# Smoothing effect of weak solutions for the spatially homogeneous Boltzmann equation without angular cutoff

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**Abstract** In this paper, we consider the spatially homogeneous Boltzmann equation without angular cutoff. We prove that every $L^1$-weak solution to the Cauchy problem with finite moments of all orders acquires the $C^\infty$-regularity in the velocity variable for all positive time.

## 1. Introduction

Consider the Cauchy problem for the spatially homogeneous Boltzmann equation,

$$\begin{cases} f_t(t,v) = Q(f,f)(t,v), & t \in \mathbb{R}^+, v \in \mathbb{R}^3, \\ f(0,v) = f_0(v), \end{cases} \tag{1.1}$$

where $f = f(t,v)$ is the density distribution function of particles with velocity $v \in \mathbb{R}^3$ at time $t$. The right-hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g,f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)\{g(v'_*)f(v') - g(v_*)f(v)\}\, d\sigma\, dv_*,$$

which is well defined for suitable functions $f$ and $g$ specified later. Notice that the collision operator $Q(\cdot,\cdot)$ acts only on the velocity variable $v \in \mathbb{R}^3$. In the following discussion, we will use the $\sigma$-representation, that is, for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

which give the relations between the post- and precollisional velocities. For monoatomic gas, the nonnegative cross section $B(z,\sigma)$ depends only on $|z|$ and the

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scalar product $\frac{z}{|z|} \cdot \sigma$. As in [6]–[8], we assume that it takes the form

$$B(v - v_*, \cos\theta) = \Phi(|v - v_*|) b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, 0 \le \theta \le \frac{\pi}{2}, \tag{1.2}$$

in which it contains a kinetic factor given by

$$\Phi(|v - v_*|) = \Phi_\gamma(|v - v_*|) = |v - v_*|^\gamma, \tag{1.3}$$

with $\gamma > -3$ and a factor related to the collision angle with singularity,

$$b(\cos\theta)\theta^{2+2s} \to K, \quad \text{when } \theta \to 0+, \tag{1.4}$$

for some positive constant $K$ and $0 < s < 1$.

The main purpose of this paper is to show the smoothing effect of the spatially homogeneous Boltzmann equation; that is, any weak solution to the Cauchy problem (1.1) acquires regularity as soon as $t > 0$. Let us recall the precise definition of weak solution for the Cauchy problem (1.1) given in [16] (see also [17]). To this end, we introduce the standard notation, $\langle v \rangle = (1 + |v|^2)^{1/2}$,

$$\|f\|_{L^p_\ell} = \Big( \int_{\mathbb{R}^3} |f(v)|^p \langle v \rangle^{\ell p} \, dv \Big)^{1/p}, \quad \text{for } p \ge 1, \ell \in \mathbb{R},$$

$$\|f\|_{H^m_\ell} = \Big( \int_{\mathbb{R}^3} \big| \langle D \rangle^m \big( \langle v \rangle^\ell f(v) \big) \big|^2 \, dv \Big)^{1/2}, \quad \text{for } m, \ell \in \mathbb{R},$$

$$\|f\|_{L \log L} = \int_{\mathbb{R}^3} |f(v)| \log\big(1 + |f(v)|\big) \, dv,$$

and we denote $a^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$.

DEFINITION 1.1

Let $f_0 \ge 0$ be a function defined on $\mathbb{R}^3$ with finite mass, energy, and entropy; that is,

$$\int_{\mathbb{R}^3} f_0(v) \big[ 1 + |v|^2 + \log\big(1 + f_0(v)\big) \big] \, dv < +\infty.$$

We say that $f$ is a weak solution of the Cauchy problem (1.1), if it satisfies the following conditions:

$$f \ge 0, \quad f \in C\big(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^3)\big) \cap L^1\big([0, T]; L^1_{2+\gamma^+}(\mathbb{R}^3)\big),$$

$$f(0, \cdot) = f_0(\cdot),$$

$$\int_{\mathbb{R}^3} f(t, v)\psi(v) \, dv = \int_{\mathbb{R}^3} f_0(v)\psi(v) \, dv \quad \text{for } \psi = 1, v_1, v_2, v_3, |v|^2,$$

$$f(t, \cdot) \in L \log L, \quad \int_{\mathbb{R}^3} f(t, v) \log f(t, v) \, dv \le \int_{\mathbb{R}^3} f_0 \log f_0 dv, \quad \forall t \ge 0,$$

$$\int_{\mathbb{R}^3} f(t, v)\varphi(t, v) \, dv - \int_{\mathbb{R}^3} f_0(v)\varphi(0, v) \, dv - \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau, v)\partial_\tau \varphi(\tau, v) \, dv$$
$$= \int_0^t d\tau \int_{\mathbb{R}^3} Q(f, f)(\tau, v)\varphi(\tau, v) \, dv,$$

where $\varphi \in C^1(\mathbb{R}^+; C_0^\infty(\mathbb{R}^3))$. Here, the last integral on the right-hand side given above is defined by

$$\begin{aligned}&\int_{\mathbb{R}^3} Q(f,f)(v)\varphi(v)\,dv\\&\qquad = \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} Bf(v_*)f(v)\big(\varphi(v')+\varphi(v'_*)-\varphi(v)-\varphi(v_*)\big)\,dv\,dv_*\,d\sigma.\end{aligned}$$

