Moduli spaces of principal bundles on singular varieties

Adrian Langer

To the memory of Professor Masaki Maruyama

Abstract Let k be an algebraically closed field of characteristic zero. Let $f : X \to S$ be a flat, projective morphism of k-schemes of finite type with integral geometric fibers. We prove the existence of a projective relative moduli space for semistable singular principal bundles on the fibers of f.

This generalizes the result of A. Schmitt, who studied the case when X is a nodal curve.

1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field k of characteristic zero. In [14] and [15] M. Maruyama, generalizing Gieseker's result from the surface case, constructed coarse moduli spaces of semistable sheaves on X. (In fact, the construction worked in some other cases.) Later these moduli spaces were also constructed for arbitrary varieties (see C. Simpson's paper [21]) and in an arbitrary characteristic (see [12] and [11]). Since the moduli space of semistable sheaves compactifies the moduli space of (semistable) vector bundles, it is an obvious problem to try to construct similar compactifications in the case of principal bundles. This problem was considered by many authors (see [19] and the references within), and it was solved in the case of smooth varieties. However, in the case of singular varieties the problem is still open in spite of some partial results (see, e.g., [3] and [18]). The aim of this paper is to solve this problem in the characteristic zero case.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a faithful k-representation of the reductive group G. In the following we assume that image of the representation ρ is contained in $\mathrm{SL}(V)$.

A *pseudo-G-bundle* is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a torsion-free $\mathcal{O}_X$-module of rank $r = \dim V$ and $\tau\colon \mathrm{Sym}^*(\mathcal{A} \otimes V)^G \to \mathcal{O}_X$ is a nontrivial homomorphism of $\mathcal{O}_X$-algebras. In [3] U. Bhosle, following earlier work of A. Schmitt [16] in the smooth case, constructed the moduli space of pseudo-G-bundles in the case when

Kyoto Journal of Mathematics, Vol. 53, No. 1 (2013), 3–23
DOI 10.1215/21562261-1966053, © 2013 by Kyoto University
Received August 26, 2011. Revised December 28, 2011. Accepted January 5, 2012.
2010 Mathematics Subject Classification: Primary 14D20, 14D22; Secondary 14H60, 14J60.
Author's work partially supported by the Alexander von Humboldt Foundation Bessel Research Award and by Polish Ministry of Science and Higher Education (MNiSW) grant N N201 420639.

X satisfies some technical condition, which she showed to hold for seminormal or S_2-varieties. However, it is easy to see that this condition is always satisfied (see Lemma 2.3).

Giving the homomorphism τ is equivalent to giving a section

$$\sigma : X \to \mathbb{H}\mathrm{om}(\mathcal{A}, V^\vee \otimes \mathcal{O}_X)//G = \mathrm{Spec}\big(\mathrm{Sym}^*(\mathcal{A} \otimes V)^G\big).$$

Let $U_{\mathcal{A}}$ denote the maximum open subset of X where $\mathcal{A}$ is locally free. We say that the pseudo-G-bundle $(\mathcal{A}, \tau)$ is a *singular principal G-bundle* if there exists a nonempty open subset $U \subset U_{\mathcal{A}}$ such that $\sigma(U) \subset \mathbb{I}\mathrm{som}(V \otimes \mathcal{O}_U, \mathcal{A}^\vee|_U)/G$.

In the case when X is smooth, A. Schmitt showed in [17] that the moduli space of δ-semistable pseudo-G-bundles parameterizes only singular principal G-bundles (for large values of the parameter polynomial δ). In a subsequent paper [18], he also showed that in the case when X is a curve with only nodes as singularities, the moduli space constructed by Bhosle parameterizes only singular principal G-bundles. Moreover, under some mild assumptions on the representation ρ, he proved that $\sigma(U_{\mathcal{A}}) \subset \mathbb{I}\mathrm{som}(V \otimes \mathcal{O}_{U_{\mathcal{A}}}, \mathcal{A}^\vee|_{U_{\mathcal{A}}})/G$. (In this case we say that $(\mathcal{A}, \tau)$ is an *honest singular principal G-bundle.*)

In this paper we prove that the same result holds for all the varieties: the moduli space constructed by Bhosle (for large values of the parameter polynomial δ) parameterizes singular principal G-bundles for all varieties X and all representations ρ. More precisely, we prove the following theorem.

THEOREM 1.1

Let $f : X \to S$ be a flat, projective morphism of k-schemes of finite type with integral geometric fibers. Assume that k has characteristic zero. Let us fix a polynomial P and a faithful representation $\rho\colon G \to \mathrm{SL}(V) \subset \mathrm{GL}(V)$ of the reductive algebraic group G.

1. *There exists a projective moduli space $M^\rho_{X/S,P} \to S$ for S-flat families of semistable singular principal G-bundles on $X \to S$ such that for all $s \in S$ the restriction $\mathcal{A}|_{X_s}$ has Hilbert polynomial P.*
2. *Let P correspond to sheaves of degree zero. If the fibers of f are Gorenstein and there exists a G-invariant nondegenerate quadratic form φ on V, then $M^\rho_{X/S,P} \to S$ parameterizes only honest singular principal G-bundles.*

Since the fiber of $M^\rho_{X/S,P} \to S$ over $s \in S$ is equal to $M^\rho_{X_s,P}$ this theorem shows that moduli spaces of singular principal bundles are compatible with degeneration.

Our approach is similar to the one used in [5] and [6] as explained in [19]: we prove a global boundedness result for swamps. (This part of our paper works in any characteristic.) Then we use this fact to prove the semistable reduction theorem in the same way as in the case of smooth varieties. The above-mentioned boundedness result is the main novelty of the paper. It is obtained by proving that the tensor product of semistable sheaves on a variety is not far from being semistable.

The second part of the theorem follows from careful computation of Hilbert polynomials of dual sheaves on Gorenstein varieties.

Unfortunately, the above approach does not work in positive characteristic because we still do not know how to construct moduli spaces of swamps for representations of type $\rho_{a,b,c}\colon \mathrm{GL}(V)\to \mathrm{GL}((V^{\otimes a})^{\oplus b}\otimes(\det V)^{-c})$ for $c\neq 0$. In the case of characteristic zero, to construct the moduli space of pseudo-G-bundles it was sufficient to use moduli spaces of $\rho_{a,b,c}$-swamps for $c=0$. But the construction used the Reynolds operator, which is not available in positive characteristic.

Moreover, in positive characteristic there appears a serious problem with defining the pullback operation for families of pseudo-G-bundles on nonnormal varieties (see [19, Remark 2.9.2.23]).

The structure of the paper is as follows. In Section 2 we recall some definitions and results, and we show that Bhosle's condition is satisfied for all varieties. In Section 3 we study Picard schemes in the relative setting, and we state some existence results for moduli spaces of swamps. Section 4 is the technical heart of the paper: we prove that the tensor product of semistable sheaves on nonnormal varieties is close to being semistable. Then in Section 5 we show that in many cases singular principal bundles of degree zero are honest. In Section 6 we use all these results to prove a semistable reduction theorem and to show the existence of projective relative moduli spaces for (honest) singular principal bundles.

Notation. All the schemes in the paper are locally Noetherian. A *variety* is an irreducible and reduced separated scheme of finite type over an algebraically closed field.

2. Preliminaries

2.1. Basic definitions

Let X be a d-dimensional projective variety over an algebraically closed field k. Let $\mathcal{O}_X(1)$ be an ample line bundle on X.

We say that a coherent sheaf E on X is *torsion-free* if it is pure of dimension d. For a torsion-free sheaf E we can write its Hilbert polynomial as

$$P(E)(m):=\chi\big(X,E\otimes\mathcal{O}_X(m)\big)=\sum_{i=0}^{d}\alpha_i(E)\frac{m^i}{i!}.$$

The *rank* of E is defined as the dimension of $E\otimes K(X)$, where $K(X)$ is the field of rational functions. It is denoted by $\mathrm{rk}\,E$, and it is equal to $\alpha_d(E)/\alpha_d(\mathcal{O}_X)$. We also define the *degree* of E as

$$\deg E=\alpha_{d-1}(E)-\mathrm{rk}\,E\cdot\alpha_{d-1}(\mathcal{O}_X)$$

(see [9, Definition 1.2.11]). The *slope* $\mu(E)$ is, as usual, defined as the quotient of the degree of E by the rank of E.

