Displaceability and the mean Euler characteristic

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Abstract In this paper we show that the mean Euler characteristic of equivariant symplectic homology is an effective obstruction against the existence of displaceable exact contact embeddings. As an application we show that certain Brieskorn manifolds do not admit displaceable exact contact embeddings.

1. Introduction

A contact manifold (Σ, ξ) is said to admit an *exact contact embedding* if there exists an embedding $\iota\colon \Sigma \to V$ into an exact symplectic manifold (V, λ) and a contact form α for (Σ, ξ) such that $\alpha - \iota^*\lambda$ is exact and such that $\iota(\Sigma) \subset V$ is bounding. In this paper we suppose, in addition, that any target manifold (V, λ) is convex; that is, there exists an exhaustion $V = \bigcup_k V_k$ of V by compact sets $V_k \subset V_{k+1}$ with smooth boundary such that $\lambda|_{\partial V_k}$ is a contact form and such that the first Chern class of (V, λ) vanishes on $\pi_2(V)$. An exact contact embedding is called *displaceable* if $\iota(\Sigma)$ can be displaced from itself by a Hamiltonian isotopy of V.

We refer to [4], [5], [11] for more details on exact contact embeddings, and for examples and obstructions to such embeddings.

The *mean Euler characteristic* of a simply connected contact manifold was introduced by van Koert [14] in terms of contact homology and was studied further in [6], [9]. Here, we consider the mean Euler characteristic of equivariant symplectic homology, which can be thought of as the mean Euler characteristic of a filling. For the definition see Section 2. Under additional assumptions, these notions coincide; see Corollary 2.2 and the subsequent remark.

The idea behind the mean Euler characteristic is that sometimes it can be computed by looking at the closed Reeb orbits of a suitable contact form, without computing the homology.

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We say that a simply connected cooriented contact manifold (Σ, α) is *index-positive* if the mean index $\Delta(\gamma)$ of every periodic Reeb orbit γ is positive. Similarly, we say that (Σ, α) is *index-negative* if the mean index $\Delta(\gamma)$ of every periodic Reeb orbit γ is negative. Finally, we say that (Σ, α) is *index-definite* if it is index-positive or index-negative. Recall that the mean index Δ is related to the Conley–Zehnder index μ_{CZ} as follows: for any nondegenerate Reeb orbit γ in a contact manifold (Σ^{2n-1}, α), its N-fold cover γ^N satisfies

$$\mu_{CZ}(\gamma^N) = N\Delta(\gamma) + e(N), \tag{1}$$

where $e(N)$ is an error term bounded by $n-1$ (see [13, Lemma 3.4]).

In this note we prove the following theorem.

THEOREM A

Assume that (Σ, ξ) is a $(2n-1)$-dimensional simply connected contact manifold which admits a displaceable exact contact embedding. Suppose furthermore that (Σ, α) is index-definite for some α defining ξ. Then the following holds.

(i) (Σ, α) *is index-positive.*

(ii) *The mean Euler characteristic of its filling W is a half-integer:*

$$\chi_m(W) = \frac{(-1)^{n+1}}{2}\chi(W, \Sigma).$$

(iii) *If, in addition, Σ is a rational homology sphere, then the mean Euler characteristic of its filling equals $(-1)^{n+1}/2$.*

REMARKS

(i) Yau proved in [19] that boundaries of subcritical Stein manifolds carry an index-positive contact form.

Ritter [12] proved that the displaceability of Σ implies the vanishing of the symplectic homology of the filling W. It is conceivable that then in fact the equivariant symplectic homology of W vanishes. This would imply that for assertions (ii) and (iii) of Theorem A the assumption that (Σ, α) is index-definite can be omitted (see Remark 3.1).

(ii) In the situation of (ii), assume in addition that (Σ, ξ) has a contact form α such that all Reeb orbits are nondegenerate and such that the number of Reeb orbits of a given index is uniformly bounded. Then the Euler characteristic $\chi(W)$ of any filling W of Σ is determined by a dynamical and a topological invariant of Σ (see Corollary 2.2(ii) below). This should be compared with the results by Oancea and Viterbo [11], where many strong constraints on the topology of a filling W are given in purely topological terms of Σ.

Brieskorn manifolds We next look at a class of examples for which the mean Euler characteristic can indeed be computed from the closed orbits of a suitable Reeb flow. Given positive integers $a_0, \dots, a_n$ one defines the Brieskorn manifold $\Sigma(a_0, \dots, a_n)$ as the link of a certain singularity. The Brieskorn manifold

$\Sigma(a_0, \dots, a_n)$ is said to be *nontrivial* if $a_i \neq 1$ for all i. If $a_0, \dots, a_n$ are pairwise relatively prime, and if $n > 2$, then $\Sigma(a_0, \dots, a_n)$ is homeomorphic to S^{2n-1}. Brieskorn manifolds carry a natural contact structure. A trivial Brieskorn manifold is a round sphere with its standard contact structure in $\mathbb{R}^{2n}$, and hence admits a displaceable exact contact embedding.

COROLLARY B

A nontrivial Brieskorn manifold $\Sigma(a_0, \dots, a_n)$ of dimension at least 5 whose exponents are pairwise relatively prime does not admit a displaceable exact contact embedding. In particular, it does not admit an exact contact embedding into a subcritical Stein manifold whose first Chern class vanishes.

The restriction to manifolds of dimension at least 5 comes from the following observations.

REMARKS

(i) In dimension 3, nontrivial Brieskorn manifolds are not simply connected. Hence Reeb orbits in these manifolds can become contractible in the filling even if they are not contractible in the Brieskorn manifold. The mean Euler characteristic of symplectic homology, on the other hand, counts Reeb orbits that are contractible in the filling, so it cannot be determined by just considering the contact manifold by itself.

(ii) In [11, Proposition 6.2(b)] it is shown that no Brieskorn manifold $\Sigma(a_0, \dots, a_n)$ of dimension at least 5 with $a_j \geq 2$ for all j admits a contact embedding into a subcritical Stein manifold.

It would be interesting to know whether Corollary B still holds true if we drop the convexity assumption on the target manifold (V, λ) or the assumption that its first Chern class vanishes.

