(\mathbf{C})$, the filtration $\varphi(z)$ on $H_{0,\mathbf{C}}$ is the direct sum of its restrictions $\varphi(z)_{(k,\bullet)}$ to $H_{0,\mathbf{C},(k,\bullet)}$ for $k \geq 0$.

The filtration $\varphi(z)_{(k,\bullet)}$ is described as follows. In the isomorphism in (2), it is given by $\varphi(0)_{(k,0)}$ on $Z_{(-k),\mathbf{C}}$ and the filtration on $A_\mathbf{C}$ whose F^0 is $A_\mathbf{C}$, whose F^2 is 0, and whose F^1 is $\mathbf{C}\cdot(ze_1+e_2)$ (resp., $\mathbf{C}e_1$) if $z \in \mathbf{C}$ (resp., if $z = \infty$).

5.3. More preparations on $\mathrm{SL}(2)$-orbits

5.3.1
Assume that we are given an $\mathrm{SL}(2)$-orbit (ρ_w, φ_w) on gr^W_w in one variable for each $w \in \mathbf{Z}$. Let

$$E = W_0\mathrm{End}_\mathbf{R}(\mathrm{gr}^W) = \bigoplus_{w \leq 0} E_w, \qquad E_w = \bigoplus_{a \in \mathbf{Z}} \mathrm{Hom}_\mathbf{R}(\mathrm{gr}^W_a, \mathrm{gr}^W_{a+w}).$$

We apply our preparations in Section 5.2 to the $\mathrm{SL}(2)$-orbit in one variable of pure weight w induced on each E_w by (ρ_a, φ_a) and $(\rho_{a+w}, \varphi_{a+w})$ $(a \in \mathbf{Z})$. By Section 5.2.2, we have a direct sum decomposition

$$E_w = \bigoplus_{k,r \geq 0} E_{w,(k,r)}.$$

LEMMA 5.3.2
We have that $E_{w,(k,r)}E_{w',(k',r')} \subset \bigoplus_{k'',r''} E_{w+w',(k'',r'')}$, *where* (k'',r'') *ranges over all elements of* $\mathbf{N}\times\mathbf{N}$ *such that* $r'' \leq r+r'$ *and* $k''-2r'' = (k+k')-2(r+r')$.

Proof
This follows from (1) and (3) of Section 5.2.3. ☐

5.3.3

Let $\mathbf{R}\{\{t\}\}$ be the ring of power series in t over $\mathbf{R}$ which absolutely converge when $|t|$ is small. We define subrings $\mathfrak{A}_0$, $\mathfrak{A}$, $\mathfrak{B}_0$, $\mathfrak{B}$ of $\mathbf{R}\{\{t\}\}\otimes_{\mathbf{R}} E$ as follows:

$$\mathfrak{A}_0 = E_{\bullet,(\bullet,0)} \subset \mathfrak{A} = \sum_{r\geq 0} t^{2r}\mathbf{R}\{\{t^2\}\}\otimes_{\mathbf{R}} E_{\bullet,(\bullet,r)},$$

$$\mathfrak{B}_0 = \sum_{k\geq 0} t^k E_{\bullet,(k,0)} \subset \mathfrak{B} = \sum_{k\geq 0} t^k \mathbf{R}\{\{t^2\}\}\otimes_{\mathbf{R}} E_{\bullet,(k,\bullet)}.$$

For $w\leq 0$, define the two-sided ideals of these rings as

$$W_w\mathfrak{A}_0 = W_wE_{\bullet,(\bullet,0)} \subset W_w\mathfrak{A} = \sum_{r\geq 0} t^{2r}\mathbf{R}\{\{t^2\}\}\otimes_{\mathbf{R}} W_wE_{\bullet,(\bullet,r)},$$

$$W_w\mathfrak{B}_0 = \sum_{k\geq 0} t^k W_wE_{\bullet,(k,0)} \subset W_w\mathfrak{B} = \sum_{k\geq 0} t^k \mathbf{R}\{\{t^2\}\}\otimes_{\mathbf{R}} W_wE_{\bullet,(k,\bullet)}.$$

LEMMA 5.3.4

We have, for $w\leq 0$,

$$\mathrm{Ad}\big(\tau^\star(t)\big)W_w\mathfrak{B} = W_w\mathfrak{A}, \qquad \mathrm{Ad}\big(\tau^\star(t)\big)W_w\mathfrak{B}_0 = W_w\mathfrak{A}_0.$$

These are direct consequences from the definitions in Section 5.3.3.

We will apply the following Lemma 5.3.6 in Section 5.4.4 (resp., in the proof of Theorem 6.2.4) by taking $A = \mathbf{C}\otimes_{\mathbf{R}}\mathfrak{B}$ (resp., $A = W_0\mathrm{End}_{\mathbf{C}}(H_{\mathbf{C}})$).

5.3.5

Let A be a $\mathbf{Q}$-algebra. For a nilpotent ideal I of A, we have bijections

$$\exp: I\to 1+I, \qquad \log: 1+I\to I,$$

$$\exp(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \qquad \log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$$

(these are finite sums, for $x\in I$ are nilpotent) which are the inverses of each other. Let $I^{(r)}$ $(r\geq 1)$ be two-sided ideals of A such that $I^{(1)}\supset I^{(2)}\supset I^{(3)}\supset\cdots$, $I^{(r)}I^{(s)}\subset I^{(r+s)}$ for any $r,s\geq 1$, and $I^{(r)}=0$ for $r\gg 1$. Let $I = I^{(1)}$. Then I is a nilpotent two-sided ideal.

LEMMA 5.3.6

Let the notation be as in Section 5.3.5. Let M_j $(1\leq j\leq m)$ be $\mathbf{Q}$-submodules of I such that

$$I^{(r)} = \bigoplus_{j=1}^{m}(M_j\cap I^{(r)})$$

for any $r\geq 1$. If $x\in I$, then there is a unique family $(x_j)_{1\leq j\leq m}$ of elements x_j of M_j such that

$$\exp(x) = \exp(x_1)\cdots\exp(x_m).$$

Proof
This follows by an easy induction on r such that $I^{(r)} = 0$. □

5.4. Proof of Theorem 5.1.2

5.4.1
We first prove Theorem 5.1.2 in the case $n = 1$. We use the following part of the SL(2)-orbit theorem in one variable of Schmid [21].

Assume that we are in the pure case. Then for $y \gg 0$, we have

$$\exp(iyN)F = \exp\big(g(y)\big)\tau(y^{-1/2})\mathbf{r} \quad \text{with } \mathbf{r} = \exp(iN)\hat{F}$$

for some convergent power series $g(y) = \sum_{m=0}^{\infty} y^{-m} a_m$ in y^{-1} with $a_m \in \mathrm{End}_{\mathbf{R}}(H_{0,\mathbf{R}})$ such that $a_0 = 0$ and such that $\mathrm{Ad}(N)^{m+1} a_m = 0$ for any m.

PROPOSITION 5.4.2
Let $(N, F) \in D^{\mathrm{mild}}_{\mathrm{nilp}}$ (see Section 5.1.1) with one N. Let $(W^{(1)}, \hat{F})$ be the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure $(W^{(1)}, F)$, and let $\mathbf{r} := \exp(iN)\hat{F}$. Then $(W, \mathbf{r})$ is an $\mathbf{R}$-split mixed Hodge structure, and the splitting $\mathrm{spl}_W(\mathbf{r})$ of W is compatible with N.

Proof
This follows from [7, Lemma 2.20]. □

5.4.3
Let $(N, F) \in D^{\mathrm{mild}}_{\mathrm{nilp}}$ with one N. Let $\mathbf{r} = \exp(iN)\hat{F}$ as in Proposition 5.4.2, and let $s = \mathrm{spl}_W(\mathbf{r}) : \mathrm{gr}^W \xrightarrow{\cong} H_{0,\mathbf{R}}$, $s^{(1)} = \mathrm{spl}_{W^{(1)}}(F) = \mathrm{spl}_{W^{(1)}}(\hat{F}) : \mathrm{gr}^{W^{(1)}} \xrightarrow{\cong} H_{0,\mathbf{R}}$. By Proposition 5.4.2, N is of weight 0 for s. Let $\tau^\star : \mathbf{G}_m \to \mathrm{Aut}(\mathrm{gr}^W)$ be the homomorphism associated to the SL(2)-orbit on gr^W in one variable associated to $(\mathrm{gr}^W(N), F(\mathrm{gr}^W))$. (In the case $\mathrm{gr}^W(N) = 0$, $\tau^\star$ is defined to be the trivial homomorphism.) Let $y \in \mathbf{R}_{>0}$, and let $t = y^{-1/2}$.

For the proof of the case $n = 1$ of Theorem 5.1.2(1) and 5.1.2(2), it is sufficient to prove that $\delta_W(\exp(iyN)F)$ and $\mathrm{Ad}(\tau^\star(t))^{-1}\delta_W(\exp(iyN)F)$ converge in $\mathcal{L}$ when $y \to \infty$. We prove it.

Note that the actions of $\tau(t)$ and $\tau^\star(t)$ on $D(\mathrm{gr}^W)$ are the same.

5.4.4
Let the notation be as in Section 5.4.3. For $y \gg 0$, let $g_w(y)$ for each $w \in \mathbf{Z}$ be as in the above result of Schmid from Section 5.4.1 for $(\mathrm{gr}^W(N), F(\mathrm{gr}^W))$, and let $g(y) = \bigoplus_w g_w(y) \in E = W_0\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)$. By the above result of Schmid, we have

(1) $\exp(iy\,\mathrm{gr}^W(N)F(\mathrm{gr}^W) = \exp(g(y))\tau^\star(t)\mathbf{r}(\mathrm{gr}^W)$, $g(y), \exp(g(y)) \in \mathfrak{A}$,

where $\mathrm{gr}^W(N)$ is the map $\mathrm{gr}^W \to \mathrm{gr}^W$ induced by N and $\mathfrak{A}$ is as in Section 5.3.3. Let $h(y) = \mathrm{Ad}(\tau^\star(t))^{-1}g(y)$. Then

(2) $h(y) \in \mathfrak{B}$

by Lemma 5.3.4.

Let $\delta^{(1)} = \delta_{W^{(1)}}(F)$, and let $\zeta^{(1)}$ be the corresponding ζ (Section 1.2.5), so that

(3) $F = s^{(1)} \exp(-\zeta^{(1)}) \exp(i\delta^{(1)})(s^{(1)})^{-1}\hat{F}$.

Write $s^{(1)} \exp(-\zeta^{(1)}) \exp(i\delta^{(1)})(s^{(1)})^{-1} = \exp(\alpha)\exp(\beta)$, where $\alpha, \beta \in W_0\mathrm{End}_{\mathbf{C}}(H_{0,\mathbf{C}}) \cap W^{(1)}_{-2}\mathrm{End}_{\mathbf{C}}(H_{0,\mathbf{C}})$, α is of s-weight at most -1, and β is of s-weight 0. By (3), we have

(4) $F(\mathrm{gr}^W) = \exp(\mathrm{gr}^W(\beta)\hat{F}(\mathrm{gr}^W)$,

where $\mathrm{gr}^W(\beta)$ is the map $\mathrm{gr}^W \to \mathrm{gr}^W$ induced by β. We have

(5) $\exp(iyN)\exp(\beta)\hat{F} = s\exp(iy\,\mathrm{gr}^W(N))\exp(\mathrm{gr}^W(\beta)\hat{F}(\mathrm{gr}^W) = s\exp(iy\,\mathrm{gr}^W(N))F(\mathrm{gr}^W) = s\exp(g(y))\tau^\star(t)\mathbf{r}(\mathrm{gr}^W)$

where the first equality follows from $N = s\,\mathrm{gr}^W(N)s^{-1}$ (see Proposition 5.4.2), $\beta = s\beta\,\mathrm{gr}^W(\beta)s^{-1}$, and $\hat{F} = s\hat{F}(\mathrm{gr}^W)$, the second equality follows from (4), and the last equality follows from (1).

Since $s^{(1)}\delta^{(1)}(s^{(1)})^{-1}$ and $s^{(1)}\zeta^{(1)}(s^{(1)})^{-1}$ commute with N, α and β commute with N. Hence, we have

$$s^{-1}\alpha s \in W_{-1}\mathfrak{A}_0.$$

We have

$$\begin{aligned}\exp(iyN)F &= \exp(iyN)\exp(\alpha)\exp(\beta)\hat{F} = \exp(\alpha)\exp(iyN)\exp(\beta)\hat{F}\\ &= \exp(\alpha)s\exp\bigl(g(y)\bigr)\tau^\star(t)\mathbf{r}(\mathrm{gr}^W)\\ &= s\exp\bigl(g(y)\bigr)\exp\bigl(\alpha(y)\bigr)\tau^\star(t)\mathbf{r}(\mathrm{gr}^W),\end{aligned}$$

where $\alpha(y) := \mathrm{Ad}(\exp(g(y)))^{-1}(s^{-1}\alpha s)$. Here the third equality follows from (5).

Since $s^{-1}\alpha s \in W_{-1}\mathfrak{A}_0$ and $\exp(g(y)) \in \mathfrak{A}$, we have $\alpha(y) \in W_{-1}\mathfrak{A}$. Hence,

$$\mathrm{Ad}\bigl(\tau^\star(t)\bigr)^{-1}\alpha(y) \in W_{-1}\mathfrak{B}.$$

To apply Lemma 5.3.6, we use the direct sum decomposition

$$\mathbf{C} \otimes_{\mathbf{R}} W_{-1}\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W) = M'_1 \oplus M'_2 \oplus M'_3,$$

where $M'_1 = W_{-1}\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)$, M'_2 is the -1-eigenspace of the complex conjugation acting on the $(\le -1, \le -1)$-Hodge component of $\mathbf{C} \otimes_{\mathbf{R}} W_{-1}\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W)$ with respect to $\mathbf{r}(\mathrm{gr}^W)$, and $M'_3 = F^0(\mathbf{C} \otimes_{\mathbf{R}} W_{-1}\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W))$ for the Hodge filtration $\mathbf{r}(\mathrm{gr}^W)$. In Lemma 5.3.6, consider the case

$$A = \mathfrak{B}, \qquad I^{(r)} = W_{-r}A, \qquad I = I^{(1)},$$

$$M_j = \bigoplus_{k\ge 0} t^k\mathbf{R}\bigl\{\{t\}\bigr\} \otimes_{\mathbf{R}} M'_{j,(k,\bullet)} \quad (j = 1,2,3).$$

Then the assumption of Lemma 5.3.6 is satisfied. By Lemma 5.3.6, we have

$$\exp\bigl(\mathrm{Ad}\bigl(\tau^\star(t)\bigr)^{-1}\alpha(y)\bigr) = \exp\bigl(a(y)\bigr)\exp\bigl(ib(y)\bigr)\exp\bigl(c(y)\bigr),$$

where $a(y) \in M_1$, $ib(y) \in M_2$, and $c(y) \in M_3$. Then

$$\exp(iyN)F = s\exp\bigl(g(y)\bigr)\tau^\star(t)\exp\bigl(a(y)\bigr)\exp\bigl(ib(y)\bigr)\mathbf{r}(\mathrm{gr}^W).$$

Hence,

$$\delta_W\bigl(\exp(iyN)F\bigr)=\mathrm{Ad}\bigl(\exp\bigl(g(y)\bigr)\bigr)\mathrm{Ad}\bigl(\tau^\star(t)\bigr)b(y)\in\mathfrak{A},$$

$$\mathrm{Ad}\bigl(\tau^\star(t)\bigr)^{-1}\delta_W\bigl(\exp(iyN)F\bigr)=\mathrm{Ad}\bigl(\exp\bigl(h(y)\bigr)\bigr)b(y)\in\mathfrak{B}.$$

Hence, $\delta_W(\exp(iyN)F)$ and $\mathrm{Ad}(\tau^\star(t))^{-1}\delta_W(\exp(iyN)F)$ converge when $y\to\infty$.

5.4.5

We prove Theorem 5.1.2 in general. Let $(N_1,\dots,N_n,F)\in D_{\mathrm{nilp}}$. Let τ (resp., $\tau^\star$) : $\mathbf{G}_{m,\mathbf{R}}^n\to\prod_w\mathrm{Aut}_{\mathbf{R}}(\mathrm{gr}_w^W)$ be the homomorphism whose w-component is the τ (resp., $\tau^\star$) (see Section 2.1.2) of the SL(2)-orbit in n variables on gr_w^W associated to $(\mathrm{gr}_w^W(N_1),\dots,\mathrm{gr}_w^W(N_n),F(\mathrm{gr}_w^W))$. Note that the action of $\mathbf{G}_{m,\mathbf{R}}^n$ on $D(\mathrm{gr}^W)$ via τ and that via $\tau^\star$ are the same.