For two coherent sheaves E,F on X we set

$$E\mathbin{\widehat{\otimes}}F=E\otimes F/\mathrm{Torsion}.$$

LEMMA 2.1

If X is a normal variety and E and F are torsion-free sheaves on X, then

$$\mu(E \widehat{\otimes} F) = \mu(E) + \mu(F).$$

Proof

If E is a torsion-free sheaf, then for a general choice of hyperplanes $H_1, \dots, H_d \in |\mathcal{O}_X(1)|$ we have

$$P(E)(m) = \sum_{i=0}^{d} \chi(E|_{\bigcap_{j\le i} H_j}) \binom{m+i-1}{i}$$

(see [9, Lemma 1.2.1]). It follows that the rank and degree of E depend only on $\chi(E|_{\bigcap_{j\le i} H_j})$ for $i=d$ and $i=d-1$.

If X is a normal variety, then by assumption E is locally free outside of a closed subset of codimension ≥ 2. For a general choice of hyperplanes $H_1, \dots, H_d \in |\mathcal{O}_X(1)|$ the intersection $\bigcap_{j\le d} H_j$ is a union of points and $\bigcap_{j\le d-1} H_j$ is a smooth curve. Therefore the sheaves $E|_{\bigcap_{j\le i} H_j}$ for $i=d$ and $i=d-1$ are locally free. Similarly, the sheaves $F|_{\bigcap_{j\le i} H_j}$ for $i=d$ and $i=d-1$ are locally free. Since in the case of points and smooth curves our assertion is clear, we get the lemma. $\square$

If X is normal, then we can define the determinant of a torsion-free sheaf E as the reflexivization of $\bigwedge^{\operatorname{rk} E} E$. In this case the degree $\deg E$ is equal to the degree of the determinant. This fact follows immediately from the proof of the above lemma.

2.2. Serre's conditions S_k

We say that a coherent sheaf E on a scheme X satisfies *condition S_k* if for all points $x \in X$ we have $\operatorname{depth}_x(E_x) \ge \min(\dim E_x, k)$.

The following lemma is quite standard, but we need a more general version than usual. In the case of smooth projective varieties it is essentially equivalent to [9, Proposition 1.1.6].

LEMMA 2.2

Let X be a Cohen–Macaulay scheme of finite type over a field. Then

1. $\mathcal{E}xt^q_X(E,\omega_X)$ *is supported on the support of E and for all points $x \in X$ we have $\mathcal{E}xt^q_X(E,\omega_X)_x = 0$ if $q < \operatorname{codim}_x E$. Moreover,* $\operatorname{codim}_x \mathcal{E}xt^q_X(E,\omega_X) \ge q$ *for $q \ge \operatorname{codim}_x E$.*
2. *E satisfies condition S_k if and only if for all points $x \in X$ we have* $\operatorname{codim}_x \mathcal{E}xt^q_X(E,\omega_X) \ge q+k$ *for all $q > \operatorname{codim}_x E$.*

Proof

By assumption X is Cohen–Macaulay, and every local ring $\mathcal{O}_{X,x}$ is a quotient of a regular local ring, so we can apply the local duality theorem (see [8, Theorem 6.7]) to prove that $\mathcal{E}xt^q_X(E,\omega_X)_x \ne 0$ if and only if $\mathcal{H}_x^{\dim_x X - q}(E) \ne 0$. But the local cohomology $\mathcal{H}_x^{\dim_x X-q}(E)$ vanishes if $\dim_x X - q > \dim_x E$, which proves the first part of (1). If $q = \operatorname{codim}_x E$, then $\operatorname{codim}_x(\mathcal{E}\mathrm{xt}^q_X(E,\omega_X)) \ge q$ is equivalent to the obvious inequality $\dim_x(\mathcal{E}\mathrm{xt}^q_X(E,\omega_X)) \le \dim_x E$. Hence, since every sheaf satisfies S_0, the second part of (1) follows from (2).

To prove (2) note that by [8, Theorem 3.8] $\operatorname{depth}_x(E_x) \ge \min(\dim E_x, k)$ if and only if $\mathcal{H}^i_x(E) = 0$ for all $i < \min(\dim E_x, k)$. By the local duality theorem this last condition is equivalent to $\mathcal{E}\mathrm{xt}^q_X(E,\omega_X)_x = 0$ for $q > \max(\operatorname{codim}_x E, \dim \mathcal{O}_{X,x} - k)$. This is equivalent to saying that for $q > \operatorname{codim}_x E$ a nonvanishing of $\mathcal{E}\mathrm{xt}^q_X(E,\omega_X)_x$ implies $\dim \mathcal{O}_{X,x} \ge q + k$. $\square$

Let k be an algebraically closed field. Let X be a d-dimensional, pure (i.e., $\mathcal{O}_X$ satisfies S_1) scheme of finite type over k. Let C be a smooth curve defined over k, and let us fix a closed point $0 \in C$. By $p_X : Z = X \times C \to X$ we denote the projection. Let Y be a nonempty proper closed subscheme of $X \times \{0\}$ (in particular, we assume that X has dimension ≥ 1), and let $i : Y \hookrightarrow Z$ denote the corresponding closed embedding. Let us also set $U = Z - Y$, and let $j : U \hookrightarrow Z$ denote the corresponding open embedding.

LEMMA 2.3

If E is a pure sheaf of dimension d on X, then we have a canonical isomorphism $p_X^* E \simeq j_* j^*(p_X^* E)$. In particular, $\mathcal{O}_Z \simeq j_* \mathcal{O}_U$, and for any locally free sheaf F on Z we have $F \simeq j_* j^* F$.

Proof

Let us set $F = p_X^* E$. Since we have a canonical map $F \to j_* j^* F$, the assertion is local and hence we can assume that X and Y are affine. By [8, Proposition 2.2] we have an exact sequence

$$0 \to i_* \mathcal{H}^0_Y(F) \to F \to j_* j^* F \to i_* \mathcal{H}^1_Y(F) \to 0.$$

To prove that $i_* \mathcal{H}^i_Y(F) = 0$ for $i = 0, 1$, it is sufficient to prove that for every point $y \in Y$, the depth of F_y is at least 2 (see [8, Theorem 3.8]). Now, let us take a local parameter $s \in \mathcal{O}_{C,0}$. Then $F_y / s F_y \simeq E_y$ has depth at least 1 (because by assumption E satisfies S_1), so the required assertion is clear. $\square$

REMARK 2.4

The above lemma shows in particular that every variety satisfies condition (2.19) in the sense of Bhosle (see [3, Definition 2.8]).

2.3. Moduli spaces of pseudo-G-bundles

Let us fix a faithful representation $\rho\colon G \to \mathrm{SL}(V) \subset \mathrm{GL}(V)$, $r = \dim V$, of a reductive algebraic group G.

A *pseudo-G-bundle* is a pair $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a torsion-free $\mathcal{O}_X$-module of rank r, and $\tau\colon \mathrm{Sym}^*(\mathcal{A} \otimes V)^G \to \mathcal{O}_X$ is a nontrivial homomorphism of $\mathcal{O}_X$-

algebras. Giving τ is equivalent to giving a section

$$\sigma : X \to \mathbb{H}\mathrm{om}(\mathcal{A}, V^\vee \otimes \mathcal{O}_X)/\!/G = \mathrm{Spec}\big(\mathrm{Sym}^*(\mathcal{A} \otimes V)^G\big).$$

A *weighted filtration* $(\mathcal{A}_\bullet, \alpha_\bullet)$ of $\mathcal{A}$ is a pair consisting of a filtration

$$\mathcal{A}_\bullet = (0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_s \subset \mathcal{A})$$

by saturated subsheaves (i.e., such that the quotients $\mathcal{A}/\mathcal{A}_i$ are torsion-free) of increasing ranks and an s-tuple

$$\alpha_\bullet = (\alpha_1, \ldots, \alpha_s)$$

of positive rational numbers. To every weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ one can associate the polynomial

$$M(\mathcal{A}_\bullet, \alpha_\bullet) := \sum_{i=1}^{s} \alpha_i \big(P(\mathcal{A}) \cdot \mathrm{rk}(\mathcal{A}_i) - P(\mathcal{A}_i) \cdot \mathrm{rk}(\mathcal{A})\big).$$

If $(\mathcal{A}_\bullet, \alpha_\bullet)$ is a weighted filtration of a pseudo-G-bundle $(\mathcal{A}, \tau)$, then one can also define the number $\mu(\mathcal{A}_\bullet, \alpha_\bullet, \tau)$ describing the stability of the $\mathrm{SL}(\mathcal{A} \otimes K(X))$-group action on $\mathrm{Hom}(\mathcal{A} \otimes K(X), V^\vee \otimes K(X))/\!/G$ (see, e.g., [20, 3.3.2]).