Hence, this integral is well defined for any test function $\varphi \in L^\infty([0,T]; W^{2,\infty}(\mathbb{R}^3))$ (see [16, p. 291]).

To state the main theorem in this paper, we introduce the entropy dissipation functional by

$$D(g,f) = -\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B(g'_*f' - g_*f)\log f\,dv\,dv_*\,d\sigma,$$

where $f=f(v), f'=f(v'), g_*=g(v_*), g'_*=g(v'_*)$.

THEOREM 1.2

*Let the cross section $B$ in the form (1.2) satisfy (1.3) and (1.4) with $0<s<1$.*

(1) *Suppose that $\gamma > \max\{-2s,-1\}$. Let $f$ be a weak solution of the Cauchy problem (1.1). For $0 \le T_0 < T_1$, if $f$ satisfies*

$$|v|^\ell f \in L^\infty\big([T_0,T_1]; L^1(\mathbb{R}^3)\big) \quad \text{for any } \ell \in \mathbb{N}, \tag{1.5}$$

*then*

$$f \in L^\infty\big([t_0,T_1]; \mathcal{S}(\mathbb{R}^3)\big),$$

*for any $t_0 \in ]T_0,T_1[$.*

(2) *When $-1 \ge \gamma > -2s$, the same conclusion as above holds if we have the following entropy dissipation estimate:*

$$\int_{T_0}^{T_1} D\big(f(t),f(t)\big)\,dt < +\infty. \tag{1.6}$$

The existence of weak solutions to the Cauchy problem (1.1) was proved by Villani [16] when $\gamma \ge -2$, assuming additionally in the case $\gamma > 0$ that $f_0 \in L^1_{2+\delta}$ for some $\delta > 0$. One important property of the weak solution for the hard potentials (namely, when $\gamma > 0$) is, according to the work by Wennberg [18] (cf. also Bobylev [9]), the moment gain property. It means that $f$ satisfies (1.5) for arbitrary $T_0 > 0$ when the initial data only satisfies finite mass, energy, and entropy. However, without assuming the moment condition (1.5), we can still consider the smoothing effect in the case of mild singularity ($0<s<1/2$) for the hard potential ($\gamma > 0$), and the argument is similar to the one used in [13] (see Theorem 5.2 in Section 5).

The regularization property given by the above theorem has been studied by many authors (cf. [2], [3], [12]–[15]). However, to our knowledge, it has not yet been completely established in the sense that the kinetic factor $\Phi(|z|)$ was modified to avoid the singularity at the origin except for the Maxwellian molecule case in previous works, and moreover, some extra conditions other than those in Definition 1.1 of weak solution were required in [3] and [12].

We would like to emphasize that the result of Theorem 1.2 gives the full regularization property for any weak solution satisfying some natural boundedness condition in some weighted $L^1$ and $L\log L$-space and do not require any differentiability assumption on the solution. Therefore, by our result, Villani's weak solutions in [16] are proved to be smooth by Theorem 1.2 together with the moment gain property when $\gamma > 0$.

To compare with a recent work [10], it was proved therein that $W_p^{1,1} \cap H^3$ (strong) solutions gain full regularity in the case $0 < s < 1/2$. Their method is based on a priori estimates of the smooth solution, together with results given in [11] about the propagation of the norm $W_p^{1,1}$ and the uniqueness of this strong solution. Departing from [10], we start from the weak solution given in Definition 1.1 without any known uniqueness result. Therefore, an a priori estimate for the smooth function is not enough to show the regularity for the weak solution in $L^1$ with moments. For the proof of Theorem 1.2, some suitable mollifier, acting on the weak solution, becomes necessary, so that its commutator with the collision operator requires some subtle analysis.