2. The mean Euler characteristic

Assume that (W, λ) is a compact exact symplectic manifold; that is, $\omega = d\lambda$ is a symplectic form on W, with convex boundary $\Sigma = \partial W$. We assume throughout that the first Chern class $c_1(W)$ of $(W, d\lambda)$ vanishes on $\pi_2(W)$ and that Σ is simply connected. For $i \in \mathbb{Z}$ we denote by

$$b_i(W) = \dim\big(SH_i^{S^1,+}(W; \mathbb{Q})\big)$$

the ith Betti number of the positive part of the equivariant symplectic homology of W (as defined in [2], [17]).

For later use, we call a homology $H_*(C_*, \partial)$ *index-positive* if there exists N such that $H_i(C_*, \partial) = 0$ for all $i < N$. Note here that if $(\Sigma, \alpha) = \partial(W, d\lambda)$ is index-positive in the previously defined sense, then $SH_*^{S^1,+}(W)$ is index-positive in the homological sense. The notions index-negative and index-definite are defined on homology level in a similar way.

DEFINITION

W is called *homologically bounded* if the Betti numbers $b_i(W)$ are uniformly bounded.

If W is homologically bounded, we define its *mean Euler characteristic* as

$$\chi_m(W) = \frac{1}{2}\Big(\liminf_{N\to\infty} \frac{1}{N} \sum_{i=-N}^{N} (-1)^i b_i(W) + \limsup_{N\to\infty} \frac{1}{N} \sum_{i=-N}^{N} (-1)^i b_i(W)\Big).$$

The uniform bound on the Betti numbers implies that the limes inferior and the limes superior exist.

A closed Reeb orbit of a contact form α on Σ is a (possibly multiply covered) closed orbit of the Reeb flow on Σ defined by α. Assume that α is a contact form on Σ with the property that all the closed Reeb orbits are nondegenerate. We recall that a closed Reeb orbit γ is called *bad* if it is the m-fold cover of a Reeb orbit γ' and the difference of Conley–Zehnder indices $\mu(\gamma) - \mu(\gamma')$ is odd. A closed Reeb orbit which is not bad is called *good*.

DEFINITION

Σ is called *dynamically bounded* if there exists a uniform bound for the number of good closed Reeb orbits of Conley–Zehnder index i for every $i \in \mathbb{Z}$.

We denote by $\mathfrak{G}_N$ the set of good closed Reeb orbits of Conley–Zehnder index lying between $-N$ and N. If Σ is dynamically bounded, we define its mean Euler characteristic by

$$\chi_m(\Sigma) = \frac{1}{2}\Big(\liminf_{N\to\infty} \frac{1}{N} \sum_{\gamma\in\mathfrak{G}_N} (-1)^{\mu(\gamma)} + \limsup_{N\to\infty} \frac{1}{N} \sum_{\gamma\in\mathfrak{G}_N} (-1)^{\mu(\gamma)}\Big).$$

REMARK

Ginzburg and Kerman [9] define the positive and negative parts of the mean Euler characteristic of contact homology by summing over all positive and all negative degrees, respectively. Their mean Euler characteristic is half of the one we define.

If W is a compact exact symplectic manifold, we say that W is dynamically bounded if its boundary $\Sigma = \partial W$ is dynamically bounded.

THEOREM 2.1

Assume that W is dynamically bounded. Then it is homologically bounded and

$$\chi_m(\partial W) = \chi_m(W).$$

COROLLARY 2.2

(i) *If W is dynamically bounded, then its mean Euler characteristic is independent of the filling.*

(ii) *If, in addition, Σ carries an index-definite contact form, then*

$$\chi(W) = 2(-1)^{n+1}\chi_m(\Sigma) + \chi(\Sigma).$$

For assertion (ii) we have also used Theorem A(ii) and the additivity $\chi(W) = \chi(\Sigma) + \chi(W, \Sigma)$ of the Euler characteristic.

REMARK

Since the generators of the positive part of equivariant symplectic homology and contact homology are the same, the mean Euler characteristic can also be expressed in terms of contact homology data. This was done in the original definition in [14]. Note, however, that the degree of a Reeb orbit γ in contact homology is defined as $\mu_{CZ}(\gamma) + n - 3$ if the dimension of the contact manifold is $2n-1$. This can result in a sign difference for the mean Euler characteristic.

Proof of Theorem 2.1

If Γ denotes the set of all closed Reeb orbits on $\Sigma = \partial W$, the critical manifold $\mathfrak{C}$ for the positive equivariant part of the action functional of classical mechanics is then given by

$$\mathfrak{C} = \bigcup_{\gamma \in \Gamma} \gamma \times_{S^1} ES^1.$$

If γ is a k-fold cover of a simple Reeb orbit, then the isotropy group of the action of S^1 on γ is $\mathbb{Z}_k$. Therefore,

$$\gamma \times_{S^1} ES^1 = B\mathbb{Z}_k$$

is the infinite-dimensional lens space. The Morse–Bott spectral sequence (see [7, Section 7.2.2]) tells us that there exists a spectral sequence converging to $SH_*^{S^1,+}(W;\mathbb{Q})$, whose second page is given by

$$E^1_{i,j} = \bigoplus_{\substack{\gamma \in \Gamma \\ \mu(\gamma)=i}} H_j(\gamma \times_{S^1} ES^1; \mathcal{O}_\gamma).$$

The twist bundle $\mathcal{O}_\gamma$ is trivial if γ is good and equals the orientation bundle of the lens space if γ is bad (see [1], [2], [16]); that is, $\mathcal{O}_\gamma$ is the nontrivial bundle if k is even. The homology of an infinite-dimensional lens space with rational coefficients equals $\mathbb{Q}$ in degree zero and vanishes otherwise. Its homology with coefficients twisted by the orientation bundle is trivial. Therefore the second page of the Morse–Bott spectral sequence simplifies to

$$E^1_{i,j} = \bigoplus_{\substack{\gamma \in \mathfrak{G} \\ \mu(\gamma)=i}} \mathbb{Q},$$

where $\mathfrak{G} \subset \Gamma$ are the good closed Reeb orbits. We conclude that the mean Euler characteristic of E^1 coincides with $\chi_m(\Sigma)$. Since the Euler characteristic is unchanged if we pass to homology, we deduce that $\chi_m(\Sigma)$ equals $\chi_m(W)$. This finishes the proof of the theorem. $\square$

In our application to Brieskorn manifolds, we will compute the mean Euler characteristic for a contact form of Morse–Bott type. Brieskorn manifolds can be thought of as Boothby–Wang orbibundles over symplectic orbifolds, since there is a contact form for which all Reeb orbits are periodic. For such special contact manifolds the mean Euler characteristic has a particularly simple form.