Let us fix a positive polynomial δ with rational coefficients and of degree $\leq \dim X - 1$. Then we say that a pseudo-G-bundle $(\mathcal{A}, \tau)$ is *δ-(semi)stable* if $\mathcal{A}$ is torsion-free, and for any weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of $\mathcal{A}$ we have the inequality

$$M(\mathcal{A}_\bullet, \alpha_\bullet) + \delta \cdot \mu(\mathcal{A}_\bullet, \alpha_\bullet, \tau) (\geq) 0.$$

To define the slope version of (semi)stability, instead of $M(\mathcal{A}_\bullet, \alpha_\bullet)$ one uses the rational number

$$L(\mathcal{A}_\bullet, \alpha_\bullet) := \sum_{i=1}^{s} \alpha_i \big(\deg \mathcal{A} \cdot \mathrm{rk}(\mathcal{A}_i) - \deg \mathcal{A}_i \cdot \mathrm{rk}(\mathcal{A})\big).$$

The next theorem follows from the results of Schmitt [16] (in the smooth case) and from the results of Bhosle [3] and Lemma 2.3 in general.

THEOREM 2.5

Let $(X, \mathcal{O}_X(1))$ be a polarized projective variety defined over an algebraically closed field of characteristic zero. Then there exists a projective moduli space $M_{X,P}^{\varrho;\delta}$ for δ-semistable pseudo G-bundles $(\mathcal{A}, \tau)$ on X, such that $\mathcal{A}$ has Hilbert polynomial P (with respect to $\mathcal{O}_X(1)$).

2.4. Semistability of singular principal G-bundles

Let $(\mathcal{A}, \tau)$ be a pseudo G-bundle. Let us recall that giving τ is equivalent to giving a section

$$\sigma : X \to \mathbb{H}\mathrm{om}(\mathcal{A}, V^\vee \otimes \mathcal{O}_X)/\!/G = \mathrm{Spec}\big(\mathrm{Sym}^*(\mathcal{A} \otimes V)^G\big).$$

Let $U_{\mathcal{A}}$ denotes the maximum open subset of X where $\mathcal{A}$ is locally free. The pseudo-G-bundle $(\mathcal{A}, \tau)$ is a *singular principal G-bundle* if there exists a nonempty open subset $U \subset U_{\mathcal{A}}$ such that

$$\sigma(U) \subset \mathbb{I}\mathrm{som}(V \otimes \mathcal{O}_U, \mathcal{A}^\vee|_U)/G.$$

If $\mathcal{A}$ has degree zero and $\sigma(U_{\mathcal{A}}) \subset \mathbb{I}\mathrm{som}(V \otimes \mathcal{O}_{U_{\mathcal{A}}}, \mathcal{A}^\vee|_{U_{\mathcal{A}}})/G$, then we say that $(\mathcal{A}, \tau)$ is an *honest singular principal G-bundle*.

Let us recall that a singular principal G-bundle $(\mathcal{A}, \tau)$, via the following pullback diagram, defines a principal G-bundle $\mathcal{P}(\mathcal{A}, \tau)$ over the open subset U:

$$\begin{tikzcd}
\mathcal{P}(\mathcal{A},\tau) \arrow[r] \arrow[d] & \mathbb{I}\mathrm{som}(V \otimes \mathcal{O}_U, \mathcal{A}^\vee|_U) \arrow[d] \\
U \arrow[r, "\sigma|_U"] & \mathbb{I}\mathrm{som}(V \otimes \mathcal{O}_U, \mathcal{A}^\vee|_U)/G.
\end{tikzcd}$$

If X is smooth, then every singular principal G-bundle is honest (see [20, Lemma 3.4.2]). Note that our definitions are slightly different from those appearing in previous literature (which changed in time to those that are close to our definitions).

Let $(\mathcal{A}, \tau)$ be a singular principal G-bundle, and let $\lambda : \mathbb{G}_m \to G$ be a one-parameter subgroup of G. Let

$$Q_G(\lambda) := \Big\{ g \in G : \lim_{t \to \infty} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G \Big\}.$$

A *reduction* of $(\mathcal{A}, \tau)$ to λ is a section $\beta : U' \to \mathcal{P}(\mathcal{A}, \tau)/Q_G(\lambda)$ defined over some nonempty open subset $U' \subset U$. This reduction defines a reduction of structure group of a principal $\mathrm{GL}(V)$-bundle associated to $\mathcal{A}|_{U'}$ to the parabolic subgroup $Q_{\mathrm{GL}(V)}(\lambda)$, so we get a weighted filtration $(\mathcal{A}'_\bullet, \alpha_\bullet)$ of $\mathcal{A}|_{U'}$.

Let $j : U' \hookrightarrow X$ denote the open embedding. Then for $i = 1, \ldots, s$ we define $\mathcal{A}_i$ as the saturation of $\mathcal{A} \cap j_*(\mathcal{A}'_i)$. In particular, we get a weighted filtration $(\mathcal{A}_\bullet, \alpha_\bullet)$ of $\mathcal{A}$.

We say that a singular principal G-bundle $(\mathcal{A}, \tau)$ is *(semi)stable* if $\mathcal{A}$ is torsion-free and for any reduction of $(\mathcal{A}, \tau)$ to a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ we have the inequality

$$M(\mathcal{A}_\bullet, \alpha_\bullet) (\geq) 0.$$

3. Moduli spaces of swamps revisited

In this section we recall and re-prove some basic results concerning the existence of the relative Picard scheme and its compactifications. Then we apply these results to the existence of moduli spaces of swamps.

We interpret the compactified Picard scheme as the coarse moduli space of stable rank 1 sheaves, and we use Simpson's construction of these moduli spaces to prove the existence of the universal family (i.e., the Poincaré sheaf) under appropriate assumptions. This approach, although very natural, seems to be hard to find in existing literature, especially in the relative case.

The notation in this section is as follows. R denotes a universally Japanese ring. We also fix a projective morphism $f : X \to S$ of R-schemes of finite type with geometrically connected fibers. We assume that f is of pure relative dimension d. By $\mathcal{O}_X(1)$ we denote an f-very ample line bundle on X. We also fix a polynomial P.

3.1. Universal families on relative moduli spaces

Let us define the moduli functor $\mathcal{M}_{X/S,P} : (\mathrm{Sch}\,/S) \longrightarrow (\mathrm{Sets})$ by sending $T \to S$ to

$$\mathcal{M}_{X/S,P}(T) = \left\{ \begin{matrix} \text{isomorphism classes of } T\text{-flat families of Gieseker} \\ \text{semistable sheaves with Hilbert polynomial } P \\ \text{on the geometric fibers of } p : T \times_S X \to T \end{matrix} \right\} \Big/ \sim,$$

where $\sim$ is the equivalence relation $\sim$ defined by $F \sim F'$ if and only if there exists an invertible sheaf K on T such that $F \simeq F' \otimes p^* K$.

THEOREM 3.1 (SEE [14], [15], [21], [12] AND [11])

There exists a projective S-scheme $M_{X/S,P}$, which uniformly corepresents the functor $\mathcal{M}_{X/S,P}$. Moreover, there is an open subscheme $M^s_{X/S,P} \subset M_{X/S,P}$ that universally corepresents the subfunctor $\mathcal{M}^s_{X/S,P}$ of families of geometrically Gieseker stable sheaves.

We are interested in when the moduli scheme $M^s_{X/S,P}$ represents the functor $\mathcal{M}^s_{X/S,P}$. This is equivalent to the existence of a universal family on $M^s_{X/S,P} \times_S X$.

Let us recall that the moduli scheme $M^s_{X/S,P}$ is constructed as a quotient of an appropriate subscheme R^s of the Quot-scheme $\mathrm{Quot}(\mathcal{H}; P)$ by $\mathrm{PGL}(V)$. Let $q^*\mathcal{H} \to \tilde{F}$ denote the universal quotient on $R^s \times_S X$.

PROPOSITION 3.2 ([9, PROPOSITION 4.6.2])

The moduli scheme $M^s_{X/S,P}$ represents the functor $\mathcal{M}^s_{X/S,P}$ if and only if there exists a $\mathrm{GL}(V)$-linearized line bundle A on R^s on which elements t of the center $Z(\mathrm{GL}(V)) \simeq \mathbb{G}_m$ act via multiplication by t. If such A exists, then $\mathcal{H}om(p^*A, \tilde{F})$ descends to a universal family and any universal family is obtained in such a way.

3.2. Existence of compactified Picard schemes in the relative case

For simplicity we assume that all geometric fibers of f are irreducible and reduced (hence they are varieties) and that S is connected.