By the SL(2)-orbit theorem in n variables (see [14, Theorem 0.5]),

$$\mathrm{Ad}\bigl(\tau(t)\bigr)^{-1}\delta_W\Bigl(\exp\Bigl(\sum_{j=1}^n iy_jN_j\Bigr)F\Bigr)\quad\bigl(t=(t_1,\dots,t_n),t_j=(y_{j+1}/y_j)^{1/2},y_{n+1}=1\bigr)$$

is a convergent series in $t_1,\dots,t_n$. Hence, $\delta_W(\exp(\sum_{j=1}^n iy_jN_j)F)$ and $\mathrm{Ad}(\tau^\star(t))^{-1}\delta_W(\exp(\sum_{j=1}^n iy_jN_j)F)$ have the shapes of Laurent series

$$\delta_W\Bigl(\exp\Bigl(\sum_{j=1}^n iy_jN_j\Bigr)F\Bigr)=\Bigl(\prod_{j=1}^n t_j\Bigr)^{-r}\cdot\sum_{m\in\mathbf{N}^n}\Bigl(\prod_{1\le j\le n}t_j^{m(j)}\Bigr)a_m,$$

$$\mathrm{Ad}\bigl(\tau^\star(t)\bigr)^{-1}\delta_W\Bigl(\exp\Bigl(\sum_{j=1}^n iy_jN_j\Bigr)F\Bigr)=\Bigl(\prod_{j=1}^n t_j\Bigr)^{-s}\cdot\sum_{m\in\mathbf{N}^n}\Bigl(\prod_{1\le j\le n}t_j^{m(j)}\Bigr)b_m$$

for some $r,s\in\mathbf{N}$ and $a_m,b_m\in\mathcal{L}$, where the sums $\sum_{m\in\mathbf{N}^n}$ are convergent series.

Now assume $(N_1,\dots,N_n,F)\in D_{\mathrm{nilp}}^{\mathrm{mild}}$. We prove that we can take $r=s=0$ (i.e., these series are actually Taylor series). It is sufficient to prove that, when we fix j and fix a sufficiently small $t_k>0$ for $k\neq j$, then these series become Taylor series in one variable in t_j.

But in this situation, the first Laurent series becomes $\delta_W(\exp(iy'N')F')$ with $(N',F')\in D_{\mathrm{nilp}}$, where

$$y'=t_j^{-2},\qquad N'=t_j^2\sum_{k=1}^j y_kN_k=\sum_{k=1}^j\Bigl(\prod_{k\le\ell\le n,\ell\neq j}t_\ell^{-2}\Bigr)N_k,$$

$$F'=\exp\Bigl(\sum_{k=j+1}^n iy_kN_k\Bigr)F=\exp\Bigl(i\sum_{k=j+1}^n\Bigl(\prod_{\ell=k}^n t_\ell^{-2}\Bigr)N_k\Bigr)F.$$

We consider the second Laurent series. Let $\tau_j^\star:\mathbf{G}_{m,\mathbf{R}}\to G_{\mathbf{R}}(\mathrm{gr}^W)$ be the restriction of $\tau^\star$ to the jth $\mathbf{G}_{m,\mathbf{R}}$. It is sufficient to prove that, when the t_k's for $k\neq j$ are fixed, $\delta(t):=\mathrm{Ad}(\tau_j^\star(t_j))^{-1}\delta_W(\exp(\sum_{j=1}^n iy_jN_j)F)$ is a Taylor series in t_j. Let $\tau_j^{\star,\prime}:\mathbf{G}_{m,\mathbf{R}}\to G_{\mathbf{R}}(\mathrm{gr}^W)$ be the $\tau^\star$ of the SL(2)-orbit in one variable associated to (N',F'), where N' and F' are as above. By the case $n=1$ applied to (N',F'), $\delta'(t):=\mathrm{Ad}(\tau_j^{\star,\prime}(t_j))^{-1}\delta_W(\exp(\sum_{j=1}^n iy_jN_j)F)$ is a Taylor series in

t_j. Let $W^{(j)}$ be the relative monodromy filtration $M(N_1+\cdots+N_j, W)$. By [14, Proposition 4.2], there is a convergent Taylor series $u=\sum_m(\prod_{k=j+1}^n t_k^{m(k)})u_m$ in $t_{j+1},\ldots,t_n$ with $u_m \in W_{-1}^{(j)}\mathfrak{g}_{\mathbf{R}}$ such that $u_0=0$ and such that

$$\tau_j^{\star,\prime}(t_j)=\exp(u)\tau_j^{\star}(t_j)\exp(-u).$$

We have

$$\delta(t)=\mathrm{Ad}\big(\exp(v)\exp(-u)\big)^{-1}\delta'(t), \quad \text{where } v=\mathrm{Ad}\big(\tau_j^{\star}(t_j)\big)^{-1}u.$$

Since $u_m \in W_{-1}^{(j)}\mathfrak{g}_{\mathbf{R}}$, v is a Taylor series in t_j. Hence, $\delta(t)$ is a Taylor series in t_j.

5.4.6

In the mild SL(2)-orbit theorem (see Theorem 5.1.2(1), 5.1.2(2)), the power series depend real analytically on $(N_1,\ldots,N_n,F)$ in the following sense. Let A be a real analytic manifold, and let $A\to\mathfrak{g}_{\mathbf{R}}$, $\alpha\mapsto N_{j,\alpha}$ $(1\le j\le n)$ and $A\to\check{D}$, $\alpha\mapsto F_\alpha$ be real analytic functions. Assume that the $N_{j,\alpha}$'s are nilpotent and commute with each other, and assume that $(N_{1,\alpha},\ldots,N_{n,\alpha},F_\alpha)$ generates a nilpotent orbit for any α. Assume further that, for each $1\le j\le n$, the relative monodromy filtration $M(N_1+\cdots+N_j,W)$ is independent of α. Then the ε in Theorem 5.1.2(1) and 5.1.2(2) can be taken to be constant locally on A, and the coefficients of the power series in Theorem 5.1.2(1) and 5.1.2(2) are real analytic functions on A.

This follows from the corresponding result [14, Proposition 10.8] (see [9, Remark 4.65(ii)] for the pure case) for the original SL(2)-orbit theorem and from the above proof in Section 5.4.5 to reduce the mild SL(2)-orbit theorem to the original one.

5.5. Proof of Theorem 5.1.7

We prove Theorem 5.1.7. We first prove the following result.

PROPOSITION 5.5.1

Let $(\sigma,Z,(\sigma_j)_{0\le j\le n},(N_j)_{1\le j\le n})\in D^{\sharp,\mathrm{mild}}_{\Sigma,[:]}$ *(see Section 4.5.5). Then for* $\tilde{N}_j$ *as in Section 4.5.6 and for* $F\in Z$*, the image of* $(\tilde{N}_1,\ldots,\tilde{N}_n,F)\in D^{\mathrm{mild}}_{\mathrm{nilp}}$ *in* $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}\times\mathcal{L}$ *(see Theorem 5.1.2) is independent of the choices of* $\tilde{N}_j$ *and* F*.*

Proof

For another choice $(\tilde{N}'_1,\ldots,\tilde{N}'_n,F')$ of $(\tilde{N}_1,\ldots,\tilde{N}_n,F)$, we have $\tilde{N}'_1=\tilde{N}_1$, $\tilde{N}'_j=\tilde{N}_j+R_{j-1}$ for $2\le j\le n$, and $F'=\exp(iR_n)F$ for some $R_j\in\sigma_{j,\mathbf{R}}$. We have

$$\exp\Big(\sum_{j=1}^n iy_j\tilde{N}'_j\Big)F'=\exp\Big(\sum_{j=1}^n iy_j\big(\tilde{N}_j+(y_{j+1}/y_j)R_j\big)\Big)F,$$

where y_{n+1} denotes 1. The limit of this for $y_j/y_{j+1}\to\infty$ coincides with the limit of that for $R_j=0$ by Section 5.4.6. □

5.5.2
By Proposition 5.5.1, we have a map $D^{\sharp,\mathrm{mild}}_{\Sigma,[:]} \to D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \times \mathcal{L}$. Let $D^{\sharp,\mathrm{mild}}_{\Sigma,\mathrm{val}} \to D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \times \mathcal{L}$ be the composition with $D^{\sharp,\mathrm{mild}}_{\Sigma,\mathrm{val}} \to D^{\sharp,\mathrm{mild}}_{\Sigma,[:]}$ (see Section 4.4.6). Since the last map is proper surjective (see Section 4.4.6), Theorem 5.1.7 is reduced to the following.

PROPOSITION 5.5.3
The map $D^{\sharp,\mathrm{mild}}_{\Sigma,\mathrm{val}} \to D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)} \times \mathcal{L}$ is continuous.

Just as [15, III, Theorem 3.3.2] was reduced to the case $y^*_{\lambda,t} = y_{\lambda,t}$ of [15, III, Proposition 3.3.4] (see Section A.3.4), Proposition 5.5.3 is reduced to (A_0) of the following Proposition 5.5.4. The proof of Proposition 5.5.4 given below is similar to the proof of [15, III, Proposition 3.3.4].

PROPOSITION 5.5.4
Let the situation and the assumptions be as in [15, III, Section 3.3.3] with $y^*_{\lambda,t} = y_{\lambda,t}$ there. Assume that there is $\varepsilon \in \mathbf{R}_{>0}$ such that, for any $(y_s)_{s\in S} \in \mathbf{R}^S$ satisfying the following condition (C), there is a splitting of W (which may depend on $(y_s)_s$) which is compatible with $\sum_{s\in S} y_s N_s$.

(C) *If $1 \le j \le n$, $s \in S_j$, and $y_s \neq 0$, then $y_t y_s^{-1} < \varepsilon$ for any $t \in S_{\ge j+1}$ and $|y_t y_s^{-1} - a_t a_s^{-1}| < \varepsilon$ for any $t \in S_j$.*

Note that $(N_1,\dots,N_n,F) \in D^{\mathrm{mild}}_{\mathrm{nilp}}$ by this assumption. Let $\tau,\tau^\star : \mathbf{G}^n_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}})$ be the homomorphisms given by the $\mathrm{SL}(2)$-orbit in n variables associated to $(N_1,\dots,N_n,F)$. Let

$$\delta = \lim \delta_W\Big(\exp\Big(\sum_{j=1}^{n} iy_jN_j\Big)F\Big), \qquad \delta' = \lim \mathrm{Ad}\big(\tau^\star(t)\big)^{-1}\delta_W\Big(\exp\Big(\sum_{j=1}^{n} iy_jN_j\Big)F\Big),$$

where $y_j/y_{j+1} \to \infty$ $(1 \le j \le n,\ y_{n+1} = 1)$ and where $t = (t_1,\dots,t_n)$, $t_j = (y_{j+1}/y_j)^{1/2}$. For $1 \le j \le n+1$, let $e_{\lambda,\ge j} := \exp(\sum_{s\in S_{\ge j}} iy_{\lambda,s}N_s) \in G_{\mathbf{C}}$. Then we have the following (A_j) for $0 \le j \le n$.

(A_j) *for $1 \le j \le n$: Let $e \ge 1$. If λ is sufficiently large, then there are $F^{(j)}_\lambda \in \check{D}$ satisfying the following (1)–(4).*

(1) $y^e_{\lambda,c_j} d(F_\lambda, F^{(j)}_\lambda) \to 0$.

(2) $((N_s)_{s\in S_{\le j}}, e_{\lambda,\ge j+1}F^{(j)}_\lambda)$ *generates a nilpotent orbit.*

(3) $\delta_W(\exp(\sum_{s\in S} iy_{\lambda,s}N_s)F^{(j)}_\lambda)$ *converges to δ.*

(4) $\mathrm{Ad}(\tau^\star(t))^{-1}\delta_W(\exp(\sum_{s\in S} iy_{\lambda,s}N_s)F^{(j)}_\lambda)$ *with $t = (t_1,\dots,t_n)$, $t_j = (y_{\lambda,c_{j+1}}/y_{\lambda,c_j})^{1/2}$ converges to δ'.*

(A_0): *Let $e \ge 1$. Then if λ is sufficiently large, we have the following (3) and (4).*

(3) $\delta_W(\exp(\sum_{s\in S} iy_{\lambda,s}N_s)F_\lambda)$ *converges to δ.*

(4) $\mathrm{Ad}(\tau^\star(t))^{-1}\delta_W(\exp(\sum_{s\in S} iy_{\lambda,s}N_s)F_\lambda)$ *with* t *as in* $(A_j)(4)$.

5.5.5

We prove Proposition 5.5.4 by downward induction on j. For $1\le j\le n$, let τ_j be the restriction of τ to the jth factor of $\mathbf{G}_{m,\mathbf{R}}$, and let $\tau_{\ge j}=\prod_{k=j}^n \tau_k((y_{\lambda,c_{k+1}}/y_{\lambda,c_k})^{1/2})\in G_{\mathbf{R}}$. Then (A_n) follows from [15, III, Section 3.3.3, (5)] for $j=n$ with $y^*_{\lambda,t}=y_{\lambda,t}$ and from Section 5.4.6.

Assume that $0\le j<n$. We prove (A_j) assuming (A_{j+1}). Take sufficiently large integers $e,e',e''\ge 0$. Take $F_\lambda^{(j+1)}$ as in (A_{j+1}) with e replaced by $e+e'+e''$. In the case $1\le j<n$ (resp., $j=0$), let $F_\lambda^{(j)}$ be F_λ^* in [15, III, Section 3.3.3, (5)] with e there replaced by $e+e'+e''$ (resp., let $F_\lambda^{(0)}=F_\lambda$).

We have

(5) $y_{\lambda,c_{j+1}}^{e+e'+e''}d(F_\lambda^{(j)},F_\lambda^{(j+1)})\to 0$.

By [15, III, Lemma 3.3.6], $\tau_{\ge j+1}^{-1}e_{\lambda,\ge j+1}F_\lambda^{(j+1)}$ converges. Hence, by (5), $\tau_{\ge j+1}^{-1}e_{\lambda,\ge j+1}F_\lambda^{(j)}$ converges, and we have

(6) $y_{\lambda,c_{j+1}}^{e+e'}d(\tau_{\ge j+1}^{-1}e_{\lambda,\ge j+1}F_\lambda^{(j)},\tau_{\ge j+1}^{-1}e_{\lambda,\ge j+1}F_\lambda^{(j+1)})\to 0$.

By the mild SL(2)-orbit theorem in Theorem 5.1.2 for

$$((N_s)_{s\in S_{\le j}},\tau_{\ge j+1}^{-1}e_{\lambda,\ge j+1}F_\lambda^{(j)})\qquad\text{and}\qquad((N_s)_{s\in S_{\le j}},\tau_{\ge j+1}^{-1}e_{\lambda,\ge j+1}F_\lambda^{(j+1)}),$$

and by Section 5.4.6, we have the following.

(7) The four sequences

$$a_\lambda:=\delta_W\Big(\tau_{\ge j+1}^{-1}\exp\Big(\sum_{s\in S} iy_{\lambda,s}N_s\Big)F_\lambda^{(j)}\Big),$$

$$b_\lambda:=\delta_W\Big(\tau_{\ge j+1}^{-1}\exp\Big(\sum_{s\in S} iy_{\lambda,s}N_s\Big)F_\lambda^{(j+1)}\Big),$$

$$a'_\lambda:=\mathrm{Ad}\big(\tau^\star(t)\big)^{-1}\delta_W\Big(\tau_{\ge j+1}^{-1}\exp\Big(\sum_{s\in S} iy_{\lambda,s}N_s\Big)F_\lambda^{(j)}\Big),$$

$$b'_\lambda:=\mathrm{Ad}\big(\tau^\star(t)\big)^{-1}\delta_W\Big(\tau_{\ge j+1}^{-1}\exp\Big(\sum_{s\in S} iy_{\lambda,s}N_s\Big)F_\lambda^{(j+1)}\Big)$$

converge in $\mathcal{L}$, and we have

$$y_{\lambda,c_{j+1}}^e(a_\lambda-b_\lambda)\to 0,\qquad y_{\lambda,c_{j+1}}^e(a'_\lambda-b'_\lambda)\to 0.$$

By the induction assumption on j, $\exp(\sum_{s\in S} iy_{\lambda,s}N_s)F_\lambda^{(j+1)}$ converges to δ and $\mathrm{Ad}(\tau^\star(t))^{-1}\exp(\sum_{s\in S} iy_{\lambda,s}N_s)F_\lambda^{(j+1)}$ converges to δ'. Hence, by (7), $\exp(\sum_{s\in S} iy_{\lambda,s}N_s)F_\lambda^{(j)}$ converges to δ and $\mathrm{Ad}(\tau^\star(t))^{-1}\exp(\sum_{s\in S} iy_{\lambda,s}N_s)F_\lambda^{(j)}$ converges to δ'.

5.6. Proofs of other results in Section 5.1

LEMMA 5.6.1

Let $x = (N_1, \dots, N_n, F) \in D^{\mathrm{mild}}_{\mathrm{nilp}}$, and let $p \in D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim$ be the image of x. Let $\mathbf{0} = (0, \dots, 0) \in \mathbf{Z}^n$.