We start with introducing some notation to state the result. Consider a contact manifold (Σ, α) with Morse–Bott contact form α having only finitely many orbit spaces, so that we have an S^1-action on Σ. Denote the periods by $T_1 < \cdots < T_k$, so all T_i divide T_k. Denote the subspace consisting of points on periodic Reeb orbits with period T_i in Σ by N_{T_i}.

LEMMA 2.3
If $H^1(N_{T_i};\mathbb{Z}_2) = 0$, then $H^1(N_{T_i} \times_{S^1} ES^1;\mathbb{Z}_2) = 0$.

Proof
Consider the Leray spectral sequence for $N_{T_i} \times_{S^1} ES^1$ as a fibration over $\mathbb{C}P^\infty$. As $\pi_1(\mathbb{C}P^\infty) = 0$, the Leray spectral sequence with $\mathbb{Z}_2$-coefficients converges to the cohomology of $N_{T_i} \times_{S^1} ES^1$. The E_2-page of this spectral sequence is given by $E_2^{pq} = H^p(\mathbb{C}P^\infty; H^q(N_{T_i};\mathbb{Z}_2))$. Since $H^1(N_{T_i};\mathbb{Z}_2) = 0$ by assumption, there are no degree 1-terms on E_2. Hence there are no degree 1-terms in E_∞ either, and $H^1(N_{T_i} \times_{S^1} ES^1;\mathbb{Z}_2) = 0$. □

Finally we introduce the function

$$\phi_{T_i;T_{i+1},\dots,T_k} = \#\big\{a \in \mathbb{N} \,\big|\, aT_i < T_k \text{ and } aT_i \notin T_j\mathbb{N} \text{ for } j = i+1,\dots,k\big\}.$$

PROPOSITION 2.4
Let (Σ, α) be a contact manifold as above and assume that it admits an exact filling $(W, d\lambda)$. Suppose that $c_1(\xi = \ker\alpha) = 0$, so that the Maslov index is well defined. Let $\mu_P := \mu(\Sigma)$ be the Maslov index of a principal orbit of the Reeb action. Assume that $H^1(N_T \times_{S^1} ES^1;\mathbb{Z}_2) = 0$ for all N_T and that there are no bad orbits.

If $\mu_P \neq 0$, then the following hold.

- (Σ, α) *is homologically bounded.*
- (Σ, α) *is index-positive if $\mu_P > 0$ and index-negative if $\mu_P < 0$.*
- *The mean Euler characteristic satisfies the following formula,*

$$\chi_m(W) = \frac{\sum_{i=1}^k (-1)^{\mu(N_{T_i}) - 1/2\dim(N_{T_i}/S^1)} \phi_{T_i;T_{i+1},\dots,T_k} \chi^{S^1}(N_{T_i})}{|\mu_P|}.$$

Here $\chi^{S^1}(N_T)$ denotes the Euler characteristic of the S^1-equivariant homology of the S^1-manifold N_T.

Proof
We use the notation

$$H_p^{S^1}(N_T;\mathbb{Q}) := H_p(N_T \times_{S^1} ES^1;\mathbb{Q}).$$

As before, there is the Morse–Bott spectral sequence converging to $SH_*^{S^1,+}(W;\mathbb{Q})$. The second page is given by

$$E^1_{pq} = \bigoplus_{\substack{N_T \\ \mu(N_T)-1/2\dim(N_T/S^1)=p}} H_p^{S^1}(N_T;\mathbb{Q}).$$

Indeed, the coefficient ring is not twisted as $H^1(N_T \times_{S^1} ES^1;\mathbb{Z}_2)=0$.

The period of a principal orbit is T_k, so we have $\phi^R_{T_k}=\mathbb{1}$. Since the Robbin–Salamon version of the Maslov index is additive under concatenations, it follows that for any set of periodic orbits N_T with return time $T>T_k$ we have

$$\mu(N_T)=\mu(N_{T_k})+\mu(N_{T-T_k}).$$

It follows that the E^1-page is periodic in the q-direction with period $|\mu(N_{T_k})|=|\mu_P|$ (as $N_{T_k}=\Sigma$). Since we have assumed that $\mu_P\neq 0$, we see that $SH^{S^1,+}(W)$ is homologically bounded.

Moreover, by the definition of the Maslov index μ_P, the sign of μ_P determines whether (Σ,α) is index-positive or index-negative.

Finally, the mean Euler characteristic can be obtained by summing all contributions in one period and dividing by the period. This gives

$$\chi_m(W)=\frac{\sum_{T\leq T_k}(-1)^{\mu(N_T)-1/2\dim(N_T/S^1)}\chi^{S^1}(N_T)}{|\mu_P|}.$$

Now observe that the definition of the functions $\phi_{T_i;T_{i+1},\dots,T_k}$ is such that it counts how often multiple covers of a set of periodic orbits N_{T_i} appear in one period of the E^1-page without being contained in a larger orbit space. We thus obtain the above formula. □

REMARK
This proposition is a generalization of [6, Example 8.2], and Espina's methods could also be used to show the above.

3. Proof of Theorem A

In the first two paragraphs of this section we prove three general statements that in particular imply assertions (i) and (ii) of Theorem A. In Sections 3.3 and 3.4 we then work out the situation for rational homology spheres.