Let us fix a polynomial P. For all locally Noetherian S-schemes $T \to S$ let us set

$$\mathcal{P}\mathrm{ic}'_{X/S,P}(T) = \left\{ \begin{matrix} \text{isomorphism classes of invertible sheaves } L \text{ on } X_T = T \times_S X \\ \text{such that } \chi(X_t, L_t(n)) = P(n) \text{ for every geometric } t \in T \end{matrix} \right\}.$$

Note that if $\mathcal{P}\mathrm{ic}'_{X/S,P}(T)$ is nonempty, then the highest coefficient of P is the same as the highest coefficient of the Hilbert polynomial of $\mathcal{O}_{X_s}$ for any $s \in S$.

As above, we introduce an equivalence relation $\sim$ on $\mathcal{P}\mathrm{ic}'_{X/S,P}(T)$ by $L \sim L'$ if and only if there exists an invertible sheaf K on T such that $L \simeq L' \otimes p^*K$. Then we can define *the Picard functor*

$$\mathcal{P}\mathrm{ic}_{X/S,P} : (\mathrm{Sch}/S) \longrightarrow (\mathrm{Sets})$$

by sending an S-scheme T to $\mathcal{P}\mathrm{ic}_{X/S,P}(T) = \mathcal{P}\mathrm{ic}'_{X/S,P}(T)/\sim$.

Let us also define the compactified relative Picard functors. There are two different methods of compactification of the Picard scheme. We can compactify the Picard scheme by adding all the rank 1 torsion-free sheaves on the fibers of X or only those rank 1 torsion-free sheaves that are locally free on the smooth locus of the fibers. The second method has the advantage of producing a smaller scheme.

Let us set

$$\overline{\mathcal{P}\mathrm{ic}'}_{X/S,P}(T) = \left\{ \begin{array}{c} \text{isomorphism classes of } T\text{-flat sheaves } L \text{ on } X_T = T \times_S X \\ \text{such that } L_t \text{ is a torsion-free, rank 1 sheaf on } X_t \\ \text{and } \chi(X_t, L_t(n)) = P(n) \text{ for every geometric } t \in T \end{array} \right\}.$$

As above, we define the *compactified Picard functor*

$$\overline{\mathcal{P}\mathrm{ic}}_{X/S,P} : (\mathrm{Sch}/S) \longrightarrow (\mathrm{Sets})$$

by sending an S-scheme T to $\overline{\mathcal{P}\mathrm{ic}}_{X/S,P}(T) = \overline{\mathcal{P}\mathrm{ic}'}_{X/S,P}(T)/\sim$.

We also define *the small compactified Picard functor*

$$\overline{\mathcal{P}\mathrm{ic}}^{\mathrm{sm}}_{X/S,P} : (\mathrm{Sch}/S) \longrightarrow (\mathrm{Sets})$$

by sending an S-scheme T to

$$\overline{\mathcal{P}\mathrm{ic}}^{\mathrm{sm}}_{X/S,P}(T) = \left\{ \begin{array}{c} L \in \overline{\mathcal{P}\mathrm{ic}'}_{X/S,P}(T) \text{ such that } L \text{ is locally free} \\ \text{on the smooth locus of } X_T/T \end{array} \right\} / \sim .$$

THEOREM 3.3

Assume that $f : X \to S$ has a section $g : S \to X$.

1. *There exists a quasi-projective S-scheme* $\mathrm{Pic}_{X/S,P}$ *that represents the Picard functor* $\mathcal{P}\mathrm{ic}_{X/S,P}$.
2. *If $g(S)$ is contained in the smooth locus of X/S, then there exists a projective S-scheme* $\overline{\mathrm{Pic}}_{X/S,P}$ *that represents the compactified Picard functor* $\overline{\mathcal{P}\mathrm{ic}}_{X/S,P}$. *Moreover,* $\overline{\mathrm{Pic}}_{X/S,P}$ *contains a closed S-subscheme* $\overline{\mathrm{Pic}}^{\mathrm{sm}}_{X/S,P}$ *that represents the small compactified Picard functor* $\overline{\mathcal{P}\mathrm{ic}}^{\mathrm{sm}}_{X/S,P}$.

Proof

First let us remark that all the Picard functors $\mathcal{P}\mathrm{ic}_{X/S,P}$, $\overline{\mathcal{P}\mathrm{ic}}_{X/S,P}$, and $\overline{\mathcal{P}\mathrm{ic}}^{\mathrm{sm}}_{X/S,P}$ are subfunctors of the moduli functor $\mathcal{M}_{X/S,P}$. In fact, from our assumptions it follows that $\overline{\mathcal{P}\mathrm{ic}}_{X/S,P} = \mathcal{M}^s_{X/S,P} = \mathcal{M}_{X/S,P}$. Now we can construct $\mathrm{Pic}_{X/S,P}$, $\overline{\mathrm{Pic}}_{X/S,P}$, and $\overline{\mathrm{Pic}}^{\mathrm{sm}}_{X/S,P}$ as geometric invariant theory quotients of appropriate subschemes $R_{\mathrm{Pic}} \subset R_{\overline{\mathrm{Pic}}^{\mathrm{sm}}} \subset R_{\overline{\mathrm{Pic}}} = R^s = R^{ss}$ of the Quot-scheme used to construct the moduli space $M^s_{X/S,P}$ by $\mathrm{GL}(V)$. In fact all these quotients are

$\mathrm{PGL}(V)$-principal bundles. To prove that $\overline{\mathrm{Pic}}^{\mathrm{sm}}_{X/S,P}$ is a closed subscheme of $\overline{\mathrm{Pic}}_{X/S,P}$ it is sufficient to see that $R_{\overline{\mathrm{Pic}}^{\mathrm{sm}}}$ is a closed subscheme of $R_{\overline{\mathrm{Pic}}}$. This follows from [2, Lemma on p. 37] applied to the universal quotient restricted to the smooth locus of $R_{\overline{\mathrm{Pic}}}\times_S X\to R_{\overline{\mathrm{Pic}}}$.

To prove (1) by (a slight generalization of) Proposition 3.2 it is sufficient to show existence of a $\mathrm{GL}(V)$-linearized line bundle A_{Pic} on R_{Pic} on which the center of $\mathrm{GL}(V)$ acts with weight 1.

Let us set $A_{\mathrm{Pic}}=\det p_*(\tilde{F}\otimes q^*\mathcal{O}_{g(S)})$, where $\tilde{F}$ comes from the universal quotient on $R_{\mathrm{Pic}}\times_S X$. The definition makes sense since $\tilde{F}$ is a line bundle on $R_{\mathrm{Pic}}\times_S X$ and $p_*(\tilde{F}\otimes q^*\mathcal{O}_{g(S)})=(\mathrm{id}_{R_{\mathrm{Pic}}}\times_S g)^*\tilde{F}$ is also a line bundle. The center of $\mathrm{GL}(V)$ acts on the fiber of A_{Pic} at $([\rho],x)\in R_{\mathrm{Pic}}\times_S X$ with weight $\chi(\mathcal{O}_{X_{f(x)}}|_x)=1$, which implies the first assertion of the theorem.

Now assume that $g(S)$ is contained in the smooth locus of X/S. Then the same argument as above gives the existence of the Poincaré sheaf on $\overline{\mathrm{Pic}}^{\mathrm{sm}}_{X/S,P}$. The existence of the Poincaré sheaf on $\overline{\mathrm{Pic}}_{X/S,P}$ is slightly more difficult. First let us show that there exists a resolution

$$0\to E_n\to\cdots\to E_0\to\mathcal{O}_{g(S)}\to 0,$$

where E_i are locally free sheaves on X. Since there are sufficiently many locally free sheaves on X, we can construct the resolution up to step E_{n-1}, where n is the relative dimension of X/S. Then the kernel of $E_{n-1}\to E_{n-2}$ is also locally free. Indeed, it is sufficient to check it on the geometric fiber X_s over $s\in S$, where one can use the fact that the homological dimension of $\mathcal{O}_{g(s)}$ is equal to n. (This follows from the smoothness assumption.)

