On the geometry of the Lehn–Lehn–Sorger–van Straten eightfold

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Abstract In this article we make a few remarks about the geometry of the holomorphic symplectic manifold Z constructed by Lehn, Lehn, Sorger, and van Straten as a two-step contraction of the variety of twisted cubic curves on a cubic fourfold $Y \subset \mathbb{P}^5$. We show that Z is birational to a component of the moduli space of stable sheaves in the Calabi–Yau subcategory of the derived category of Y. Using this description we deduce that the twisted cubics contained in a hyperplane section $Y_H = Y \cap H$ of Y give rise to a Lagrangian subvariety $Z_H \subset Z$. For a generic choice of the hyperplane, Z_H is birational to the theta-divisor in the intermediate Jacobian $\mathrm{J}(Y_H)$.

1. Introduction

We work over the field of complex numbers. Throughout the article $Y \subset \mathbb{P}^5$ is a smooth cubic fourfold not containing a plane. In [14] the variety $M_3(Y)$ of generalized twisted cubic curves on Y was studied. It was shown that $M_3(Y)$ is 10-dimensional, smooth, and irreducible. By starting from this variety, an 8-dimensional irreducible holomorphic symplectic (IHS) manifold Z was constructed. More precisely, it was shown that there exist morphisms

$$M_3(Y) \xrightarrow{a} Z' \xrightarrow{\sigma} Z \tag{1.1}$$

and

$$\mu\colon Y \hookrightarrow Z, \tag{1.2}$$

where a is a $\mathbb{P}^2$-fiber bundle and σ is the blowup along the image of μ. It was later shown in [1] that Z is birational—and hence deformation equivalent—to a Hilbert scheme of four points on a K3 surface.

In this article we present another point of view on Z. We show that an open subset of Z can be described as a moduli space of Gieseker-stable torsion-free sheaves of rank 3 on Y.

Kuznetsov and Markushevich [12] have constructed a closed two-form on any moduli space of sheaves on Y. Properties of the Kuznetsov–Markushevich form are known to be closely related to the structure of the derived category

Kyoto Journal of Mathematics, Vol. 57, No. 4 (2017), 789–806
First published online 9, June 2017.
DOI 10.1215/21562261-2017-0014,
Received April 5, 2016. Accepted June 10, 2016.
2010 Mathematics Subject Classification: Primary 14F05; Secondary 53C26.

of Y. The bounded derived category $\mathcal{D}^b(Y)$ of coherent sheaves on Y has an exceptional collection $\mathcal{O}_Y$, $\mathcal{O}_Y(1)$, $\mathcal{O}_Y(2)$ with right-orthogonal $\mathcal{A}_Y$, so that $\mathcal{D}^b(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle$. The category $\mathcal{A}_Y$ is a Calabi–Yau category of dimension 2, meaning that its Serre functor is the shift by 2 (see [11, Section 4]).

It was shown in [12] that the two-form on moduli spaces of sheaves on Y is nondegenerate if the sheaves lie in $\mathcal{A}_Y$. The torsion-free sheaves mentioned above lie in $\mathcal{A}_Y$. This gives an alternative description of the symplectic form on Z.

THEOREM 2.8

The component $\mathcal{M}_F$ of the moduli space of Gieseker-stable rank 3 sheaves on Y with Hilbert polynomial $\frac{3}{8}n^4 + \frac{9}{4}n^3 + \frac{33}{8}n^2 + \frac{9}{4}n$ is birational to the IHS manifold Z. Under this birational equivalence the symplectic form on Z defined in [14] corresponds to the Kuznetsov–Markushevich form on $\mathcal{M}_F$.

A similar approach relying on the description of an open part of Z as a moduli space was used by Addington and Lehn [1] to prove that the variety Z is a deformation of a Hilbert scheme of four points on a K3 surface. Ouchi [16] considered the case of cubic fourfolds containing a plane. He proved that one can describe (a birational model of) the Lehn–Lehn–Sorger–van Straten variety as a moduli space of Bridgeland-stable objects in the derived category of a twisted K3 surface. Moreover, in this situation one also has a Lagrangian embedding of the cubic fourfold into the Lehn–Lehn–Sorger–van Straten variety as in (1.2). Another similar construction has been proposed by Lahoz, Macrì, and Stellari [13], who proved that Z is birational to a component of the moduli space of stable vector bundles of rank 6 on Y.

Using the birational equivalence between Z and the moduli space of sheaves on Y, we show that twisted cubics lying in hyperplane sections Y_H of Y give rise to Lagrangian subvarieties in Z, and we discuss the geometry of these subvarieties.

THEOREM

Denote by Z_H the image in Z of twisted cubics lying in a hyperplane section $Y_H = Y \cap H$ under the map a from (1.1). If Y and H are generic, then Z_H is a Lagrangian subvariety of Z which is birational to the theta-divisor of the intermediate Jacobian of Y_H.

Proof

See Proposition 2.9 and Theorem 3.3. □

This is analogous to the case of lines on Y: it is well known that lines on Y form an IHS fourfold, and lines contained in hyperplane sections of Y form Lagrangian surfaces in this fourfold (see, e.g., [17]).

2. Twisted cubics and sheaves on a cubic fourfold

2.1. Twisted cubics on cubic surfaces and determinantal representations

Let us recall the structure of the general fiber of the map $a\colon M_3(Y) \to Z'$ in (1.1). We follow [14] in notation and terminology, and we refer to [5] and [14] for all details about the geometry of twisted cubics.

Consider a cubic surface $S = Y \cap \mathbb{P}^3$, where $\mathbb{P}^3$ is a general linear subspace in $\mathbb{P}^5$. There exist several families of generalized twisted cubics on S. Each of the families is isomorphic to $\mathbb{P}^2$, and these are the fibers of the map a. The number of families depends on S. If the surface is smooth, then there are 72 families, corresponding to 72 ways to represent S as a blowup of $\mathbb{P}^2$ (and to the 72 roots in the lattice E_6). Each of the families is a linear system which gives a map to $\mathbb{P}^2$. If S is singular, then generalized twisted cubics on it can be of two different types. Curves of the first type are arithmetically Cohen–Macaulay (aCM), and those of the second type are non-CM. A detailed description of their geometry on surfaces with different singularity types can be found in [14, Section 2]. For our purposes it is enough to recall that the image in Z' of non-CM curves under the map a is exactly the exceptional divisor of the blowup $\sigma\colon Z' \to Z$ in (1.1) (see [14, Proposition 4.1]).

In this section we deal only with aCM curves, and we also assume that the surface S has only ADE singularities. In this case every aCM curve belongs to a 2-dimensional linear system with smooth general member, just as in the case of smooth S (see [14, Theorem 2.1]). Moreover, these linear systems are in one-to-one correspondence with the determinantal representations of S. Let us explain this in detail.

Let S be a cubic surface in $\mathbb{P}^3$ with at most ADE singularities. Let $\alpha\colon S \hookrightarrow \mathbb{P}^3$ denote the embedding, and let $p\colon \tilde{S} \to S$ be the minimal resolution of singularities. Take a general aCM twisted cubic C on S, and let $\tilde{C} \subset \tilde{S}$ be its proper preimage. Let $\tilde{L} = \mathcal{O}_{\tilde{S}}(\tilde{C})$ be the corresponding line bundle, and let $L = p_*\tilde{L}$ be its direct image.

LEMMA 2.1

The sheaf L has the following properties:

(1) $H^0(S,L) = \mathbb{C}^3$, $H^k(S,L) = 0$ *for* $k \geq 1$*;* $H^k(S,L(-1)) = H^k(S,L(-2)) = 0$ *for* $k \geq 0$*;*

(2) *we have the following resolution:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}^{\oplus 3} \longrightarrow \alpha_* L \longrightarrow 0, \tag{2.1}$$

where A is given by a (3×3)-matrix of linear forms on $\mathbb{P}^3$, and the surface S is the vanishing locus of $\det A$*;*

(3) $\mathcal{E}\mathrm{xt}^k(L,L) = 0$ *for* $k \geq 1$.

Proof
We note that the map $\alpha \circ p \colon \tilde{S} \to \mathbb{P}^3$ is given by the anticanonical linear system on $\tilde{S}$, so we will use the notation $K_{\tilde{S}} = \mathcal{O}_{\tilde{S}}(-1)$.

