Miyawaki's F_{12} spinor L-function conjecture

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Abstract In this paper we prove the Miyawaki conjecture related to the spinor L-function of a Siegel cusp form of weight 12 and degree 3 as a special example of results related to Miyawaki lifts of odd degree. We work with the Fourier–Jacobi model of Ikeda lifts discovered by Yamazaki and Hayashida.

1. Introduction

At a time when no one had a clear idea how to systematically generalize *Saito–Kurokawa liftings* to cuspidal automorphic forms, Miyawaki [8] made two precise conjectures. We follow Miyawaki's notation. Let F_{12} be the unique (up to scalar) Siegel cusp form of degree 3 and weight 12. Armed with rich numerical data and certain insights, Miyawaki determined the local factors of the spinor L-function $L(s, F_{12})$ at the primes $p = 2, 3$ (see [8, Theorem 4.2]). He further showed that this implies a certain degeneration of the standard L-function of F (see [8], Section 6). The conjecture related to the standard L-function was recently proven by Ikeda [6]. Miyawaki states in his paper that "the author believes that the above theorem (Theorem 4.2) will be true for any prime p (Conjecture 4.3)."

In this paper we manifest his belief in a theorem. Our proof builds on results of Andrianov [1], [2], Yamazaki [11], Ikeda [5], [6], Hayashida [4] and others.

THEOREM

Let F_{12} be the Siegel cusp form of degree 3 and weight 12, as given in [8]. Then the spinor L-function $L(s, F_{12})$ is given by

$$(1.1) \qquad L(s, F_{12}) = L(s - 9, \Delta) L(s - 10, \Delta) L(s, \Delta \otimes g_{20}).$$

Here $L(s, \Delta)$ is the Hecke L-function of the Ramanujan function Δ, and $L(s, \Delta \otimes g_{20})$ is the Rankin L-function of the convolution of Δ with the newform g_{20} of weight 20.

We also want to mention that the spinor L-function $L(s, F_{12})$ has a holomorphic continuation to the whole complex plane and is the first example of a spinor L-function of a Siegel cusp form of degree 3 which satisfies a functional equation.

Since Ikeda lifts are Hecke eigenforms for the even Hecke algebra and not necessarily for the full Hecke algebra, we use a quite general approach (Theorem 2.3) to prove the Miyawaki conjecture. It contains implicitly a pullback formula for Ikeda lifts.

2. Ikeda's lifts and Hayashida's construction

Let R be any subring of the real numbers $\mathbb{R}$. Then

$$\mathrm{G}^+\mathrm{Sp}_n(R) := \left\{ \gamma \in \mathrm{GL}_{2n}(R) \, \middle| \, \gamma J_n \gamma^t = \mu(\gamma) J_n, \mu(\gamma) > 0 \right\},$$

where $J_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix} \right)$. The group $\mathrm{G}^+\mathrm{Sp}_n(\mathbb{R})$ acts on the Siegel upper half-plane $\mathfrak{H}_n$ of degree n by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)(z) := (az + b)(cz + d)^{-1}$. Let n, k be positive integers. For a holomorphic complex valued function F on $\mathfrak{H}_n$ and $\gamma \in \mathrm{G}^+\mathrm{Sp}_n(\mathbb{R})$ the Petersson slash operator $|_k$ is given by

$$F|_k\gamma(z) := \det(cz + d)^{-k} F\big(\gamma(z)\big).$$

The space of Siegel modular forms M_k^n of weight k and degree n is given by all F with the invariance property $F|_k\gamma = F$ for $\gamma \in \Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ and certain well-known growth conditions in the case $n = 1$. The subspace of cusp forms we denote by S_k^n. For $F, G \in S_k^n$ we have the Petersson scalar product $\langle F, G \rangle$. Let $\mathcal{H}^n$ be the Hecke algebra associated to the Hecke pair

$$\big(\Gamma_n, \mathrm{M}_{2n}(\mathbb{Z}) \cap \mathrm{G}^+\mathrm{Sp}_n(\mathbb{Q})\big).$$

Then it is well known that this Hecke algebra decomposes in an infinite restricted tensor product $\otimes_p \mathcal{H}_p^n$ of local Hecke algebras generated by

$$T(p)^{(n)} := \Gamma_n \operatorname{diag}(1_n; p1_n)\Gamma_n,$$

$$T_r(p^2)^{(n)} := \Gamma_n \operatorname{diag}(p1_r, 1_{n-r}; p1_r, p^2 1_{n-r})\Gamma_n \quad (0 \le r \le n).$$

The action of the Hecke algebra on S_k^n is induced by the action of $T = \Gamma_n g \Gamma_n = \bigcup_i \Gamma_n g_i$ (disjoint) via

$$(2.1) \qquad\qquad F|_kT := \sum_i \mu(g_i)^{nk-n(n+1)/2} F|_k g_i.$$

Let $F \in S_k^n$ be an eigenform for all Hecke operators T with eigenvalue $\lambda_F(T)$. For $T(p^d)^{(n)} := \sum_{g, \mu(g) = p^d} \Gamma_n g \Gamma_n$, where g runs through $\mathrm{M}_{2n}(\mathbb{Z}) \cap \mathrm{G}^+\mathrm{Sp}_n(\mathbb{Q})$ with different elementary divisors, we set $\lambda_F(T) = \lambda_F(p^d)$.