(1) *Let δ be the image of x in $\mathcal{L}$. Then δ is of $\mathrm{Ad}(\tau_p^\star)$-weight at most $\mathbf{0}$.*

(2) *Let $\delta' \in \mathcal{L}$ be the limit of $\mathrm{Ad}(\tau_p^\star(t))^{-1}\delta_W(\exp(\sum_{j=1}^n iy_jN_j)F)$ $(t = (t_1, \dots, t_n), t_j = (y_{j+1}/y_j)^{1/2}$, y_{n+1} denotes 1, $t_j \to 0)$. Then δ' coincides with the component of δ of $\mathrm{Ad}(\tau_p^\star)$-weight $\mathbf{0}$.*

Proof
Let $\delta(y) = \delta_W(\exp(\sum_{j=1}^n iy_jN_j)F)$, and let $\delta'(y) = \mathrm{Ad}(\tau_p^\star(t))^{-1}\delta_W(\exp(\sum_{j=1}^n iy_jN_j)F)$, where t_j is as above. Then $\delta(y)$ and $\delta'(y)$ are convergent series in $t_1, \dots, t_n$. For $a \in \mathbf{Z}^n$, let δ_a (resp., δ'_a, $\delta(y)_a$, $\delta'(y)_a$) be the component of δ (resp., δ', $\delta(y)$, $\delta'(y)$) of $\mathrm{Ad}(\tau_p^\star)$-weight a. Then $\delta(y)_a = (\prod_{j=1}^n t_j^{a(j)})\delta'(y)_a$. Hence, $\delta(y)_a$ is divisible by $\prod_{j=1}^n t_j^{\max(a(j),0)}$. Hence, the constant term δ_a of $\delta(y)_a$ is 0 unless $a \le \mathbf{0}$. On the other hand, by the reduction to the case of one N, we have

$$\delta'(y) \in \sum_{k \in \mathbf{N}^n} \Big(\prod_{j=1}^n t_j^{k(j)}\Big)\mathbf{R}\{\{t_1, \dots, t_n\}\} \cdot \Big(\bigcap_{j=1}^n W^{(j)}_{k(j)}\mathcal{L}\Big).$$

Hence, the constant term of $\delta'(y)$ belongs to $\bigcap_{j=1}^n W_0^{(j)}\mathcal{L}$. That is, δ' is of $\mathrm{Ad}(\tau_p^\star)$-weight at most $\mathbf{0}$. For $a \in \mathbf{Z}^n$ such that $a \le \mathbf{0}$, the constant term δ'_a of $\delta'(y)_a = \prod_{j=1}^n t_j^{-a(j)}\delta(y)_a$ is 0 unless $a = \mathbf{0}$. This argument also shows that $\delta_{\mathbf{0}} = \delta'_{\mathbf{0}}$. □

5.6.2
We prove Theorem 5.1.10(1). By Lemma 5.6.1, it is sufficient to prove that the element $\delta' \in \mathcal{L}$ in Lemma 5.6.1(2) belongs to $\mathcal{L}(\mathbf{r})$, where $\mathbf{r} = (\varphi_w(i, \dots, i))_w$. Since $\mathbf{r}(y) := \tau_p^\star(t)^{-1}\exp(\sum_{j=1}^n iy_jN_j)F(\mathrm{gr}^W)$ converges to $\mathbf{r}$, $\mathcal{L}(\mathbf{r}(y))$ converges to $\mathcal{L}(\mathbf{r})$. Since $\delta'(y) \in \mathcal{L}(\mathbf{r}(y))$, its limit δ' belongs to $\mathcal{L}(\mathbf{r})$.

5.6.3
We prove Theorem 5.1.10(2). Let $s \in \mathrm{spl}(W)$ be the image of x (see Lemma 5.6.1) in $\mathrm{spl}(W)$. Consider the element (p, Z) of $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ (see Section 2.3.2), where Z is the subset of D whose image in $D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ is $(Z(p), s, \delta_{\mathbf{0}})$. This element exists uniquely by Section 5.6.2. We show that $F_y := \exp(\sum_{j=1}^n iy_jN_j)F$ converges to (p, Z) in $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$.

Let Φ be the set $\{W^{(1)}(\mathrm{gr}^W), \dots, W^{(n)}(\mathrm{gr}^W)\}$ of weight filtrations on gr^W associated to p. Fix a distance $\beta : D(\mathrm{gr}^W) \to \mathbf{R}^n_{\ge 0}$ to Φ-boundary. Let $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^\sim \times \mathrm{spl}(W) \times \mathcal{L}$ be the map $\nu_{\alpha,\beta}$ in Proposition 2.3.9, where $\alpha = \tau_p$. Then $\nu_{\alpha,\beta}(p, Z) = (p, s, \mathrm{Ad}(\tau_p(\beta(\mathbf{r})))^{-1}\delta_{\mathbf{0}})$. Hence, it is sufficient to prove that $\nu_{\alpha,\beta}(F_y) \in D(\mathrm{gr}^W) \times \mathrm{spl}(W) \times \mathcal{L}$ converges to $(p, s, \mathrm{Ad}(\tau_p^\star(\beta(\mathbf{r})))^{-1}\delta_{\mathbf{0}})$. It is sufficient to prove that $\mathrm{Ad}(\tau_p^\star(\beta(F_y(\mathrm{gr}^W))))^{-1}\delta(y)$ converges to

$\mathrm{Ad}(\tau_p^\star(\beta(\mathbf{r})))^{-1}\delta_{\mathbf{0}}$. But this is deduced from the fact that $\beta(F_y(\mathrm{gr}^W))t^{-1}$ converges to $\beta(\mathbf{r})$.

5.6.4

We prove Theorem 5.1.10(3). By Theorem 5.1.10(2), it is sufficient to prove the compatibility of the map $D^{\sharp,\mathrm{mild}}_{\Sigma,[:]} \to D^{\diamond}_{\mathrm{SL}(2)}$ with log structures. This is reduced to the pure case treated in Section 4.5.12, because the log structure of $D^{\diamond}_{\mathrm{SL}(2)}$ is the inverse image of that of $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)^{\sim}$.

5.6.5

Theorem 5.1.11 follows from Lemma 5.6.1 and Theorem 5.1.10(1). These complete the proofs of the results in Section 5.1.

6. Complements

In Section 6.1, we give properties of the extended period domains. In Section 6.2, we show that for nilpotent orbits in one variable, we have stronger results (see Theorems 6.2.2, 6.2.4) which connect the world of nilpotent orbits with the world of SL(2)-orbits and Borel–Serre orbits. In Section 6.3, we consider extended period maps.

6.1. Global properties of the extended period domains

THEOREM 6.1.1

Let X be one of $D^\star_{\mathrm{SL}(2)}$, $D^{\star,+}_{\mathrm{SL}(2)}$, $D^{\star,-}_{\mathrm{SL}(2)}$, $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$, $D^{\diamond}_{\mathrm{SL}(2)}$, $D_{\mathrm{BS},\mathrm{val}}$, $D^{I}_{\mathrm{SL}(2),\mathrm{val}}$, $D^{II}_{\mathrm{SL}(2),\mathrm{val}}$, $D^\star_{\mathrm{SL}(2),\mathrm{val}}$, $D^{\diamond}_{\mathrm{SL}(2),\mathrm{val}}$, $D^\sharp_{\Sigma,[:]}$, or $D^\sharp_{\Sigma,[\mathrm{val}]}$. Let Γ be a subgroup of $G_{\mathbf{Z}}$.

(1) *The action of Γ on X is proper, and the quotient space $\Gamma\setminus X$ is Hausdorff.*

(2) *Assume that Γ is neat. Let $\gamma\in\Gamma$, let $p\in X$, and assume $\gamma p=p$. Then $\gamma=1$.*

(3) *Assume that Γ is neat. Then the projection $X\to\Gamma\setminus X$ is a local homeomorphism. Further, for $X=D^\star_{\mathrm{SL}(2)}$, $D^{\star,+}_{\mathrm{SL}(2)}$, $D^{\star,-}_{\mathrm{SL}(2)}$, or $D^{\star,\mathrm{BS}}_{\mathrm{SL}(2)}$, there is a structure on the quotient such that the projection is a local isomorphism in $\mathcal{B}'_{\mathbf{R}}(\log)$.*

Note that the corresponding results for D_{BS} , $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$, and for $D^\sharp_\Sigma$ and $D^\sharp_{\Sigma,\mathrm{val}}$ have already been proved in [15, I, Theorem 9.1], [15, II, Theorem 3.5.17], and [15, III, Theorem 4.3.6], respectively.

Proof

Statement (3) for X follows from (1) and (2) for X. Hence, it is sufficient to prove (1) and (2). Since we have continuous maps $D^\star_{\mathrm{SL}(2),\mathrm{val}}\to D_{\mathrm{BS},\mathrm{val}}\to D_{\mathrm{BS}}$ and $D_{\Sigma,[\mathrm{val}]}\to D_{\Sigma,[:]}\to\Gamma\setminus D_\Sigma$ which are compatible with the actions of Γ, the results for $D^\star_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{BS},\mathrm{val}}$, $D_{\Sigma,[\mathrm{val}]}$, and $D_{\Sigma,[:]}$ follow from the results for D_{BS} and $\Gamma\setminus D_\Sigma$. Since $D^\star_{\mathrm{SL}(2),\mathrm{val}}\to D^\star_{\mathrm{SL}(2)}$ is proper and surjective, the properness of the action of Γ on $D^{\star}_{\mathrm{SL}(2)}$ follows from that for $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$. Statement (2) for $D^{\star}_{\mathrm{SL}(2)}$ follows from the $\bar{L}$-bundle property (i.e., Theorem 2.3.14 for situation (a) in Section 2.3.5) and the result for the pure case. Since there are continuous maps $D^{\diamond}_{\mathrm{SL}(2),\mathrm{val}} \to D^{\diamond}_{\mathrm{SL}(2)} \to D^{\star}_{\mathrm{SL}(2)}$ which are compatible with the actions of Γ, the results for $D^{\diamond}_{\mathrm{SL}(2)}$ and $D^{\diamond}_{\mathrm{SL}(2),\mathrm{val}}$ follow from the result for $D^{\star}_{\mathrm{SL}(2)}$. □

COROLLARY 6.1.2

The space X in the above theorem is Hausdorff.

This is obtained from the above theorem by taking $\Gamma = \{1\}$.

COROLLARY 6.1.3

Let $X = D^{I}_{\mathrm{SL}(2)}$, $D^{II}_{\mathrm{SL}(2)}$, $D^{\star}_{\mathrm{SL}(2)}$, or $D^{\diamond}_{\mathrm{SL}(2)}$. Let Γ be a neat subgroup of $G_{\mathbf{Z}}$. Then there is a unique structure on $\Gamma\setminus X$ as an object of $\mathcal{B}'_{\mathbf{R}}(\log)^+$ (see Section 2.7.2) such that the projection $X \to \Gamma\setminus X$ is a morphism in $\mathcal{B}'_{\mathbf{R}}(\log)^+$ which is locally an isomorphism.

Proof

This follows from Theorem 6.1.1(3) and the corresponding results for $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ in [15, II, Theorem 3.5.17]. □

6.2. Results on nilpotent orbits in one variable

We prove Theorems 6.2.2 and 6.2.4 on nilpotent orbits in one variable. In Sections 6.2.5–6.2.14, we give a counterexample for the extension of Theorem 6.2.2 to nilpotent orbits in many variables.

6.2.1

Let $(D^{\sharp}_{\Sigma,[:]})' \subset D^{\sharp}_{\Sigma,[:]}$ be the union of the two open sets $D^{\sharp,\mathrm{mild}}_{\Sigma,[:]}$ (see Section 5.1.4) and the inverse image of $D_{\mathrm{SL}(2),\mathrm{nspl}}$ by $D^{\sharp}_{\Sigma,[:]} \to D^{I}_{\mathrm{SL}(2)}$ in Theorem 4.5.2(1). Then $(D^{\sharp}_{\Sigma,[:]})'$ is the union of $D^{\sharp,\mathrm{mild}}_{\Sigma,[:]}$ and the set of the points p of $D^{\sharp}_{\Sigma,[:]}$ such that if $N_1, \dots, N_n$ (ordered) is the monodromy logarithms associated to p, then (W, N_1) does not split. The morphisms $D^{\sharp,\mathrm{mild}}_{\Sigma,[:]} \to D^{\star}_{\mathrm{SL}(2)}$ (see Theorem 5.1.10, Section 0.2.3) and $D_{\mathrm{SL}(2),\mathrm{nspl}} \to D^{\star}_{\mathrm{SL}(2)}$ (see Section 2.5.6) induce a morphism $(D^{\sharp}_{\Sigma,[:]})' \to D^{\star}_{\mathrm{SL}(2)}$.

Let $(D^{\sharp}_{\Sigma,[\mathrm{val}]})'$ be the inverse image of $(D^{\sharp}_{\Sigma,[:]})'$ in $D^{\sharp}_{\Sigma,[\mathrm{val}]}$. Then we obtain the induced morphism $(D^{\sharp}_{\Sigma,[\mathrm{val}]})' \to D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$ and a commutative diagram

$$\begin{array}{ccc} (D^{\sharp}_{\Sigma,[\mathrm{val}]})' & \overset{\psi}{\to} & D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \\ \downarrow & & \downarrow \\ (D^{\sharp}_{\Sigma,[:]})' & \overset{\psi}{\to} & D^{\star}_{\mathrm{SL}(2)} \end{array}$$

Let Ξ be as in Section 0.1.5. Since $(D^{\sharp}_{\Xi})' = D^{\sharp}_{\Xi}$, we have the following.

THEOREM 6.2.2

The identity map of D extends uniquely to a continuous map

$$D_{\Xi}^{\sharp} \to D_{\mathrm{SL}(2),\mathrm{val}}^{\star}$$

and, hence, extends uniquely to a continuous map $D_{\Xi}^{\sharp} \to D_{\mathrm{BS},\mathrm{val}}$.

REMARK 6.2.3

(1) The image of $D_{\Xi}^{\sharp}$ in $D_{\mathrm{SL}(2)}$ is contained in $D_{\mathrm{SL}(2),\leq 1}$ for both structures I, II of $D_{\mathrm{SL}(2)}$. (We denote by ≤ 1 the part where the log structure is of rank at most 1.)

(2) However, the image of $D_{\Xi}^{\sharp}$ in $D_{\mathrm{SL}(2)}^{\star}$ is not necessarily contained in $D_{\mathrm{SL}(2),\leq 1}^{\star}$. (This is seen in Section 7.3.9 below.) Hence, the morphism in Theorem 6.2.2 cannot be obtained as the composition $D_{\Xi}^{\sharp} \to D_{\mathrm{SL}(2),\leq 1}^{\star} \cong D_{\mathrm{SL}(2),\leq 1,\mathrm{val}}^{\star}$. (The first arrow here need not exist.) For $p \in D_{\Xi}^{\sharp}$, it can happen that the image of p in $D_{\mathrm{SL}(2),\mathrm{val}}^{\star}$ has some information about p which the image of p in $D_{\mathrm{SL}(2)}^{\star}$ does not have (see Sections 7.1.11, 7.3.8 below).

(3) The image of $D_{\Xi}^{\sharp,\mathrm{mild}} \to D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}}$ (see Section 2.1.4) is contained in $D_{\mathrm{SL}(2),\leq 1}^{\star,\mathrm{mild}}$.

THEOREM 6.2.4

Let $p = (\mathbf{R}_{\geq 0}N, \exp(i\mathbf{R}N)F) \in D_{\Xi}^{\sharp}$ with $N \neq 0$. Let $W' = W^{(1)}$ be the relative monodromy filtration of N with respect to W. Let $(W', \hat{F})$ be the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure (W', F), that is, $\mathrm{spl}_{W'}(F)(F(\mathrm{gr}^{W'}))$ *(see Section 1.2). Then the following conditions are equivalent.*

(i) *p belongs to $D_{\Xi}^{\sharp,\mathrm{mild}}$.*
(ii) $\exp(iyN)F$ *converges in $D_{\mathrm{SL}(2)}^{\diamond}$ when $y \to \infty$.*
(iii) $\delta_W(\exp(iyN)F)$ *converges in $\mathcal{L}$ when $y \to \infty$.*
(iv) *The image of p in $D_{\mathrm{SL}(2)}^{\star}$ (see Theorem 6.2.2) belongs to $D_{\mathrm{SL}(2)}^{\star,\mathrm{mild}}$.*
(v) *The image of p in D_{BS} (see Theorem 6.2.2) belongs to $D_{\mathrm{BS}}^{\mathrm{mild}}$.*
(vi) *The image of p in $D_{\mathrm{SL}(2)}$ belongs to $D_{\mathrm{SL}(2),\mathrm{spl}}$ (see Section 2.5.6).*
(vii) $\delta_W(\exp(iN)\hat{F}) = 0$.
(viii) *The splitting* $\mathrm{spl}_W(\exp(iN)\hat{F})$ *of W is compatible with N.*

Proof

We have proved (i) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (iii) is clear. We know (ii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v), (v) $\Rightarrow$ (vi) $\Leftrightarrow$ (vii). (viii) $\Rightarrow$ (i) is clear. It is sufficient to prove the implications (iii) $\Rightarrow$ (vii) and (vii) $\Rightarrow$ (viii).

Let $s = \mathrm{spl}_W(\exp(iN)\hat{F})$, $\bar{N} = \mathrm{gr}^W(N) \in \bigoplus_w \mathrm{Hom}(\mathrm{gr}_w^W, \mathrm{gr}_w^W)$, and $N_0 = s\bar{N}s^{-1}$. We prove (vii) $\Rightarrow$ (viii). Assume (vii). Then $\exp(iN)\hat{F} = s(\exp(i\bar{N})\hat{F}(\mathrm{gr}^W)) = \exp(iN_0)\hat{F}$. For the mixed Hodge structure $(W', \exp(iN)\hat{F}) =$

$(W', \exp(iN_0)\hat{F})$, we have $N = \delta_{W'}(\exp(iN)\hat{F}) = \delta_{W'}(\exp(iN_0)\hat{F}) = N_0$ and (viii) holds.