3.1. Two general statements

PROPOSITION 3.1
Assume that (Σ,ξ) is a $(2n-1)$-dimensional simply connected contact manifold admitting a displaceable exact contact embedding into $(V,d\lambda)$. Denote the compact

component of $V \setminus \Sigma$ by W. Suppose furthermore that (Σ, α) is index-positive. Then

$$SH_*^{S^1,+}(W) \cong H_{*+n-1}^{S^1}(W, \Sigma).$$

COROLLARY 3.2

Under the assumptions of Proposition 3.1,

$$\chi_m(W) = (-1)^{n+1} \frac{\chi(W, \Sigma)}{2}.$$

Proof of Proposition 3.1

Consider the S^1-equivariant version of the Viterbo long exact sequence,

$$\cdots \longrightarrow H_{*+n}^{S^1}(W, \Sigma) \longrightarrow SH_*^{S^1}(W) \longrightarrow SH_*^{S^1,+}(W) \longrightarrow H_{*+n-1}^{S^1}(W, \Sigma) \longrightarrow \cdots$$

from [2], [17]. By assumption $SH_*^{S^1,+}(W)$ is index-positive. The group homology $H_*^{S^1}(W, \Sigma)$ is also index-positive, so we conclude that $SH_*^{S^1}(W)$ must be index-positive as this group is sandwiched between zeros for sufficiently negative $*$.

By Ritter's theorem [12, Theorem 97] displaceability of Σ implies $SH_*(W) = 0$. The Gysin sequence for equivariant and nonequivariant symplectic homology from [2] reads

$$\cdots \longrightarrow SH_*(W) \longrightarrow SH_*^{S^1}(W) \xrightarrow{D_*} SH_{*-2}^{S^1}(W) \longrightarrow SH_{*-1}(W) \longrightarrow \cdots$$

so all maps D_* are isomorphisms. Since we just showed that $SH_*^{S^1}(W)$ is index-positive, it must vanish in all degrees.

Finally consider the equivariant version of the Viterbo sequence once again. Since $SH_*^{S^1}(W)$ vanishes, it follows that

$$H_{*+n}^{S^1}(W, \Sigma) \cong SH_{*+1}^{S^1,+}(W). \qquad \square$$

Corollary 3.2 follows from Proposition 3.1 by observing that $H_*^{S^1}(W, \Sigma) \cong H_*(W, \Sigma) \otimes H_*(\mathbb{C}P^\infty)$, since the S^1-action on (W, Σ) is trivial (by construction of Viterbo's long exact sequence). In other words, $H_*^{S^1}(W, \Sigma)$ consists of infinitely many copies of $H_*(W, \Sigma)$ which are degree-shifted by $0, 2, 4, \ldots$.

REMARK

It is conceivable that the displaceability of W implies that $SH_*^{S^1}(W)$ vanishes. (This is known if W is a subcritical Stein domain; see [3] and [2, p. 5].) The conclusion of Proposition 3.1, without the assumption that (Σ, α) is index-positive, would then follow at once from Viterbo's S^1-equivariant long exact sequence. Assertions (ii) and (iii) of Theorem A would then hold without the assumption that (Σ, α) is index-definite.

3.2. Index-positivity

LEMMA 3.3

Assume that (Σ, ξ) is a $(2n-1)$-dimensional simply connected contact manifold *admitting a displaceable exact contact embedding into* $(V, d\lambda)$*. Denote the compact component of* $V \setminus \Sigma$ *by* W*. Suppose furthermore that* $(\Sigma, \xi = \ker\alpha)$ *is index-definite. Then* (Σ, α) *is index-positive.*

Proof
Again by Ritter's theorem [12, Theorem 97] we conclude that $SH_*(W) = 0$. Hence the Viterbo long exact sequence from [2], [17] reduces to

$$\cdots \longrightarrow 0 \longrightarrow SH_*^+(W) \xrightarrow{\cong} H_{*+n-1}(W, \Sigma) \longrightarrow 0 \longrightarrow \cdots,$$

so we see that $SH_{n+1}^+(W) \cong H_{2n}(W, \Sigma) \cong H^0(W) \neq 0$.

Now suppose that (Σ, α) is index-negative. On one hand, our previous observation shows that there is a generator of degree $n+1$. On the other hand, if α is a nondegenerate contact form, then the iteration formula (1) tells us that an N-fold cover of a Reeb orbit γ satisfies

$$|\mu_{CZ}(\gamma^N) - N\Delta(\gamma)| \leq n - 1,$$

where $\Delta(\gamma)$ denotes the mean index of the Reeb orbit γ. Since (Σ, α) is index-negative, $\Delta(\gamma) < 0$, so $\mu_{CZ}(\gamma^N) < n-1$. In particular, no generator of $SH_{n+1}^+(W)$ can be realized by a Reeb orbit. This contradiction shows that (Σ, α) must be index-positive. □

3.3. Displaceability and splitting the sequence of the pair

In the following lemma, (V, Ω) is a connected manifold endowed with a volume form, and $W \subset V$ is a compact connected submanifold with connected boundary of the same dimension as V with the property that the volume of the complement of W in V is infinite. We say that the hypersurface $\Sigma = \partial W \subset V$ is *volume-preserving displaceable* if there exists a compactly supported smooth family of volume-preserving vector fields X_t, $t \in [0, 1]$, on V such that the time-1 map ϕ of its flow satisfies $\phi(\Sigma) \cap \Sigma = \emptyset$.

LEMMA 3.4
Assume that $\Sigma = \partial W$ *is volume-preserving displaceable in* V*. Then the projection homomorphism* $p_*\colon H_*(W; \mathbb{Q}) \to H_*(W, \Sigma; \mathbb{Q})$ *vanishes.*

Proof
We prove the lemma in two steps. For the first step we need the assumption about volume preservation.

STEP 1
The volume-preserving diffeomorphism ϕ *displacing* Σ *displaces the whole filling; that is,* $\phi(W) \cap W = \emptyset$*.*

We divide the proof of Step 1 into three substeps.

STEP 1a

There exists a point $x \in W$ such that $\phi(x) \notin W$.