For the reader's convenience we use Miyawaki's notation for the spinor and standard L-function. Hence, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$ the spinor L-function is denoted by $L(s, F) = \prod_p L_p(s, F)$, and the standard L-function is denoted by $L(s, F, st) = \prod_p L_p(s, F, st)$. Let $F \in S_k$ be a primitive newform (normalized Hecke eigenform) with Fourier coefficients $a(n)$. Then $L(s, F) = \sum_m a(m)m^{-s}$ and $L(s, F, st) = \zeta(2s - 2k + 2)/\zeta(s - k + 1) \sum_m a(m)^2 m^{-s}$. We drop the index $n = 1$ in the context of elliptic modular forms.

Let $g \in S_{2k}$ be a primitive newform with parameters $\{\alpha_p^{\pm 1}\}_p$ (determined by the eigenvalues $a(p) = (\alpha_p + \alpha_p^{-1})p^{(2k-1)/2}$ and $|\alpha_p| = 1$). Then the p-part of the

Hecke L-function $L(s, g)$ is given by $((1 - \alpha_p p^{-s+k-(1/2)})(1 - \alpha_p^{-1} p^{-s+k-(1/2)}))^{-1}$. In [5], for every positive integer $m \equiv k \pmod 2$, Ikeda explicitly constructed a Hecke eigenform $G = I_{k+m}^{2m}(g) \in S_{k+m}^{2m}$, whose standard L-function is equal to

$$(2.2) \qquad \zeta(s) \prod_{j=1}^{2m} L(s + k + m - j, g).$$

Recently Ikeda [6] introduced the concept of *Miyawaki lifts* and by using techniques from representation theory obtained the following main result ([6, Theorem 1.1]). Let $m, r, k \in \mathbb{N}$, and let n be a nonnegative integer with $m = n + r$ and $m \equiv k \pmod 2$. Let $g \in S_{2k}$ be a primitive newform, and let $G \in S_{k+m}^{2m}$ be the Ikeda lift of g. We fix the imbedding $j_1 \times j_2 : \mathfrak{H}_{2n+r} \times \mathfrak{H}_r \longrightarrow \mathfrak{H}_{2n+2r}$, given by $(\tau, \tilde{\tau}) \mapsto \left(\begin{smallmatrix} \tau & 0 \\ 0 & \tilde{\tau} \end{smallmatrix} \right)$. Then the pullback $G \circ (j_1 \times j_2)$ can be viewed as a cusp form on $\mathfrak{H}_r$ if we fix the variable on $\mathfrak{H}_{2n+r}$. Hence the function

$$(2.3) \qquad \mathcal{G}_{g,h}(z) := \langle G \circ (j_1 \times j_2)(z, *), h \rangle,$$

where $h \in S_{k+m}^r$ is well defined. Let g be a Hecke eigenform, and let $\mathcal{G}_{g,h}$ not be identically zero. Then $\mathcal{G}_{g,h}$ is a Hecke eigenform with standard L-function

$$(2.4) \qquad L(s, h, st) \prod_{i=1}^{2n} L(s + k - n - i, g).$$

Applications

Let $g_{20} \in S_{20}$ and $\Delta \in S_{12}$ be the unique primitive newforms. By numerical calculations Ikeda has checked that $\mathcal{G}_{g_{20}, \Delta}$ is not identically zero. Hence this function has the same eigenvalues as the function F_{12} introduced by Miyawaki, since $\dim S_{12}^3 = 1$. So, finally, the standard L-function of F_{12} is equal to

$$(2.5) \qquad L(s, \Delta, st) L(s + 10, g_{20}) L(s + 9, g_{20}).$$

This proves Miyawaki's conjecture related to the standard L-function of F_{12}.

Ikeda's proof [6] of Theorem 1.1 depends upon properties of unramified principal series of Sp_{2m} over p-adic fields. It is directly related to the "even" Hecke algebra of the Hecke pair

$$\left(\Gamma_n, \mathrm{M}_{2n}(\mathbb{Z}) \cap \mathrm{Sp}_n(\mathbb{Q}) \right).$$

Hence it does not cover the eigenvalues $\lambda_{F_{12}}(p)$ necessary to study the spinor L-function of F_{12}. To avoid this gap we choose a different approach to study the spinor and standard L-functions of a Hecke eigenform; namely, we use Hayashida's [4] description of Ikeda lifts in the frame work of Jacobi forms and Fourier–Jacobi expansions of Siegel modular forms. It is in the spirit of Yamazaki's proof [11] of the Maass relations of Siegel Eisenstein series and Eichler and Zagier's [3] description of the Maass Spezialschar.