(1) First we observe that $\mathrm{R}^m p_* \tilde{L} = 0$ for $m \geq 1$. This follows from the long exact sequence of higher direct images for the triple

$$0 \longrightarrow \mathcal{O}_{\tilde{S}} \longrightarrow \tilde{L} \longrightarrow \mathcal{O}_{\tilde{C}} \otimes \tilde{L} \longrightarrow 0, \tag{2.2}$$

because the singularities of S are rational, so that $\mathrm{R}^m p_* \mathcal{O}_{\tilde{S}} = 0$ for $m \geq 1$, and the map p induces an embedding of $\tilde{C}$ into S, so that $\mathrm{R}^m p_*$ vanishes on sheaves supported on $\tilde{C}$ for $m \geq 1$.

Analogously, $\mathrm{R}^m p_* \tilde{L}(-1) = \mathrm{R}^m p_* \tilde{L}(-2) = 0$ for $m \geq 1$. Hence, it is enough to verify the cohomology vanishing for $\tilde{L}$.

The linear system $|\tilde{L}|$ is 2-dimensional and basepoint free (we refer to [14, Section 2, in particular, Proposition 2.5]). We also know the intersection products $\tilde{L} \cdot \tilde{L} = 1$, $\tilde{L} \cdot K_{\tilde{S}} = -3$, and $K_{\tilde{S}} \cdot K_{\tilde{S}} = 3$. Using Riemann–Roch we find $\chi(\tilde{L}) = 3$ and $\chi(\tilde{L}(-1)) = \chi(\tilde{L}(-2)) = 0$. We have $H^0(\tilde{S}, \tilde{L}(-1)) = H^0(\tilde{S}, \tilde{L}(-2)) = 0$, which is clear from (2.2), since $\tilde{L}|_{\tilde{C}} = \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\tilde{S}}(1)|_{\tilde{C}} = \mathcal{O}_{\mathbb{P}^1}(3)$. By Serre duality we have $H^2(\tilde{S}, \tilde{L}) = H^0(\tilde{S}, \tilde{L}^\vee(-1))^* = 0$, $H^2(\tilde{S}, \tilde{L}(-1)) = H^0(\tilde{S}, \tilde{L}^\vee)^* = 0$ because $\tilde{L}^\vee$ is the ideal sheaf of $\tilde{C}$, and $H^2(\tilde{S}, \tilde{L}(-2)) = H^0(\tilde{S}, \tilde{L}^\vee(1))^* = 0$. The last vanishing follows from the fact that C is not contained in any hyperplane in $\mathbb{P}^3$. It follows that $H^1(\tilde{S}, \tilde{L}) = H^1(\tilde{S}, \tilde{L}(-1)) = H^1(\tilde{S}, \tilde{L}(-2)) = 0$.

(2) We decompose the sheaf $\alpha_* L$ with respect to the full exceptional collection $\mathcal{D}^b(\mathbb{P}^3) = \langle \mathcal{O}_{\mathbb{P}^3}(-1), \mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(2) \rangle$. From part (1) it follows that $\alpha_* L$ is right-orthogonal to $\mathcal{O}_{\mathbb{P}^3}(2)$ and $\mathcal{O}_{\mathbb{P}^3}(1)$. The left mutation of $\alpha_* L$ through $\mathcal{O}_{\mathbb{P}^3}$ is given by a cone of the morphism $\mathcal{O}_{\mathbb{P}^3}^{\oplus 3} \to \alpha_* L$ induced by the global sections of L. This cone is contained in the subcategory generated by the exceptional object $\mathcal{O}_{\mathbb{P}^3}(-1)$. Hence, it must be equal to $\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3}[1]$, and we obtain the resolution (2.1) for $\alpha_* L$.

(3) Since L is a vector bundle outside of the singular points of S, the sheaves $\mathcal{E}xt^k(L, L)$ for $k \geq 1$ must have 0-dimensional support. It follows that it will be sufficient to prove that $\mathrm{Ext}^k(L, L) = 0$ for $k \geq 0$.

We first compute $\mathrm{Ext}^k(\alpha_* L, \alpha_* L)$. Applying $\mathrm{Hom}(-, \alpha_* L)$ to (2.1) we get the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(\alpha_* L, \alpha_* L) \longrightarrow H^0(\mathbb{P}^3, \alpha_* L)^{\oplus 3} \longrightarrow H^0\big(\mathbb{P}^3, \alpha_* L(1)\big)^{\oplus 3} \\ &\longrightarrow \mathrm{Ext}^1(\alpha_* L, \alpha_* L) \longrightarrow 0, \end{aligned}$$

where we use that $H^k(\mathbb{P}^3, \alpha_* L(m)) = 0$ for $k \geq 1$ and $m \geq 0$, which is clear from (2.1). This also shows that $\mathrm{Ext}^k(\alpha_* L, \alpha_* L) = 0$ for $k \geq 2$. We have $\dim \mathrm{Hom}(\alpha_* L, \alpha_* L) = 1$, and from the sequence above and (2.1), we compute $\dim \mathrm{Ext}^1(\alpha_* L, \alpha_* L) = 19$.

The object $\mathrm{L}\alpha^* \alpha_* L$ is included in the triangle $\mathrm{L}\alpha^* \alpha_* L \to L \to L(-3)[2] \to \mathrm{L}\alpha^* \alpha_* L[1]$ (see [12, Lemma 1.3.1]). Applying $\mathrm{Hom}(-, L)$ to this triangle and using $\mathrm{Ext}^k(\mathrm{L}\alpha^* \alpha_* L, L) = \mathrm{Ext}^k(\alpha_* L, \alpha_* L)$ we get the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(L, L) \longrightarrow \mathrm{Ext}^1(\alpha_* L, \alpha_* L) \longrightarrow \mathrm{Hom}\big(L, L(3)\big) \longrightarrow \mathrm{Ext}^2(L, L) \longrightarrow 0.$$

The arrow in the middle is an isomorphism. To see this, note that $\mathrm{Hom}(L, L(3)) = H^0(S, N_{S/\mathbb{P}^3}) = \mathbb{C}^{19}$ and that all the deformations of $\alpha_* L$ are induced by the deformations of its support S. It follows that $\mathrm{Ext}^1(L, L) = \mathrm{Ext}^2(L, L) = 0$. As we have mentioned above, the sheaves $\mathcal{E}\mathrm{xt}^k(L, L)$ have 0-dimensional support for $k \geq 1$, and from the local-to-global spectral sequence we see that $\mathrm{Ext}^k(L, L) = H^0(S, \mathcal{E}\mathrm{xt}^k(L, L))$ for $k \geq 1$. It follows that $\mathcal{E}\mathrm{xt}^1(L, L) = \mathcal{E}\mathrm{xt}^2(L, L) = 0$. To prove the vanishing of higher $\mathcal{E}\mathrm{xt}$'s we construct a quasiperiodic free resolution for L. From (2.1) we see that the restriction of the complex $\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}^{\oplus 3}$ to S will have cohomology L in degree 0 and $L(-3)$ in degree -1. Hence, L is quasi-isomorphic to the complex of the form

$$\begin{aligned} \cdots \longrightarrow \mathcal{O}_S(-7)^{\oplus 3} \longrightarrow \mathcal{O}_S(-6)^{\oplus 3} \longrightarrow \mathcal{O}_S(-4)^{\oplus 3} \longrightarrow \mathcal{O}_S(-3)^{\oplus 3} \\ \longrightarrow \mathcal{O}_S(-1)^{\oplus 3} \longrightarrow \mathcal{O}_S^{\oplus 3} \longrightarrow 0. \end{aligned}$$

This complex is quasiperiodic of period 2, with subsequent entries obtained by tensoring by $\mathcal{O}_S(-3)$. Applying $\mathcal{H}\mathrm{om}(-, L)$ to this complex we see that the $\mathcal{E}\mathrm{xt}^k(L, L)$'s are also quasiperiodic, and vanishing of the first two of these sheaves implies vanishing of the rest. □

Starting from L, we have constructed the determinantal representation of S. Conversely, given a sequence (2.1), generalized twisted cubics corresponding to this determinantal representation can be recovered as vanishing loci of sections of L. A more detailed discussion of determinantal representations of cubic surfaces with different singularity types can be found in [14, Section 3].