For the proof of (iii) $\Rightarrow$ (vii), we first prove the following claim.

CLAIM

$\delta_W(\exp(iN)\hat{F})$ is of W'-weight at most -1.

Proof of Claim

Let $A = W_0\mathrm{End}_{\mathbf{C}}(H_{0,\mathbf{C}})$. For $r \geq 1$, let $I^{(r)}$ be the two-sided ideal $W_{-1}A \cap W'_{-r}A$ of A, and let $I = I^{(1)}$. Let $M_1 = I \cap \mathrm{End}_{\mathbf{R}}(H_{\mathbf{R}})$. Let M_2 be the part of $iM_1 \subset I$ consisting of all elements which belong to the $(\leq -1, \leq -1)$-Hodge component of A with respect to $\exp(iN_0)\hat{F}$. Let M_3 be the part of I consisting of all elements which belong to F^0A with respect to $\exp(iN)\hat{F}$. Then we have $I^{(r)} = (I^{(r)} \cap M_1) \oplus (I^{(r)} \cap M_2) \oplus (I^{(r)} \cap M_3)$ for any $r \geq 1$. We have $\exp(iN)\hat{F} = \exp(x)\exp(iN_0)F$ for some $x \in I$. Hence, by Lemma 5.3.6, there are $x_j \in M_j$ $(j = 1, 2, 3)$ such that $\exp(iN)\hat{F} = \exp(x_1)\exp(x_2)\exp(x_3)\exp(iN_0)\hat{F} = \exp(x_1)\exp(x_2)\hat{F} = s'\exp(i\delta)\exp(i\bar{N})\hat{F}(\mathrm{gr}^W)$, where $s' = \exp(x_1)s \in \mathrm{spl}(W)$ and $i\delta = s^{-1}x_2s$. We have $\delta_W(\exp(iN)\hat{F}) = \delta \in W'_{-1}\mathrm{End}_{\mathbf{R}}(\mathrm{gr}^W_{\mathbf{R}})$. □

Now we prove (iii) $\Rightarrow$ (vii). Assume that $\delta_W(\exp(iN)\hat{F}) \neq 0$. By the claim, there is $w \leq -1$ such that the component of $\delta_W(\exp(iN)\hat{F})$ of τ-weight w is nonzero. Since $\mathrm{Ad}(\tau(\sqrt{y}))\delta_W(\exp(iyN)F)$ converges to $\delta_W(\exp(iN)\hat{F})$ when $y \to \infty$, $\delta_W(\exp(iyN)F)$ is $\mathrm{Ad}(\tau(\sqrt{y})^{-1})B(y)$, where $B(y)$ converges to an element whose part of weight w is nonzero. Hence, the part of the τ-weight w of $\delta_W(\exp(iyN)F)$ is $y^{-w/2}C(y)$, where $C(y)$ converges to a nonzero element, and hence diverges. □

6.2.5

We have constructed CKS maps $D^{\sharp,\mathrm{mild}}_{\Sigma,[:]} \to D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}$ (see Theorem 5.1.10) and $D^{\sharp}_{\Xi} \to D^{\star}_{\mathrm{SL}(2)}$ (see Theorem 6.2.2). In the rest of Section 6.2, we show an example of σ of rank 2 such that there is no continuous map $D^{\sharp}_{\sigma,\mathrm{val}} \to D^{\star}_{\mathrm{SL}(2)}$ which extends the identity map of D.

6.2.6

Take an integer $m \geq 1$. (The case $m \geq 3$ will be a crucial example.) Let H_0 be of rank $2m+1$ with base e'_j $(1 \leq j \leq m)$, e_j $(1 \leq j \leq m)$, and e.

The weight filtration is as follows. $W_{-m-1} = 0$. W_{-m} is generated by e'_j and e_j $(1 \leq j \leq m)$. $W_{-1} = W_{-m}$. W_0 is the total space.

We have N_1, N_2 defined as follows. $N_1e = 0$, $N_1e_j = e'_j$. $N_1e'_j = 0$. $N_2e = e'_m$, $N_2e_j = e_{j-1}$ and $N_2e'_j = e'_{j-1}$ for $2 \leq j \leq m$, and $N_2e_1 = N_2e'_1 = 0$. Let σ be the cone generated by N_1 and N_2. Note that (W, N_1) splits, but (W, N_2) does not split.

6.2.7
For $j=1,2$, let $W^{(j)}$ be the relative monodromy filtration of N_j with respect to W.

We give a splitting of $W^{(1)}$, which is compatible with N_1, as follows. e is of weight 0. e_j is of weight $-m+1$, and e'_j is of weight $-m-1$ $(1 \le j \le m)$.

We give a splitting of $W^{(2)}$, which is compatible with N_2, as follows. e is of weight 0. e_j is of weight $-2(m-j)$, and e'_j is of weight $-2(m-j+1)$ $(1 \le j \le m)$.

6.2.8
Define $\alpha_1, \alpha_2 : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}})$ by using the above splittings of $W^{(1)}$ and $W^{(2)}$, respectively. Then α_1 and α_2 commute. Define $\alpha_1^\star, \alpha_2^\star : \mathbf{G}_{m,\mathbf{R}} \to \mathrm{Aut}_{\mathbf{R}}(H_{0,\mathbf{R}})$ by $\alpha_j^\star(t)e = e$ and $\alpha_j^\star(t)x = t^m \alpha_j(t)x$ for $x \in W_{-m}$.

Let

$$t(y) = \alpha_1\big((y_2/y_1)^{1/2}\big)\alpha_2\big((1/y_2)^{1/2}\big), \qquad t^\star(y) = \alpha_1^\star\big((y_2/y_1)^{1/2}\big)\alpha_2^\star\big((1/y_2)^{1/2}\big).$$

6.2.9
We have

$$\mathrm{Ad}\big(t(y)\big)^{-1}(y_1N_1 + y_2N_2) = N_1 + N_{2,y},$$

where $N_{2,y}$ coincides with N_2 on e_j and e'_j $(1 \le j \le m)$, but $N_{2,y}e = (y_2/y_1)^{(m+1)/2}e'_m$ and the last element converges to 0 when $y_1/y_2 \to \infty$. We have

$$\mathrm{Ad}\big(t^\star(y)\big)^{-1}(y_1N_1 + y_2N_2) = N_1 + N'_{2,y},$$

where $N'_{2,y}$ coincides with N_2 on e_j and e'_j $(1 \le j \le m)$, but $N'_{2,y}e = u_y e'_m$, where

$$u_y = (y_2/y_1)^{1/2} y_2^{m/2} = y_1^{-1/2} y_2^{(m+1)/2}.$$

6.2.10
Note that u_y need not converge when $y_2 \to \infty$ and $y_1/y_2 \to \infty$.

6.2.11
Let F be as follows. $F^1 = 0$. F^0 is generated by e and e_m. F^{-j}, for $1 \le j \le m-1$, is generated by F^{-j+1}, e_{m-j}, and e'_{m-j+1}. F^{-m} is the total space. Then (N_1, N_2, F) generates a nilpotent orbit. Hence, $\exp(iy_1N_1 + iy_2N_2)F$, as $y_2 \to \infty$ and $y_1/y_2 \to \infty$, converges in $D^\sharp_{\sigma,\mathrm{val}}$.

LEMMA 6.2.12
Let the notation be as above. If $m \ge 3$, then $\exp(iy_1N_1 + iy_2N_2)F$, as $y_2 \to \infty$ and $y_1/y_2 \to \infty$, need not converge in $D^\star_{\mathrm{SL}(2)}$.

This follows from the following result.

LEMMA 6.2.13
Let the notation and the assumption be as above. Let $F_y := t^\star(y)^{-1}\exp(iy_1N_1 + iy_2N_2)F$. Then, $\delta_W(F_y)$ does not converge in $\bar{\mathcal{L}}$ when $y_2 \to \infty$ and $y_1/y_2 \to \infty$.

6.2.14

We prove Lemma 6.2.13. Since $F_y = \exp(\mathrm{Ad}(t^\star(y))^{-1}(iy_1N_1 + iyN_2))F$, F_y is described as follows. $F_y^1 = 0$, F_y^0 is generated by $e + \sum_{k=1}^m i^k \cdot k!^{-1} \cdot u_y \cdot e'_{m-j+1}$ and $\exp(iN_1 + iN_2)e_m$, F_y^{-j} $(1 \le j \le m-1)$ is generated by F_y^{-j+1}, $\exp(iN_1 + iN_2)e_{m-j}$, and $\exp(iN_1 + iN_2)e'_{m-j+1}$, and F_y^{-m} is the total space.

The Hodge type of gr^W_{-m} of F_y is that the $(j, -m-j)$-Hodge component is 1-dimensional if $j = 0, -m$, is 2-dimensional if $-1 \ge j \ge 1-m$, and is 0 otherwise. $\delta_W(F_y)$ sends e to the sum of the $(j, -m-j)$ components of $v_y := \sum_{1 \le k \le m, k:\mathrm{odd}} (-1)^{(k-1)/2} \cdot k!^{-1} \cdot t \cdot e_{m-k+1}$ for $-1 \ge j \ge 1-m$.

CLAIM

v_y does not belong to the $((0,-m) + (-m,0))$-Hodge component of gr^W_{-m}.

By the claim, v_y is u_y times a *nonzero* element which is independent of y_1, y_2. Hence, when $y_1/y_2, y_2 \to \infty$, v_y need not converge in $\bar{\mathcal{L}}$.

We prove the claim. Assume that v_y belongs to the $((0,-m) + (-m,0))$-Hodge component of gr^W_{-m}. Then we should have

$$\begin{aligned}\sum_{1 \le k \le m, k:\mathrm{odd}} &(-1)^{(k-1)/2} \cdot k!^{-1} \cdot e'_{m-k+1} \\ &= a \exp(iN_1 + iN_2)e_m + b \exp(-iN_1 - iN_2)e_m\end{aligned}$$

for some $a, b \in \mathbf{C}$. If V denotes the $\mathbf{C}$-vector space generated by e_j $(1 \le j \le m)$ and e'_j $(1 \le j \le m-3)$, we should have

$$\begin{aligned}e'_m - (1/6)e'_{m-2} \equiv\ & a\big(ie'_m - e'_{m-1} - (i/2)e'_{m-2}\big) \\ &+ b\big(-ie'_m - e'_{m-1} + (i/2)e'_{m-2}\big) \bmod V.\end{aligned}$$

(To get this, use $(N_1 + N_2)^k = kN_1N_2^{k-1} + N_2^k$, and hence, $\exp(iN_1 + iN_2) = 1 + \sum_{k=1}^\infty (i^k \cdot (k-1)!^{-1} \cdot N_1N_2^{k-1} + i^k \cdot k!^{-1} \cdot N_2^k)$.) By comparing the coefficients of e'_{m-1}, we have $a + b = 0$. Hence,

$$e'_m - (1/6)e'_{m-2} \equiv a \cdot 2i \cdot \big(e'_m - (1/2)e'_{m-2}\big) \bmod V.$$

This is impossible.

REMARK 6.2.15

We do not know whether the identity map of D always extends to a continuous map $D^\sharp_{\Sigma,[\mathrm{val}]} \to D^\star_{\mathrm{SL}(2),\mathrm{val}}$ or not.

6.3. Extended period maps

The following is a modified version of [15, III, Theorem 7.5.1(i)].

THEOREM 6.3.1

Let S be a connected, log smooth, fs log analytic space, and let U be the open subspace of S consisting of all points of S at which the log structure of S is trivial. Let H be a variation of mixed Hodge structure on U with polarized graded quotients for the weight filtration and with unipotent local monodromy along

$S \smallsetminus U$. Assume that H extends to a log mixed Hodge structure (see [15, III, Section 1.3]) on S (i.e., H is admissible along $S \smallsetminus U$ as a variation of mixed Hodge structure). Fix a basepoint $u \in U$, and let $\Lambda = (H_0, W, (\langle\cdot,\cdot\rangle_w)_w, (h^{p,q})_{p,q})$ be $(H_{\mathbf{Z},u}, W, (\langle\cdot,\cdot\rangle_{w,u})_w$(the Hodge numbers of H)). Let Γ be a subgroup of $G_{\mathbf{Z}}$ which contains the global monodromy group $\mathrm{Image}(\pi_1(U,u) \to G_{\mathbf{Z}})$, and assume that Γ is neat. Let $\varphi : U \to \Gamma \setminus D$ be the associated period map. Let $S^{\log}_{[:]} = S^{\log} \times_S S_{[:]}$, let $S^{\log}_{[\mathrm{val}]} = S^{\log} \times_S S_{[\mathrm{val}]}$, and regard U as open sets of these spaces.

Then we have the following.

(1) The map $\varphi : U \to \Gamma \setminus D$ extends uniquely to continuous maps

$$S^{\log}_{[:]} \to \Gamma \setminus D^I_{\mathrm{SL}(2)}, \qquad S^{\log}_{[\mathrm{val}]} \to \Gamma \setminus D^I_{\mathrm{SL}(2),\mathrm{val}}.$$

(2) Assume that the complement $S \smallsetminus U$ of U is a smooth divisor on S. Then the map $\varphi : U \to \Gamma \setminus D$ extends uniquely to a continuous map

$$S^{\log} \to \Gamma \setminus D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$$

and hence extends uniquely to a continuous map $S^{\log} \to \Gamma \setminus D_{\mathrm{BS},\mathrm{val}}$.

Proof

Statement (1) is a modified version of [15, III, Theorem 7.5.1(i)], which treated the extended period map $S^{\log}_{\mathrm{val}} \to D^I_{\mathrm{SL}(2)}$, where $S^{\log}_{\mathrm{val}}$ is the topological space defined in [17, Section 3.6.26]. This map factors through the quotient space $S^{\log}_{[:]}$ of $S^{\log}_{\mathrm{val}}$ as is seen by the arguments in Section 4.5.9. Since $S^{\log}_{\mathrm{val}} \to S_{[:]}$ is a proper surjective continuous map, the map $S_{[:]} \to \Gamma \setminus D^I_{\mathrm{SL}(2)}$ is continuous. The last map is compatible with log structures as is seen by the arguments in Section 5.6.4 and, hence, induces a continuous map $S^{\log}_{[\mathrm{val}]} \to \Gamma \setminus D^I_{\mathrm{SL}(2),\mathrm{val}}$.

By using Theorem 6.2.2, (2) is proved similarly to (1). □

In the rest of this section, we consider mild log mixed Hodge structures.

PROPOSITION 6.3.2

Let σ be a rational nilpotent cone (it is an $\mathbf{R}_{\geq 0}$-cone generated by rational elements) in $\mathfrak{g}_{\mathbf{R}}$. Assume that there is $F \in \check{D}$ such that (σ, F) generates a nilpotent orbit. If (W,N) splits for any rational element N of σ, then (W,N) splits for any element N of the cone σ.

Proof

We may assume that N is in the interior $\sigma_{>0}$ of σ. This is because if we denote by σ' the face of σ such that N belongs to the interior of σ', then $(\sigma', \exp(iN')F)$ generates a nilpotent orbit for some $N' \in \sigma_{>0}$ and hence we can replace σ by σ'.

Assume $N \in \sigma_{>0}$. Let $\hat{F}$ be the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure $(M(W,\sigma), F)$. Then $\exp(iN')\hat{F} \in D$ for any $N' \in \sigma_{>0}$. Hence, as the composition of the continuous map $D \to \mathcal{L}$, $x \mapsto \delta_W(x)$ and the continuous map $\sigma_{>0} \to D$, $N' \mapsto \exp(iN')\hat{F}$, the map $\sigma_{>0} \to \mathcal{L}$, $N' \mapsto$

$\delta_W(\exp(iN')\hat{F})$ is continuous. By the part (i) $\Rightarrow$ (vii) of Theorem 6.2.4, the last map sends all rational elements of $\sigma_{>0}$ to 0. Hence, it also sends N to 0. By the part (vii) $\Rightarrow$ (i) of Theorem 6.2.4, this shows that (W,N) splits. $\square$

6.3.3

Let $\mathcal{B}(\log)$ be the category of locally ringed spaces over $\mathbf{C}$ endowed with fs log structures satisfying a certain condition, defined in [17] (see [15, III, Section 1.1] for a review). Let S be an object of $\mathcal{B}(\log)$, and let H be a log mixed Hodge structure on S with polarized graded quotients for the weight filtration. By Proposition 6.3.2, for $s \in S$ and for $t \in S^{\log}$ lying over s, the following two conditions (i) and (ii) below are equivalent. Let

$$\pi_1^+(s^{\log}) := \mathrm{Hom}\big((M_S/\mathcal{O}_S^\times)_s, \mathbf{N}\big) \subset \pi_1(s^{\log}) = \mathrm{Hom}\big((M_S/\mathcal{O}_S^\times)_s, \mathbf{Z}\big),$$

$$\begin{aligned}\pi_1(s^{\log}, \mathbf{R}_{\geq 0}) &:= \mathrm{Hom}\big((M_S/\mathcal{O}_S^\times)_s, \mathbf{R}_{\geq 0}^{\mathrm{add}}\big) \subset \mathbf{R} \otimes \pi_1(s^{\log}) \\ &= \mathrm{Hom}\big((M_S/\mathcal{O}_S^\times)_s, \mathbf{R}^{\mathrm{add}}\big).\end{aligned}$$

Consider the action ρ of $\pi_1(s^{\log})$ on $H_{\mathbf{Z},t}$, and consider the homomorphism

$$\log(\rho) : \mathbf{R} \otimes \pi_1(s^{\log}) \to \mathrm{End}_{\mathbf{R}}(H_{\mathbf{R},t}), \qquad a \otimes \gamma \mapsto a \log\big(\rho(\gamma)\big).$$

Let W be the weight filtration on $H_{\mathbf{R},t}$.

