A metric linear space is an open cone

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Abstract In this paper we show that a metrizable topological vector space over $\mathbf{R}$ is topologically an open cone. This generalizes the partial results obtained by Henderson.

If X, Y are topological spaces and if A closed $\subseteq X$, then the reduced product of X and Y over A, which we denote by $(X \times Y)_A$, is defined by $(X \times Y)_A = ((X \backslash A) \times Y) \cup A$ with the topology given by $\{U \times V : U \text{ open} \subseteq X \backslash A,\ V \text{ open} \subseteq Y\} \cup \{((U \backslash A) \times Y) \cup (A \cap U) : U \text{ open} \subseteq X\}$ (see [3, p. 25]). In particular, an open cone $C(X)$ over a topological space X is defined as $C(X) = (\mathbf{R}_+ \times X)_{\{0\}}$.

A metric topological vector space T over $\mathbf{R}$ has an F-norm; that is, a function $|\cdot| : T \to \mathbf{R}_+$ satisfying

(1) $|x| = 0$ iff $x = 0$,
(2) $|x + y| \leq |x| + |y|$,
(3) $|\boldsymbol{\lambda} x| \leq |x|$ for all $\lambda \in \mathbf{R}, -1 \leq \lambda \leq 1$,

and $d_0(x, y) = |x - y|$ is a translation invariant metric defining the topology of T (see [6]). Replacing the F-norm by the F-norm $|x|' = \int_0^1 |tx|\,\mathrm{dt}$, we obtain another metric defining the T-topology, and it has the Eidelheit–Mazur property; that is, for all $0 \neq x \in T$, $d(t_1 x, 0) < d(t_2 x, 0)$ for all $t_1, t_2 \in \mathbf{R}$ with $t_1 < t_2$, and we may assume that $d < 1$ (see [4], [7]). In this paper we shall always assume that the F-norm on T and its induced metric have these properties. T is said to admit arbitrary short lines if for all $\varepsilon > 0$, there exists $0 \neq x \in T$ such that $\sup_{t \in R} |tx| < \varepsilon$.

In [8, Lemma 1.1] and [9, Lemma 2] Henderson showed that some special classes of metric topological vector spaces over $\mathbf{R}$ are topologically open cones. In particular, this result holds for infinite-dimensional locally convex metric linear spaces (see [8, Lemma 1.1]) by virtue of a peculiar topological property of these spaces (see [2, Proposition 1.1]). In this paper, we show that this property holds for any metric linear space that admits arbitrary short lines (see Proposition 3). This will enable us to establish that every metric topological vector space over $\mathbf{R}$ is topologically an open cone (see Theorem 4).

Recall that if T is any Hausdorff topological vector space over $\mathbf{R}$, then an open neighborhood of 0 in T is called *shrinkable* if $[0, 1)\overline{U} \subseteq U$, and the Minkowski

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functional μ of U is the function $\mu : T \to \mathbf{R}_+$ defined by $\mu(x) = \inf\{t > 0 : x \in tU\}$. It follows that

(1) $\mu(tx) = t\mu(x)$ for all $t \in \mathbf{R}_+$,
(2) $\mu^{-1}([0,1)) = U$ and $\mu^{-1}([0,1]) = \overline{U}$,
(3) $t\overline{U} \subseteq t_1 U$ for $0 < t < t_1$,

so that $\mu^{-1}([0,t)) = \bigcup_{t_1<t} t_1 U$ and $\mu^{-1}((t,\infty)) = \bigcup_{t_1>t}(T - t_1\overline{U})$; hence, μ is continuous. Note that open shrinkable neighborhoods of 0 in T form a local basis at 0 (see [10, Theorem 4]) and that shrinkability is the precise condition that guarantees the continuity of the Minkowski functional of an open star-shaped neighborhood of 0 (see [10, Theorem 5]).

We shall need two lemmas.