2.2. Moduli spaces of sheaves on a cubic fourfold

Let $S = Y \cap \mathbb{P}^3$ be a linear section of Y with ADE singularities, and let L be a sheaf which gives a determinantal representation of S as in (2.1). Denote by $i \colon S \hookrightarrow Y$ the embedding. We consider the moduli space of torsion sheaves on Y of the form $i_* L$ to get a description of an open subset of Z.

LEMMA 2.2

For any $u \in \mathrm{Ext}^1(i_* L, i_* L)$ its Yoneda square $u \circ u \in \mathrm{Ext}^2(i_* L, i_* L)$ is zero, so that the deformations of $i_* L$ are unobstructed.

Proof

Recall that L is a rank 1 sheaf on S. The unobstructedness is clear when S is smooth, because L is a line bundle in this case. Then the local $\mathcal{E}\mathrm{xt}$'s are given by $\mathcal{E}\mathrm{xt}^k(i_* L, i_* L) = i_* \Lambda^k N_{S/Y}$ (see [12, Lemma 1.3.2] for the proof of this). In the case when S is singular and L is not locally free we can use the same argument as in [12, Lemma 1.3.2] to obtain a spectral sequence $E_2^{p,q} = i_*(\mathcal{E}\mathrm{xt}^p(L, L) \otimes \Lambda^q N_{S/Y}) \Rightarrow \mathcal{E}\mathrm{xt}^{p+q}(i_* L, i_* L)$. Now we can use Lemma 2.1(2) to conclude that in this case $\mathcal{E}\mathrm{xt}^k(i_* L, i_* L) = i_* \Lambda^k N_{S/Y}$ as well.

We have $N_{S/Y} = \mathcal{O}_S(1)^{\oplus 2}$ and $H^m(S, \mathcal{O}_S(k)) = 0$ for $k \geq 0$, $m \geq 1$, and from the local-to-global spectral sequence we deduce that $\mathrm{Ext}^k(i_* L, i_* L) =$

$H^0(S, \Lambda^k N_{S/Y})$. The algebra structure is induced by the exterior product $\Lambda^k N_{S/Y} \otimes \Lambda^m N_{S/Y} \to \Lambda^{k+m} N_{S/Y}$ (see [12, Lemma 1.3.3]). The exterior square of any section of $N_{S/Y}$ is zero, and unobstructedness follows. $\square$

The sheaf i_*L has Hilbert polynomial $P(i_*L, n) = \frac{3}{2}n^2 + \frac{9}{2}n + 3$, which is easy to compute from (2.1). Denote by $\mathcal{M}_L$ the irreducible component of the moduli space of semistable sheaves with this Hilbert polynomial containing i_*L.

Let us denote by V the 6-dimensional vector space, so that $Y \subset \mathbb{P}(V) = \mathbb{P}^5$. Denote by G the Grassmannian $\mathrm{Gr}(4, V)$. Recall from [14] that we have a closed embedding $\mu\colon Y \hookrightarrow Z$, and the open subset $Z \backslash \mu(Y)$ corresponds to aCM twisted cubics. There exists a map $\pi : Z\backslash\mu(Y) \to G$ which sends a twisted cubic to its linear span in $\mathbb{P}^5$. If we consider linear sections $S = Y \cap \mathbb{P}^3$, then S can have non-ADE singularities, but the codimension in G of such linear subspaces is at least 4 by [14, Propositions 4.2, 4.3]. Denote by $G^\circ \subset G$ the open subset consisting of $U \in G$ such that $Y \cap \mathbb{P}(U)$ has only ADE singularities. Let $Z^\circ = \pi^{-1}(G^\circ)$ be the corresponding open subset in $Z\backslash\mu(Y)$. The complement of this open subset has codimension 4.

LEMMA 2.3

There exists an open subset $\mathcal{M}_L^\circ \hookrightarrow \mathcal{M}_L$ isomorphic to Z°. The sheaves on Y corresponding to points of $\mathcal{M}_L^\circ$ are of the form i_*L, where L gives a determinantal representation for a linear section $S = Y \cap \mathbb{P}^3$ with ADE singularities.

Proof

Denote by $\mathcal{U}$ the universal subbundle of $\mathcal{O}_G \otimes V$. Let $p : \mathbb{P}(\mathcal{U}) \to G$ be the projection, and let $\mathcal{H} = \mathcal{H}om_p(\mathcal{O}_{\mathbb{P}(\mathcal{U})}(-1)^{\oplus 3}, \mathcal{O}_{\mathbb{P}(\mathcal{U})}^{\oplus 3})$. We have $\mathcal{H} \simeq (\mathcal{U}^\vee)^{\oplus 9}$. We will denote by the same letter $\mathcal{H}$ the total space of the bundle $\mathcal{H}$. By construction, over $\mathcal{H} \times_G \mathbb{P}(\mathcal{U})$ we have the universal morphism

$$\mathcal{O}_{\mathbb{P}(\mathcal{U})}(-1)^{\oplus 3} \xrightarrow{\mathcal{A}} \mathcal{O}_{\mathbb{P}(\mathcal{U})}^{\oplus 3}.$$

Denote by $\mathcal{H}^\circ$ the open subset in the total space of $\mathcal{H}$ where $\det(\mathcal{A}) \neq 0$. Consider the closed embedding $j : \mathcal{H}^\circ \times_G \mathbb{P}(\mathcal{U}) \hookrightarrow \mathcal{H}^\circ \times \mathbb{P}(V)$ and the sheaf $\mathcal{M} = \mathrm{coker}(j_*\mathcal{A})$ on $\mathcal{H}^\circ \times \mathbb{P}(V)$. Let $q : \mathcal{H}^\circ \times \mathbb{P}(V) \to \mathcal{H}^\circ$ be the projection. For a point $A \in \mathcal{H}^\circ$ the restriction $\mathcal{M}|_{q^{-1}(A)}$ is a sheaf that defines a determinantal representation of a cubic surface in $\mathbb{P}(U) \subset \mathbb{P}(V)$. The condition that this surface is contained in Y defines a closed subvariety $\mathcal{W} \subset \mathcal{H}^\circ$.

Let $\beta : \mathcal{W} \times Y \hookrightarrow \mathcal{H}^\circ \times \mathbb{P}(V)$ be the closed embedding. Define $\mathcal{L} = \mathcal{M}|_{\mathcal{W}\times Y}$, and consider the open subset $G^\circ \subset G$ of subspaces $U \subset V$ such that $\mathbb{P}(U) \cap Y$ has ADE singularities. Let $\mathcal{W}^\circ$ be the preimage of G° under the natural map $\mathcal{W} \to G$. The sheaf $\mathcal{L}$ on $\mathcal{W}^\circ \times Y$ is flat over $\mathcal{W}^\circ$, since Hilbert polynomials of its restrictions to the fibers are the same (see [8, Chapter III, Theorem 9.9]). We obtain a morphism $\psi : \mathcal{W}^\circ \to \mathcal{M}_L$. Denote its image by $\mathcal{M}_L^\circ$. Consider the fiber $\mathcal{W}_U$ of the map $\mathcal{W}^\circ \to G^\circ$ over a point $U \in G$ and the restriction of $\mathcal{L}$ to $\mathcal{W}_U \times Y$. Over a point $w \in \mathcal{W}_U$ the sheaf $\mathcal{L}$ defines a determinantal representation of the surface $Y \cap \mathbb{P}(U)$. The general structure of determinantal representations (see [14, Section 3]) implies that each connected component of the fiber $\mathcal{W}_U$ is a single $(\mathrm{GL}_3 \times \mathrm{GL}_3)/\mathbb{C}^*$ orbit (see [14, Corollary 3.7]). Connected components of $\mathcal{W}_U$ are in one-to-one correspondence with nonisomorphic determinantal representations of $Y \cap \mathbb{P}(U)$. The restriction of $\mathcal{L}$ to each connected component of $\mathcal{W}_U \times Y$ is a constant family of sheaves, so the map ψ contracts connected components of the fiber $\mathcal{W}_U$. From the explicit description of Z° given above, we see that $\mathcal{M}_L^\circ$ is isomorphic to Z°. The properties stated in the lemma are clear from construction. We also see that $\mathcal{W}^\circ$ is a $(\mathrm{GL}_3 \times \mathrm{GL}_3)/\mathbb{C}^*$-fiber bundle over Z°. □

The sheaves i_*L are not contained in the subcategory $\mathcal{A}_Y$. To show that the closed two-form described in [12] is a symplectic form on $\mathcal{M}_L^\circ$, we are going to project the sheaves i_*L to $\mathcal{A}_Y$ and then show that this projection induces an isomorphism of open subsets of moduli spaces respecting the two-forms (up to a sign).