We argue by contradiction and assume that $\phi(W) \subset W$. In particular, the restriction of ϕ to W gives a diffeomorphism between the two manifolds with boundary W and $\phi(W) \subset W$. Therefore, if $y \in W$ satisfies $\phi(y) \in \partial W$, it follows that $y \in \partial W$. We conclude

$$\phi(W) \cap \partial W \subset \phi(\partial W).$$

Since ϕ displaces the boundary from itself, we obtain

$$\phi(W) \cap \partial W = \emptyset.$$

Denoting by int the interior of a set, we can write this equivalently as

$$\phi(W) \subset \mathrm{int}(W).$$

Hence $\phi(W)$ is a strict subset of W. Since W is compact, its volume is finite. Therefore, the volume of $\phi(W)$ is strictly less than the volume of W. This contradicts the fact that ϕ is volume preserving. Therefore the assertion of Step 1a has to hold true.

STEP 1b

We have $\phi(\partial W) \subset W^c$.

Since ϕ displaces ∂W from itself, we have $\partial W \subset \mathrm{int}(W) \cup W^c$. Since ∂W is connected by assumption, we either have $\phi(\partial W) \subset \mathrm{int}(W)$ or $\phi(\partial W) \subset W^c$. Therefore it suffices to show that $\phi(\partial W) \cap W^c$ is not empty. Since W^c has infinite volume but $\phi(W)$ has finite volume by assumption, we conclude that there exists a point $y_0 \in W^c$ such that $y_0 \notin \phi(W)$. Step 1a implies the existence of a point $y_1 \in W^c$ satisfying $y_1 \in \phi(W)$. Since V, W, and ∂W are connected by assumption, we obtain from the Mayer–Vietoris long exact sequence that W^c is connected as well. Therefore there exists a path $y \in C^0([0,1], W^c)$ satisfying $y(0) = y_0$ and $y(1) = y_1$. Since W^c is Hausdorff, there exists $t \in (0,1)$ such that $y(t) \in \partial(\phi(W)) = \phi(\partial W)$. Therefore $\phi(\partial W) \cap W^c$ is not empty, which finishes the proof of Step 1b.

STEP 1c

We prove Step 1.

We assume by contradiction that there exists a point $x_0 \in W \cap \phi(W)$. By Step 1a and the fact that ϕ is volume preserving, we conclude that W cannot be a subset of $\phi(W)$. Therefore there has to exist a point $x_1 \in W \cap (\phi(W))^c$ as well. Since W is connected by assumption, there exists a path $x \in C^0([0,1], W)$ satisfying $x(0) = x_0$ and $x(1) = x_1$. As in Step 1b there has to exist $t \in (0,1)$ such that $x(t) \in \phi(\partial W)$. But this contradicts the assertion of Step 1b. The proof of Step 1 is complete.

STEP 2

If a diffeomorphism ϕ isotopic to the identity satisfies $\phi(W)\cap W=\emptyset$, then the projection homomorphism $p_*\colon H_*(W;\mathbb{Q})\to H_*(W,\partial W;\mathbb{Q})$ vanishes.

We prove the dual version in de Rham cohomology; that is, we show that the inclusion homomorphism from the compactly supported de Rham cohomology of W to the de Rham cohomology of W vanishes. To see this, pick $\omega\in\Omega^k(W)$, which is compactly supported and closed. We show that there exists $\eta\in\Omega_{k-1}(W)$ not necessarily compactly supported such that $\omega=d\eta$. Since ω is compactly supported, we can extend it trivially to a closed k-form on V which we refer to as $\widetilde{\omega}$. Since ϕ is isotopic to the identity, we have $\phi=\phi^1$ for a flow $\{\phi^t\}_{t\in[0,1]}$ generated by a time-dependent vector field X_t. By the Cartan formula and the fact that $\widetilde{\omega}$ is closed we obtain

$$\begin{aligned}\frac{d}{dt}(\phi^t)^*\widetilde{\omega}&=L_{X_t}(\phi^t)^*\widetilde{\omega}\\&=(di_{X_t}+i_{X_t}d)(\phi^t)^*\widetilde{\omega}\\&=di_{X_t}(\phi^t)^*\widetilde{\omega}.\end{aligned}$$

We define a $(k-1)$-form on V by the formula

$$\widetilde{\eta}=-\int_0^1 i_{X_t}(\phi^t)^*\widetilde{\omega}.$$

By the previous computation we get

$$\widetilde{\omega}-\phi^*\widetilde{\omega}=d\widetilde{\eta}.$$

Now set

$$\eta=\widetilde{\eta}|_W\in\Omega^{k-1}(W).$$

Since ϕ displaces W we obtain

$$\omega=d\eta.$$

This finishes the proof of Step 2 and hence of Lemma 3.4. □

3.4. Rational homology spheres and completion of the proof of Theorem A(iii)

In the case of rational homology spheres, the homology of the filling is completely determined.

LEMMA 3.5

Suppose (Σ,ξ) is a $(2n-1)$-dimensional simply connected rational homology sphere admitting a displaceable exact contact embedding into $(V,d\lambda)$. Let W denote the compact component of $V\setminus\Sigma$. Then

$$H_*(W,\Sigma;\mathbb{Q})=\begin{cases}\mathbb{Q} & \textit{if } *=2n,\\ \{0\} & \textit{else.}\end{cases}$$

Proof
We can assume that V has infinite volume. Indeed, if V has finite volume, we choose a compact convex manifold V_k in the exhaustion of V such that $W \subset V_k$, and we replace V by the manifold $\widehat{V}$ obtained by attaching cylindrical ends to the boundary of V_k. Notice that $\widehat{V}$ is also an exact convex manifold whose first Chern class vanishes on $\pi_2(\widehat{V})$.

In view of Lemma 3.4 the long exact homology sequence for the pair (W, Σ) splits for every $k \in \mathbb{Z}$ into short exact sequences

$$0 \longrightarrow H_k(W, \Sigma; \mathbb{Q}) \xrightarrow{\partial} H_{k-1}(\Sigma; \mathbb{Q}) \xrightarrow{i_*} H_{k-1}(W; \mathbb{Q}) \longrightarrow 0.$$

By using the fact that Σ is a rational homology sphere as well as $H_0(W; \mathbb{Q}) = \mathbb{Q}$ we conclude that $H_*(W, \Sigma; \mathbb{Q}) = \{0\}$ for $* \neq 2n$. Since $H^{2n}(W, \Sigma; \mathbb{Q})$ is Poincaré dual to $H_0(W; \mathbb{Q})$, the result for $* = 2n$ also follows. □

REMARK
If (Σ, ξ), as above, admits an exact contact embedding into a subcritical Stein manifold, then much more is known: any symplectically aspherical filling W of Σ is a rational homology ball (see [11, Corollary 2.14]).