Tensoring with a high tensor power $\mathcal{O}_X(m)$ we can assume that all the higher direct images of $\tilde{F}\otimes q^*(E_i(m))$ under the projection p vanish. In particular, all sheaves $p_*\big(\tilde{F}\otimes q^*(E_i(m))\big)$ are locally free. Then we can set

$$A_{\overline{\mathrm{Pic}}}=\det p_!\big(\tilde{F}\otimes q^*(\mathcal{O}_{g(S)}(m))\big)=\bigotimes_i\big(\det p_*(\tilde{F}\otimes q^*(E_i(m)))\big)^{(-1)^i}.$$

Obviously, the center of $\mathrm{GL}(V)$ still acts on the fibers of $A_{\overline{\mathrm{Pic}}}$ with weight 1. Hence the theorem follows from Proposition 3.2. $\square$

REMARK 3.4

Note that the second part of Theorem 3.3 does not immediately follow from [1] and [2]. The representability of (compactified) Picard functors is proven there only in étale topology or after rigidification (see, e.g., [2, Theorems 3.2, 3.4]). Rigidification of the compactified Picard functor amounts in our case to restricting to the open subset of $R_{\overline{\mathrm{Pic}}}$, where the restriction of $\tilde{F}$ to $g(S)$ is invertible. Then by the same argument as in the proof of Theorem 3.3(1) we can construct the scheme representing the corresponding rigidified Picard functor obtaining [2, Theorem 3.4]. However, we prefer to make a stronger assumption as in Theorem 3.3(2) to construct the projective Picard scheme.

3.3. Moduli spaces of swamps

Let us fix nonnegative integers a and b, and consider a $\mathrm{GL}(V)$-module $(V^{\otimes a})^{\oplus b}$. Let $\rho_{a,b}\colon \mathrm{GL}(V)\to \mathrm{GL}((V^{\otimes a})^{\oplus b})$ be the corresponding representation. If $\mathcal{A}$ is a sheaf of rank $r=\dim V$, then we can associate to it a sheaf $\mathcal{A}_{\rho_{a,b}}=(\mathcal{A}^{\otimes a})^{\oplus b}$. On the open set where $\mathcal{A}$ is locally free, $\mathcal{A}_{\rho_{a,b}}$ is a locally free sheaf associated to the principal bundle obtained by extension from the frame bundle of $\mathcal{A}$.

Let us recall that a $\rho_{a,b}$*-swamp* is a triple $(\mathcal{A},L,\varphi)$ consisting of a torsion-free sheaf $\mathcal{A}$ on X, a rank 1 torsion-free sheaf L on X, and a nonzero homomorphism $\varphi:\mathcal{A}_{\rho_{a,b}}\to L$.

Let us fix a positive polynomial δ of degree $\le d-1$ with rational coefficients. Let us write $\delta(m)=\overline{\delta}\frac{m^{d-1}}{(d-1)!}+O(m^{d-2})$.

For a weighted filtration $(\mathcal{A}_\bullet,\alpha_\bullet)$ of $\mathcal{A}$, we set $r_i=\operatorname{rk}\mathcal{A}_i$ and we consider a vector $\gamma\in\mathbb{Q}^r$ defined by

$$\gamma=\sum\alpha_i(\underbrace{r_i-r,\dots,r_i-r}_{r_i\times},\underbrace{r_i,\dots,r_i}_{(r-r_i)\times}).$$

Let γ_j denote the jth component of γ. We set

$$\mu(\mathcal{A}_\bullet,\alpha_\bullet;\varphi)=-\min\{\gamma_{i_1}+\cdots+\gamma_{i_a}\mid (i_1,\dots,i_a)\in I:\varphi_{|(\mathcal{A}_{i_1}\otimes\cdots\otimes\mathcal{A}_{i_a})^{\oplus b}}\not\equiv 0\},$$

where $I=\{1,\dots,s+1\}^{\times a}$ is the set of all multi-indices.

Let us recall that a $\rho_{a,b}$-swamp $(\mathcal{A},L,\varphi)$ is δ*-(semi)stable* if for all weighted filtrations $(\mathcal{A}_\bullet,\alpha_\bullet)$ we have

$$M(\mathcal{A}_\bullet,\alpha_\bullet)+\mu(\mathcal{A}_\bullet,\alpha_\bullet;\varphi)\delta(\ge)0.$$

A $\rho_{a,b}$-swamp $(\mathcal{A},L,\varphi)$ is *slope* $\overline{\delta}$*-(semi)stable* if for all weighted filtrations $(\mathcal{A}_\bullet,\alpha_\bullet)$ we have

$$L(\mathcal{A}_\bullet,\alpha_\bullet)+\mu(\mathcal{A}_\bullet,\alpha_\bullet;\varphi)\overline{\delta}(\ge)0.$$

Now we can state the most general existence result for moduli spaces of swamps. We keep the notation from the beginning of this section.

THEOREM 3.5

Let us fix an S-flat family $\mathcal{L}$ of pure sheaves of dimension d on the fibers of $f:X\to S$. Assume either that $d=1$ or that f has only irreducible and reduced geometric fibers. Then there exists a coarse S-projective moduli space for δ-semistable S-flat families of $\rho_{a,b}$-swamps $(\mathcal{A},\mathcal{L},\varphi)$ such that for every $s\in S$ the restriction $\mathcal{A}|_{X_s}$ has Hilbert polynomial P.

In the case when X is a smooth complex projective variety this theorem was proved by Gómez and Sols in [7] and later generalized by Bhosle to singular complex varieties satisfying Bhosle's condition in [3]. Note that in [7] and [3] the authors considered only the case when $\mathcal{L}$ is locally free. However, this is not necessary due to Lemma 2.3, and it is sufficient to assume that $\mathcal{L}$ is torsion-free. Generalization to the relative case in arbitrary characteristic follows from [12] and [11]. We need only to comment on why one does need to require that the fibers of f are irreducible or reduced in the curve case. This fact follows from [9, Remark 4.4.9]: torsion submodules for sheaves on curves are detected by any twist of its global sections. This allows one to omit using [3, Proposition 2.12] in the curve case. In particular, this shows that all the results of Sorger [22] are now a part of the more general theory.

We also have another variant of the above theorem (cf. [19, Theorem 2.3.2.5]).

THEOREM 3.6

Let us fix a Hilbert polynomial Q. Assume that all geometric fibers of f are irreducible and reduced, and assume that $f : X \to S$ has a section $g : S \to X$ such that $g(S)$ is contained in the smooth locus of X/S. Then there exists a coarse moduli space for δ-semistable S-flat families of $\rho_{a,b}$-swamps $(\mathcal{A}, \mathcal{L}, \varphi)$ such that for every $s \in S$ the restriction $\mathcal{A}|_{X_s}$ has Hilbert polynomial P and the restriction $\mathcal{L}|_{X_s}$ has Hilbert polynomial Q. This moduli space is projective over $\overline{\mathrm{Pic}}_{X/S,Q}$.

4. Tensor product of semistable sheaves on nonnormal varieties

Let $(X, \mathcal{O}_X(1))$ be a d-dimensional polarized projective variety defined over an algebraically closed field k.

Let $\nu : \tilde{X} \to X$ denote the normalization of X, and let E be a coherent $\mathcal{O}_X$-module. Since ν is a finite morphism, there exists a well-defined coherent $\mathcal{O}_{\tilde{X}}$-module $\nu^! E$ corresponding to the $\nu_* \mathcal{O}_{\tilde{X}}$-module $\mathcal{H}om(\nu_* \mathcal{O}_{\tilde{X}}, E)$. If E is torsion-free, then we have $\mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E) = 0$. Hence

$$\nu_*(\nu^! E) = \mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, E) \subset \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, E) = E,$$

and $\nu^! E$ is also torsion-free.

LEMMA 4.1

There exists a constant α (depending only on the variety X) such that for any rank r torsion-free sheaf E on X we have

$$0 \le \mu(E) - \mu\big(\mathcal{H}om(\nu_* \mathcal{O}_{\tilde{X}}, E)\big) \le \alpha.$$

Proof

We have an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, E) \to E \to \mathcal{E}xt^1_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E).$$

For large m we have

$$P\big(\mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, E)\big)(m) \le P(E)(m),$$

and, since $\mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, E)$ and E have the same rank, we have

$$\mu\big(\mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, E)\big) \le \mu(E).$$

On the other hand we have

$$\alpha_{d-1}(E) \le \alpha_{d-1}\big(\mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, E)\big) + \alpha_{d-1}\big(\mathcal{E}xt^1_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E)\big).$$

Note that $\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E)$ is supported on the support of $\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$. Let $Y_1,\dots,Y_k$ denote codimension 1 irreducible components of the support of $\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$. Then $\alpha_{d-1}(\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E))$ can be bounded from above using the ranks of $\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E)$ at $Y_1,\dots,Y_k$. Hence by the above inequality, to prove the lemma it is sufficient to bound these ranks.