LEMMA 2.4

The sheaves i_*L are globally generated and lie in the subcategory $\langle \mathcal{A}_Y, \mathcal{O}_Y \rangle$. The space of global sections $H^0(Y, i_*L)$ is 3-dimensional, and the sheaf F_L, defined by the exact triple

$$0 \longrightarrow F_L \longrightarrow \mathcal{O}_Y^{\oplus 3} \longrightarrow i_*L \longrightarrow 0, \tag{2.3}$$

lies in $\mathcal{A}_Y$.

Proof

From Lemma 2.1 we deduce that i_*L is right-orthogonal to $\mathcal{O}_Y(1)$ and $\mathcal{O}_Y(2)$, so that i_*L lies in $\langle \mathcal{A}_Y, \mathcal{O}_Y \rangle$. It also follows from Lemma 2.1 that i_*L is globally generated, that the global sections are 3-dimensional, and that the higher cohomology groups of L vanish. Thus, F_L is (up to a shift) the left mutation of i_*L through the exceptional bundle $\mathcal{O}_Y$, and in particular, it lies in $\mathcal{A}_Y$. □

LEMMA 2.5

Consider the exact triple (2.3) where i_*L is in $\mathcal{M}_L^\circ$. Then F_L is a Gieseker-stable rank 3 sheaf contained in $\mathcal{A}_Y$ with Hilbert polynomial $P(F_L, n) = \frac{3}{8}n^4 + \frac{9}{4}n^3 + \frac{33}{8}n^2 + \frac{9}{4}n$.

Proof

By Lemma 2.3 the sheaf i_*L is right-orthogonal to $\mathcal{O}_Y(2)$ and $\mathcal{O}_Y(1)$. The sheaf F_L is a shift of the left mutation of i_*L through $\mathcal{O}_Y$; hence, it is contained if $\mathcal{A}_Y$ is. The Hilbert polynomial can be computed using the Hirzebruch–Riemann–Roch formula. It remains to check the stability of F_L.

The sheaf F_L is a subsheaf of $\mathcal{O}_Y^{\oplus 3}$; hence, it has no torsion. In order to check the stability we consider all proper saturated subsheaves $\mathcal{G} \subset F_L$. We have to make sure that $p(\mathcal{G}, n) < p(F_L, n)$, where p is the reduced Hilbert polynomial (see [9] for all the relevant definitions). We use the convention that the inequalities between polynomials are supposed to hold for $n \gg 0$.

We denote by P the nonreduced Hilbert polynomial. We have $P(\mathcal{O}_Y, n) = a_0 n^4 + a_1 n^3 + \cdots + a_4$, with the leading coefficient $a_0 = \frac{3}{4!}$. From the exact sequence (2.3) we see that $P(F_L, n) = 3P(\mathcal{O}_Y, n) - P(i_*L, n)$. Since i_*L has 2-dimensional support, the degree of $P(i_*L, n)$ is 2, and hence, the leading coefficient of $P(F_L, n)$ equals $3a_0$. So we have

$$p(F_L, n) = p(\mathcal{O}_Y, n) - \frac{1}{3a_0} P(i_*L, n). \tag{2.4}$$

Let $\tilde{\mathcal{G}}$ be the saturation of $\mathcal{G}$ inside $\mathcal{O}_Y^{\oplus 3}$. Then $\tilde{\mathcal{G}}$ is a reflexive sheaf and we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \tilde{\mathcal{G}} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_L & \longrightarrow & \mathcal{O}_Y^{\oplus 3} & \longrightarrow & i_*L & \longrightarrow & 0 \end{array}$$

In this diagram $\mathcal{H}$ is a torsion sheaf which injects into i_*L because $F_L/\mathcal{G}$ is torsion-free. Note that $\mathcal{O}_Y^{\oplus 3}$ is Mumford polystable, so $c_1(\mathcal{G}) \leq c_1(\tilde{\mathcal{G}}) \leq 0$. If $c_1(\mathcal{G}) < 0$, then $\mathcal{G}$ is not destabilizing in F_L because $c_1(F_L) = 0$.

Next we consider the case $c_1(\mathcal{G}) = c_1(\tilde{\mathcal{G}}) = 0$. In this case $\tilde{\mathcal{G}} = \mathcal{O}_Y^{\oplus m}$ where $m = 1$ or $m = 2$. This is clear if $\operatorname{rk} \tilde{\mathcal{G}} = 1$, since a reflexive sheaf of rank 1 is a line bundle. If $\operatorname{rk} \tilde{\mathcal{G}} = 2$, then we can consider the quotient $\mathcal{O}_Y^{\oplus 3}/\tilde{\mathcal{G}}$ which is torsion-free, globally generated, and of rank 1 and has zero first Chern class. It follows that the quotient is isomorphic to $\mathcal{O}_Y$ and then $\tilde{\mathcal{G}} = \mathcal{O}_Y^{\oplus 2}$.

We have an exact triple $0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_Y^{\oplus m} \longrightarrow \mathcal{H} \longrightarrow 0$ with m equal to 1 or 2. We see that $p(\mathcal{G}, n) = p(\mathcal{O}_Y, n) - \frac{1}{ma_0} P(\mathcal{H}, n)$. Note that $\mathcal{H}$ is a nonzero sheaf which injects into i_*L, and the sheaf L on the surface S is torsion-free of rank 1. Hence, the leading coefficient of $P(\mathcal{H}, n)$ is the same as that for $P(i_*L, n)$, and this implies that $\frac{1}{ma_0} P(\mathcal{H}, n) > \frac{1}{3a_0} P(i_*L, n)$. From this and (2.4) we conclude that $p(\mathcal{G}, n) < p(F_L, n)$; hence, $\mathcal{G}$ is not destabilizing. This completes the proof. □

Let us consider the moduli space of rank 3 semistable sheaves on Y with Hilbert polynomial $P(F_L, n)$. Denote by $\mathcal{M}_F$ its irreducible component which contains the sheaves F_L from (2.3).

LEMMA 2.6

The left mutation of i_*L through $\mathcal{O}_Y$ gives an open embedding $\mathcal{M}_L^\circ \to \mathcal{M}_F$.

Proof

Recall from the proof of Lemma 2.3 that $\mathcal{M}_L^\circ$ was defined as the image of a map $\mathcal{W}^\circ \to \mathcal{M}_L$, where $\mathcal{W}^\circ$ was a fiber bundle over Z°. On $X = \mathcal{W}^\circ \times Y$ a universal sheaf $\mathcal{L}$ flat over $\mathcal{W}^\circ$ was constructed. Denote by $\pi\colon X \to \mathcal{W}^\circ$ the projection.

By the definition of $\mathcal{M}_L^\circ$ and from Lemma 2.1 it follows that $\pi_*\mathcal{L}$ is a rank 3 vector bundle and we have an exact sequence $0 \to \mathcal{F}_\mathcal{L} \to \pi^*\pi_*\mathcal{L} \to \mathcal{L} \to 0$. The family of sheaves $\mathcal{F}_{\mathcal{L}}$ defines a map $\mathcal{W}^\circ \to \mathcal{M}_F$ which factors through $\mathcal{M}_L^\circ \to \mathcal{M}_F$. We will show that the differential of the latter map is an isomorphism.