(i) For any $\gamma \in \pi_1(s^{\log}, \mathbf{R}_{\geq 0})$, $(W, \log(\rho)(\gamma))$ splits.
(ii) For any $\gamma \in \pi_1^+(s^{\log})$, $(W, \log(\rho(\gamma)))$ splits.

We say that H is *mild* (we also say H is *of mild degeneration*) if the equivalent conditions (i) and (ii) are satisfied for any s and t.

LEMMA 6.3.4

Let S and H be as above, and assume H is mild. Let $S' \to S$ be a morphism in $\mathcal{B}(\log)$. Then the pullback of H to S' is mild.

This is clear.

PROPOSITION 6.3.5

Let S be a log smooth fs log analytic space over $\mathbf{C}$, and let H be a log mixed Hodge structure on S with polarized graded quotients for the weight filtration W. Then the following two conditions (i) and (ii) are equivalent.

(i) *H is mild.*

(ii) *For any smooth analytic curve C over $\mathbf{C}$ and any analytic map $f : C \to S$ such that the subset $f^{-1}(S \smallsetminus U)$ of C is finite, the pullback f^*H on C is mild. Here we endow C with the log structure associated to the finite subset $f^{-1}(S \smallsetminus U)$.*

If S is an algebraic variety over $\mathbf{C}$, then these conditions are equivalent to the modified version of the condition (ii) in which we take only smooth algebraic curves C in it.

Proof

By Lemma 6.3.4, we have (i) $\Rightarrow$ (ii). We prove (ii) $\Rightarrow$ (i). Assume (ii). Let $s \in S \smallsetminus U$, and let t be a point of $S^{\log}$ lying over s. Let $\gamma \in \pi_1^+(s^{\log})$. We prove that $(W, \log(\rho)(\gamma))$ splits.

Let σ be the face of $\pi_1^+(s^{\log})$, regarded as a monoid, such that γ belongs to the interior of σ. Then there are $s' \in S$ and $t' \in S^{\log}$ lying over s' and isomorphisms $\pi^+((s')^{\log}) \cong \sigma$ and $H_{\mathbf{R},t'} \cong H_{\mathbf{R},t}$ such that the action of $\pi_1^+(s^{\log})$ on $H_{\mathbf{R},t}$ and the action of $\pi_1^+((s')^{\log})$ on $H_{\mathbf{R},t'}$ are compatible via these isomorphisms. By this we are reduced to the case where γ belongs to the interior of $\pi_1^+(s^{\log})$.

Assume γ belongs to the interior of $\pi_1^+(s^{\log})$. Then there are a smooth analytic curve over $\mathbf{C}$, a morphism $f: C \to S$, and $s' \in C$ satisfying the following conditions (1)–(3). (1) $f(s') = s$. (2) $f^{-1}(S \smallsetminus U)$ is finite. (3) The image of $\pi_1^+((s')^{\log}) \to \pi_1^+(s^{\log})$ contains γ. By the condition (ii), this proves that $(W, \log(\rho)(\gamma))$ splits.

In the case where S is an algebraic variety, the same arguments show that the modified version of (ii) implies (i). $\square$

THEOREM 6.3.6

Let the assumptions be as in Theorem 6.3.1. Assume furthermore that H is mild.

(1) *The period map $\varphi: S \to \Gamma \backslash D$ extends uniquely to continuous maps*

$$S^{\log}_{[:]} \to \Gamma \backslash D^{\diamond}_{\mathrm{SL}(2)}, \qquad S^{\log}_{[:]} \to \Gamma \backslash D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)},$$

$$S^{\log}_{[\mathrm{val}]} \to \Gamma \backslash D^{\diamond}_{\mathrm{SL}(2),\mathrm{val}}, \qquad S^{\log}_{[\mathrm{val}]} \to \Gamma \backslash D^{\star,\mathrm{mild}}_{\mathrm{SL}(2),\mathrm{val}}, \qquad S^{\log}_{[\mathrm{val}]} \to \Gamma \backslash D^{\mathrm{mild}}_{\mathrm{BS},\mathrm{val}}.$$

(2) *For any point $s \in S$, there exist an open neighborhood V of s, a log modification V' of V (see [17, Definition 3.6.12]), a commutative subgroup Γ' of Γ, and a fan Σ in $\mathfrak{g}_{\mathbf{Q}}$ which is strongly compatible with Γ' such that the period map $\varphi|_{U \cap V}$ lifts to a morphism $U \cap V \to \Gamma' \backslash D$ which extends uniquely to a morphism $V' \to \Gamma' \backslash D^{\mathrm{mild}}_{\Sigma}$ of log manifolds.*

$$\begin{array}{ccccc} U & \supset & U \cap V & \subset & V' \\ {\scriptstyle\varphi}\downarrow & & \downarrow & & \downarrow \\ \Gamma \backslash D & \leftarrow & \Gamma' \backslash D & \subset & \Gamma' \backslash D^{\mathrm{mild}}_{\Sigma} \end{array}$$

Furthermore, we have the following.

(2.1) *Assume $S \smallsetminus U$ is a smooth divisor. Then we can take $V = V' = S$ and $\Gamma' = \Gamma$. That is, we have a commutative diagram*

$$\begin{array}{ccc} U & \subset & S \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ \Gamma \backslash D & \subset & \Gamma \backslash D^{\mathrm{mild}}_{\Sigma}. \end{array}$$

(2.2) *Assume that Γ is commutative. Then we can take $\Gamma' = \Gamma$.*

(2.3) *Assume that Γ is commutative and that the following condition (i) is satisfied.*

(i) *There is a finite family* $(S_j)_{1\le j\le n}$ *of connected locally closed analytic subspaces of* S *such that* $S=\bigcup_{j=1}^n S_j$ *as a set and such that, for each* j*, the inverse image of the sheaf* $M_S/\mathcal{O}_S^\times$ *on* S_j *is locally constant.*

Then we can take $\Gamma'=\Gamma$ *and* $V=S$*.*

Statements (1) and (2) are modified versions of [15, III, Theorem 7.5.1(1) and 7.5.1(2)], respectively, and (2) is proved in the same way as [15, III, Theorem 7.5.1(2)]. We can deduce (1) from (2) by using $D_\Sigma^{\sharp,\mathrm{mild}}\to D_{\mathrm{SL}(2)}^\diamond$ (see Theorem 5.1.10) by the arguments in the above proof of Theorem 6.3.1(1).

7. Relations with asymptotic behaviors of regulators and local height pairings

In this section, we show examples to describe the relations of this work to the work [5] on the asymptotic behaviors of regulators and local height pairings.

7.1. Example III

This is Example III in[15, I and II]. It appeared in [15, III] as the case $b=2$ of Section 7.1.3. As in Section 7.2 below, this example is related to the regulator of K_2 of a degenerating elliptic curve.

In this Example III and also in Example IV in Section 7.3 below, we compare D_{BS}, $D_{\mathrm{SL}(2)}^I$, $D_{\mathrm{SL}(2)}^{II}$, $D_{\mathrm{SL}(2)}^\star$, $D_{\mathrm{SL}(2)}^\diamond$, and their associated valuative spaces, by regarding them as topological spaces; that is, we forget the real analytic structures.

7.1.1

Let $H_0=\mathbf{Z}^3$ with basis e_1,e_2,e_3. The weight filtration is given by

$$W_{-4}=0\subset W_{-3}=\mathbf{R}e_1+\mathbf{R}e_2=W_{-1}\subset W_0=H_{0,\mathbf{R}}.$$

The intersection form on gr_{-3}^W is the antisymmetric form characterized by $\langle e_2, e_1\rangle=1$.

7.1.2

We have that $D(\mathrm{gr}^W)\cong\mathfrak{h}$, the upper half-plane, and $D_{\mathrm{SL}(2)}(\mathrm{gr}^W)=D_{\mathrm{BS}}(\mathrm{gr}^W)=\mathfrak{h}_{\mathrm{BS}}$.

7.1.3

We have a homeomorphism

$$D_{\mathrm{SL}(2),\mathrm{val}}^\star\overset{\cong}{\to}D_{\mathrm{BS},\mathrm{val}},$$

and this induces a homeomorphism

$$D_{\mathrm{SL}(2)}^\star\cong D_{\mathrm{BS}}$$

of quotient spaces. Let W' be the increasing filtration on gr^W given by

$$W'_{-5}=0\subset W'_{-4}=\mathbf{R}e_1=W'_{-3}\subset W'_{-2}=\mathrm{gr}_{-3}^W\subset W'_0=\mathrm{gr}^W,$$

and let $\Phi = \{W'\}$. Let P be the parabolic subgroup of $G_{\mathbf{R}}$ consisting of elements which preserve W'. Then, $D_{\mathrm{BS}}(P) = D^{\star}_{\mathrm{SL}(2)}(\Phi)$, and it is the inverse image of the open set $\{x+iy \mid x \in \mathbf{R}, y \in (0,\infty]\}$ of $\mathfrak{h}_{\mathrm{BS}}$ under the projection $D_{\mathrm{BS}} = D^{\star}_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}(\mathrm{gr}^W) = D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = \mathfrak{h}_{\mathrm{BS}}$.

We have (see [15, II, Section 3.6.2])

$$D^{I}_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)}.$$

So we denote $D^{I}_{\mathrm{SL}(2)}$ and $D^{II}_{\mathrm{SL}(2)}$ simply by $D_{\mathrm{SL}(2)}$.

7.1.4
Let

$$V := \mathbf{R}e_1 + \mathbf{R}e_2.$$

We have

$$\mathrm{spl}(W) \cong V, \qquad \mathcal{L} \cong V,$$

where $v \in V$ corresponds in the first isomorphism to the splitting of W given by $e_3 + v$, that is, $s \in \mathrm{spl}(W)$ such that $s(e_3(\mathrm{gr}^W_0)) = e_3 + v$, and $v \in V$ corresponds in the second isomorphism to $\delta \in \mathcal{L}$ such that $\delta(e_3(\mathrm{gr}^W_0)) = v$. We have $\mathcal{L}(F) = \mathcal{L}$ for any $F \in D(\mathrm{gr}^W)$.

7.1.5
We have homeomorphisms

$$D \cong \mathfrak{h} \times \mathcal{L} \times \mathrm{spl}(W) \cong \mathbf{R}_{>0} \times V \times \mathbf{R} \times V,$$

where the first isomorphism is $F \mapsto (F(\mathrm{gr}^W), \delta_W(F), \mathrm{spl}_W(F))$, and the second isomorphism sends $(x+iy, \delta, s)$ to (t, δ, x, s), where $x, y \in \mathbf{R}$, $y > 0$, and $t := 1/\sqrt{y}$, and we identify both $\mathcal{L}$ and $\mathrm{spl}(W)$ with V via the isomorphisms in Section 7.1.4. We call the composition $D \cong \mathbf{R}_{>0} \times V \times \mathbf{R} \times V$ the *standard isomorphism for D*. Let $\bar{V} = \bar{\mathcal{L}}$ be as in Section 1.3.8(4).

We have a commutative diagram of homeomorphisms

$$\begin{array}{ccc}
D^{\diamond,\mathrm{weak}}_{\mathrm{SL}(2)}(\Phi) & \cong & (\mathbf{R}_{\ge 0} \times V \times \mathbf{R} \times V)' \\
\uparrow & & \uparrow_{(1)} \\
D^{\diamond}_{\mathrm{SL}(2)}(\Phi) & \cong & (\mathbf{R}_{\ge 0} \times V \times \mathbf{R} \times V)' \\
\downarrow & & \downarrow_{(2)} \\
D^{\star}_{\mathrm{SL}(2)}(\Phi) & \cong & \mathbf{R}_{\ge 0} \times \bar{V} \times \mathbf{R} \times V \\
\uparrow & & \uparrow \\
D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi) & \cong & (\mathbf{R}_{\ge 0} \times \bar{V})_{\mathrm{val}} \times \mathbf{R} \times V \\
\downarrow & & \downarrow_{(3)} \\
D_{\mathrm{SL}(2),\mathrm{val}}(\Phi) & \cong & (\mathbf{R}_{\ge 0} \times \bar{V})_{\mathrm{val}} \times \mathbf{R} \times V \\
\downarrow & & \downarrow \\
D_{\mathrm{SL}(2)}(\Phi) & \cong & \mathbf{R}_{\ge 0} \times \bar{V} \times \mathbf{R} \times V,
\end{array}$$

where

$$(\mathbf{R}_{\geq 0}\times V\times\mathbf{R}\times V)':=\{(t,\delta,x,s)\in\mathbf{R}_{\geq 0}\times V\times\mathbf{R}\times V\mid \delta\in\mathbf{R}e_1 \text{ if } t=0\}$$

and where $(\mathbf{R}_{\geq 0}\times\bar{V})_{\mathrm{val}}$ is the valuative space of $\mathbf{R}_{\geq 0}\times\bar{V}$ associated to the canonical log structure (see a description below and also [17, Section 0.5.21]). The homeomorphism for $D^{\diamond,\mathrm{weak}}_{\mathrm{SL}(2)}(\Phi)$ is compatible with the standard isomorphism for D, but other homeomorphisms are *not* compatible with the standard isomorphism.

The homeomorphism for $D^{\diamond}_{\mathrm{SL}(2)}(\Phi)$ (resp., $D^{\star}_{\mathrm{SL}(2)}(\Phi)$, $D_{\mathrm{SL}(2)}(\Phi)$) sends a point of D corresponding to $(t,c_1e_1+c_2e_2,x,u)\in\mathbf{R}_{>0}\times V\times\mathbf{R}\times V$ under the standard isomorphism to $(t,c_1e_1+t^{-1}c_2e_2,x,u)$ (resp., $(t,tc_1e_1+t^{-1}c_2e_2,x,u)$, $(t,t^4c_1e_1+t^2c_2e_2,x,u)$). The homeomorphism for $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$ (resp., $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$) is compatible with the homeomorphism for $D^{\star}_{\mathrm{SL}(2)}(\Phi)$ (resp., $D_{\mathrm{SL}(2)}(\Phi)$).

Concerning the vertical arrows on the right-hand side, they are described as follows. The arrows without labels are the canonical projections (see Section 3.1). The map (1) sends $(t,c_1e_1+c_2e_2,x,u)$ to $(t,c_1e_1+tc_2e_2,x,u)$. The map (2) sends $(t,c_1e_1+c_2e_2,x,u)$ to $(t,tc_1e_1+c_2e_2,x,u)$. The map (3) is explained below.

The valuative space $(\mathbf{R}_{\geq 0}\times\bar{V})_{\mathrm{val}}$ is described as follows. Over $U=(\mathbf{R}_{>0}\times\bar{V})\cup(\mathbf{R}_{\geq 0}\times V)\subset\mathbf{R}_{\geq 0}\times\bar{V}$, it is U. The inverse image of $\{0\}\times(\bar{V}\smallsetminus V)$ in $(\mathbf{R}_{\geq 0}\times\bar{V})_{\mathrm{val}}$ consists of points

(a) $p(0,\lambda)$ $(\lambda\in\bar{V}\smallsetminus V)$,
(b) $p(c,\lambda)$ $(c\in\mathbf{R}_{>0}\smallsetminus\mathbf{Q}_{>0},\ \lambda\in\bar{V}\smallsetminus V)$,
(c) $p(c+,\lambda)$ $(c\in\mathbf{Q}_{\geq 0},\ \lambda\in\bar{V}\smallsetminus V)$,
(d) $p(c-,\lambda)$ $(c\in\mathbf{Q}_{>0},\ \lambda\in\bar{V}\smallsetminus V)$,
(e) $p(c,\mu)$ $(c\in\mathbf{Q}_{>0},\ \mu\in V\smallsetminus\{0\})$.

Write $\lambda=0\circ\mu$ with $\mu\in V\smallsetminus\{0\}$ (see Section 1.3.8(4)). Then the above point is the limit of $t^{c'}\mu$, where $t>0$, $t\to 0$, and in the cases of (b) and (e) (resp., case (a), case (c), case (d)), $c'=c$ (resp., $c'\to\infty$, $c'>c$ and $c'\to c$, $c'<c$ and $c'\to c$).