Let $G \in S_\kappa^{2m}$ be a nontrivial cusp form of integer weight κ and degree $2m$. We are mainly interested in the case $\kappa = k + n + 1$ and $m = n + 1$. We further

consider the Fourier–Jacobi expansion

$$(2.6) \qquad G \begin{pmatrix} \tau & z^t \\ z & \tilde{\tau} \end{pmatrix} = \sum_{l=1}^{\infty} \Phi_{\kappa,l}^G(\tau, z) \tilde{q}^l,$$

where $\tilde{q} := e^{2\pi i \tilde{\tau}}$. It is well known that the Fourier–Jacobi coefficients $\Phi_{\kappa,l}^G$ are Jacobi cusp forms $J_{\kappa,l}^{\mathrm{cusp},2m-1}$ of weight κ, index l, and degree $2m-1$ on $\mathfrak{H}_{2m-1}^J :=$ $\mathfrak{H}_{2m-1} \times \mathbb{C}^{2m-1}$. Let $G = I^{2m}(g)$ be an Ikeda lift of the primitive newform $g \in S_{2k}$ with $\kappa = m + k$. Then Hayashida [4] proved that there exists an operator

$$(2.7) \qquad D_{2m-1}(l, \{\alpha_p\}_p) : J_{\kappa,1}^{\mathrm{cusp},2m-1} \longrightarrow J_{\kappa,l}^{\mathrm{cusp},2m-1},$$

where $\{\alpha_p\}_p$ are the parameters of g, with the fundamental property

$$(2.8) \qquad \Phi_{\kappa,l}^G = \Phi_{\kappa,1}^G \mid D_{2m-1}(l, \{\alpha_p\}_p).$$

In the following we briefly recall the explicit definition of this operator. We also would like to refer to the work of Katsurada and Kawamura [7] for a further application of Hayashida's Fourier–Jacobi model of Ikeda lifts. Let R be any subfield of $\mathbb{R}$. Let

$$G_{2m-1}^J(R) := \left\{ \gamma = \begin{pmatrix} * & * & * & * \\ * & \alpha & * & * \\ * & * & * & * \\ 0 & 0 & 0 & \beta \end{pmatrix} \in \mathrm{G}^+\mathrm{Sp}_{2m}(R) \mid \alpha, \beta \in R \text{ and positive} \right\}.$$

Then we set $\nu(\gamma) := \alpha/\beta$ and $\Gamma_{2m-1}^J := \Gamma_{2m} \cap G_{2m-1}^J(\mathbb{R})$. With $\Phi \in J_{\kappa,l}^{\mathrm{cusp},2m-1}$, put $\widehat{\Phi} := \Phi \tilde{q}^l$. Define the Petersson slash operator for $\gamma \in G_{2m-1}^J(\mathbb{R})$ via

$$\Phi|_{\kappa,l}\gamma := (\tilde{q})^{-l\nu(\gamma)} \widehat{\Phi}|_{\kappa}\gamma.$$

Yamazaki [11] introduced the two Hecke operators $T^J(p)$ and $T_{0,2m}^J(p^2)$ on the space $J_{\kappa,l}^{\mathrm{cusp},2m-1}$ by the double cosets

$$(2.9) \qquad \Gamma_{2m-1}^J \operatorname{diag}(1_{2m-1}, p; p1_{2m-1}, 1)\Gamma_{2m-1},$$

$$(2.10) \qquad \Gamma_{2m-1}^J \operatorname{diag}(p1_{2m-1}, p^2; p1_{2m-1}, 1)\Gamma_{2m-1}.$$

More generally (we mainly follow Hayashida's normalization) let $M \in G_{2m-1}^J(\mathbb{Z})$, and let the disjoint decomposition

$$T^J := \Gamma_{2m-1}^J M \Gamma_{2m-1}^J = \bigcup_i \Gamma_{2m-1}^J M_i$$

be given. Then we define

$$(2.11) \qquad \Phi|_{\kappa,l} T^J := \nu(M)^{m\kappa - (2m-1)m} \sum_i \Phi|_{\kappa,l} M_i.$$

Now we recall the definition of the operators $D_{2m-1}(\{\alpha_p\}_p)$, where $\{\alpha_p\}_p$ are the parameters of $g \in S_{2k}$, through the following formal Dirichlet series:

$$\sum_{l=1}^{\infty} D_{2m-1}(l, \{\alpha_p\}_p) l^{-s} := \prod_p \left\{ 1 - G_p^{(m)}(\alpha_p) T^J(p) p^{(m-1)(m+2)/2-s} \right.$$

$$\left. + T_{0,2m-1}^J(p^2) p^{2m(2m-1)-1-2s} \right\}^{-1}.$$

Here $G_p^{(1)}(\alpha_p) = 1$ and

$$(2.12) \qquad G_p^{(m)}(\alpha_p) = \prod_{1 \leq i \leq m-1} \left\{ (1 + \alpha_p p^{(1-2i)/2})(1 + \alpha_p^{-1} p^{(1-2i)/2}) \right\}^{-1}$$

otherwise. Finally, we have the following.