There exists a subsheaf $G\subset E$ such that G is locally free (we need only locally free in codimension 1) and E/G is torsion (i.e., equal to zero at the generic point of X). This can be constructed by taking r general sections of $E(m)$ for large m and twisting the image of $\mathcal{O}_X^r\subset H^0(E(m))\otimes\mathcal{O}_X\to E(m)$ by $\mathcal{O}_X(-m)$.

Then we have an exact sequence

$$0=\mathcal{H}\mathrm{om}(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E)\to\mathcal{H}\mathrm{om}(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E/G)\to\mathcal{E}\mathrm{xt}^1(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, G).$$

Note that the sheaves in this sequence are supported on $\bigcup Y_i$ and the rank of $\mathcal{E}\mathrm{xt}^1(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, G)$ on Y_i is the same as the rank of $\mathcal{E}\mathrm{xt}^1(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, \mathcal{O}_X^r)$ on Y_i. In particular, it depends only on the rank r, and it is independent of E. Hence the dimensions of $\mathcal{H}\mathrm{om}(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E/G)$ at the generic points of $Y_1,\dots,Y_k$ are bounded from above by a linear function of r. But this implies that the ranks of E/G, and hence also of $\mathcal{E}\mathrm{xt}^1(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E/G)$, on $Y_1,\dots,Y_k$ are bounded independently of E. Now we can use the sequence

$$\mathcal{E}\mathrm{xt}^1(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, G)\to\mathcal{E}\mathrm{xt}^1(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E)\to\mathcal{E}\mathrm{xt}^1(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E/G)$$

to bound the ranks of $\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(\nu_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X, E)$ on $Y_1,\dots,Y_k$. □

COROLLARY 4.2

Let us set $\beta=\alpha_{d-1}(\mathcal{O}_{\tilde{X}})-\alpha_{d-1}(\mathcal{O}_X)$. Then for any rank r torsion-free sheaf E on X we have

$$\beta\le\mu(E)-\mu(\nu^!E)\le\alpha+\beta,$$

where the slopes are computed with respect to $\mathcal{O}_X(1)$ on X and $\nu^*\mathcal{O}_X(1)$ on $\tilde{X}$.

Proof

For any sheaf F on $\tilde{X}$ we have

$$\chi\big(\tilde{X}, F\otimes\nu^*\mathcal{O}_X(m)\big)=\chi\big(X, \nu_*F\otimes\mathcal{O}_X(m)\big).$$

This implies that

$$\mu(\nu_*F)-\mu(F)=\alpha_{d-1}(\mathcal{O}_{\tilde{X}})-\alpha_{d-1}(\mathcal{O}_X)=\beta.$$

Therefore, since

$$\nu_*(\nu^!E)=\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\nu_*\mathcal{O}_{\tilde{X}}, E),$$

we have

$$\begin{aligned}\mu(E)-\mu(\nu^!E)&=\big(\mu(E)-\mu(\mathcal{H}\mathrm{om}(\nu_*\mathcal{O}_{\tilde{X}}, E))\big)+\big(\mu(\nu_*(\nu^!E))-\mu(\nu^!E)\big)\\&=\big(\mu(E)-\mu(\mathcal{H}\mathrm{om}(\nu_*\mathcal{O}_{\tilde{X}}, E))\big)+\beta.\end{aligned}$$

Now the corollary follows from Lemma 4.1. □

COROLLARY 4.3

For any rank r torsion-free sheaf E on X we have

$$\beta \le \mu_{\max}(E) - \mu_{\max}(\nu^! E) \le \alpha + \beta.$$

Proof

If $G \subset E$ is a subsheaf of E, then $\nu^! G \subset \nu^! E$, and hence

$$\mu(G) \le \mu(\nu^! G) + \alpha + \beta \le \mu_{\max}(\nu^! E) + \alpha + \beta.$$

This proves that

$$\mu_{\max}(E) \le \mu_{\max}(\nu^! E) + \alpha + \beta.$$

Now if $F \subset \nu^! E$, then $\nu_* F \subset \nu_*(\nu^! E) \subset E$. Therefore

$$\mu(F) = \mu(\nu_* F) - \beta \le \mu_{\max}(E) - \beta,$$

which implies that

$$\mu_{\max}(\nu^! E) \le \mu_{\max}(E) - \beta. \qquad \square$$

For a torsion-free sheaf E on X we set $\nu^\sharp E = \nu^* E/\mathrm{Torsion}$. Then $\nu_* \nu^\sharp E = (\nu_* \nu^* E)/\mathrm{Torsion}$.

Note that $\nu^!$ is an equivalence of categories of sheaves on X and $\tilde{X}$, whereas $\nu^\sharp$ has much worse properties. But $\nu^\sharp$ has the following important property: since $\nu^*(E_1 \otimes E_2) = \nu^* E_1 \otimes \nu^* E_2$ we have $\nu^\sharp(E_1 \widehat{\otimes} E_2) = \nu^\sharp E_1 \widehat{\otimes} \nu^\sharp E_2$.

Let $\mathcal{C} = \mathrm{Ann}(\nu_* \mathcal{O}_{\tilde{X}}/\mathcal{O}_X) \subset \mathcal{O}_X$ and $\mathcal{C}_{\tilde{X}} = \mathcal{C} \cdot \mathcal{O}_{\tilde{X}} \subset \mathcal{O}_{\tilde{X}}$ denote conductor ideals of the normalization.

LEMMA 4.4

For any torsion-free sheaf E on X we have

$$\mu(\nu^\sharp E) \le \mu(\nu^! E) - \mu(\mathcal{C}_{\tilde{X}}).$$

Proof

Note that $\mathcal{C} = \mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, \mathcal{O}_X)$. Therefore for any coherent $\mathcal{O}_X$-module E we have a canonical map

$$\mathcal{C} \otimes E = \mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, \mathcal{O}_X) \otimes \mathcal{H}om(\mathcal{O}_X, E) \to \mathcal{H}om_{\mathcal{O}_X}(\nu_* \mathcal{O}_{\tilde{X}}, E) = \nu_*(\nu^! E)$$

given by composition of homomorphisms. Since ν^* and ν_* are adjoint functors this map induces

$$\nu^* \mathcal{C} \otimes \nu^* E \to \nu^! E.$$

Since E is torsion-free and $\mathcal{C}_{\tilde{X}} = \nu^\sharp \mathcal{C}$ we get

$$\mathcal{C}_{\tilde{X}} \widehat{\otimes} \nu^\sharp E \simeq \mathcal{C}_{\tilde{X}} \cdot \nu^\sharp E \hookrightarrow \nu^! E.$$

Since this inclusion is an isomorphism at the generic point of $\tilde{X}$ we have the inequality

$$\mu(\mathcal{C}_{\tilde{X}} \widehat{\otimes} \nu^\sharp E) \le \mu(\nu^! E).$$

Now Lemma 2.1 gives

$$\mu(\mathcal{C}_{\tilde{X}} \widehat{\otimes} \nu^{\sharp} E) = \mu(\nu^{\sharp} E) + \mu(\mathcal{C}_{\tilde{X}}),$$

which implies the required inequality. □

COROLLARY 4.5

For any rank r torsion-free sheaf E on X we have

$$-\beta \le \mu(\nu^{\sharp} E) - \mu(E) \le -\beta - \mu(\mathcal{C}_{\tilde{X}}),$$

where the slopes are computed with respect to $\mathcal{O}_X(1)$ on X and $\nu^* \mathcal{O}_X(1)$ on $\tilde{X}$.

Proof

The canonical map $E \to \nu_*(\nu^* E)$ leads to the inclusion

$$E \hookrightarrow \nu_*(\nu^{\sharp} E).$$

This gives

$$\mu(E) \le \mu\big(\nu_*(\nu^{\sharp} E)\big) = \mu(\nu^{\sharp} E) + \beta,$$

where the last equality follows from the proof of Lemma 4.2. This bounds the difference $\mu(\nu^{\sharp} E) - \mu(E)$ from below. To get the bound from above it is sufficient to use Lemma 4.4 and Corollary 4.2. □

REMARK 4.6

By Lemma 4.4 and Corollary 4.5 we have

$$\mu(\nu^! E) \ge \mu(\nu^{\sharp} E) + \mu(\mathcal{C}_{\tilde{X}}) \ge \mu(E) - \beta + \mu(\mathcal{C}_{\tilde{X}}).$$

This allows us to take $\alpha = -\mu(\mathcal{C}_{\tilde{X}})$ in Lemma 4.1. The proof of Lemma 4.1 also gives a related and explicit bound on α.

Corollary 4.1 can be used to prove the following corollary.