For a sheaf i_*L corresponding to a point of $\mathcal{M}_L^\circ$ and any tangent vector $u \in \mathrm{Ext}^1(i_*L, i_*L)$, we have the unique morphism of triangles

$$\begin{tikzcd} F_L \arrow[r] \arrow[d, "u'"] & \mathcal{O}_Y^{\oplus 3} \arrow[r] \arrow[d, "0"] & i_*L \arrow[r] \arrow[d, "u"] & F_L[1] \arrow[d, "u'[1]"] \\ F_L[1] \arrow[r] & \mathcal{O}_Y^{\oplus 3}[1] \arrow[r] & i_*L[1] \arrow[r] & F_L[2] \end{tikzcd} \tag{2.5}$$

The uniqueness of u' follows from $\mathrm{Ext}^1(\mathcal{O}_Y, F_L) = 0$. Moreover, u is uniquely determined by u' because $\mathrm{Ext}^1(i_*L, \mathcal{O}_Y) = \mathrm{Ext}^3(\mathcal{O}_Y, i_*L(-3))^* = 0$. This shows that the mutation induces an isomorphism of $\mathrm{Ext}^1(i_*L, i_*L)$ and $\mathrm{Ext}^1(F_L, F_L)$.

Finally, let us prove that the map $\mathcal{M}_L^\circ \to \mathcal{M}_F$ is injective. It follows from Grothendieck–Verdier duality that $\mathcal{E}\mathrm{xt}^2(i_*L, \mathcal{O}_Y) = i_*L^\vee(2)$. Then from (2.3) we see that $\mathcal{E}\mathrm{xt}^1(F_L, \mathcal{O}_Y) = i_*L^\vee(2)$, and hence, L can be reconstructed from F_L. $\square$

2.3. The symplectic form and Lagrangian subvarieties

Let us recall the description of the two-form on the moduli spaces of sheaves on Y from [12]. Given a coherent sheaf $\mathcal{F}$ on Y we can define its Atiyah class $\mathrm{At}_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega_Y)$. The Atiyah class is functorial, meaning that for any morphism of sheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ we have $\mathrm{At}_{\mathcal{G}} \circ \alpha = (\alpha \otimes \mathrm{id}) \circ \mathrm{At}_{\mathcal{F}}$.

We define a bilinear form σ on the vector space $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$. Given two elements $u, v \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ we consider the composition $\mathrm{At}_{\mathcal{F}} \circ u \circ v \in \mathrm{Ext}^3(\mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ and apply the trace map $\mathrm{Tr}\colon \mathrm{Ext}^3(\mathcal{F}, \mathcal{F} \otimes \Omega_Y) \to \mathrm{Ext}^3(\mathcal{O}_Y, \Omega_Y) = H^{1,3}(Y) = \mathbb{C}$ to it:

$$\sigma(u, v) = \mathrm{Tr}(\mathrm{At}_{\mathcal{F}} \circ u \circ v). \tag{2.6}$$

Note that, when the Kuranishi space of $\mathcal{F}$ is smooth, then for any $u \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$ we have $u \circ u = 0$ and then $\sigma(u, u) = 0$. In this case σ is antisymmetric. Hence, the formula (2.6) defines a two-form at smooth points of moduli spaces of sheaves on Y. This form is closed by [12, Theorem 2.2].

LEMMA 2.7

The formula (2.6) defines a symplectic form on $\mathcal{M}_L^\circ$ which coincides up to a nonzero constant with the restriction of the symplectic form on Z under the isomorphism $\mathcal{M}_L^\circ \simeq Z^\circ$.

Proof

By Lemma 2.2 the sheaves i_*L from $\mathcal{M}_L^\circ$ have unobstructed deformations, so that (2.6) indeed defines a two-form.

Recall from Lemma 2.6 that we have an open embedding $\mathcal{M}_L^\circ \hookrightarrow \mathcal{M}_F$. Let us show that this embedding respects (up to a sign) symplectic forms on $\mathcal{M}_L$ and $\mathcal{M}_F$ given by (2.6). Note that by the functoriality of Atiyah classes the following diagram gives a morphism of triangles:

$$\begin{tikzcd} F_L \arrow[r] \arrow[d, "\mathrm{At}_{F_L}"] & \mathcal{O}_Y^{\oplus 3} \arrow[r] \arrow[d, "\mathrm{At}_{\mathcal{O}_Y^{\oplus 3}}=0"] & i_*L \arrow[r] \arrow[d, "\mathrm{At}_{i_*L}"] & F_L[1] \arrow[d, "\mathrm{At}_{F_L}[1]"] \\ F_L \otimes \Omega_Y[1] \arrow[r] & \Omega_Y^{\oplus 3}[1] \arrow[r] & i_*L \otimes \Omega_Y[1] \arrow[r] & F_L \otimes \Omega_Y[2] \end{tikzcd}$$

For any pair of tangent vectors $u, v \in \mathrm{Ext}^1(i_*L, i_*L)$ we have two morphisms of triangles as in (2.5). If we compose these two morphisms of triangles with the one induced by Atiyah classes, then we get the following:

$$\begin{tikzcd} F_L \arrow[r] \arrow[d, "\mathrm{At}_{F_L} \circ u' \circ v'"] & \mathcal{O}_Y^{\oplus 3} \arrow[r] \arrow[d, "0"] & i_*L \arrow[r] \arrow[d, "\mathrm{At}_{i_*L} \circ u \circ v"] & F_L[1] \arrow[d, "\mathrm{At}_{F_L} \circ u' \circ v'[1]"] \\ F_L \otimes \Omega_Y[3] \arrow[r] & \Omega_Y^{\oplus 3}[3] \arrow[r] & i_*L \otimes \Omega_Y[3] \arrow[r] & F_L \otimes \Omega_Y[4] \end{tikzcd}$$

This diagram is a morphism of triangles, and the additivity of traces implies that $\sigma(u,v) = -\sigma(u', v')$.

By [12, Theorem 4.3] the form σ on $\mathcal{M}_F$ is symplectic, because the sheaves F_L are contained in $\mathcal{A}_Y$. Hence, σ is a symplectic form on $\mathcal{M}_L^\circ$. But $\mathcal{M}_L^\circ$ is embedded into Z as an open subset with complement of codimension 4. This implies that the symplectic form on $\mathcal{M}_L^\circ$ is unique up to a constant, because Z is IHS. This completes the proof. □

THEOREM 2.8

The component $\mathcal{M}_F$ of the moduli space of Gieseker-stable sheaves with Hilbert polynomial $P(F_L, n)$ is birational to the IHS manifold Z. Under this birational equivalence the symplectic form on Z defined in [14] corresponds to the Kuznetsov–Markushevich form on $\mathcal{M}_F$.

Proof

This follows from Lemmas 2.3, 2.5, 2.6, and 2.7. □

Now we explain how hyperplane sections of Y give rise to Lagrangian subvarieties of Z. Let $H \subset \mathbb{P}^5$ be a generic hyperplane, so that $Y_H = Y \cap H$ is a smooth cubic threefold. Twisted cubics contained in Z form a subvariety $M_3(Y)_H \subset M_3(Y)$ whose image in Z we denote by Z_H. Its open subset $Z_H^\circ = Z_H \cap Z^\circ$ consists of sheaves i_*L whose support is contained in H.

PROPOSITION 2.9

We have that Z_H is a Lagrangian subvariety of Z.

Proof

It is clear that Z_H has dimension 4, since the Grassmannian of 3-dimensional subspaces in H is $\mathbb{P}^4$. Consider a sheaf i_*L whose support S is smooth and contained in Y_H. Since L is a locally free sheaf on S we have $\mathcal{E}xt^k(i_*L, i_*L) =$

$i_*\Lambda^k N_{S/Y}$ (see, e.g., [12, Lemma 1.3.2]). The higher cohomologies of the sheaves $\mathcal{E}\mathrm{xt}^k(i_*L, i_*L)$ vanish for $k \geq 0$, because $N_{S/Y} = \mathcal{O}_S(1)^{\oplus 2}$ and the sheaves $\mathcal{O}_S(k)$ have no higher cohomologies for $k \geq 0$. Hence, from the local-to-global spectral sequence we find that $T_{i_*L}\mathcal{M}_L = \mathrm{Ext}^1(i_*L, i_*L) = H^0(S, N_{S/Y})$. Moreover, the Yoneda multiplication on the Ext's is given by the map $H^0(S, N_{S/Y}) \times H^0(S, N_{S/Y}) \to H^0(S, \Lambda^2 N_{S/Y})$, which is induced from the exterior product morphism $N_{S/Y} \otimes N_{S/Y} \to \Lambda^2 N_{S/Y}$ (see [12, Lemma 1.3.3]). Now, the tangent space to Z_H at i_*L is $H^0(S, N_{S/Y_H})$. But the exterior product $N_{S/Y_H} \otimes N_{S/Y_H} \to \Lambda^2 N_{S/Y_H} = 0$ vanishes because N_{S/Y_H} is of rank 1. So the Yoneda product vanishes on the corresponding subspace of $\mathrm{Ext}^1(i_*L, i_*L)$, and from the definition of the symplectic form (2.6) we conclude that the tangent subspace to Z_H is Lagrangian. This holds on an open subset of Z_H, so Z_H is a Lagrangian subvariety. □

In the next section we give a description of the subvarieties Z_H in terms of intermediate Jacobians of the threefolds Y_H.