The map (3) sends (t,δ,x,u) $((t,\delta)\in(\mathbf{R}_{>0}\times\bar{V})\cup(\mathbf{R}_{\geq 0}\times V))$ to $(t,t^3\delta,x,u)$; $(p(0,\lambda),x,u)$ to $(p(0,\lambda),x,u)$; $(p(c,\alpha),x,u)$ $(\alpha\in\bar{V})$ to $(p(c-3,\alpha),x,u)$ if $c>3$, to $(0,\alpha,x,u)$ if $c=3$, and to $(0,0,x,u)$ if $0<c<3$; $(p(c+,\lambda),x,u)$ to $(p((c-3)+,\lambda),x,u)$ if $c\geq 3$ and to $(0,0,x,u)$ if $0\leq c<3$; and $(p(c-,\lambda),x,u)$ to $(p((c-3)-,\lambda),x,u)$ if $c>3$ and to $(0,0,x,u)$ if $0<c\leq 3$.

7.1.6

We describe for Example III

(1) that there is no continuous map $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)\to D_{\mathrm{BS}}(P)=D^{\star}_{\mathrm{SL}(2)}(\Phi)$ which extends the identity map of D, and

(2) how $\eta: D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)\to D_{\mathrm{BS},\mathrm{val}}(P)=D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$ is not continuous.

Fixing $c_1,c_2\in\mathbf{R}$, for $t>0$, let $p(t)$ be the point of D corresponding to $(t,c_1e_1+c_2e_2,0,0)$ via the homeomorphism for $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$ in Section 7.1.5.

Then, via the homeomorphism for $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$, $p(t)$ corresponds to $(t, t^3c_1e_1 + t^3c_2e_2, 0, 0)$.

Hence, when $t \to 0$, $p(t)$ converges in $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$ to the point p corresponding to $(0,0,0,0)$, but it converges in $D^{\star}_{\mathrm{SL}(2)}(\Phi)$ to the point that corresponds to $(0, c_1e_1 + c_2e_2, 0, 0)$, which depends on (c_1, c_2). This explains (1). For (2), the image of p under η is the point p' of $D^{\star}_{\mathrm{SL}(2)}(\Phi)$ corresponding to $(0,0,0,0)$. If $(c_1, c_2) \neq (0,0)$, then $p(t)$ does not converge to p' in $D^{\star}_{\mathrm{SL}(2)}(\Phi)$.

7.1.7

As is mentioned in Section 5.1.12, the topology of $D^{\diamond,\mathrm{weak}}_{\mathrm{SL}(2)}$ does not coincide with the one of $D^{\diamond}_{\mathrm{SL}(2)}$. Fixing $c \in \mathbf{R}$, for $t > 0$, let $p(t)$ be the point of D corresponding to $(t, tce_2, 0, 0)$ via the homeomorphism for $D^{\diamond,\mathrm{weak}}_{\mathrm{SL}(2)}(\Phi)$ in Section 7.1.5. Then, when $t \to 0$, $p(t)$ converges in $D^{\diamond,\mathrm{weak}}_{\mathrm{SL}(2)}(\Phi)$ to the point corresponding to $(0,0,0,0)$. On the other hand, $p(t)$ corresponds to $(t, ce_2, 0, 0)$ under the homeomorphism for $D^{\star}_{\mathrm{SL}(2)}(\Phi)$ and hence converges in $D^{\star}_{\mathrm{SL}(2)}(\Phi)$, but the limit depends on the choice of c.

7.1.8

The open set $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}(\Phi)$ of $D^{\star}_{\mathrm{SL}(2)}(\Phi)$ is the part consisting of elements corresponding to (t, δ, x, u) such that $\delta \in V \subset \bar{V}$. The map $D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}(\Phi) \to D_{\mathrm{SL}(2)}(\Phi)$ corresponds to $(t, \delta, x, u) \mapsto (t, t^3\delta, x, u)$. It does not extend to a continuous map $D^{\star}_{\mathrm{SL}(2)} \to D_{\mathrm{SL}(2)}$. In fact, fixing $v \in V \smallsetminus \{0\}$, let $p(t)$ for $t > 0$ be the point of D corresponding to $(t, t^{-3}v, 0, 0)$ via the homeomorphism for $D^{\star}_{\mathrm{SL}(2)}(\Phi)$. Then when $t \to 0$, $p(t)$ converges to the point of $D^{\star}_{\mathrm{SL}(2)}(\Phi)$ corresponding to $(0, 0 \circ v, 0, 0)$ (see Section 1.3.8(4)). But $p(t)$ converges to the point of $D_{\mathrm{SL}(2)}(\Phi)$ corresponding to $(0, v, 0, 0)$, which depends on the choice of v.

7.1.9

Let $a \in \mathbf{Q}_{>0}$, and define $N_a \in \mathfrak{g}_{\mathbf{Q}}$ by $N_a(e_3) = ae_2$, $N_a(e_2) = e_1$, and $N_a(e_1) = 0$. For $b \in \mathbf{R}$, let $F_b \in \check{D}$ be the decreasing filtration defined as follows: $F_b^1 = 0$, F_b^0 is generated by $e_3 + ibe_1$, F_b^{-1} is generated by F_b^0 and e_2, and F_b^{-2} is the total space. Then (N_a, F_b) generates a nilpotent orbit. Let $\sigma_a = \mathbf{R}_{\geq 0}N_a$. Then $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}})F_b) \in D^{\sharp}_{\sigma_a}$ is the limit of $\exp(iyN_a)F_b$ for $y \to \infty$. This $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}})F_b)$ belongs to $D^{\sharp,\mathrm{mild}}_{\sigma_a}$ (see Section 5.1.4) if and only if $a = 0$.

We consider the image of $\exp(iyN_a)F_b \in D$ in $\mathbf{R}_{>0} \times V \times \mathbf{R} \times V$ under the isomorphism in Section 7.1.5. Let $t = 1/\sqrt{y}$. In the standard isomorphism for D, the image is $(t, at^{-2}e_2 + be_1, 0, -(b/2)t^2e_2)$. (The last component is computed by using the relation of δ and ζ; see Section 1.2.5.) In the homeomorphism for $D^{\diamond}_{\mathrm{SL}(2)}$, the image is $(t, at^{-3}e_2 + be_1, 0, -(b/2)t^2e_2)$. In the homeomorphism for $D^{\star}_{\mathrm{SL}(2)}$,the image is $(tat^{-3}e_2 + bte_1, 0, -(b/2)t^2e_2)$. In the homeomorphism for $D_{\mathrm{SL}(2)}$, the image is $(t, ae_2 + bt^4e_1, 0, -(b/2)t^2e_2)$. By taking the limit for $t \to 0$, we have the following.

LEMMA 7.1.10

(1) *If* $a \neq 0$, *then the image of* $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}})F_b) \in D^{\sharp}_{\sigma_a}$ *in* $D^{\star}_{\mathrm{SL}(2)}(\Phi)$ *(resp.,* $D_{\mathrm{SL}(2)}(\Phi)$, $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$, $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$*) has the coordinate* $(0, \infty e_2, 0, 0)$ *(resp.,* $(0, ae_2, 0, 0)$, $(p(3, ae_2), 0, 0)$, $(0, ae_2, 0, 0)$*).*

(2) *If* $a = 0$, *then the image of* $(\sigma_0, \exp(i\sigma_{0,\mathbf{R}})F_b) \in D^{\sharp,\mathrm{mild}}_{\sigma_0}$ *in* $D^{\diamond}_{\mathrm{SL}(2)}(\Phi)$ *(resp.,* $D^{\star}_{\mathrm{SL}(2)}(\Phi)$, $D_{\mathrm{SL}(2)}(\Phi)$, $D^{\star}_{\mathrm{SL}(2)}(\Phi)$, $D_{\mathrm{SL}(2)}(\Phi)$*) has the coordinate* $(0, be_1, 0, 0)$ *(resp.,* $(0,0,0,0)$, $(0,0,0,0)$, $(0,0,0,0)$ *if* $b \neq 0$ *and* $(0,0,0,0)$ *if* $b = 0$, $(0,0,0,0)$*).*

7.1.11

By Lemma 7.1.10, we have the following. Consider the image p of $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}}) F_b) \in D^{\sharp}_{\sigma_a}$ in one of $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D^{\star}_{\mathrm{SL}(2)}$, or $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$. In the case $a = 0$, consider also the image in $D^{\diamond}_{\mathrm{SL}(2)}$.

(1) p remembers a in the cases of $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, and $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$, but p does not remember a in the other cases. In the case $a \neq 0$, p does not remember b in any of these cases.

(2) Assume $a = 0$. Then p remembers b in the case of $D^{\diamond}_{\mathrm{SL}(2)}$, but p does not remember b in all other cases.

$$\begin{array}{ccc} D^{\sharp,\mathrm{mild}}_{\Xi} & \to & D^{\diamond}_{\mathrm{SL}(2)} \\ \downarrow & & \downarrow \\ D^{\sharp}_{\Xi} & \to & D_{\mathrm{SL}(2)} \end{array}$$

7.2. Degeneration and regulator maps

7.2.1

Let X be a proper smooth variety over $\mathbf{C}$. Let $n \geq 1$ and $r \geq 0$. Then we have the (rth) regulator map (see [1])

$$\mathrm{reg}_X : K_n(X) \to \bigoplus_{p,q} \bigl(H^m(X)(r)_{\mathbf{C},p,q}\bigr)^-,$$

where $m = 2r - n - 1$ and (p,q) ranges over all elements of $\mathbf{Z}^2$ such that $p + q = m - 2r$, $p < 0$, $q < 0$, $H^m(X)(r)_{\mathbf{C},p,q}$ is the (p,q)-Hodge component of $H^m(X, \mathbf{C})$ with respect to the Hodge structure $H^m(X)(r)$, and $(\cdot)^-$ denotes the minus part for the complex conjugation which fixes the image of $H^m(X, \mathbf{Z}(r)) = H^m(X, \mathbf{Z}) \otimes (2\pi i)^r \mathbf{Z}$.

This regulator map is understood as δ (see Section 1.2) of a mixed Hodge structure as follows. An element $Z \in K_n(X)$ determines a mixed Hodge structure H_Z with an exact sequence $0 \to H^m(X)(r) \to H_Z \to \mathbf{Z} \to 0$. We have

$$\mathrm{reg}_X(Z) = \delta_W(H_Z),$$

where W is the weight filtration of H_Z.

7.2.2

Let $X \to S$, $0 \in S$, and n, r, m be as in Section 0.3, and let $Z \in K_n(X \smallsetminus X_0)$. For $t \in S \smallsetminus \{0\}$, let $Z(t) \in K_n(X_t)$ be the pullback of Z.

Then the regulator $\mathrm{reg}_{X_t}(Z(t))$ is understood as $\delta_W(H_Z(t))$, where H_Z denotes the variation of mixed Hodge structure on $S \smallsetminus \{0\}$ defined by Z which has an exact sequence $0 \to H^m(X/S)(r) \to H_Z \to \mathbf{Z} \to 0$ with $H^m(X/S)$ as in Section 0.3 and whose fiber $H_Z(t)$ at t is the mixed Hodge structure in Section 7.2.1 associated to $Z(t)$. This H_Z is admissible along X_0 and extends uniquely to a log mixed Hodge structure on S, which we denote by the same letter H_Z. Hence, the behavior of $t \mapsto \mathrm{reg}_{X_t}(Z(t))$ in the degeneration is explained by the theory of degeneration of mixed Hodge structure as in this article. For the details of what follows, see [5].

PROPOSITION 7.2.3

Assume Z comes from $K_n(X)$. Then the log mixed Hodge structure H_Z on S is mild.

Proof

The Clemens–Schmid sequence $H^m(X_0, \mathbf{Q}) \to H^m(X/S)_{\mathbf{Q},t} \xrightarrow{N} H^m(X/S)_{\mathbf{Q},t} \to H_{2d-m}(X_0, \mathbf{Q})$ ($t \in S \smallsetminus \{0\}$ is near to 0) induces an injection $H^m(X/S)_{\mathbf{Q},t}/NH^m(X/S)_{\mathbf{Q},t} \to H_{2d-m}(X_0, \mathbf{Q})$. Here N is the monodromy logarithm of $H_{Z,\mathbf{Q}}$ at $0 \in S$. We have a commutative diagram

$$\begin{array}{ccc} K_n(X \smallsetminus X_0) & \xrightarrow{\partial} & K'_{n-1}(X_0) \\ \downarrow & & \downarrow \\ H^m(X/S)_{\mathbf{Q},t}/NH^m(X/S)_{\mathbf{Q},t} & \hookrightarrow & H_{2d-m}(X_0, \mathbf{Q}) \end{array}$$

Here the left vertical arrow sends $Z \in K_n(X \smallsetminus X_0)$ to Ne, where e is the lifting of $1 \in \mathbf{Q}$ to $H_{Z,\mathbf{Q},t}$ under the exact sequence $0 \to H^m(X/S)_{\mathbf{Q},t} \to H_{Z,\mathbf{Q},t} \to \mathbf{Q} \to 0$. K'_{n-1} denotes the K-group of coherent sheaves. The right vertical arrow is the topological Chern class map. By the localization theory of K-theory, we have an exact sequence $K_n(X) \to K_n(X \smallsetminus X_0) \xrightarrow{\partial} K'_{n-1}(X_0)$.

Assume $Z \in K_n(X \smallsetminus X_0)$ comes from $K_n(X)$. Then $\partial(Z) = 0$, and hence, the above diagram shows that the image of Z in $H^m(X/S)_{\mathbf{Q},t}/NH^m(X/S)_{\mathbf{Q},t}$ is 0. This proves that (W, N) splits. □

By Proposition 7.2.3 and Theorem 5.1.7, we have the following result.

THEOREM 7.2.4

If $Z \in K_n(X \smallsetminus X_0)$ comes from $K_n(X)$, then the regulator $\mathrm{reg}_{X_t} Z(t)$ ($t \in S \smallsetminus \{0\}$) converges in $\mathcal{L}$ when $t \to 0$.

REMARK 7.2.5

In [5], this result will be generalized to the situation in which S need not be of dimension at most 1. This generalization will be reduced to Theorem 7.2.4 by using Proposition 6.3.5.

7.2.6

Let $X \to S$ and $0 \in S$ be as in Section 0.3. Take $H_0 = \mathbf{Z} \oplus H^m(X/S)(r)_{\mathbf{Z},t}$. The extended period maps in Section 6.3 give a commutative diagram

$$\begin{array}{ccc} S^{\log} \times K_n(X) & \to & \Gamma \backslash D^{\diamond}_{\mathrm{SL}(2)} \\ \downarrow & & \downarrow \\ S^{\log} \times K_n(X \smallsetminus X_0) & \to & \Gamma \backslash D_{\mathrm{SL}(2)} \end{array}$$

Here Γ is the group of all elements γ of $\mathrm{Aut}_{\mathbf{Z}}(H_0)$ satisfying the following conditions. (i) γ preserves $H^m(X/S)(r)_{\mathbf{Z},t}$. (ii) $\gamma e \equiv e \bmod H^m(X/S)(r)_{\mathbf{Z},t}$, where e denotes $(1,0) \in H_0$. (iii) The action of γ on $H^m(X/S)_{\mathbf{Z},t}$ is contained in the monodromy group of $H^m(X/S)_{\mathbf{Z}}$.

7.2.7

We give an explicit example. Assume that $X \to S$ is a family of elliptic curves which degenerates at $0 \in S$, and assume $n = r = 2$. Then the period domain and the extended period domains which appear here are those of Example III (see Section 7.1). Let the notation be as in Section 7.1.

We discuss an explicit example of $Z \in K_2(X \smallsetminus X_0)$. Let α_j and β_k be a finite number of torsion sections of $X \smallsetminus X_0$ over $S \smallsetminus \{0\}$, let $m_j, n_k \in \mathbf{Z}$ such that $\sum_j m_j = \sum_k n_k = 0$, and consider the divisors $\alpha = \sum_j m_j(\alpha_j)$, $\beta = \sum_k n_k(\beta_k)$ on $X \smallsetminus X_0$ of degree 0. Then we have an element $Z_{\alpha,\beta} \in K_2(X \smallsetminus X_0)$ (see [4], [20]). It is essentially the Steinberg symbol $\{f_\alpha, f_\beta\}$, where f_α (resp., f_β) is an element of $\mathbf{Q} \otimes \mathbf{C}(X)^\times$ whose divisor is α (resp., β). When t tends to 0 in S, there are $a, b \in W_{-3}H_{0,\mathbf{R}}$ such that we have

$$\mathrm{reg}_{X_t}\bigl(Z_{\alpha,\beta}(t)\bigr) = ay + b + O(y^{-1}),$$

where y is defined by $q(t) = e^{2\pi i(x+iy)}$ $(x, y \in \mathbf{R})$ with $q(t)$ the q-invariant of the elliptic curve X_t.

We have

$$a \equiv \sum_{j,k} m_j n_k B_3\bigl(\{r(\alpha_j) - r(\beta_k)\}\bigr) e_2 \bmod \mathbf{R}e_1,$$

where B_3 is the Bernoulli polynomial of degree 3; $r(\mu)$ for a torsion section μ is the element of $\mathbf{Q}/\mathbf{Z}$ such that, as a section of the Tate curve $\mathbf{G}_m/q^{\mathbf{Z}}$, μ is expressed as $sq^{r(\mu)} \bmod q^{\mathbf{Z}}$ with s a root of 1; and $\{\cdot\} : \mathbf{Q}/\mathbf{Z} \to [0,1) \subset \mathbf{Q}$ is the lifting.