Lemma 3.5 shows that the Euler characteristic of the relative homology is given by

$$\chi(W, \Sigma) = 1.$$

Assertion (iii) of Theorem A follows from this and Corollary 3.2.

4. Brieskorn manifolds

Choose positive integers $a_0, \dots, a_n$. The Brieskorn variety $V(a_0, \dots, a_n)$ is defined as the following subvariety of $\mathbb{C}^{n+1}$,

$$V_\epsilon(a_0, \dots, a_n) = \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=0}^{n} z_i^{a_i} = \epsilon \right\}.$$

For $\epsilon = 0$, this variety is singular unless one of the exponents a_i is equal to 1. For $\epsilon \neq 0$, we have a complex submanifold of $\mathbb{C}^{n+1}$.

Given a Brieskorn variety $V_0(a_0, \dots, a_n)$ we define the Brieskorn manifold as

$$\Sigma(a_0, \dots, a_n) := V_0(a_0, \dots, a_n) \cap S_R^{2n+1},$$

where S_R^{2n+1} is the sphere of radius $R > 0$ in $\mathbb{C}^{n+1}$. For the diffeomorphism type, the precise value of R does not matter. Brieskorn manifolds carry a natural contact structure, which comes from the following construction.

LEMMA 4.1
Let (W, i) be a complex variety together with a function f that is plurisubharmonic away from singular points. Then regular level sets $M = f^{-1}(c)$ carry a contact structure $\xi = TM \cap iTM = \ker(-df \circ i)|_M$.

Applying this lemma with the plurisubharmonic function $f = \sum_j \frac{a_j}{8}|z_j|^2$ we obtain the particularly nice contact form

$$\alpha = \frac{i}{8}\sum_j a_j(z_j\, d\bar{z}_j - \bar{z}_j\, dz_j)$$

for this natural contact structure. Its Reeb vector field at radius $R = 1$ is given by

$$R_\alpha = 4i\sum_j \frac{1}{a_j}(z_j\partial_{z_j} - \bar{z}_j\partial_{\bar{z}_j}).$$

The Reeb flow therefore is

$$\phi_t^{R_\alpha}(z_0,\dots,z_n) = (e^{4it/a_0}z_0,\dots,e^{4it/a_n}z_n).$$

We thus see that all Reeb orbits are periodic. This allows us to interpret Brieskorn manifolds as Boothby–Wang bundles over symplectic orbifolds.

PROPOSITION 4.2

Brieskorn manifolds admit a Stein filling, and their contactomorphism type does not depend on the radius R of the sphere used to define them.

Indeed, by definition, Brieskorn manifolds are singularly fillable. One can smooth this filling by taking $\epsilon \neq 0$, and consider V_ϵ rather than V_0. The resulting contact structure is contactomorphic by Gray stability. Furthermore, V_ϵ gives then the Stein filling. Gray stability can also be used to show independence of the radius R (see [8, Theorem 7.1.2]).

4.1. Brieskorn manifolds and homology spheres

Let us start by citing some theorems from [10]. This book gives precise conditions for Brieskorn manifolds to be integral homology spheres. However, we shall restrict ourselves to the following case.

PROPOSITION 4.3

If $a_0,\dots,a_n$ are pairwise relatively prime, then $\Sigma(a_0,\dots,a_n)$ is an integral homology sphere.

Furthermore, higher-dimensional Brieskorn manifolds, that is, $\dim\Sigma > 3$, are always simply connected, so we in fact find the following.

THEOREM 4.4

If $a_0,\dots,a_n$ are pairwise relatively prime and if $n > 2$, then $\Sigma(a_0,\dots,a_n)$ is homeomorphic to S^{2n-1}.

REMARK

If one of the exponents a_j is equal to 1, then the resulting Brieskorn manifold $(\Sigma(a_0,\dots,a_n),\alpha)$ is contactomorphic to the standard sphere (S^{2n-1},α_0). Indeed, in this case the Brieskorn variety $V_\epsilon(a_0,\dots,a_n)$ is biholomorphic to $\mathbb{C}^n$, as we can regard the variety as a graph.

4.2. Formula for the mean Euler characteristic for Brieskorn manifolds

We can think of Brieskorn manifolds as Boothby–Wang orbibundles over symplectic orbifolds. However, all the essential data are contained in the S^1-equivariant homology groups associated with the Reeb action. The following lemma will hence be useful.

LEMMA 4.5

Let N be a rational homology sphere of dimension $2n-1$ with a fixed-point free S^1-action $N\times S^1\to N$. Then

$$H_*^{S^1}(N;\mathbb{Q})\cong H_*(\mathbb{CP}^{n-1};\mathbb{Q}).$$

In particular,

$$\chi^{S^1}(N)=n.$$

Proof

Note that $N\times ES^1$ carries a free S^1-action, so we can think of $N\times ES^1$ as an S^1-bundle over $N\times_{S^1}ES^1$. We consider the Gysin sequence for this space with $\mathbb{Q}$-coefficients. Since N is a rational homology sphere of dimension $2n-1$ and ES^1 is contractible, all homology groups of $N\times ES^1$ except in dimension 0 and $2n-1$ vanish. Hence the Gysin sequence reduces to

$$\underset{\cong 0}{H_*(N)}\xrightarrow{\pi_*}\underset{=H_*^{S^1}(N)}{H_*(N\times_{S^1}ES^1)}\xrightarrow{\cap e}\underset{=H_{*-2}^{S^1}(N)}{H_{*-2}(N\times_{S^1}ES^1)}\longrightarrow\underset{\cong 0}{H_{*-1}(N)}$$

for $1<*<2n-1$. This shows that $H_*^{S^1}(N;\mathbb{Q})\cong H_*(\mathbb{CP}^{n-1};\mathbb{Q})$ for $*<2n-1$. To see that there are no other terms, we shall argue that $H_*^{S^1}(N;\mathbb{Q})$ is bounded. For this, choose an S^1-equivariant Morse–Bott function $f\colon N\to\mathbb{R}$ (see [18, Lemma 4.8] for the existence of such a function). Define a Morse–Bott function

$$\tilde f\colon N\times_{S^1}ES^1\longrightarrow\mathbb{R},$$
$$[x,v]\longmapsto f(x).$$