3. Twisted cubics on a cubic threefold

In this section we assume that the cubic fourfold Y and its hyperplane section Y_H are chosen generically, so that Y_H is smooth and all the surfaces obtained by intersecting Y_H with 3-dimensional subspaces have at worst ADE singularities. For general Y and H this indeed will be the case, because for a general cubic threefold in $\mathbb{P}^4$ its hyperplane sections have only ADE singularities. One can see this from dimension count by considering the codimensions of loci of cubic surfaces with different singularity types (see, e.g., [14, Sections 2.2, 2.3]).

The cubic threefold Y_H has an intermediate Jacobian $\mathrm{J}(Y_H)$, which is a principally polarized abelian variety. We will show that if we choose a general hyperplane H, then the Abel–Jacobi map

$$\mathrm{AJ}\colon Z_H \to \mathrm{J}(Y_H)$$

defines a closed embedding on an open subset Z_H° and the complement $Z_H \backslash Z_H^\circ$ is contracted to a point. The image of AJ is the theta-divisor $\Theta \subset \mathrm{J}(Y_H)$.

Recall from the description of Z that we have an embedding $\mu\colon Y \hookrightarrow Z$. We have $Z_H^\circ \simeq Z_H \setminus \mu(Y)$ and $Z_H \cap \mu(Y) \simeq Y_H$. Hence, the Abel–Jacobi map $\mathrm{AJ}\colon Z_H \to \mathrm{J}(Y_H)$ gives a resolution of the unique singular point of the theta-divisor, and the exceptional divisor of this map is isomorphic to Y_H. This explicit description of the singularity of the theta-divisor first obtained in [2] implies Torelli's theorem for cubic threefolds. The fact that Z_H is birational to the theta-divisor in $\mathrm{J}(Y_H)$ also follows from [10] (see also [3, Proposition 4.2]).

3.1. Differential of the Abel–Jacobi map

As before, we will identify the open subset Z_H° with an open subset in the moduli space of sheaves of the form i_*L, where $i\colon S \hookrightarrow Y_H$ is a hyperplane section and L is a sheaf which gives a determinantal representation (2.1) of this section.

The Abel–Jacobi map $\mathrm{AJ}\colon Z_H^\circ \to \mathrm{J}(Y_H)$ can be described as follows. We use the Chern classes with values in the Chow ring $\mathrm{CH}(Y_H)$. The second Chern class $c_2(i_*L) \in \mathrm{CH}^2(Y_H)$ is a cycle class of degree 3. Let $h \in \mathrm{CH}^1(Y_H)$ denote the class of a hyperplane section. Then $c_2(i_*L) - h^2$ is a cycle class homologous to zero, and it defines an element in the intermediate Jacobian. Since $c_2(i_*L)$ can be represented by corresponding twisted cubics, the map above extends to $\mathrm{AJ} : Z_H \to \mathrm{J}(Y_H)$.

LEMMA 3.1

The differential of the Abel–Jacobi map $d\mathrm{AJ}_{i_*L}\colon \mathrm{Ext}^1(i_*L, i_*L) \to H^{1,2}(Y_H)$ *at the point corresponding to the sheaf* i_*L *is given by*

$$d\mathrm{AJ}_{i_*L}(u) = \frac{1}{2}\mathrm{Tr}(\mathrm{At}_{i_*L} \circ u), \tag{3.1}$$

for any $u \in \mathrm{Ext}^1(i_*L, i_*L)$.

Proof

We apply the general formula for the derivative of the Abel–Jacobi map (see Appendix, Proposition A.1). We have $c_1(i_*L) = 0$, so that the second Segre class equals $s_2(i_*L) = -2c_2(i_*L)$, which yields the $\frac{1}{2}$ factor in the statement. □

It will be convenient for us to rewrite (3.1) in terms of the linkage class of a sheaf (see [12]). We recall its definition in our particular case of the embedding $j\colon Y_H \hookrightarrow \mathbb{P}^4$. If $\mathcal{F}$ is a sheaf on Y_H, then the object $j^*j_*\mathcal{F} \in \mathcal{D}^b(Y_H)$ has nonzero cohomologies only in degrees -1 and 0. They are equal to $\mathcal{F} \otimes N^\vee_{Y/\mathbb{P}^4} = \mathcal{F}(-3)$ and $\mathcal{F}$, respectively. Hence, the triangle

$$\mathcal{F}(-3)[1] \longrightarrow \mathrm{L}j^*j_*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}(-3)[2].$$

The last morphism in this triangle is called the *linkage class* of $\mathcal{F}$ and will be denoted by $\epsilon_\mathcal{F}\colon \mathcal{F} \to \mathcal{F}(-3)[2]$. The linkage class can also be described as follows (see [12, Theorem 3.2]): let us denote by $\kappa \in \mathrm{Ext}^1(\Omega_{Y_H}, \mathcal{O}_{Y_H}(-3))$ the extension class of the conormal sequence $0 \to \mathcal{O}_{Y_H}(-3) \to \Omega_{\mathbb{P}^4}|_{Y_H} \to \Omega_{Y_H} \to 0$; then we have $\epsilon_\mathcal{F} = (\mathrm{id}_\mathcal{F} \otimes \kappa) \circ \mathrm{At}_\mathcal{F}$.

Note that composition with κ gives an isomorphism of vector spaces $H^{1,2}(Y_H) = \mathrm{Ext}^2(\mathcal{O}_{Y_H}, \Omega_{Y_H})$ and $\mathrm{Ext}^3(\mathcal{O}_{Y_H}, \mathcal{O}_{Y_H}(-3)) = H^0(Y_H, \mathcal{O}_{Y_H}(1))^*$. Composing the right-hand side of (3.1) with κ and using the fact that taking traces commutes with compositions, we obtain the following expression for $d\mathrm{AJ}(u)$ where $u \in \mathrm{Ext}^1(i_*L, i_*L)$:

$$\kappa \circ d\mathrm{AJ}_{i_*L}(u) = \frac{1}{2}\mathrm{Tr}(\epsilon_{i_*L} \circ u) \in H^0\big(Y_H, \mathcal{O}_H(1)\big)^*. \tag{3.2}$$

PROPOSITION 3.2

The differential of the Abel–Jacobi map (3.1) is injective.

Proof
As before, we will denote by $i\colon S\hookrightarrow Y_H$ and $j\colon Y_H\hookrightarrow\mathbb{P}^4$ the embeddings. A point of Z_H° is represented by a sheaf i_*L. Let us also use the notation $\mathcal{F}=i_*L$. It suffices to show that the map $u\mapsto\kappa\circ d\mathrm{AJ}_{i_*L}(u)$ is injective. The proof is done in three steps.