LEMMA 1

Let T be a metrizable topological vector space over $\mathbf{R}$, let U be an open shrinkable neighborhood of 0 in T, and let μ be the Minkowski functional of U. Then $T \backslash \mu^{-1}(0) \cup \{0\}$ is topologically an open cone.

Proof

Let d be a translation-invariant metric bounded by 1 that defines the T topology and that has the Eidelheit–Mazur property. Then $d_1(x,y) = \max(|\mu(x) - \mu(y)|, d(x,y))$ is a metric that defines the T topology and again it has the Eidelheit–Mazur property. Note that for $x \in T\backslash\mu^{-1}(0)$, we have $\sup\{d_1(tx,0) : t \in \mathbf{R}_+\} = +\infty$. We have a homeomorphism $g : T\backslash\mu^{-1}(0) \cup \{0\} \to C(\mu^{-1}(1))$ defined by

$$g(z) = \begin{cases} (d_1(z,0), z/\mu(z)), & z \neq 0 \\ 0, & z = 0 \end{cases}$$

whose inverse is defined by $g^{-1} : C(\mu^{-1}(1)) \to T\backslash\mu^{-1}(0) \cup \{0\}$,

$$\{g^{-1}(\lambda, z)\} = \mathbf{R}_+ z \cap \{w \in T : d_1(w,0) = \lambda\}$$
$$\text{for all } \lambda \in \mathbf{R}, \lambda > 0 \text{ and } g^{-1}(0) = 0. \qquad \square$$

LEMMA 2

There exists a homeomorphism

$$h = (h_1, h_2) : \big(\mathbf{R}^N \times (0,\infty)\big) \cup \big\{(\mathbf{0}, 0))\big\} \to \mathbf{R}^n \times (0,\infty)$$

such that $h_1((\mathbf{x},t)) - \mathbf{x} \in \sum \mathbf{R}$ *for all* $\mathbf{x} \in \mathbf{R}^N, h((\mathbf{0},0)) = (\mathbf{0},1)$ *and* $h = \mathrm{id}$ *on* $\mathbf{R}^N \times (2,\infty)$.

Proof

Let $C_0 = \mathbf{R}^N \times (-\infty, 2)$, and let

$$C_n = \Big\{\mathbf{x} \in \mathbf{R}^N : -\frac{1}{2^n} < x_i < \frac{1}{2^n} \text{ for } 1 \le i \le n \text{ and } x_{n+1} > -\frac{1}{2^n}\Big\} \times \Big(-\frac{1}{2^n}, \frac{1}{2^n}\Big).$$

Then $(C_n)_{n\geq 0}$ is a fundamental system of open (convex) neighborhoods of $(\mathbf{0},0)$ in $\mathbf{R}^N\times\mathbf{R}$. Now [1, Lemma 1.4] provides a homeomorphism

$$\begin{aligned} p_n &= \big((p_n)_1,(p_n)_2\big):(\overline{C}_n\backslash C_{n+1},\partial C_n,\partial C_{n+1}) \\ &\quad\to(\mathbf{R}^N\times[1-n,2-n],\mathbf{R}^N\times\{2-n\},\mathbf{R}^N\times\{1-n\}) \end{aligned}$$

such that $(p_n)_1(\mathbf{x},t)-\mathbf{x}\in\sum\mathbf{R}$ for all $(\mathbf{x},t)\in\overline{C}_n\backslash C_{n+1}$, for all $n\geq 0$.