Consider the Morse–Bott spectral sequence for $H_*(N\times_{S^1}ES^1;\mathbb{Q})$ with respect to the Morse–Bott function $\tilde f$. Its E^1-page is given by $E^1_{pq}=H_q(R_p;\mathbb{Q})$, where R_p are the critical manifolds of $\tilde f$ with index p. Again, by [7, Section 7.2.2] this sequence converges to $H_*(N\times_{S^1}ES^1;\mathbb{Q})$. Note that the critical manifolds form infinite-dimensional lens spaces, so $H_q(R_p;\mathbb{Q})\cong\mathbb{Q}$ if $q=0$ and 0 otherwise. Since there are only finitely many critical manifolds (because N is compact), it follows that $H_*^{S^1}(N;\mathbb{Q})$ is bounded.

With this in mind, we reexamine the Gysin sequence. Assume that $H_k^{S^1}(N;\mathbb{Q})$ is nonzero for some $k\geq 2n-1$. Then $H_{k+2}^{S^1}(N;\mathbb{Q})$ is nonzero either, and so forth. Hence $H_*^{S^1}(N;\mathbb{Q})$ is not bounded, which contradicts our previous term. The lemma follows. □

REMARK

Strictly speaking, $N \times_{S^1} ES^1$ has no manifold structure. Recalling $ES^1 = S^\infty$, we can, however, approximate this space by $N \times_{S^1} S^{2M+1}$ for large M. For the latter space, the above argument works and can be adapted to show triviality of $H_i(N \times_{S^1} S^{2M+1}; \mathbb{Q})$ for $i \geq 2n-1$ and $i < 2M$.

PROPOSITION 4.6

The Brieskorn manifold $\Sigma(a_0, \dots, a_n)$ with its natural contact form α is index-positive if $\sum_j \frac{1}{a_j} > 1$, and index-negative if $\sum_j \frac{1}{a_j} < 1$. Furthermore, if the exponents $a_0, \dots, a_n$ are pairwise relatively prime, then the mean Euler characteristic of $\Sigma(a_0, \dots, a_n)$ is given by

$$\begin{aligned}(2)\quad &\chi_m\big(\Sigma(a_0, \dots, a_n), \alpha\big) = (-1)^{n+1} \\ &\times \frac{n + (n-1)\sum_{i_0}(a_{i_0} - 1) + \cdots + 1\sum_{i_0 < \cdots < i_{n-2}}(a_{i_0} - 1)\cdots(a_{i_{n-2}} - 1)}{2\big|\big(\sum_j a_0 \cdots \widehat{a_j} \cdots a_n\big) - a_0 \cdots a_n\big|}.\end{aligned}$$

Proof

The proof is a direct application of Proposition 2.4. The principal orbits have period $a_0 \cdots a_n$. Exceptional orbits have periods $a_0, \dots, a_n, a_0a_1, \dots, a_{n-1}a_n, \dots, a_1 \cdots a_n$. Given a collection of exponents $I = \{a_{i_1}, \dots, a_{i_k}\} \subset \{a_0, \dots, a_n\}$ we denote the associated subset of periodic orbits with period $a_{i_1} \cdots a_{i_k}$ by N_I.

In [14] the Maslov index of all periodic Reeb orbits is computed. For the principal orbit, the result is

$$\mu_P := 2\,\mathrm{lcm}_i\, a_i\Big(\sum_j \frac{1}{a_j} - 1\Big) = 2\sum_j a_0 \cdots \widehat{a_j} \cdots a_n - a_0 \cdots a_n.$$

We check that the conditions of Proposition 2.4 are satisfied. By Proposition 4.3 it follows that $H^1(N_I; \mathbb{Z}_2) = 0$ if the index set I has more than 2 elements (i.e. $\dim N_T > 1$), so Lemma 2.3 applies. Furthermore, the index computations in [14] show that there are no bad orbits.

Hence Proposition 2.4 applies, so $\Sigma(a_0, \dots, a_n)$ is index-positive if $\sum_j \frac{1}{a_j} > 1$ and index-negative if $\sum_j \frac{1}{a_j} < 1$. Furthermore, the S^1-equivariant Euler characteristics needed in Proposition 2.4 are obtained from Lemma 4.5.

The formula for the Maslov index of the exceptional orbits is slightly more complicated (see [15, (3.1)]), but we only need to observe that the parity of $\mu(N_{T_i}) - 1/2 \dim(N_{T_i}/S^1)$ is the same as the one of $n+1$.

We conclude the proof by determining the coefficients $\phi_{T_i; T_{i+1}, \dots, T_n}$. We do this by counting how often multiple covers of an orbit space appear in one period. The full orbit space $N_{\{a_0, \dots, a_n\}}$ appears once. The orbit space $N_{\{a_0, \dots, a_{n-1}\}}$ appears a_n times, but the last time it contributes, it is part of $N_{\{a_0, \dots, a_n\}}$, which we already considered. Therefore $N_{\{a_0, \dots, a_{n-1}\}}$ contributes $a_n - 1$ times. By downward induction on the cardinality of I, we conclude that N_I appears $\prod_j (a_j - 1)/\prod_{a \in I}(a - 1)$ times in one period. $\square$

REMARK

The mean Euler characteristic of S^1-equivariant symplectic homology coincides with the mean Euler characteristic of contact homology. This means that the above computation amounts to an application of the algorithm in [15]. However, there are still many issues with the foundations of contact homology, so we do not pursue this line of thought.

5. Proof of Corollary B

We start by some general observations that will be needed in the proof.