COROLLARY 4.7

For any rank r torsion-free sheaf E on X we have

$$-\beta \le \mu_{\max}(\nu^{\sharp} E) - \mu_{\max}(E) \le -\beta - \mu(\mathcal{C}_{\tilde{X}}).$$

Proof

If $G \subset E$ is a subsheaf of E, then $\nu^{\sharp} G \subset \nu^{\sharp} E$, and hence

$$\mu(G) \le \mu(\nu^{\sharp} G) + \beta \le \mu_{\max}(\nu^{\sharp} E) + \beta.$$

This proves that

$$\mu_{\max}(E) \le \mu_{\max}(\nu^{\sharp} E) + \beta.$$

Now if $F \subset \nu^{\sharp} E$, then by the proof of Lemma 4.4 we have

$$\mathcal{C}_{\tilde{X}} \widehat{\otimes} F \subset \mathcal{C}_{\tilde{X}} \widehat{\otimes} \nu^{\sharp} E \hookrightarrow \nu^! E.$$

Together with Lemma 2.1 and Corollary 4.3, this gives

$$\mu(F) \le \mu_{\max}(\nu^! E) - \mu(\mathcal{C}_{\tilde{X}}) \le \mu_{\max}(E) - \beta - \mu(\mathcal{C}_{\tilde{X}}),$$

which implies that

$$\mu_{\max}(\nu^\sharp E) \le \mu_{\max}(E) - \beta - \mu(\mathcal{C}_{\tilde{X}}). \qquad \square$$

Since $\nu^*(E_1 \otimes E_2) = \nu^* E_1 \otimes \nu^* E_2$ we have $\nu^\sharp(E_1 \widehat{\otimes} E_2) = \nu^\sharp E_1 \widehat{\otimes} \nu^\sharp E_2$. Therefore [13, Introduction] or [6, Lemma 3.2.1] imply the following proposition.

PROPOSITION 4.8

There exists an explicit constant γ (depending only on the polarized variety $(X, \mathcal{O}_X(1))$) such that for any two torsion-free sheaves E_1 and E_2 on X of ranks r_1, r_2, respectively, we have

$$\mu_{\max}(E_1 \widehat{\otimes} E_2) \le \mu_{\max}(E_1) + \mu_{\max}(E_2) + (r_1 + r_2)\gamma.$$

5. Honest singular principal bundles

In this section X is a d-dimensional projective variety defined over an algebraically closed field k with a fixed ample line bundle $\mathcal{O}_X(1)$.

The main aim of this section is proof of the following generalization of [18, Proposition 3.4].

PROPOSITION 5.1

Assume that X is Gorenstein (i.e., a Cohen–Macaulay scheme with invertible dualizing sheaf ω_X), and assume that there exists a G-invariant nondegenerate quadratic form φ on V. Then every degree zero singular principal bundle is an honest singular principal bundle.

Proof

Let $(\mathcal{A}, \tau)$ be a degree zero singular principal bundle. As in the proof of [18, Proposition 3.4] one can easily show that there exists an injective map $\mathcal{A} \to \mathcal{A}^\vee$ induced by the form φ. By Lemma 5.3 we see that the Hilbert polynomials of $\mathcal{A}$ and $\mathcal{A}^\vee$ are the same up to the terms of order $O(m^{d-2})$. Hence $\mathcal{A} \to \mathcal{A}^\vee$ is an isomorphism in codimension 1. Now let us recall that for each $x \in X$ two finitely generated modules over a local ring $\mathcal{O}_{X,x}$ satisfying S_2 that coincide in codimension 1 are equal. In particular, at each point x where $\mathcal{A}$ is locally free the map $\mathcal{A} \to \mathcal{A}^\vee$ is an isomorphism. As in the proof of [18, Proposition 3.4] this implies that

$$\sigma(U_{\mathcal{A}}) \subset \mathbb{I}\mathrm{som}(V \otimes \mathcal{O}_{U_{\mathcal{A}}}, \mathcal{A}^\vee|_{U_{\mathcal{A}}})/G. \qquad \square$$

The following lemma generalizes a well-known equality from smooth varieties to singular ones.

LEMMA 5.2

For any rank r coherent sheaf E and a line bundle L we have

$$\deg(E \otimes L) = \deg E + r\bigl(L \cdot \mathcal{O}_X(1)^{d-1}\bigr).$$

Proof

We use the notation from Kollár's book [10, Chapter VI.2]. In particular, $K_i(X)$ stands for the subgroup of the Grothendieck group of X generated by subsheaves supported in dimension at most i. We have

$$L \otimes E(m) = \sum_{i=0}^{d} c_1(L)^i \cdot E(m)$$

(see, e.g., [10, Chapter VI.2, Lemma 2.12]). On the other hand, by [10, Chapter VI.2, Corollary 2.3] we have

$$E \equiv r\mathcal{O}_X \bmod K_{d-1}(X).$$

Note that

$$L \otimes E(m) = E(m) + rc_1(L) \cdot \mathcal{O}_X(m) + c_1(L) \cdot (E - r\mathcal{O}_X)(m) + \sum_{i \geq 2} c_1(L)^i \cdot E(m)$$

and $c_1(L) \cdot (E - r\mathcal{O}_X) + \sum_{i\geq 2} c_1(L)^i \cdot E \in K_{d-2}(X)$ by [10, Chapter VI.2, Proposition 2.5]. Therefore by [10, Chapter VI.2, Corollary 2.13] we have

$$\chi\bigl(X, L \otimes E(m)\bigr) = \chi\bigl(X, E(m)\bigr) + r\chi\bigl(X, c_1(L) \cdot \mathcal{O}_X(m)\bigr) + O(m^{d-2}).$$

By the Riemann–Roch theorem for singular varieties (see [4, Corollary 18.3.1]), we have

$$\begin{aligned}\chi\bigl(X, c_1(L) \cdot \mathcal{O}_X(m)\bigr) &= \chi\bigl(X, \mathcal{O}_X(m)\bigr) - \chi\bigl(X, L^{-1}(m)\bigr)\\ &= \int_X \bigl(\mathrm{ch}(\mathcal{O}_X(m)) - \mathrm{ch}(L^{-1}(m))\bigr)\mathrm{T}\,\mathrm{d}X\\ &= \bigl(L \cdot \mathcal{O}_X(1)^{d-1}\bigr)\frac{m^{d-1}}{(d-1)!} + O(m^{d-2})\end{aligned}$$

which, together with the previous equality, implies the lemma. □

LEMMA 5.3

If X is Gorenstein and E is a torsion-free sheaf on X, then

$$\deg E^{\vee} = -\deg E.$$

Proof

Since X is Cohen–Macaulay, Serre's duality gives the equality

$$\chi(X, E) = (-1)^d \sum_{i=0}^{d} (-1)^i \dim \mathrm{Ext}^i(E, \omega_X).$$

The local to global Ext spectral sequence

$$H^p\big(X, \mathcal{E}\mathrm{xt}^q(E,\omega_X)\big) \Rightarrow \mathrm{Ext}^{p+q}(E,\omega_X)$$

implies that

$$\sum_{i=0}^{d}(-1)^i \dim \mathrm{Ext}^i(E,\omega_X) = \sum_{0\le p,q\le d}(-1)^{p+q}\dim H^p\big(X,\mathcal{E}\mathrm{xt}^q(E,\omega_X)\big)$$
$$= \sum_{q=0}^{d}(-1)^q \chi\big(X,\mathcal{E}\mathrm{xt}^q_X(E,\omega_X)\big).$$

Therefore we obtain

$$\chi\big(X,E(m)\big) = (-1)^d\sum_{q=0}^{d}(-1)^q\chi\big(X,\mathcal{E}\mathrm{xt}^q_X(E,\omega_X)\otimes\mathcal{O}_X(-m)\big).$$

By Lemma 2.2 we have $\dim \mathcal{E}\mathrm{xt}^q_X(E,\omega_X)\le d-2$ for $q>0$, so by [10, Chapter VI, Corollary 2.14],

$$\chi\big(X,\mathcal{E}\mathrm{xt}^q_X(E,\omega_X)\otimes\mathcal{O}_X(-m)\big) = O(m^{d-2})$$

for $q>0$. Since ω_X is invertible, $\mathcal{H}\mathrm{om}(E,\omega_X)=E^\vee\otimes\omega_X$ and we get

$$\chi\big(X,E(m)\big) = (-1)^d\chi\big(X,E^\vee\otimes\omega_X(-m)\big)+O(m^{d-2}).$$

In particular, we have

$$\alpha_{d-1}(E^\vee\otimes\omega_X) = -\alpha_{d-1}(E).$$

Therefore by Lemma 5.2,

$$\deg E^\vee = \deg(E^\vee\otimes\omega_X) - rc_1(\omega_X)\cdot c_1\big(\mathcal{O}_X(1)\big)^{d-1}$$
$$= \alpha_{d-1}(E^\vee\otimes\omega_X) - r\alpha_{d-1}(\mathcal{O}_X) - rc_1(\omega_X)\cdot c_1\big(\mathcal{O}_X(1)\big)^{d-1}$$
$$= -\deg E - 2r\alpha_{d-1}(\mathcal{O}_X) - rc_1(\omega_X)\cdot c_1\big(\mathcal{O}_X(1)\big)^{d-1}.$$

Applying this equality for $E=\mathcal{O}_X$ we see that

$$-2\alpha_{d-1}(\mathcal{O}_X) - c_1(\omega_X)\cdot c_1\big(\mathcal{O}_X(1)\big)^{d-1} = 0,$$

so $\deg E^\vee = -\deg E$. □

6. Semistable reduction for singular principal G-bundles

The following global boundedness of swamps on singular varieties can be proven in the same way as in the case of smooth varieties (see [5, Theorem 4.2.1], [6, Theorem 3.2.2], or [19, Theorem 2.3.4.3]). The only difference is that we need Proposition 4.8 (instead of, e.g., [6, Lemma 3.2.1]).