Define inductively the sequence of homeomorphisms p'_n, $n\geq 0$,

$$\begin{aligned} p'_n &= \big((p'_n)_1,(p'_n)_2\big):(\overline{C}_n\backslash C_{n+1},\partial C_n,\partial C_{n+1}) \\ &\quad\to(\mathbf{R}^N\times[1-n,2-n],\mathbf{R}^N\times\{2-n\},\mathbf{R}^N\times\{1-n\}) \end{aligned}$$

such that $p'_n=p'_{n+1}$ on ∂C_{n+1}, $(p'_n)_1(\mathbf{x},t)-\mathbf{x}\in\sum\mathbf{R}$ for all $(\mathbf{x},t)\in\overline{C}_0\backslash C_{n+1}$, and $p'_0=\mathrm{id}$ on ∂C_0. For $n=0$, let P_0 be the homeomorphism of $\mathbf{R}^N\times[1,2]$ defined by $P_0(\mathbf{x},t)=((p_0^{-1})_1(\mathbf{x},2),t)$, and set $p'_0=P_0\circ p_0$ so that $p'_0=\mathrm{id}$ on ∂C_0 and $(p'_0)_1(\mathbf{x},t)-\mathbf{x}\in\sum\mathbf{R}$ for all $(\mathbf{x},t)\in\overline{C}_0\backslash C_1$. Assume that p'_k has been defined satisfying the required conditions for all $0\leq k\leq n$. Let P_{n+1} be the homeomorphism of $\mathbf{R}^N\times[-n,1-n]$ defined by $P_{n+1}(\mathbf{x},t)=((p'_n\circ p_{n+1}^{-1})_1(x,1-n),t)$, and set $p'_{n+1}=P_{n+1}\circ p_{n+1}$ so that $p'_{n+1}=p'_n$ on ∂C_{n+1}, and $(p'_{n+1})_1(\mathbf{x},t)-\mathbf{x}\in\sum\mathbf{R}$ for all $(x,t)\in\overline{C}_{n+1}\backslash C_{n+2}$. The sequence of homeomorphisms $p'_n, n\geq 0$ patch up to form a homeomorphism $a=(a_1,a_2):(\mathbf{R}^N\times\mathbf{R})\backslash\{(\mathbf{0},0)\}\to\mathbf{R}^N\times\mathbf{R}$ such that $a_1(\mathbf{x},t)-\mathbf{x}\in\sum\mathbf{R}$ for all $(\mathbf{x},t)\in\mathbf{R}^N\times\mathbf{R}$ and $a=\mathrm{id}$ on $\mathbf{R}^N\times(2,\infty)$.

Let $D_0=(\mathbf{R}^N\times(0,2))\cup\{(\mathbf{0},0)\}$, and let

$$\begin{aligned} D_n &= \left(\left\{\mathbf{x}\in\mathbf{R}^N:-\frac{1}{2^n}<x_i<\frac{1}{2^n}\text{ for }1\leq i\leq n\text{ and }x_{n+1}>-\frac{1}{2^n}\right\}\times\left(0,\frac{1}{2^n}\right)\right) \\ &\quad\cup\big\{(\mathbf{0},0)\big\}. \end{aligned}$$

Then $(D_n)_{n\geq 0}$ is a fundamental system of open (convex) neighborhood of $(\mathbf{0},0)$ in $(\mathbf{R}^N\times(0,\infty))\cup\{(\mathbf{0},0)\}$. We have a homeomorphism, for all $n\geq 0$,

$$\begin{aligned} q_n &= \big((q_n)_1,(q_n)_2\big):(\overline{D}_n\backslash D_{n+1},\partial D_n,\partial D_{n+1}) \\ &\quad\to(\mathbf{R}^N\times[1-n,2-n],\mathbf{R}^N\times\{2-n\},\mathbf{R}^N\times\{1-n\}) \end{aligned}$$

defined by

$$\begin{aligned} &(q_n)_1(\mathbf{x},t) \\ &\quad=\left(\frac{1}{t}\left(1+\frac{1-(\mu_{n+1}((x,t)))^{-1}}{(\mu_n((x,t)))^{-1}-(\mu_{n+1}((x,t)))^{-1}}\right)(x_1,\dots,x_{n+2}),x_{n+3},x_{n+4},\dots\right), \end{aligned}$$

$$(q_n)_2(\mathbf{x},t)=1+\frac{1-(\mu_{n+1}((x,t)))^{-1}}{(\mu_n((x,t)))^{-1}-(\mu_{n+1}((x,t)))^{-1}}-n,$$

where μ_n is the Minkowski functional of C_n (note that $\mu_n((\mathbf{x},t))$ depends only on t and the first $n+1$ coordinates of $\mathbf{x}$).