COROLLARY 2.1

Let $m \in \mathbb{N}$, and let p be a prime. Then

$$(2.13) \qquad D_{2m-1}(p, \{\alpha_p\}_p) = G_p^{(m)}(\alpha_p) T^J(p) p^{(m-1)(m+2)/2}.$$

In the cases $m = 1$ and $m = 2$ we have

$$(2.14) \qquad D_1(p, \{\alpha_p\}) = T^J(p) \qquad \text{and} \qquad D_3(p, \{\alpha_p\}_p) = p^2 G_p^{(2)}(\alpha_p) T^J(p).$$

Moreover, let $b(n)$ be the Fourier coefficients of the primitive newform $g \in S_{2k}$. Then we have

$$G_p^{(2)}(\alpha_p) = p^k \left\{ \left(b(p) + p^k + p^{k-1} \right) \right\}^{-1}.$$

Let $\mathbb{W}$ be the Witt operator related to the imbedding $\mathfrak{H}_{2m-1} \times \mathfrak{H} \longrightarrow \mathfrak{H}_{2m}$, $(\tau, \tilde{\tau}) \mapsto \left(\begin{smallmatrix} \tau & 0 \\ 0 & \tilde{\tau} \end{smallmatrix} \right)$. Then $\mathbb{W}\Phi_{k+m,l}^G$ is an element of S_{k+m}^{2m-1} and $\mathbb{W}G \in S_{k+m}^{2m-1} \otimes S_{k+m}$. We consider the Witt map with respect to the Fourier–Jacobi expansion of an Ikeda lift $G \in S_\kappa^{2m}$, where $m = n + 1$ and $\kappa = k + n + 1$,

$$\mathbb{W}\Big(\sum_{l=1}^{\infty} \Phi_{\kappa,l}^G(\tau, z) e^{2\pi i l \tilde{\tau}} \Big).$$

Here G is an Ikeda lift related to $g \in S_{2k}$ with Satake parameter α_p, α_p^{-1} of absolute value one. Using Hayashida's discovery of a certain operator $D_{2n-1}(m, \{\alpha_p\}_p)$ with the property $\Phi_{\kappa,l}^G = \Phi_{\kappa,1} \mid D_{2m-1}(l, \{\alpha_p\}_p)$, our approach is reduced to the determination of

$$(2.15) \qquad \sum_{l=1}^{\infty} \mathbb{W}\big(\Phi_{\kappa,1}^G \mid D_{2m-1}(l, \{\alpha_p\}_p) \big)(\tau, z) \tilde{q}^l.$$

Directly from the definition of the Hecke operator $T^J(p)$ we can calculate the action of the Witt operator restriction. Then it is obvious that there exist symplectic Hecke operators $\mathcal{T}_l^{(2m-1)} \in \mathcal{H}^{2m-1}$ such that

$$(\mathbb{W}G)(\tau, \tilde{\tau}) = \sum_{l=1}^{\infty} (\mathbb{W}\Phi_{\kappa,l}^G)(\tau) \tilde{q}^l$$

$$= (\mathbb{W}\Phi_{\kappa,1}^G)|_\kappa \sum_{l=1}^{\infty} \mathcal{T}_l^{(2m-1)}(\tau) \tilde{q}^l.$$

After a straightforward calculation we obtain the following lemma.

LEMMA 2.2

Let $G \in S_\kappa^{2m}$ be an Ikeda lift of $g \in S_{2k}$, where $\kappa = k+m$. Then for every prime p:

$$(2.16) \qquad \mathbb{W}\Phi_{\kappa,p}^G = p^{(m-1)(m+2)/2} p^{-(m-1)\kappa} G_p^{(m)}(\alpha_p)(\mathbb{W}\Phi_{\kappa,1}^G)|_\kappa T^{(2m-1)}(p).$$

REMARK

For $m = 1$ we have $\mathbb{W}\Phi_{\kappa,p}^G = (\mathbb{W}\Phi_{\kappa,1}^G)|_\kappa T(p)$ and for $m = 2$ we have

$$(2.17) \qquad \mathbb{W}\Phi_{\kappa,p}^G = p^{2-\kappa} G_p^{(2)}(\alpha_p)(\mathbb{W}\Phi_{\kappa,1}^G)|_\kappa T^{(3)}(p).$$

In the more general case $m = 3$ we have

$$(2.18) \qquad \mathbb{W}\Phi_{\kappa,p}^G = p^{5-2\kappa} G_p^{(3)}(\alpha_p)(\mathbb{W}\Phi_{\kappa,1}^G)|_\kappa T^{(5)}(p).$$

THEOREM 2.3

Let $k > m$, and let $G \in S_{k+m}^{2m}$ be an Ikeda lift attached to the primitive newform $g \in S_{2k}$. For a Hecke eigenform $H \in S_{k+m}^{2m-1}$ and a primitive newform $h \in S_{k+m}$ we have

$$\langle \mathbb{W}G, H \otimes h \rangle \neq 0,$$

if and only if $\langle \mathbb{W}\Phi_{k+m,1}^G, H \rangle \neq 0$, and $\lambda_h(p)$ is equal to $\lambda_H(p)$ times
(2.19)
$$p^{(m-1)(m+2)/2} p^{-(m-1)(k+m)} \prod_{1 \leq i \leq m-1} \left\{ (1+\alpha_p p^{(1-2i)/2})(1+\alpha_p^{-1} p^{(1-2i)/2}) \right\}^{-1}.$$

Here α_p, α_p^{-1} are the parameters of g.