THEOREM 6.1

Let us fix a polynomial P, integers a, b, and a class l in the Néron–Severi group of X. Then the set of isomorphism classes of torsion-free sheaves $\mathcal{A}$ on X with

Hilbert polynomial P and such that there exists a positive rational number $\overline{\delta}$ and a slope $\overline{\delta}$-semistable $\rho_{a,b}$-swamp $(\mathcal{A}, L, \varphi)$ with L of class l is bounded.

This boundedness result implies the following semistable reduction theorem (see [5, Theorem 5.4.4], [6, Theorem 4.4.1], or [19, Theorem 2.4.4.1]). We skip the proof as it is the same as in the smooth case.

THEOREM 6.2

Assume that k has characteristic zero. Then there exists a polynomial δ_∞ such that for every positive polynomial $\delta > \delta_\infty$, every δ-semistable pseudo-G-bundle $(\mathcal{A}, \tau)$ is a singular principal G-bundle.

Let us recall that a singular principal G-bundle is semistable if and only if the associated pseudo-G-bundle is δ-semistable for $\delta > \delta_\infty$ (see [5, Theorem 5.4.1]). Therefore the above semistable reduction theorem and Theorem 2.5 imply the following corollary.

COROLLARY 6.3

Assume that k has characteristic zero, and let us fix a polynomial P. Then there exists a projective moduli space $M^{\rho}_{X,P}$ for semistable principal G-bundles $(\mathcal{A}, \tau)$ on X such that $\mathcal{A}$ has Hilbert polynomial P.

Now let us consider the relative case. Let $f : X \to S$ be a flat, projective morphism of k-schemes of finite type with integral geometric fibers. Assume that k has characteristic zero, and fix a polynomial P.

THEOREM 6.4

Let us fix a faithful representation $\rho\colon G \to \mathrm{GL}(V)$ of the reductive algebraic group G.

1. *There exists a projective moduli space $M^{\rho}_{X/S,P} \to S$ for S-flat families of semistable singular principal G-bundles on $X \to S$ such that for all $s \in S$ the restriction $\mathcal{A}|_{X_s}$ has Hilbert polynomial P.*
2. *Let P correspond to sheaves of degree zero. If the fibers of f are Gorenstein and there exists a G-invariant nondegenerate quadratic form φ on V, then $M^{\rho}_{X/S,P} \to S$ parameterizes only honest singular principal G-bundles.*

The first part of this theorem follows directly from the above corollary (rewritten in the relative setting). The second part is a direct consequence of Proposition 5.1. Since the proof in the relative setting is essentially the same as usual (cf. [9, Theorem 4.3.7]) we skip the details.

Acknowledgments. The author would like to thank Alexander Schmitt for useful conversations. Most of this paper was written during his visit to the University of Duisburg-Essen.

References

[1] A. Altman and S. Kleiman, *Compactifying the Picard scheme*, Adv. Math. **35** (1980), 50–112. MR 0555258. DOI 10.1016/0001-8708(80)90043-2.

[2] ———, *Compactifying the Picard scheme, II*, Amer. J. Math. **101** (1979), 10–41. MR 0527824. DOI 10.2307/2373937.

[3] U. N. Bhosle, *Tensor fields and singular principal bundles*, Int. Math. Res. Not. **2004**, no. 57, 3057–3077. MR 2098029. DOI 10.1155/S1073792804133114.

[4] W. Fulton, *Intersection Theory*, Ergeb. Math. Grenzgeb. (3) **2**, Springer, Berlin, 1984. MR 0732620.

[5] T. Gómez, A. Langer, A. Schmitt, and I. Sols, *Moduli spaces for principal bundles in arbitrary characteristic*, Adv. Math. **219** (2008), 1177–1245. MR 2450609. DOI 10.1016/j.aim.2008.05.015.

[6] ———, "Moduli spaces for principal bundles in large characteristic" in *Teichmüller Theory and Moduli Problem*, Ramanujan Math. Soc. Lect. Notes Ser. **10**, Ramanujan Math. Soc., Mysore, 281–371. MR 2667560.

[7] T. Gómez and I. Sols, *Stable tensors and moduli space of orthogonal sheaves*, preprint, arXiv:math/0103150v4 [math.AG]

[8] R. Hartshorne, *Local Cohomology, A Seminar Given by A. Grothendieck, Harvard University, Fall, 1961*, Lecture Notes in Math. **41**, Springer, Berlin, 1967. MR 0224620.

[9] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, 2nd ed., Cambridge Math. Lib., Cambridge Univ. Press, Cambridge, 2010. MR 2665168. DOI 10.1017/CBO9780511711985.

[10] J. Kollár, *Rational Curves on Algebraic Varieties*, Ergeb. Math. Grenzgeb. (3) **32**, Springer, Berlin, 1996. MR 1440180.

[11] A. Langer, *Moduli spaces of sheaves in mixed characteristic*, Duke Math. J. **124** (2004), 571–586. MR 2085175. DOI 10.1215/S0012-7094-04-12434-0.

[12] ———, *Semistable sheaves in positive characteristic*, Ann. of Math. (2) **159** (2004), 251–276. MR 2051393. DOI 10.4007/annals.2004.159.251.

[13] ———, *Semistable principal G-bundles in positive characteristic*, Duke Math. J. **128** (2005), 511–540. MR 2145742. DOI 10.1215/S0012-7094-04-12833-7.

[14] M. Maruyama, *Moduli of stable sheaves, I*, J. Math. Kyoto Univ. **17** (1977), 91–126. MR 0450271.

[15] ———, *Moduli of stable sheaves, II*, J. Math. Kyoto Univ. **18** (1978), 557–614. MR 0509499.

[16] A. Schmitt, *Singular principal bundles over higher-dimensional manifolds and their moduli spaces*, Int. Math. Res. Not. **2002**, no. 23, 1183–1209. MR 1903952. DOI 10.1155/S1073792802107069.

[17] ———, *A closer look at semistability for singular principal bundles*, Int. Math. Res. Not. **2004**, no. 62, 3327–3366. MR 2097106. DOI 10.1155/S1073792804132984.

[18] ———, *Moduli spaces for semistable honest singular principal bundles on a nodal curve which are compatible with degeneration: A remark on U. N. Bhosle's paper "Tensor fields and singular principal bundles,"* Int. Math. Res. Not. **2005**, no. 23, 1427–1437. MR 2152237. DOI 10.1155/IMRN.2005.1427.

[19] ———, *Geometric Invariant Theory and Decorated Principal Bundles*, Zur. Lect. Adv. Math., Eur. Math. Soc. (EMS), Zürich, 2008. MR 2437660. DOI 10.4171/065.

[20] ———, "Moduli spaces for principal bundles" in *Moduli Spaces and Vector Bundles*, London Math. Soc. Lecture Note Ser. **359**, Cambridge Univ. Press, Cambridge, 2009, 388–423. MR 2537075. DOI 10.1017/CBO9781139107037.013.

[21] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety, I*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 47–129. MR 1307297.

[22] C. Sorger, *Theta-caractéristiques des courbes tracées sur une surface lisse*, J. Reine Angew. Math. **435** (1993), 83–118. MR 1203912. DOI 10.1515/crll.1993.435.83.

Institute of Mathematics, Warsaw University, ulica Banacha 2, 02-097 Warszawa, Poland; alan@mimuw.edu.pl