Similar to the previous inductive construction of the homeomorphisms $p_n', n \geq 0$, we obtain a sequence of homeomorphisms $q_n', n \geq 0$,

$$\begin{aligned} q_n' &= \big((q_n')_1, (q_n')_2\big) : (\overline{D}_n \backslash D_{n+1}, \partial D_n, \partial D_{n+1}) \\ &\to (\mathbf{R}^N \times [1-n, 2-n], \mathbf{R}^N \times \{2-n\}, \mathbf{R}^N \times \{1-n\}) \end{aligned}$$

such that $q_n' = q_{n+1}'$ on ∂D_{n+1}, $(q_n')_1(\mathbf{x}, t) - \mathbf{x} \in \sum \mathbf{R}$ for all $(\mathbf{x}, t) \in \overline{D}_n \backslash D_{n+1}$ and $q_0' = \mathrm{id}$ on ∂D_0. The sequence of homeomorphisms $q_n', n \geq 0$ patch up to form a homeomorphism $b = (b_1, b_2) : \mathbf{R}^N \times (0, \infty) \to \mathbf{R}^N \times \mathbf{R}$ such that $b_1(\mathbf{x}, t) - \mathbf{x} \in \sum \mathbf{R}$ for all $(\mathbf{x}, t) \in \mathbf{R}^N \times (0, \infty)$ and $b = \mathrm{id}$ on $\mathbf{R}^N \times (2, \infty)$. Note that the homeomorphism $a^{-1} \circ b : \mathbf{R}^N \times (0, \infty) \to (\mathbf{R}^N \times \mathbf{R}) \backslash \{(\mathbf{0}, 0)\}$ extends to a homeomorphism $K : (\mathbf{R}^N \times (0, \infty)) \cup \{(\mathbf{0}, 0)\} \to \mathbf{R}^N \times \mathbf{R}$, which on composing with the homeomorphism $L : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}^{\mathbf{N}} \times (0, \infty)$ given by

$$L(x,t) = \begin{cases} (x, e^t), & t \leq 0, \\ \big(x, 1 + \frac{t}{2}\big), & 0 \leq t \leq 2, \\ (x, t), & t \geq 2, \end{cases}$$

we obtain the sought homeomorphism h satisfying all the required properties. $\square$

The following proposition shows that the property established in [2, Proposition 1.1] for infinite-dimensional locally convex metric linear spaces holds also for metric linear spaces that admit arbitrary short lines.

PROPOSITION 3

Let T be a metric topological vector space over $\mathbf{R}$ that admits arbitrary short lines. Then $(T \times (0, \infty)) \cup \{(\mathbf{0}, 0)\}$ is homeomorphic to $T \times (0, \infty)$.

Proof

Let $|\cdot|$ be the F-norm on T, and let $\hat{T}$ be the completion of T (see [6]). By assumption, there exists $(e_n)_{n \geq 1} \subseteq T \backslash \{0\}$ such that $\sup\{|te_n| : t \in \mathbf{R}\} < 1/4^n$ for all $n \geq 1$, so that $\hat{E} = \sum_{n \leq 1} \mathbf{R} e_n$ is a subspace of $\hat{T}$ topologically isomorphic to $\mathbf{R}^N$ (see [5]). We identify $\hat{E}$ and $\mathbf{R}^N$ by this isomorphism. Note that Michael selection theorem provides a continuous section $s : (T + \hat{E})/\hat{E} \to T + \hat{E}$ such that $s(\mathbf{0}) = \mathbf{0}$ (see [3, p. 87]) so that T is homeomorphic to $\{(w, z) \in \hat{E} \times (T + \hat{E})/\hat{E} : w + s(z) \in T\}$.