Proof

Assume that $k > m$, so we can extend H to an orthogonal Hecke eigenbasis $(H_i)_I$ of S_κ^{2m-1}, where $\kappa = k+m$ and $H = H_1$. Similarly we extend h to a primitive newform basis $(h_j)_j$ of S_κ with $h = h_1$. Then we have

$$\mathbb{W}G = \sum_{i,j} \alpha_{i,j}^G H_i \otimes h_j.$$

Hence, $\langle \mathbb{W}G, H_i \otimes h_j \rangle$ is equal to $\alpha_{i,j}^G \|H_i\|^2 \|h_j\|^2$. On the other side, we can also employ the Fourier–Jacobi model of Ikeda lifts in the style of the Maass Spezialschar. Then we obtain for $\langle \mathbb{W}G(\,,\tilde{\tau}), H_i \rangle$ the expression

$$(2.20) \qquad \langle \mathbb{W}\Phi_{\kappa,1}^G, H_i \rangle \sum_{l=1}^\infty \lambda_{H_i}(\mathcal{T}_l^{2m-1}) \tilde{q}^l.$$

Here we use several results related to the Fourier–Jacobi expansion of Ikeda lifts and the fact that the operators $\mathcal{T}_l^{2m-1}$ have been chosen and normalized in such a way that they are self-adjoint with respect to the Petersson scalar product. For $l = 1$ we have the identity operator. Let $\langle \mathbb{W}\Phi_{\kappa,1}^G, H_i \rangle \neq 0$. Then it follows from [6] that $\sum_{l=1}^\infty \lambda_{H_i}(\mathcal{T}_l^{2m-1}) \tilde{q}^l$ is a primitive newform, since multiplicity-one for SL_2 is available (see [10]). Hence, it is equal to one of the newforms among the basis $(h_j)_j$. Since the eigenvalues $\lambda_{H_i}(\mathcal{T}_p^{2m-1})$ already determine the newform, the theorem is proven. $\qquad \square$

COROLLARY 2.4

Let $G \in S_\kappa^{2m}$ be an Ikeda lift with nontrivial Witt image $\mathbb{W}G$. Then there exists a Hecke eigenform $H \in S_\kappa^{2m-1}$ and a primitive newform $g \in S_\kappa$ such that $a_h(p)$ is equal to $\lambda_H(p)$ times the expression (2.19). If we normalize H in the case $m = 1$, then $H = h$.

Let $G \in S_\kappa^4$ be an Ikeda lift $(\kappa = k + 2)$ associated to the primitive newform $g \in S_{2k}$ with nontrivial Witt image. Then there exists a Hecke eigenform $H \in S_\kappa^3$ and a primitive newform $h \in S_\kappa$ such that

$$(2.21) \qquad \lambda_H(p) = \lambda_h(p)\big(\lambda_g(p) + p^k + p^{k-1}\big).$$

Applications

Let $G \in S_{12}^4$ be the Ikeda lift associated to $g_{20} \in S_{20}$. Then numerical observations show that the Witt image of F is nontrivial (see, e.g., [6]). Since $\dim S_{12}^3 = \dim S_{12} = 1$ we obtain

$$(2.22) \qquad \lambda_{F_{12}}(p) = a_\Delta(p)\big(a_{g_{20}}(p) + p^{10} + p^9\big).$$

3. The spinor L-function of Miyawaki lifts of degree 3

Miyawaki formulated his conjectures very explicitly. Hence, we restrict ourselves in this section exclusively to Siegel cusp forms of degree 3. We do not consider Eisenstein series, since their L-functions are related to modular forms of lower degree via the Siegel Φ-operator.

Let $F \in S_\kappa^3$ be a Miyawaki lift associated to an Ikeda lift of degree 4. We prove that the spinor L-function of F satisfies the Miyawaki property (P) given in [8, p. 326]. This can be stated in the following way: there exist two primitive newforms $f \in S_{k_1}$ and $g \in S_{k_2}$ with $k_1 = \kappa$ and $k_2 = 2\kappa - 4$ such that the spinor L-function $L(s, F)$ is given by

$$(3.1) \qquad L(s, F) = L(s - \kappa + 2, f)L(s - \kappa + 3, f)L(s, f \otimes g).$$

Here $L(s, f)$ is the Hecke L-function of f, and $L(s, f \otimes g)$ is the Rankin L-function of f and g. For the reader's convenience we give a precise definition of these L-functions. Let $h \in S_k$ be a primitive newform with local parameters $\{\alpha_p^h\}_p$. Then

$$(3.2) \quad L(s, h) := \sum_{m=1}^{\infty} a_h(m)m^{-s} = \prod_p \det\left(1_2 - p^{(k-1)/2}\begin{pmatrix} \alpha_p & 0 \\ 0 & \alpha_p^{-1} \end{pmatrix} p^{-s}\right)^{-1}$$

is the Hecke L-function of h. Further, let $f \in S_{k_1}$ and $g \in S_{k_2}$ be primitive newforms. Then the Rankin convolution of f and g is given by

$$(3.3) \quad \begin{aligned} &L(s, f \otimes g) \\ &:= \prod_p \det\left(1_4 - p^{(k_1+k_2-2)/2}\begin{pmatrix} \alpha_p^f & 0 \\ 0 & (\alpha_p^f)^{-1} \end{pmatrix} \otimes \begin{pmatrix} \alpha_p^g & 0 \\ 0 & (\alpha_p^g)^{-1} \end{pmatrix} p^{-s}\right)^{-1}. \end{aligned}$$