Step 1. Let us construct a locally free resolution of $j_*\mathcal{F}$. We decompose $j_*\mathcal{F}$ with respect to the exceptional collection $\mathcal{O}_{\mathbb{P}^4}(-2)$, $\mathcal{O}_{\mathbb{P}^4}(-1)$, $\mathcal{O}_{\mathbb{P}^4}$, $\mathcal{O}_{\mathbb{P}^4}(1)$, $\mathcal{O}_{\mathbb{P}^4}(2)$. The sheaf $j_*\mathcal{F}$ is already left-orthogonal to $\mathcal{O}_{\mathbb{P}^4}(2)$ and $\mathcal{O}_{\mathbb{P}^4}(1)$ (see Lemma 2.1). It is globally generated by (2.1), and its left mutation is the shift of the sheaf $\mathcal{K}$ from the exact triple $0\longrightarrow\mathcal{K}\longrightarrow\mathcal{O}_{\mathbb{P}^4}^{\oplus 3}\longrightarrow j_*\mathcal{F}\longrightarrow 0$. From the cohomology exact sequence we see that $H^0(\mathbb{P}^4,\mathcal{K}(1))=\mathbb{C}^6$ and $H^k(\mathbb{P}^4,\mathcal{K}(1))=0$ for $k\geq 1$. We can also check that $\mathcal{K}(1)$ is globally generated. (It is in fact Castelnuovo–Mumford 0-regular, as one can see by using (2.1).) The left mutation of $\mathcal{K}$ through $\mathcal{O}_{\mathbb{P}^4}(-1)$ is the cone of the surjection $\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6}\to\mathcal{K}$, and it lies in the subcategory generated by $\mathcal{O}_{\mathbb{P}^4}(-2)$. Since it has rank 3, this completes the construction of the resolution for $j_*\mathcal{F}$. The resulting resolution is

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^4}(-2)^{\oplus 3}\longrightarrow\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 6}\longrightarrow\mathcal{O}_{\mathbb{P}^4}^{\oplus 3}\longrightarrow j_*\mathcal{F}\longrightarrow 0. \tag{3.3}$$

Step 2. Let us show that the linkage class $\epsilon_{\mathcal{F}}$ induces an isomorphism

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{F})\to\mathrm{Ext}^3\big(\mathcal{F},\mathcal{F}(-3)\big).$$

The object $\mathrm{L}j^*j_*\mathcal{F}$ is included in the triangle

$$\mathrm{L}j^*j_*\mathcal{F}\longrightarrow\mathcal{F}\xrightarrow{\epsilon_{\mathcal{F}}}\mathcal{F}(-3)[2]\longrightarrow\mathrm{L}j^*j_*\mathcal{F}[1].$$

Applying $\mathrm{Hom}(\mathcal{F},-)$ to this triangle we find the following exact sequence:

$$\mathrm{Ext}^1(\mathcal{F},\mathrm{L}j^*j_*\mathcal{F})\longrightarrow\mathrm{Ext}^1(\mathcal{F},\mathcal{F})\xrightarrow{\epsilon_{\mathcal{F}}\circ-}\mathrm{Ext}^3\big(\mathcal{F},\mathcal{F}(-3)\big)\longrightarrow\mathrm{Ext}^2(\mathcal{F},\mathrm{L}j^*j_*\mathcal{F}).$$

Note that by (3.3) the object $\mathrm{L}j^*j_*\mathcal{F}$ is represented by a complex of the form $0\to\mathcal{O}_{Y_H}(-2)^{\oplus 3}\to\mathcal{O}_{Y_H}(-1)^{\oplus 6}\to\mathcal{O}_{Y_H}^{\oplus 3}\to 0$. Let us check that $\mathrm{Ext}^2(\mathcal{F},\mathrm{L}j^*j_*\mathcal{F})=0$. By Serre duality $\mathrm{Ext}^q(\mathcal{F},\mathcal{O}_{Y_H}(-p))=\mathrm{Ext}^{3-q}(\mathcal{O}_{Y_H}(-p),\mathcal{F}(-2))^*=H^{3-q}(Y_H,\mathcal{F}(p-2))^*$. From (2.1) we see that for $p=0$ and 1 these cohomology groups vanish, and for $p=2$ the only nonvanishing group corresponds to $q=3$. The spectral sequence computing $\mathrm{Ext}^k(\mathcal{F},\mathrm{L}j^*j_*\mathcal{F})$, obtained from the complex representing $\mathrm{L}j^*j_*\mathcal{F}$, implies that $\mathrm{Ext}^k(\mathcal{F},\mathrm{L}j^*j_*\mathcal{F})=0$ for $k\neq 1$ and $\mathrm{Ext}^1(\mathcal{F},\mathrm{L}j^*j_*\mathcal{F})=H^0(Y_H,\mathcal{F})^*=\mathbb{C}^3$.

We conclude that the map $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})\xrightarrow{\epsilon_{\mathcal{F}}\circ-}\mathrm{Ext}^3(\mathcal{F},\mathcal{F}(-3))$ is surjective. It is actually an isomorphism, because the vector spaces are of the same dimension. The dimensions can be computed in the same way as in the proof of Lemma 2.7.

Step 3. Let us show that $\mathrm{Tr}:\mathrm{Ext}^3(\mathcal{F},\mathcal{F}(-3))\to H^3(Y_H,\mathcal{O}_{Y_H}(-3))$ is injective. Using Serre duality we identify the dual to the trace map with

$$\mathrm{Tr}^*:H^0\big(Y_H,\mathcal{O}_{Y_H}(1)\big)\to\mathrm{Hom}\big(\mathcal{F},\mathcal{F}(1)\big).$$

One can show as in the proof of Lemma 2.2 that $\mathrm{Hom}(\mathcal{F},\mathcal{F}(1))=H^0(S,\mathcal{O}(1))$, and postcomposing Tr^* with this isomorphism gives the restriction map

$$H^0\big(Y_H,\mathcal{O}_{Y_H}(1)\big)\to H^0\big(S,\mathcal{O}_S(1)\big),$$

which is surjective. We see that the composition

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \to \mathrm{Ext}^3\big(\mathcal{F},\mathcal{F}(-3)\big) \to H^3\big(Y_H,\mathcal{O}(-3)\big)$$

is injective, and the proof is finished by means of formula (3.2). □

3.2. Image of the Abel–Jacobi map

THEOREM 3.3

Assume that Y_H is smooth and all its hyperplane sections have at worst ADE singularities. Then the image of the Abel–Jacobi map $\mathrm{AJ}\colon Z_H \to \mathrm{J}(Y_H)$ is the theta-divisor $\Theta \subset \mathrm{J}(Y_H)$. The map AJ is an embedding on Z_H° and contracts the divisor $Y_H = Z_H \backslash Z_H^\circ$ to the unique singular point of Θ.

Proof

The divisor Y_H is contracted by the Abel–Jacobi map to a point, because Y_H is a cubic threefold which has no global one-forms. To identify the image of AJ it is enough to check that a general point of Z_H is mapped to a point of Θ. The general point $z \in Z_H$ is represented by a smooth twisted cubic C on a smooth hyperplane section $S \subset Y_H$. Denote by $C_0 \subset S$ a hyperplane section of S. Then $C - C_0$ is a degree 0 cycle on Y_H, and z is mapped to the corresponding element of the intermediate Jacobian. The cohomology class $[C - C_0] \in H^2(S,\mathbb{Z})$ is orthogonal to the class of the canonical bundle K_S and has square -2. Hence, it is a root in the E_6-lattice. All such cohomology classes can be represented by differences of pairs of lines $l_1 - l_2$ in six different ways.

Recall that the Fano variety of lines on the cubic threefold Y_H is a surface, which we will denote by X. It was shown in [4] that the theta-divisor $\Theta \subset \mathrm{J}(Y_H)$ can be described as the image of the map $X \times X \to \mathrm{J}(Y_H)$ which sends a pair of lines (l_1, l_2) to the point in $\mathrm{J}(Y_H)$ corresponding to the degree 0 cycle $l_1 - l_2$. The map $X \times X \to \Theta$ has degree 6. We get a commutative diagram:

$$\begin{tikzcd}
X \times X \arrow[r, "6:1"] \arrow[d, dashed, "6:1"'] & \Theta \\
Z_H \arrow[ur, "\mathrm{AJ}"'] &
\end{tikzcd}$$

It follows from the diagram above that AJ is generically of degree 1. Since AJ is étale on Z_H° by Proposition 3.2 and the theta-divisor Θ is a normal variety (see [2, Section 3, Proposition 2]), we deduce that $\mathrm{AJ} : Z_H^\circ \to \Theta$ is an open embedding. This completes the proof. □

Appendix: Differential of the Abel–Jacobi map

Let X be a smooth complex projective variety of dimension n. Recall that the pth intermediate Jacobian of X is the complex torus

$$\mathrm{J}^p(X) = H^{2p-1}(X,\mathbb{C})/\big(F^p H^{2p-1}(X,\mathbb{C}) + H^{2p-1}(X,\mathbb{Z})\big),$$

where $F^\bullet$ denotes the Hodge filtration. We use the Abel–Jacobi map (see [7, Appendix A])

$$\mathrm{AJ}^p\colon \mathrm{CH}^p(X,\mathbb{Z})_h \to \mathrm{J}^p(X),$$

where $\mathrm{CH}^p(X)_h$ is the group of homologically trivial codimension p algebraic cycles on X up to rational equivalence.