Fixing a homeomorphism $m : \mathbf{R}^N \to l^2$ such that $m(\mathbf{0}) = \mathbf{0}$ (see [3, p. 189]) and appealing to [9, Proposition 3.1], we obtain a homeomorphism $K : \mathbf{R}^N \times [0, 1] \to (\mathbf{R}^N \backslash \{\mathbf{0}\} \times (0, 1]) \cup (\mathbf{R}^N \times \{0\})$ defined by $K(x, t) = (f_t(x), t), f_0 = \mathrm{id}$. Therefore, with the above identification of $\hat{E}$ and $\mathbf{R}^N$, we see that T is homeomorphic to $\{(w, z) \in \hat{E} \backslash \{\mathbf{0}\} \times (T + \hat{E})/\hat{E} \cup \{(\mathbf{0}, \mathbf{0})\} : p_1 \circ K^{-1}(w, d(z, \mathbf{0})) + s(z) \in T\}$, where $p_1 : \mathbf{R}^N \times [0, 1] \to \mathbf{R}^N$ denotes the projection onto the first factor and d is a metric defining the topology of $(T + \hat{E})/\hat{E}$ and is bounded by 1.

Using the homeomorphism h of Lemma 2 and the above identification of $\hat{E}$ and $\mathbf{R}^N$, the following homeomorphism establishes our claim:

$$
\begin{aligned}
g: \big\{(w,z) \in \hat{E} \times (T+\hat{E})/\hat{E} : w + s(z) \in T\big\} \times (0,\infty) \cup \big\{\big((\mathbf{0},\mathbf{0}),0\big)\big\} \\
\to \big\{(w,z) \in \hat{E}\backslash\{\mathbf{0}\} \times (T+\hat{E}) \\
/\hat{E} \cup \big\{(\mathbf{0},\mathbf{0})\big\} : p_1 \circ K^{-1}\big(w, d(z,\mathbf{0})\big) + s(z) \in T\big\} \times (0,\infty)
\end{aligned}
$$

defined by $g((w,z),t) = \big((p_1 \circ K(h_1(w,t), d(z,\mathbf{o})), z), h_2(w,t)\big)$ whose inverse is given by

$$
\begin{aligned}
& g^{-1}\big((w,z),t\big) \\
& \quad = \big(((h^{-1})_1(p_1 \circ K^{-1}(w, d(z,\mathbf{o})), t), z), (h^{-1})_2(p_1 \circ K^{-1}(w, d(z,\mathbf{o})), t)\big),
\end{aligned}
$$

where $h^{-1} = ((h^{-1})_1, (h^{-1})_2)$. □

Now we can establish our theorem.

THEOREM 4

A metric topological vector space over $\mathbf{R}$ *is topologically an open cone.*

Proof

Let T be a metrizable topological vector space over $\mathbf{R}$.

If T admits arbitrary short lines, then by virtue of Michael selection theorem (see [3, p. 87]) we may assume that T has the form $T_0 \times \mathbf{R}$, where T_0 is a metric linear space that admits arbitrary short lines. Note that if $U = T_0 \times (-\infty, 1)$ has a Minkowski functional μ, then $(T_0 \times \mathbf{R})\backslash\mu^{-1}(0) \cup \{(\mathbf{0},0)\} = T_0 \times (0,\infty) \cup \{(\mathbf{0},0)\}$, which is homeomorphic to $T_0 \times \mathbf{R}$ by Proposition 3, and by Lemma 1 it is an open cone.

If T does not admit arbitrary short lines, then there exists an open shrinkable neighborhood V of $\mathbf{0}$ in T whose Minkowski functional η has the property that $\eta^{-1}(0) = \{\mathbf{0}\}$ so that $T = T\backslash\eta^{-1}(0) \cup \{\mathbf{0}\}$ and we are again done according to Lemma 1. □

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