More generally, let $\mu_0, \mu_1, \ldots, \mu_n$ be the p-Satake parameter of the Hecke eigenform $F \in S_k^n$. These are complex numbers, which are unique up to the action of the Weyl group of the symplectic group and $\mu_0^2 \mu_1 \cdot \ldots \cdot \mu_n = p^{nk - n(n+1)/2}$. This is compatible with our normalization of the Hecke operators. Because of the Satake isomorphism we can also define the spinor and the standard L-function via these parameters. The local L-factors are

$$(3.4) \qquad L_p(s, F) := (1 - \mu_0 p^{-s})^{-1} \prod_{r=1}^{n} \prod_{i_1 < \cdots < i_r} (1 - \mu_0 \mu_{i_1} \cdot \ldots \cdot \mu_{i_r} p^{-s})^{-1},$$

$$(3.5) \quad L_p(s, F, st) := (1 - p^{-s})^{-1} \prod_{i=1}^{n} (1 - \mu_i p^{-s})^{-1} (1 - \mu_i^{-1} p^{-s})^{-1}.$$

Then the spinor L-function $L(s, F)$ is equal to $\prod_p L_p(s, F)$, and the standard L-function is defined via

$$(3.6) \qquad L(s, F, st) := \prod_p L_p(s, F, st).$$

The spinor L-function can also be defined by formal power series related to the Hecke operators $T(p^d)^{(n)}$ in the following way:

$$(3.7) \qquad \sum_{d=0}^{\infty} T(p^d)^{(n)} X^d = \frac{P_p(X)}{Q_p(X)},$$

where $P_p(X)$ and $Q_p(X)$ are polynomials of degree $2^n - 2$ and 2^n with coefficients in the Hecke algebra. The denominator polynomial $Q_p(X)$ is directly related to the local L-factor of the spinor L-function. In the case $n = 1$ we have

$$(3.8) \qquad Q_p(X) = 1 - T(p)X + pT_1(p^2)X^2.$$

If we replace the operators by the eigenvalues we obtain the polynomial $Q_{p,F}(X)$. Here we drop the index (n) to simplify notation. Hence $L_p(s, F) = Q_{p,F}(p^{-s})^{-1}$. For $n = 2$ we have

$$1 - T(p)X + \big(T(p)^2 + p(p^2 + 1)T_2(p^2)\big)X^2 - p^3 T(p) T_2(p^2) X^3 + p^6 \big(T_2(p^2)\big)^2 X^4.$$

The case $n = 3$ is complicated. It had been first given by Andrianov [1], in connection with a proof of the conjecture of Shimura related to the symplectic group of genus 3. We have $Q_p(X) = \sum_{m=0}^{8} (-1)^m c(m) X^m$ with

$$(3.9) \quad c(0) = 1,$$

$$(3.10) \quad c(1) = T(p),$$

$$(3.11) \quad c(2) = p\big(T_1(p^2) + (p^2 + 1)T_2(p^2) + (p^2 + 1)^2 T_3(p^2)\big),$$

$$(3.12) \quad c(3) = p^3 T(p)\big(T_2(p^2) + T_3(p^2)\big),$$

$$(3.13) \quad c(4) = p^6 \big(T(p)^2 T_3(p^2) + T_2(p^2)^2 - 2p T_1(p^2) T_3(p^2)$$
$$\qquad\qquad - 2(p-1)T_2(p^2)T_3(p^2) - (p^6 + 2p^5 + 2p^3 + 2p - 1)T_3(p^2)^2\big),$$

$$(3.14) \quad c(5) = p^6 T_3(p^2) c(3),$$

(3.15) $c(6) = p^{12}T_3(p^2)^2 c(2),$

(3.16) $c(7) = p^{18}T_3(p^2)^3 c(1),$

(3.17) $c(8) = p^{24}T_3(p^2)^4.$

We have listed these coefficients explicitly, and double-checked them, since we need the exact values to prove the Miyawaki conjecture.