Assume now that $r(\alpha_j) = r(\beta_k) = 0$ for any j, k, that is, these torsion sections α_j and β_k are roots of 1 in the Tate curve $\mathbf{G}_m/q^{\mathbf{Z}}$. Then $Z_{\alpha,\beta}$ comes from $K_2(X)$, $a = 0$, and the degeneration is mild. In this case,

$$b = \sum_{j,k} m_j n_k D(\alpha_j/\beta_k) e_1,$$

where we regard α_j and β_k as roots of 1 and D is the real analytic modified dilogarithm function of Bloch–Wigner (see [4]).

These things will be explained in [5] by using results in [20], [4], and [10] and using the results of this article.

7.3. Example IV

This is Example IV in [15, II]. As is explained in [15, II, Section 4.4] and also in Section 7.4 below, this example is related to the local height pairing of points of a degenerating elliptic curve.

7.3.1

Let $H_0 = \mathbf{Z}^4$ with basis e_1, e_2, e_3, e_4. The weight filtration is given by

$$W_{-3} = 0 \subset W_{-2} = \mathbf{R}e_1 \subset W_{-1} = \bigoplus_{j=1}^{3} \mathbf{R}e_j \subset W_0 = H_{0,\mathbf{R}}.$$

The intersection form on gr^W_{-1} is the antisymmetric form characterized by $\langle e_3, e_2\rangle = 1$.

7.3.2

We have $D(\mathrm{gr}^W) \cong \mathfrak{h}$, the upper half-plane, and $D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = D_{\mathrm{BS}}(\mathrm{gr}^W) = \mathfrak{h}_{\mathrm{BS}}$.

7.3.3

We have a homeomorphism

$$D^{\star}_{\mathrm{SL}(2),\mathrm{val}} \overset{\cong}{\to} D_{\mathrm{BS},\mathrm{val}},$$

and this induces a homeomorphism

$$D^{\star}_{\mathrm{SL}(2)} \cong D_{\mathrm{BS}}$$

of quotient spaces. Let W' be the increasing filtration on gr^W given by

$$W'_{-3} = 0 \subset W'_{-2} = \mathrm{gr}^W_{-2} + \mathbf{R}(e_2 \bmod W_{-2}) = W'_{-1} \subset W'_0 = \mathrm{gr}^W,$$

and let $\Phi = \{W'\}$. Let P be the parabolic subgroup of $G_{\mathbf{R}}$ consisting of elements which preserve W'. Then, $D_{\mathrm{BS}}(P) = D^{\star}_{\mathrm{SL}(2)}(\Phi)$, and it is the inverse image of the open set $\{x + iy \mid x \in \mathbf{R}, y \in (0,\infty]\}$ of $\mathfrak{h}_{\mathrm{BS}}$ under the projection $D_{\mathrm{BS}} = D^{\star}_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}(\mathrm{gr}^W) = D_{\mathrm{SL}(2)}(\mathrm{gr}^W) = \mathfrak{h}_{\mathrm{BS}}$.

We have (see [15, II, Section 4.4])

$$D^{I}_{\mathrm{SL}(2)} = D^{II}_{\mathrm{SL}(2)}.$$

We will denote both of them by $D_{\mathrm{SL}(2)}$. We have a canonical homeomorphism

$$D^{\diamond}_{\mathrm{SL}(2)} \overset{\cong}{\to} D^{\star,\mathrm{mild}}_{\mathrm{SL}(2)}.$$

7.3.4

We have

$$\mathrm{spl}(W) \cong \mathbf{R}^5, \qquad \mathcal{L} \cong \mathbf{R},$$

where in the first isomorphism $(s_{3,4}, s_{2,4}, s_{1,4}, s_{1,3}, s_{1,2}) \in \mathbf{R}^5$ corresponds to the splitting of W, which is given by $e_4 + \sum_{j=1}^{3} s_{j,4}e_j$ and $e_k + s_{1,k}e_1$ $(k = 2, 3)$, and the second isomorphism is given by $\delta \mapsto r$ $(\delta \in \mathcal{L},\ r \in \mathbf{R})$, $\delta e_4 = re_1$. We have $\mathcal{L}(F) = \mathcal{L}$ for any $F \in D(\mathrm{gr}^W)$.

7.3.5
We have homeomorphisms

$$D \cong \mathfrak{h} \times \mathcal{L} \times \mathrm{spl}(W) \cong \mathbf{R}_{>0} \times \mathbf{R} \times \mathbf{R}^6,$$

where the left isomorphism is $F \mapsto (F(\mathrm{gr}^W), \delta_W(F), \mathrm{spl}_W(F))$, and the second isomorphism sends $(x+iy, \delta, s)$ to $(1/\sqrt{y}, \delta, x, s)$, where $x, y \in \mathbf{R}$, $y > 0$. We call the composite homeomorphism $D \cong \mathbf{R}_{>0} \times \mathbf{R} \times \mathbf{R}^6$ the *standard isomorphism for D*.

We have a commutative diagrams of homeomorphisms

$$\begin{array}{ccc}
D^{\star}_{\mathrm{SL}(2)}(\Phi) & \cong & \mathbf{R}_{\geq 0} \times [-\infty, \infty] \times \mathbf{R}^6 \\
\uparrow & & \uparrow \\
D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi) & \cong & (\mathbf{R}_{\geq 0} \times [-\infty, \infty])_{\mathrm{val}} \times \mathbf{R}^6 \\
\downarrow & & \downarrow (1) \\
D_{\mathrm{SL}(2),\mathrm{val}}(\Phi) & \cong & (\mathbf{R}_{\geq 0} \times [-\infty, \infty])_{\mathrm{val}} \times \mathbf{R}^6 \\
\downarrow & & \downarrow \\
D_{\mathrm{SL}(2)}(\Phi) & \cong & \mathbf{R}_{\geq 0} \times [-\infty, \infty] \times \mathbf{R}^6,
\end{array}$$

where the upper two homeomorphisms are compatible with the standard isomorphism for D, but via the lower two homeomorphisms, a point of D corresponding to $(t, \delta, u) \in \mathbf{R}_{>0} \times \mathbf{R} \times \mathbf{R}^6$ under the standard isomorphism for D is sent to $(t, t^2\delta, u)$. For the vertical arrows on the right-hand side, all the arrows except (1) are the canonical projections, and the arrow (1) is as follows. The map (1) sends (t, δ, u) $((t, \delta) \in (\mathbf{R}_{\geq 0} \times \mathbf{R}) \cup (\mathbf{R}_{>0} \times [-\infty, \infty]))$ to $(t, t^2\delta, u)$; $(p(c, \pm\infty), u)$ $(c \in \mathbf{R}_{>0} \smallsetminus \mathbf{Q}_{>0})$ to $(p(c-2, \pm\infty), u)$ if $c > 2$ and to $(0, 0, u)$ if $c < 2$; $(p(c+, \pm\infty), u)$ $(c \in \mathbf{Q}_{\geq 0})$ to $(p((c-2)+, \pm\infty), u)$ if $c \geq 2$ and to $(0, 0, u)$ if $c < 2$; $(p(c-, \pm\infty), u)$ $(c \in \mathbf{Q}_{>0})$ to $(p((c-2)-, \pm\infty), u)$ if $c > 2$ and to $(0, 0, u)$ if $c \leq 2$; and $(p(c, \delta), u)$ $(c \in \mathbf{Q}_{>0},\ \delta \in \mathbf{R} \smallsetminus \{0\})$ to $(p(c-2, \delta), u)$ if $c > 2$, to $(0, \delta, u)$ if $c = 2$, and to $(0, 0, u)$ if $c < 2$.

Here the notation $p(c, \delta)$ and so on are understood as in Section 7.1.5 by replacing $\bar{V} \supset V$ by $[-\infty, \infty] \supset \mathbf{R}$.

7.3.6
Let $a \in \mathbf{Q}_{\geq 0}$, and define $N_a \in \mathfrak{g}_{\mathbf{Q}}$ by $N_a(e_4) = ae_1$, $N_a(e_3) = e_2$, and $N_a(e_1) = N_a(e_2) = 0$. Let σ_a be the cone generated by N_a. For $b \in \mathbf{R}$, let $F_b \in \check{D}$ be the decreasing filtration defined as follows: $F_b^1 = 0$, F_b^0 is generated by e_3 and $e_4 + ibe_1$, and F_b^{-1} is the total space.

In $D^{\sharp}_{\sigma_a}$, we have the limit $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}})F_b) \in D^{\sharp}_{\sigma_a}$ of $\exp(iyN_a)F_b$ for $y \to \infty$. This $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}})F_b)$ belongs to $D^{\sharp,\mathrm{mild}}_{\sigma_a}$ if and only if $a = 0$.

For $y \in \mathbf{R}_{>0}$, via the first homeomorphism in the diagram in Section 7.3.5, $\exp(iyN_a)F_b \in D$ is sent to $(1/\sqrt{y}, ay + b, \mathbf{0})$, and hence, the limit $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}})F_b) \in D^{\sharp}_{\sigma_a}$ is sent to $(0, \infty, \mathbf{0})$ in the case $a \neq 0$ and to $(0, b, \mathbf{0})$ in the case $a = 0$. Here $\mathbf{0}$ denotes $(0, \dots, 0) \in \mathbf{R}^6$. On the other hand, for $y \in \mathbf{R}_{>0}$, via the last homeomorphism in the diagram in Section 7.3.5, $\exp(iyN_a)F_b \in D$ is sent to $(1/\sqrt{y}, a + y^{-1}b, \mathbf{0})$, and hence, the limit $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}})F_b) \in D^{\sharp}_{\sigma_a}$ is sent to $(0, a, \mathbf{0})$. By taking the limit for $y \to \infty$, we have the following result.

LEMMA 7.3.7

The limit of $\exp(iyN_a)F_b$ *for* $y \to \infty$ *exists both in* $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$ *and in* $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$. *In the case* $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$, *the* $(\mathbf{R}_{\geq 0} \times [-\infty, \infty])_{\mathrm{val}}$*-component of the limit is* $p(2,a)$ *if* $a \neq 0$ *and is* $(0,b)$ *if* $a = 0$. *In the case* $D_{\mathrm{SL}(2),\mathrm{val}}(\Phi)$, *the* $(\mathbf{R}_{\geq 0} \times [-\infty, \infty])_{\mathrm{val}}$*-component of the limit is* $p(0,a)$.

7.3.8

By Lemma 7.3.7, we have the following. Let p be the image of $(\sigma_a, \exp(i\sigma_{a,\mathbf{R}})F_b) \in D^{\sharp}_{\sigma_a}$ in one of $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D^{\star}_{\mathrm{SL}(2)}$, or $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$. We have the following.

(1) p remembers a in the cases of $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, and $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$, whereas p does not remember a in the case of $D^{\star}_{\mathrm{SL}(2)}$. In the case $a \neq 0$, p does not remember b in any of these cases.

(2) Assume $a = 0$. Then p remembers b in the cases of $D^{\star}_{\mathrm{SL}(2)}$ and $D^{\star}_{\mathrm{SL}(2),\mathrm{val}}$, but p does not remember b in the cases of $D_{\mathrm{SL}(2)}$ and $D_{\mathrm{SL}(2),\mathrm{val}}$.

7.3.9

The following is mentioned in Remark 6.2.3. Though σ_a is of rank 1, the image p of $D^{\sharp}_{\sigma_a}$ in $D^{\star}_{\mathrm{SL}(2)}$ is not contained in $D^{\star}_{\mathrm{SL}(2),\leq 1}$ (i.e., the part of $D^{\star}_{\mathrm{SL}(2)}$ at which the log structure M satisfies $\mathrm{rank}(M^{\mathrm{gp}}/\mathcal{O}^{\times})_p \leq 1$) if $a \neq 0$. Indeed, in the case $a \neq 0$, the image of class $(N_a, F_b) \in D^{\sharp}_{\sigma_a}$ in $D^{\star}_{\mathrm{SL}(2)}$ has the coordinate $(0, \infty, \mathbf{0})$, which shows that $(M/\mathcal{O}^{\times})_p \cong \mathbf{N}^2$.

7.4. Degeneration and height pairings

We explain that our spaces are related to the asymptotic behavior of the Archimedean height pairing for algebraic cycles in degeneration (cf. [18], [12], [8]).

7.4.1

Let X be a proper smooth algebraic variety over $\mathbf{C}$ of dimension d, and let Y and Z be algebraic cycles on X of codimension r and s, respectively. We assume that $r + s = d + 1$, that their supports are disjoint $|Y| \cap |Z| = \emptyset$, and that both Y and Z are homologically equivalent to 0. Then we have a height pairing $\langle Y, Z\rangle_X \in \mathbf{R}$ (the local version of the height pairing for a number field at an Archimedean place; see [2], [3]).

This height pairing is understood as δ (see Section 1.2) of a mixed Hodge structure. We have

$$\langle Y, Z\rangle_X = \delta_W(H_{Y,Z}),$$

where $H_{Y,Z}$ is the mixed Hodge structure whose weight filtration W has the following properties: $W_0H_{Y,Z} = H_{Y,Z}$, $W_{-3} = 0$, $\mathrm{gr}^W_0 = \mathbf{Z}$, $\mathrm{gr}^W_{-2} = \mathbf{Z}(1)$, $\mathrm{gr}^W_{-1} = H^{2r-1}(X)(r)$, constructed in [2] and [3]. The exact sequence $0 \to H^{2r-1}(X)(r) \to W_0/W_{-2} \to \mathbf{Z} \to 0$ is given by the class of Y, and the exact sequence $0 \to \mathbf{Z}(1) \to W_{-1} \to H^{2r-1}(X)(r) \to 0$ is given by the class of Z.

7.4.2
Let $X \to S$ and $0 \in S$ be as in Section 0.3. Let Y and Z be algebraic cycles on X of codimension r and s, respectively, such that $r+s=d+1$, where d is the relative dimension of $X \to S$, such that $|Y| \cap |Z| = \emptyset$, and such that both $Y(t)$ and $Z(t)$ are homologically equivalent to 0 for any $t \in S \smallsetminus \{0\}$.

7.4.3
Since the height pairing $\langle Y(t), Z(t)\rangle$ is understood as δ of a mixed Hodge structure (see Section 7.4.1), its behavior in the degeneration is explained by the theory of degeneration of mixed Hodge structure as in this article.

When $t \to 0$ with x fixed, there are $a, b \in \mathbf{R}$ such that we have

$$\big\langle Y(t), Z(t)\big\rangle_{X_t} = ay + b + O(y^{-1}),$$

where taking a local coordinate q on S at 0 such that $q(0)=0$, we define y by $q = e^{2\pi i(x+iy)}$ $(x, y \in \mathbf{R}, y>0)$. Here, $a=0$ if and only if the degeneration of $H_{Y,Z}$ at $0 \in S$ is mild. It is known that a is the local geometric intersection number of Y and Z over $0 \in S$.

7.4.4
We give an explicit example. Assume that $X \to S$ is a family of degenerating elliptic curves, and assume that $r=s=1$. Let $Y = \sum_j m_j(\alpha_j)$ and $Z = \sum_k n_k(\beta_k)$, where α_j and β_k are closures in X of torsion sections of $X \smallsetminus X_0 \to S \smallsetminus \{0\}$, $m_j, n_k \in \mathbf{Z}$, $\sum_j m_j = \sum_k n_k = 0$. We assume that the divisors α_j and β_k of X do not intersect for any pair (j,k).

This is an example discussed at the end of [15, II, Section 4.4]. The extended period domains which appear here are those of Example IV (see Section 7.3).

We have

$$a = \sum_{j,k} m_j n_k B_2\big(\{r(\alpha_j) - r(\beta_k)\}\big),$$

where B_2 is the Bernoulli polynomial of degree 2. (The notation is as in Section 7.2.7.) This was explained in [15, II, Proposition 4.4.8].

If $r(\alpha_j) = r(\beta_k) = 0$ for any j, k, then $a=0$ and the degeneration is mild. In this case,

$$b = \sum_{j,k} m_j n_k l(\alpha_j/\beta_k),$$

where we regard α_j and β_k as roots of 1 and $l(t) = \log(|1-t|)$. These things are surprisingly similar to Section 7.2.7. These things will be further explained in [5] by using the results of this article.

Appendix: Corrections to [17], supplements to [15, III]

Corrections of errors in the book [17] have been put on the home page of Princeton University Press. In this appendix, we update some important parts of them in Sections A.1 and A.2. In Section A.3, we give supplements to [15, III]. Sections A.1 and A.3 are important for Section 5.5 in the text.

A.1 Changes in [17, Section 6.4]

A.1.1

The following errors (1) and (2) are in [17, Sections 6.4 and 7.1], respectively.

(1) Proposition 3.1.6 is used in 6.4.12 (the second line from the end), but Proposition 3.1.6 is not strong enough for the arguments in 6.4.12.

(2) We cannot have the second convergence in 7.1.2(3).

A.1.2

We make the following changes to the book [17]. Change 1 solves the above problem (1). Changes 2 and 3 solve the above problem (2).

Change 1. We replace [17, Section 7.1.2] by Sections A.1.3–A.1.9 below.

Change 2. We move [17, Section 7.1], revised as in the above change 1, to the place just before [17, Section 6.4]. That is, we exchange the order of [17, Sections 6.4 and 7.1].

Change 3. We make the change to [17, Section 6.4.12] explained in Section A.1.10.