For $n \in \mathbb{N}$ we define

$$f(n) = \sum_{j=0}^{n} (-1)^j (n-j) \binom{n+1}{j}.$$

We claim the following identity

$$f(n) = (-1)^{n+1}. \tag{3}$$

We prove (3) by induction. It holds that $f(1) = 1$, and for the induction step we compute

$$\begin{aligned}
f(n+1) &= \sum_{j=0}^{n+1} (-1)^j (n+1-j) \binom{n+2}{j} \\
&= \sum_{j=0}^{n} (-1)^j (n+1-j) \binom{n+2}{j} \\
&= \sum_{j=0}^{n} (-1)^j (n+1-j) \left(\binom{n+1}{j} + \binom{n+1}{j-1} \right) \\
&= \sum_{j=0}^{n} (-1)^j (n+1-j) \binom{n+1}{j} + \sum_{j=0}^{n} (-1)^j (n+1-j) \binom{n+1}{j-1} \\
&= \sum_{j=0}^{n} (-1)^j (n-j) \binom{n+1}{j} + \sum_{j=0}^{n} (-1)^j \binom{n+1}{j} \\
&\quad + \sum_{j=1}^{n} (-1)^j \big(n-(j-1)\big) \binom{n+1}{j-1} \\
&= (-1)^{n+1} + \sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} - (-1)^{n+1} \\
&\quad + \sum_{j=0}^{n-1} (-1)^{j+1} (n-j) \binom{n+1}{j} \\
&= (-1)^{n+1} + (1-1)^{n+1} - (-1)^{n+1} - (-1)^{n+1} \\
&= -(-1)^{n+1} \\
&= (-1)^{n+2}.
\end{aligned}$$

This proves the induction step and hence (3) follows.

Alternatively, we can compute

$$\begin{aligned}
0 &= \frac{d}{dx}(-1+x)^{n+1}\Big|_{x=1} \\
&= \sum_{j=0}^{n}(-1)^j(n+1-j)x^{n-j}\binom{n+1}{j}\Big|_{x=1} \\
&= f(n) + \sum_{j=0}^{n}(-1)^j\binom{n+1}{j} + (-1)^{n+1}\binom{n+1}{n+1} - (-1)^{n+1} \\
&= f(n) + (-1+1)^{n+1} - (-1)^{n+1} = f(n) - (-1)^{n+1}.
\end{aligned}$$

PROPOSITION 5.1

Let $\Sigma(a_0,\dots,a_n)$ be a Brieskorn manifold whose exponents are pairwise relatively prime. Suppose that $\sum_j \frac{1}{a_j} > 1$. Then $\chi_m(\Sigma(a_0,\dots,a_n),\alpha) = (-1)^{n+1}/2$ if and only if one of the exponents is equal to 1.

Proof

The condition $\sum_j \frac{1}{a_j} > 1$ implies that the denominator of (2) (without $|\,|$) is positive, so

$$\begin{aligned}
&\chi_m\big(\Sigma(a_0,\dots,a_n),\alpha\big) = (-1)^{n+1} \\
&\qquad\times \frac{n + (n-1)\sum_{i_0}(a_{i_0}-1) + \cdots + \sum_{i_0<\dots<i_{n-2}}(a_{i_0}-1)\cdots(a_{i_{n-2}}-1)}{2\big(\big(\sum_j a_0\cdots\widehat{a_j}\cdots a_n\big) - a_0\cdots a_n\big)}.
\end{aligned}$$

Let us now try to solve the equation $\chi_m = (-1)^{n+1}/2$. We obtain

$$\begin{aligned}
n + (n-1)\sum_{i_0}(a_{i_0}-1) + \cdots + \sum_{i_0<\cdots<i_{n-2}}(a_{i_0}-1)\cdots(a_{i_{n-2}}-1) \\
= \Big(\sum_j a_0\cdots\widehat{a_j}\cdots a_n\Big) - a_0\cdots a_n.
\end{aligned}$$

We multiply out all terms on the left-hand side and organize them as linear combinations of elementary symmetric polynomials $e_d(a_0,\dots,a_n)$ of degree d, for $d = 0,\dots,n-2$. Using (3) repeatedly we obtain

$$\sum_{k=0}^{n-1}(-1)^{n-1-k}e_k(a_0,\dots,a_n) = e_n(a_0,\dots,a_n) - e_{n+1}(a_0,\dots,a_n).$$

Moving all terms to the left-hand side and collecting them yields the equation

$$\prod_{j=0}^{n}(a_j-1) = 0,$$

which can only hold if one of the exponents is equal to 1. $\square$

Observe that the remark after Theorem 4.4 implies that the mean Euler characteristic has to be equal to $(-1)^{n+1}/2$ if one of the exponents equals 1.

Proof of Corollary B
Let $\Sigma(a_0,\ldots,a_n)$ be a Brieskorn manifold with pairwise relatively prime exponents $a_0,\ldots,a_n$. If the exponents $a_0,\ldots,a_n$ satisfy $\sum_j \frac{1}{a_j} < 1$, then Proposition 4.6 tells us that $(\Sigma(a_0,\ldots,a_n),\alpha)$ is index-negative. Theorem A(i) implies that such manifolds do not admit a displaceable exact contact embedding.

If the exponents $a_0,\ldots,a_n$ are pairwise relatively prime, then $\sum_j \frac{1}{a_j} \neq 1$. Indeed, suppose that $\sum_j \frac{1}{a_j} = 1$. Then

$$\frac{1}{a_0} = 1 - \sum_{j=1}^{n} \frac{1}{a_j} = \frac{a_1 \cdots a_n - \sum_{j=1}^{n} a_1 \cdots \widehat{a_j} \cdots a_n}{a_1 \cdots a_n}.$$

If we invert the left- and right-hand side, we see that a_0 divides $a_1 \cdots a_n$, which shows that $a_0,\ldots,a_n$ are not pairwise relatively prime. This leaves the case that $\sum_j \frac{1}{a_j} > 1$. For this case, Proposition 5.1 applies, so together with Theorem A(iii) we conclude that nontrivial Brieskorn manifolds with pairwise relatively prime exponents do not admit exact displaceable contact embeddings.

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