For a coherent sheaf $\mathcal{F}_0$ on X we consider integral Segre characteristic classes

$$s_p(\mathcal{F}_0) = p! \cdot ch_p(\mathcal{F}_0) \in \mathrm{CH}^p(X,\mathbb{Z}),$$

where $ch_p(\mathcal{F}_0)$ is the p'th component of the Chern character $ch(\mathcal{F}_0)$. Segre classes can be expressed in terms of the Chern classes using Newton's formula (see [15, Section 16]).

Let us consider a deformation of $\mathcal{F}_0$ over a smooth base B with base point $0 \in B$, that is, a coherent sheaf $\mathcal{F}$ on $X \times B$ flat over B and with $\mathcal{F}_0 \simeq \mathcal{F}|_{\pi_B^{-1}(0)}$. We will denote by π_B and π_X the two projections from $X \times B$ and denote by $\mathcal{F}_t$ the restriction of $\mathcal{F}$ to $\pi_B^{-1}(t)$, $t \in B$. In this setting the difference of Segre classes $s_p(\mathcal{F}_t) - s_p(\mathcal{F}_0)$ is contained in $\mathrm{CH}^p(X,\mathbb{Z})_h$, and we get an induced Abel–Jacobi map

$$\mathrm{AJ}^p_{\mathcal{F}} : B \to J^p(X).$$

Since Segre classes are additive, it follows that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves on $X \times B$ flat over B, then

$$\mathrm{AJ}^p_{\mathcal{F}} = \mathrm{AJ}^p_{\mathcal{F}'} + \mathrm{AJ}^p_{\mathcal{F}''}. \tag{A.1}$$

Recall that a coherent sheaf $\mathcal{F}_0$ has an Atiyah class $\mathrm{At}_{\mathcal{F}_0} \in \mathrm{Ext}^1(\mathcal{F}_0, \mathcal{F}_0 \otimes \Omega_X)$ (see [12, Section 1.6]). The vector space $\bigoplus_{p,q\geq 0} \mathrm{Ext}^q(\mathcal{F}_0, \mathcal{F}_0 \otimes \Omega^p_X)$ has the structure of a bi-graded algebra with multiplication induced by the Yoneda product of the Ext's and the exterior product of differential forms, and this defines the p'th power of the Atiyah class

$$\mathrm{At}^p_{\mathcal{F}_0} \in \mathrm{Ext}^p(\mathcal{F}_0, \mathcal{F}_0 \otimes \Omega^p_X).$$

Given any tangent vector $v \in T_0 B$ we shall denote its Kodaira–Spencer class by $\mathrm{KS}_{\mathcal{F}_0}(v) \in \mathrm{Ext}^1(\mathcal{F}_0, \mathcal{F}_0)$, and we consider the composition $\mathrm{At}^p_{\mathcal{F}_0} \circ \mathrm{KS}_{\mathcal{F}_0}(v) \in \mathrm{Ext}^{p+1}(\mathcal{F}_0, \mathcal{F}_0 \otimes \Omega^p_X)$. We will also use the trace maps (see [12, Section 1.2])

$$\mathrm{Tr}\colon \mathrm{Ext}^q(\mathcal{F}_0, \mathcal{F}_0 \otimes \Omega^p_X) \to \mathrm{Ext}^q(\mathcal{O}_X, \Omega^p_X) = H^{p,q}(X).$$

PROPOSITION A.1

In the above setting the differential of the Abel–Jacobi map $\mathrm{AJ}^p_{\mathcal{F}} : B \to \mathrm{J}^p(X)$, $p \geq 2$, *at* $0 \in B$ *is given by*

$$d\mathrm{AJ}^p_{\mathcal{F},0}(v) = \mathrm{Tr}\big((-1)^{p-1} \mathrm{At}^{p-1}_{\mathcal{F}_0} \circ \mathrm{KS}_{\mathcal{F}_0}(v)\big), \tag{A.2}$$

for any $v \in T_0 B$. *The right-hand side is an element of* $H^{p-1,p}(X) \subset H^{2p-1}(X,\mathbb{C})/F^p H^{2p-1}(X,\mathbb{C})$.

Proof
We argue by induction on the length of a locally free resolution of $\mathcal{F}$. The base of induction is the case when $\mathcal{F}_0$ is a vector bundle. Then the result is essentially contained in the work of Griffiths [6] (in particular, [6, (6.8)]). We will show how to do the induction step. We note that the statement is local, so we may replace the base B by an open neighborhood of $0 \in B$ every time it is necessary. In particular, we assume that B is affine.

By our assumptions X is projective, and we denote by $\mathcal{O}_X(1)$ an ample line bundle. Then we can find k big enough, so that $\mathcal{F}(k)$ is generated by global sections and has no higher cohomology. We define a sheaf $\mathcal{G}$ on $X \times B$ as the kernel of the natural map

$$0 \longrightarrow \mathcal{G} \longrightarrow \pi_B^* \pi_{B*}\big(\mathcal{F}(k)\big) \otimes \mathcal{O}_X(-k) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Since $\mathcal{F}$ is flat over B and $\pi_{B*}(\mathcal{F}_0(k))$ is a vector bundle on B for k large enough (see [8, proof of Theorem 9.9]), the sheaf $\mathcal{G}$ is flat over B.

It follows from (A.1) that $\mathrm{AJ}^p_{\mathcal{G}} = -\mathrm{AJ}^p_{\mathcal{F}}$. Since the homological dimension of $\mathcal{G}$ has dropped by 1, the induction hypothesis yields the formula (A.2) for $\mathcal{G}$. It remains to relate the right-hand side of (A.2) for $\mathcal{G}_0$ and for $\mathcal{F}_0$.

Using the functoriality of the Kodaira–Spencer classes we obtain the following morphism of triangles:

$$\begin{tikzcd}
\mathcal{G}_0 \arrow[r] \arrow[d, "u'"] & H^0(X, \mathcal{F}_0(k)) \otimes \mathcal{O}_X(-k) \arrow[r] \arrow[d, "0"] & \mathcal{F}_0 \arrow[r] \arrow[d, "u"] & \mathcal{G}_0[1] \arrow[d, "{u'[1]}"] \\
\mathcal{G}_0[1] \arrow[r] & H^0(X, \mathcal{F}_0(k)) \otimes \mathcal{O}_X(-k)[1] \arrow[r] & \mathcal{F}_0[1] \arrow[r] & \mathcal{G}_0[2]
\end{tikzcd}$$

where $u = \mathrm{KS}_{\mathcal{F}_0}(v) \in \mathrm{Ext}^1(\mathcal{F}_0, \mathcal{F}_0)$ and $u' = \mathrm{KS}_{\mathcal{G}_0}(v) \in \mathrm{Ext}^1(\mathcal{G}_0, \mathcal{G}_0)$. Composing the vertical arrows with $\mathrm{At}^{p-1}_{\mathcal{F}_0}$, $\mathrm{At}^{p-1}_{\mathcal{O}_X(-k)}$, and $\mathrm{At}^{p-1}_{\mathcal{F}_0}$, respectively, and using the additivity of traces, we get $\mathrm{Tr}(\mathrm{At}^{p-1}_{\mathcal{F}_0} \circ \mathrm{KS}_{\mathcal{F}_0}(v)) = -\mathrm{Tr}(\mathrm{At}^{p-1}_{\mathcal{G}_0} \circ \mathrm{KS}_{\mathcal{G}_0}(v))$ because the map in the middle is zero. This completes the induction step. $\square$

Acknowledgments. The authors would like to thank Alexander Kuznetsov, Daniel Huybrechts, Christoph Sorger, Manfred Lehn, and Yukinobu Toda for useful discussions and remarks.

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