A.1.3

We will prove Theorem A(i), that E_σ is open in $\tilde{E}_\sigma$ for the strong topology. Since $\tilde{E}_{\sigma,\mathrm{val}} \to \tilde{E}_\sigma$ is proper surjective and $E_{\sigma,\mathrm{val}} \subset \tilde{E}_{\sigma,\mathrm{val}}$ is the inverse image of $E_\sigma \subset \tilde{E}_\sigma$, it is sufficient to prove that $E_{\sigma,\mathrm{val}}$ is open in $\tilde{E}_{\sigma,\mathrm{val}}$. Assume $x_\lambda = (q_\lambda, F'_\lambda) \in \tilde{E}_{\sigma,\mathrm{val}}$ converges in $\tilde{E}_{\sigma,\mathrm{val}}$ to $x = (q, F') \in E_{\sigma,\mathrm{val}}$. We prove that $x_\lambda \in E_{\sigma,\mathrm{val}}$ for any sufficiently large λ.

A.1.4

We fix notation. Let $|\cdot| : \mathrm{toric}_{\sigma,\mathrm{val}} \to |\,\mathrm{toric}\,|_{\sigma,\mathrm{val}}$ be the canonical projection induced by $\mathbf{C} \to \mathbf{R}$, $z \mapsto |z|$.

Let $(A, V, Z) \in D^\sharp_{\sigma,\mathrm{val}}$ be the image of $(|q|, F') \in E^\sharp_{\sigma,\mathrm{val}}$ under $E^\sharp_{\sigma,\mathrm{val}} \to D^\sharp_{\sigma,\mathrm{val}}$ (5.3.7), and take an excellent basis $(N_s)_{s\in S}$ for (A, V, Z) such that $N_s \in \sigma(q)$ for any s (6.3.9). Let S_j $(1 \le j \le n)$ be as in 6.3.3. Take an $\mathbf{R}$-subspace B of $\sigma_{\mathbf{R}}$ such that $\sigma_{\mathbf{R}} = A_{\mathbf{R}} \oplus B$.

We have a unique injective open continuous map

$$(\mathbf{R}^S_{\ge 0})_{\mathrm{val}} \times B \to |\,\mathrm{toric}\,|_{\sigma,\mathrm{val}}$$

which sends $((e^{-2\pi y_s})_{s\in S}, b)$ $(y_s \in \mathbf{R}, b \in B)$ to $\mathbf{e}((\sum_{s\in S} i y_s N_s) + ib)$ (cf. 3.3.5). Let U be the image of this map. Define the maps $t_s : U \to \mathbf{R}_{\ge 0}$ $(s \in S)$ and $b : U \to B$ by (an abuse of notation b)

$$(t, b) = \big((t_s)_{s\in S}, b\big) : U \simeq (\mathbf{R}^S_{\ge 0})_{\mathrm{val}} \times B \to \mathbf{R}^S_{\ge 0} \times B.$$

We have $|q| \in U$ and $t(|q|) := (t_s(|q|))_{s\in S} = 0$. Since $|q_\lambda| \to |q|$, we may assume $|q_\lambda| \in U$. Let

$$F_\lambda = \exp\big(ib(q_\lambda)\big)F'_\lambda, \qquad F = \exp\big(ib(q)\big)F'.$$

Then $((N_s)_{s\in S}, F)$ generates a nilpotent orbit.

A.1.5

We may assume that, for some m $(1 \le m \le n+1)$, $t_s(q_\lambda) = 0$ for any λ and $s \in S_{\le m-1}$ and $t_s(q_\lambda) \ne 0$ for any λ and $s \in S_{\ge m}$ (see 6.3.11). ($S_{\le 0}$ and $S_{\ge n+1}$ are defined as the empty set.)

Take $c_j \in S_j$ for each j. For $s \in S_{\ge m}$, define $y_{\lambda,s} \in \mathbf{R}$ by $t_s(q_\lambda) = e^{-2\pi y_{\lambda,s}}$. For each $j \in \mathbf{Z}$ such that $m \le j \le n$, let $N_j = \sum_{s\in S_j} a_s N_s$, where $a_s \in \mathbf{R}$ is the limit of $y_{\lambda,s}/y_{\lambda,c_j}$.

Then $(N_1, \dots, N_n, F)$ generates a nilpotent orbit. Let $\tilde\rho : \mathbf{G}^n_{m,\mathbf{R}} \to G_{\mathbf{R}}$ be the homomorphism of the SL(2)-orbit (5.2.2) associated to $(N_1, \dots, N_n, F)$. For $m \le j \le n$, let

$$e_{\lambda,\ge j} = \exp\Big(\sum_{s\in S_{\ge j}} iy_{\lambda,s}N_s\Big) \in G_{\mathbf{C}},$$

$$\tau_{\lambda,j} = \tilde\rho_j\big(\sqrt{y_{\lambda,c_{j+1}}/y_{\lambda,c_j}}\big) \in G_{\mathbf{R}}, \qquad \tau_{\lambda,\ge j} = \prod_{k=j}^{n} \tau_{\lambda,k} \in G_{\mathbf{R}},$$

where $y_{\lambda,c_{n+1}}$ denotes 1. Here $\tilde\rho_j$ is the restriction of $\tilde\rho$ to the jth factor of $\mathbf{G}^n_{m,\mathbf{R}}$. Let $\hat F_{(j)}$ $(1 \le j \le n)$ be as in 6.1.3 associated to $(N_1, \dots, N_n, F)$.

By 3.1.6 applied to $S = E_\sigma \subset X = \check E_\sigma$, we have the following result.

LEMMA A.1.6

Let the situation and the notation be as above. Let $m \le j \le n$, and let $e \ge 0$. Then for any sufficiently large λ, there exist $F^*_\lambda \in \check D$ satisfying the following (i) and (ii).

(i) $y^e_{\lambda,s} d(F_\lambda, F^*_\lambda) \to 0$ $(\forall s \in S_j)$.

(ii) (N_s, F^*_λ) *satisfies Griffiths transversality for any $s \in S_{\le j}$.*

Furthermore, in the case $j = n$, there is F^*_λ as above satisfying the following condition (ii), which is stronger than the above condition (ii).*

(ii)* $((N_s)_{s\in S}, F^*_\lambda)$ *generates a nilpotent orbit.*

PROPOSITION A.1.7

Let the situation and the assumption be as above. Then the following assertions (A_j) $(m-1 \le j \le n)$, (B_j) $(m \le j \le n)$, and (C_j) $(m \le j \le n)$ are true.

(A_j) (resp., (B_j), (C_j)) for $m \le j \le n$: Let $e \ge 1$. Then for any sufficiently large λ, there are $F^{(j)}_\lambda \in \check D$ satisfying the following (1)–(3).

(1) $y^e_{\lambda,j} d(F_\lambda, F^{(j)}_\lambda) \to 0$.

(2) $((N_s)_{s\in S_{\le j}}, e_{\lambda,\ge j+1}F^{(j)}_\lambda)$ *generates a nilpotent orbit.*

(3) $\tau^{-1}_{\lambda,\geq j+1}e_{\lambda,\geq j+1}F^{(j)}_\lambda \to \exp(iN_{j+1})\hat{F}_{(j+1)}$ *(resp., (3)* $\tau^{-1}_{\lambda,\geq j}e_{\lambda,\geq j+1}F^{(j)}_\lambda \to \hat{F}_{(j)}$*; resp., (3)* $\tau^{-1}_{\lambda,\geq j}e_{\lambda,\geq j}F^{(j)}_\lambda \to \exp(iN_j)\hat{F}_{(j)}$*).*

Here (A_n) *is formulated by understanding* $N_{n+1} = 0$ *and* $\hat{F}_{(n+1)} = F$*.*

(A_{m-1})*: For any sufficiently large* λ*, we have the following (2) and (3).*

(2) $((N_s)_{s\in S_{\leq m-1}}, e_{\lambda,\geq m}F_\lambda)$ *generates a nilpotent orbit.*

(3) $\tau^{-1}_{\lambda,\geq m}e_{\lambda,\geq m}F_\lambda \to \exp(iN_m)\hat{F}_{(m)}$.

A.1.8

We prove Proposition A.1.7 by using the downward induction of the form $(A_j) \Rightarrow (B_j) \Rightarrow (C_j) \Rightarrow (A_{j-1})$. (Here $m \leq j \leq n$.) $(B_j) \Rightarrow (C_j)$ is clear. $(A_j) \Rightarrow (B_j)$ is easy. (A_n) follows from Lemma A.1.6.

We prove $(C_{j+1}) \Rightarrow (A_j)$. By Lemma A.1.6, if $m \leq j \leq n$ (resp., $j = m-1$), then there are $F^{(j)}_\lambda \in \check{D}$ satisfying (1) and (resp., $F^{(j)}_\lambda := F_\lambda$ satisfies) the condition

(2′) $(N_s, F^{(j)}_\lambda)$ satisfies Griffiths transversality for any $s \in S_{\leq j}$.

By (C_{j+1}), there are $F^{(j+1)}_\lambda \in \check{D}$ satisfying the following.

(1″) $y^e_{\lambda,j+1}d(F_\lambda, F^{(j+1)}_\lambda) \to 0$.

(2″) $((N_s)_{s\in S_{\leq j+1}}, e_{\lambda,\geq j+2}F^{(j+1)}_\lambda)$ generates a nilpotent orbit.

(3″) $\tau^{-1}_{\lambda,\geq j+1}e_{\lambda,\geq j+1}F^{(j+1)}_\lambda \to \exp(iN_{j+1})\hat{F}_{(j+1)}$.

By (1″) and (3″), we have

(4) $\tau^{-1}_{\lambda,\geq j+1}e_{\lambda,\geq j+1}F_\lambda \to \exp(iN_{j+1})\hat{F}_{(j+1)}$.

By (4) and by (1), we have

(5) $\tau^{-1}_{\lambda,\geq j+1}e_{\lambda,\geq j+1}F^{(j)}_\lambda \to \exp(iN_{j+1})\hat{F}_{(j+1)}$.

For the left-hand side of (5), by (2′), $(N_s, \tau^{-1}_{\lambda,\geq j+1}e_{\lambda,\geq j+1}F^{(j)}_\lambda)$ satisfies Griffiths transversality for any $s \in S_{\leq j}$. On the other hand, concerning the right-hand side of (5), $((N_s)_{s\in S_{\leq j}}, \exp(iN_{j+1})\hat{F}_{(j+1)})$ generates a nilpotent orbit. Hence, (5) and 7.1.1 show that $((N_s)_{s\in S_{\leq j}}, \tau^{-1}_{\lambda,\geq j+1}e_{\lambda,\geq j+1}F^{(j)}_\lambda)$ generates a nilpotent orbit. This proves that $((N_s)_{s\in S_{\leq j}}, e_{\lambda,\geq j+1}F^{(j)}_\lambda)$ generates a nilpotent orbit for any sufficiently large λ. Hence, for any sufficiently large λ, $(W^{(j)}, e_{\lambda,\geq j+1}F^{(j)}_\lambda)$ is a mixed Hodge structure, where $W^{(j)}$ denotes the relative monodromy filtration of $N_1 + \cdots + N_j$ with respect to W. By this and by (5), we have (A_j).

A.1.9

By $(A_{m-1})(2)$ of Proposition A.1.7, x_λ belongs to $E_{\sigma,\mathrm{val}}$ if λ is sufficiently large. This proves that $E_{\sigma,\mathrm{val}}$ is open in $\tilde{E}_{\sigma,\mathrm{val}}$ and, hence, proves that E_σ is open in $\tilde{E}_\sigma$.

A.1.10

In [17, Section 6.4.12], in the second line from the end, we replace the part "by Proposition 3.1.6" with "by the case $m = 0$ and $x_\lambda \in E^\sharp_{\sigma,\mathrm{val}}$ of Proposition A.1.7" of the present article.

REMARK A.1.11
By these changes, we have the following simplification in [17]. We can assume $y^*_{\lambda,t} = y_{\lambda,t}$ in [17, Proposition 6.4.1]. It is claimed in the original [17, Section 6.4] that the proofs of [17, Theorems 5.4.3(ii), 5.4.4] are reduced to [17, Proposition 6.4.1], but actually they are reduced to the case $y^*_{\lambda,t} = y_{\lambda,t}$ of [17, Proposition 6.4.1].

A.2 Change to [17, Section 7.2]

A.2.1
Professor J.-P. Serre kindly pointed out that our book should not use [6, Section 10], for there are errors in it (cf. the version of [6, Remark 10.10] contained in "Armand Borel Oeuvres—Collected Papers, Vol. III, Springer, 1983"). We used a result [6, Proposition 10.4] in the proof of Lemma 7.2.12 of our book. To correct our argument, we change the following. We put the following assumption in Theorem 7.2.2(i): "Assume that σ is a nilpotent cone associated to a nilpotent orbit." We replace Lemma 7.2.12 and its proof in our book by the following proposition and its proof, which does not use [6, Section 10].

PROPOSITION A.2.2
Let σ be a nilpotent cone associated to a nilpotent orbit, and let $W(\sigma)$ be the associated weight filtration. Then, by assigning the Borel–Serre splitting, we have a continuous map $E^\sharp_{\sigma,\mathrm{val}} \to \mathrm{spl}(W(\sigma))$.

Proof
The composite map $E^\sharp_{\sigma,\mathrm{val}} \to D^\sharp_{\sigma,\mathrm{val}} \xrightarrow{\psi} D_{\mathrm{SL}(2)}$ is continuous by the definition of the first map and by 6.4.1 for the CKS map ψ. Let $N_1, \dots, N_n$ be a set of generators of the cone σ. Let s be a bijection $\{1,\dots,n\} \to \{1,\dots,n\}$. Then the image of the map $E^\sharp_{\sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ is contained in the union U of $D_{\mathrm{SL}(2)}(\{W(N_{s(1)} + \cdots + N_{s(j)}) \mid j = 1,\dots,n\})$, where s runs over all bijections $\{1,\dots,n\} \to \{1,\dots,n\}$. Since $N_{s(1)} + \cdots + N_{s(n)} = N_1 + \cdots + N_n$, the filtration $W(N_1 + \cdots + N_n) = W(\sigma)$ appears for any s. By [15, II, Proposition 3.2.12], the Borel–Serre splitting gives a continuous map $U \to \mathrm{spl}(W(\sigma))$. Thus, we get our assertion. □

A.2.3
We replace the third paragraph in 7.2.13 by the following: "Since the action of $\sigma_{\mathbf{R}}$ on $\mathrm{spl}(W(\sigma))$ is proper and $E^\sharp_{\sigma,\mathrm{val}}$ is Hausdorff, the action of $\sigma_{\mathbf{R}}$ on $E^\sharp_{\sigma,\mathrm{val}}$ is proper by applying Lemma 7.2.6(ii) to the continuous map $E^\sharp_{\sigma,\mathrm{val}} \to \mathrm{spl}(W(\sigma))$ in Proposition 7.2.12. Hence, $\mathrm{Re}(h_\lambda)$ converges in $\sigma_{\mathbf{R}}$ by Lemma 7.2.7."

A.2.4
Add the following sentence at the top of the fourth paragraph in 7.2.13: "Let $|\ | : E_{\sigma,\mathrm{val}} \to E^\sharp_{\sigma,\mathrm{val}}$ be the continuous map $(q, F) \to (|q|, F)$ in 7.1.3."

A.3 Supplements to [15, III]

We add explanations to [15, III, Section 3.3].

A.3.1

We put the following explanation from Section A.3.2 just after the statement of [15, III, Theorem 3.3.1].

A.3.2

This [15, III, Theorem 3.3.1] is the mixed Hodge version of [17, Theorem 5.4.3] for the pure case and is proved in the same way.

A.3.3

We replace the two lines "As in [KU09] 6.4, a key step ... We only prove this proposition." just before [15, III, Section 3.3.3] by the following Section A.3.4.

A.3.4

This [15, III, Theorem 3.3.2] is proved in the following way. We can prove the evident mixed Hodge version of Proposition A.1.6 by using the same arguments in the proof of Proposition A.1.6.

Just as [17, Theorem 5.4.4] was reduced to the case $y^*_{\lambda,t} = y_{\lambda,t}$ of [17, Lemma 6.4.1] by using the case $m = 0$ and $x_\lambda \in E^\sharp_{\sigma,\mathrm{val}}$ of Proposition A.1.6 (see Section A.1.10, Remark A.1.11), [15, III, Theorem 3.3.2] is reduced to the case $y^*_{\lambda,t} = y_{\lambda,t}$ of [15, III, Proposition 3.3.4] by using the case $m = 0$ of this mixed Hodge version of Proposition A.1.6.

A.3.5

We replace the part "proposition implies" in the second line of the remark after [15, III, Proposition 3.3.4] with "proposition and [KU09], 6.4.1 for the pure case imply."

Acknowledgments. We thank Spencer Bloch. Theorems 5.1.10 and 7.2.4 in this article were obtained in joint efforts with him related to the joint work [5].

Kato's work is supported in part by National Science Foundation grant DMS 1001729. Nakayama's work is supported in part by Japan Society for the Promotion of Science (JSPS) KAKENHI grants (C) No. 18540017, (C) No. 22540011 and (C) No. 16K05093. Usui's work is supported in part by JSPS KAKENHI (B) No. 23340008, and (C) No. 17K05200.

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