The degeneration of the spinor L-function $L(s, F)$ given in (3.1) of the Hecke eigenform $F \in S_\kappa^3$, where $\kappa = k + 2$ is equivalent to the following formulas of the eigenvalues:

$$\lambda_F(p) = a_f(p)\big(a_g(p) + p^{k_2/2} + p^{k_2/2-1}\big),$$

$$\lambda_F\big(T_3(p^2)\big) = p^{3\kappa-12},$$

$$\lambda_F\big(T_2(p^2)\big) = a_f(p)^2 p^{k_2-4} + a_g(p)p^{k_1+k_2/2-5}(p+1) - p^{3\kappa-12}(p^3+1),$$

$$\lambda_F\big(T_1(p^2)\big) = a_f(p)^2 a_g(p)p^{k_2/2-2}(p+1) + a_f(p)^2 p^{k_2-4}(p^2-1)$$
$$+ a_g(p)^2 p^{k_1-2} - a_g(p)p^{k_1+k_2/2-5}(p^2+1)(p+1)$$
$$+ p^{3\kappa-10}(p^3+1)(p-1).$$

We have already proven the formula for $\lambda_F(p)$. For this we employed the Fourier–Jacobi model of Ikeda lifts introduced by Hayashida, in the style of the Maass lifts in the setting of Saito–Kurokawa lifts. Moreover we discovered Hecke duality properties of the Witt operator on Ikeda lifts of degree 4 and the involved pullback components. The formula for $\lambda_F(T_3(p^2))$ is obvious and does not depend on the Miyawaki property. The operator $T_3(p^2)$ has only one left coset $\Gamma_3 p 1_6$. Hence $F|_\kappa T_3(p^2)$ is equal to $p^{3\kappa-12}F$.

Next we want to prove the formula for the eigenvalue $\lambda_F(T_2(p^2))$. This may be possible by studying the formal power series $\sum_{l=1}^\infty \mathcal{T}_l^{(3)} X^l$ in more detail. We also choose a different way that works for Miyawaki lifts for degrees larger than 3.

Recall the following notation. Let $F \in S_\kappa^3$ be the Miyawaki lift of the Ikeda lift $G \in S_\kappa^4$, and let the primitive newform $f \in S_\kappa$ be given. Here $\kappa = k + 2$, and G is the lift of the primitive newform $g \in S_{2k}$. Let $\{\alpha_p^g\}_p$ be the parameters of g, and let $(\beta_{p,0}^f, \beta_{1,p}^f)_p$ be the Satake parameters of f. We drop the index p to simplify notation. Further, the Satake parameters of G related to the standard zeta function had been determined by Ikeda [6] using representation theory. Let $\mu_0^G, \mu_1^G, \mu_2^G, \mu_3^G$ be the Satake parameters of G. Then we can choose uniquely (up to the action of the symplectic Weyl group):

$$\mu_1^G = \beta_1^f,$$

$$\mu_2^G = \alpha^g p^{1/2},$$

$$\mu_3^G = (\alpha^g)^{-1} p^{1/2}.$$

These equations determine μ_0^G for every prime p up to sign

(3.18) $$(\mu_0^G)^2 = p^{3\kappa-7}/\beta_1^f.$$

Next we claim that we can determine the Miyawaki formula for $T_2(p^2)$ from (3.10) and (3.12). Let $S(a,b,c) := 1 + a + b + c + ab + ac + bc + abc$, then

$$(3.19) \qquad \mu_0^G S\big(\beta_1^f, \alpha^g p^{1/2}, (\alpha^g)^{-1} p^{1/2}\big) = \lambda_F(p).$$

Further, let the polynomial $T(a,b,c)$ be given by

$$ab(c + ab + ac + bc + abc) + ac(ab + ac + bc + abc) + aab(ac + bc + abc)$$

$$+ aac(bc + abc) + abcabc + bc(ab + ac + bc + abc) + bab(ac + bc + abc)$$

$$+ bac(bc + abc) + bbcabc + cab(ac + bc + abc) + cac(bc + abc)$$

$$+ cbcabc + abac(bc + abc) + abbcabc + acbcabc$$

$$+ a(b + c + ab + ac + bc + abc) + b(c + ab + ac + bc + abc)$$

$$+ c(ab + ac + bc + abc) + ab(ac + bc + abc) + ac(bc + abc) + bcabc.$$

Then we obtain from (3.13) the equation

$$(3.20) \qquad \begin{aligned} & (\mu_0^G)^2 T\big(\beta_1^f, \alpha^g p^{1/2}, (\alpha^g)^{-1} p^{1/2}\big) \\ & \qquad = p^3 S\big(\beta_1^f, \alpha^g p^{1/2}, (\alpha^g)^{-1} p^{1/2}\big) \big(\lambda_F(T_2(p^2)) + p^{3\kappa - 12}\big). \end{aligned}$$

Let $\lambda_F(p) \neq 0$, then we can conclude

$$(3.21) \qquad \lambda_F\big(T_2(p^2)\big) = p^{3\kappa - 12} \Big(\frac{p^2}{\beta_1^f} \frac{T(\beta_1^f, \alpha^g p^{1/2}, (\alpha^g)^{-1} p^{1/2})}{S(\beta_1^f, \alpha^g p^{1/2}, (\alpha^g)^{-1} p^{1/2})} - 1 \Big).$$

After a straightforward calculation we obtain for the right side of this equation

$$p^{3\kappa - 9}\big(\beta_1^f + (\beta_1^f)^{-1} + 2\big) + \big(\alpha^g + (\alpha^g)^{-1}\big) p^{(2\kappa - 5)/2} p^{2\kappa - 7}(p + 1) - p^{3\kappa - 12}(p^3 + 1).$$

On the other side, the Fourier coefficients $a_f(p), a_g(p)$ are related to the above parameters. We have

$$(3.22) \qquad \begin{aligned} a_f(p)^2 &= p^{\kappa - 1}\big(\beta_1^f + (\beta_1^f)^{-1} + 2\big) \qquad \text{and} \\ a_g(p) &= p^{(2\kappa - 5)/2}\big(\alpha^g + (\alpha^g)^{-1}\big). \end{aligned}$$

Hence the conjecture of Miyawaki has been proven for $\lambda_F(T_2(p^2))$ if the eigenvalue $\lambda_F(p) \neq 0$. Similarly we obtain the expected formula for $\lambda_F(T_1(p^2))$ if we evaluate (3.11) in two ways and equate them. One way is via the description by Satake parameters and the other one is by plugging in the formula already obtained for $T_2(p^2)$.

So finally it remains to treat the degenerate case $\lambda_F(p) = 0$ which perhaps does not occur, but which we cannot omit, since the Lehmer conjecture has not been proven yet. We have $\lambda_F(p) = a_f(p)(a_g(p) + p^{k_2/2} + p^{k_2/2 - 1})$. But since the primitive newform $g \in S_{k_2}$ has the property $|a_g(p)| \leq 2p^{(k_2 - 1)/2}$ we have $\lambda_F(p) = 0$ if and only if $a_f(p) = 0$. Assume that $a_f(p) = 0$. Then the Satake parameters $(\mu_0^F, \mu_1^F, \mu_2^F, \mu_3^F)$ of F are given by

$$(3.23) \qquad \big(\varepsilon(\beta_1^f)^{-1/2} p^{(3\kappa - 7)/2}, \beta_1^f, \alpha^g p^{1/2}, (\alpha^g)^{-1} p^{1/2}\big),$$

with $\beta_1^f = -1$ and $\varepsilon = \pm 1$. Since the Satake parameters are only unique up to the action of the related symplectic Weyl group, we can choose $\varepsilon = 1$, because β_1^f is degenerate and has the value (-1). The p-local factor of $L(s, F)$ in this special case can be carried out directly via the Satake parameters. This leads to the predicted formulas for the spinor L-function and for the eigenvalues of F.

REMARK

In the degree 3 case after suitable normalization, let

$$1 - b_1 x + b_2 x^2 - b_3 x^3 + b_4 x^4 - b_3 x^5 + b_2 x^6 - b_1 x^7 + x^8$$

$$(1 - x)(1 - c_1 x + c_2 x^2 - c_3 x^3 + c_2 x^4 - c_1 x^5 + x^6)$$

be the local spinor and standard polynomial. Then

$$b_1^2 = 2 + 2c_1 + 2c_2 + c_3,$$

$$b_2 = 1 + 2c_1 + c_2,$$

$$b_3 = b_1(1 + c_1),$$

$$b_4 = 2 + 2c_1 + c_1^2 + c_3.$$

Hence some parts of the proof can also be formulated in terms of the formal coefficients. This was suggested by a referee.

3.1. Final remarks

In the paper [9], Murakawa gives two theorems related to the spinor L-function of Ikeda lifts (Theorem 3.1) and Miyawaki lifts (Theorem 5.1). Since our approach is different it may be worthwhile to make some comments.

The Satake parameters of the related standard L-functions are given by Ikeda [5],[6]. To talk about spinor L-function one has first to show that Ikeda lifts and Miyawaki lifts are Hecke eigenform for the full Hecke algebra. This is not proven in Ikeda's paper [5], since Ikeda has only been interested in the Hecke algebra related to $\mathrm{Sp}_{2n}(\mathbb{Q})$ and not $\mathrm{G}^+\mathrm{Sp}_{2n}(\mathbb{Q})$. Further one has to determine the sign of the Satake parameter μ_0 in the case of the Ikeda and Miyawaki lifts (for Miyawaki lifts one also has to be aware of the nonvanishing).

Murakawa first gives a proof of Theorem 3.1. He assumes that μ_0 exists (i.e., that the lift is a Hecke eigenform not only the even Hecke algebra). Since Murakawa's paper is an extraction of his master's thesis in Japanese we may assume that the related proof is given there. Then he calculates μ_0 by a method indicated by Ikeda in the case of the standard L-function. This works since the related Satake parameter can be obtained via equating certain equations in which the involved terms are nonzero.

After stating Theorem 5.1 he indicated that the proof can be given in the same way as the proof of Theorem 3.1, if one exchanges the Siegel-type Eisenstein series for a Klingen-type Eisenstein series. Again one first would have to show that the Miyawaki lift is a Hecke eigenform for the full Hecke algebra (in our proof we had to use the multiplicity-one theorem of SL_2; hence, one could give a

new proof of this by applying Murakawa's result, if fully available). Moreover we found that only if Lehmer's conjecture is true, then one might be able to transfer [9, Lemma 4.1], since in contrast to Ikeda lifts the eigenvalues $\lambda(p)$ of Miyawaki lifts could be zero. In the case of F_{12} we actually proved that all $\lambda(p)$ are nonzero if and only if the Lehmer conjecture is true. That means even when the Satake parameters μ_1, μ_2, μ_3 of the Miyawaki lift F_{12} are fixed, it may happen that μ_0 is not unique.

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