More precisely, the mollifier with symbol having time-dependent order (see Section 4) was first used in [14], where the Maxwellian molecule case ($\gamma = 0$) was studied together with the Gevrey regularity of solutions for the linearized Boltzmann equation, and in [13] the same mollifier led to the smoothing effect of weak solutions for the spatially homogeneous Boltzmann equation with the modified kinetic factor $\tilde{\Phi}(|z|) = \langle z\rangle^{\gamma}$, using the pseudodifferential calculus on the commutator between $\tilde{\Phi}$ and the mollifier. In the spatially inhomogeneous case, the regularity and the existence of classical solutions were studied by [4] for the modified kinetic factor $\tilde{\Phi}$, and by [5]–[8] for the singular kinetic factor $\Phi$. As stated in the preceding paragraph, the commutator between the mollifier and the collision operator with the singular kinetic factor requires some subtle analysis developed in [8] and [7].

Throughout this paper, we will use the following notation: $f \lesssim g$ means that there exists a generic positive constant $C$ such that $f \leq Cg$; while $f \gtrsim g$ means that $f \geq Cg$. And $f \sim g$ means that there exist two generic positive constants $c_1$ and $c_2$ such that $c_1 f \leq g \leq c_2 f$.

The rest of the paper will be organized as follows. In the next section, we will prove a uniform coercivity estimate which improves on the one given in [1] and which has its own interest. The mollifier and the commutator estimate will be given in Section 3. In Section 4 we will prove the smoothing effect of a weak solution with extra $L^2$-assumption. Finally, the last section will be devoted to the proof of Theorem 1.2.

## 2. A uniform coercive estimate

In this section, we will improve the coercive estimate for the collision operator obtained in [1], where local $H^s_{\mathrm{loc}}$-estimates were discussed (cf. (2.2), (2.3) below).

In view of the definition of the weak solution, for $D_0, E_0 > 0$ we set

$$\mathcal{U}(D_0,E_0) = \{g \in L^1_2 \cap L\log L; g \geq 0, \|g\|_{L^1} \geq D_0, \|g\|_{L^1_2} + \|g\|_{L\log L} \leq E_0\}.$$

Set $B(R) = \{v \in \mathbb{R}^3; |v| \leq R\}$ for $R > 0$, and let $B_0(R,r) = \{v \in B(R); |v - v_0| \geq r\}$ for a $v_0 \in \mathbb{R}^3$ and $r \geq 0$. It follows from the definition of $\mathcal{U}(D_0,E_0)$ that there exist positive constants $R > 1 > r_0$ depending only on $D_0, E_0$ such that

$$g \in \mathcal{U}(D_0,E_0) \quad \text{implies } \chi_{B_0(R,r_0)} g \in \mathcal{U}(D_0/2, E_0), \tag{2.1}$$

where $\chi_A$ denotes a characteristic function of the set $A \subset \mathbb{R}^3$. In fact, noticing that for $R, M > 0$,

$$R^2 \int_{\{|v|>R\}} g\, dv + \log(1+M) \int_{\{g>M\}} g\, dv \leq E_0,$$

we have

$$\int_{\{|v|\leq R\}\cap\{g\leq M\}} g\, dv \geq 3D_0/4$$

if $R \geq 2\sqrt{2E_0/D_0}$ and $\log(1+M) \geq 8E_0/D_0$, and moreover, we have

$$\int_{\{|v-v_0|<r_0\}\cap\{g\leq M\}} g\, dv \leq D_0/4$$

if $r_0 \leq \big(3D_0/(16\pi \exp(8E_0/D_0))\big)^{1/3}$.

PROPOSITION 2.1

*Suppose that the cross section $B$ of the form (1.2) satisfies (1.3) and (1.4) with $0 < s < 1$ and $\gamma > -3$. Let $g \in \mathcal{U}(D_0,E_0)$ for $D_0, E_0 > 0$.*

(1) *If $\gamma + 2s > 0$, then there exist positive constants $c_0, C$ depending only on $D_0, E_0$ such that for any $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$-\big(Q(g,f), f\big)_{L^2} \geq c_0 \|\langle v\rangle^{\gamma/2} f\|^2_{H^s} - C\|\langle v\rangle^{\gamma/2} f\|^2_{L^2}. \tag{2.2}$$

(2) *If $\gamma + 2s \leq 0$ and if $g$ belongs to $L^{3/(3+\gamma+2s')}_{|\gamma|}$ for $s' \in ]0, s[$, then there exists a $C_1 > 0$ independent of $g$ such that for any $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$-\big(Q(g,f), f\big)_{L^2} \geq c_0 \|\langle v\rangle^{\gamma/2} f\|^2_{H^s} - (C + C_1\|g\|_{L^{3/(3+\gamma+2s')}_{|\gamma|}})\|\langle v\rangle^{\gamma/2} f\|^2_{H^{s'}}, \tag{2.3}$$

*where $c_0, C$ are similar constants to those in (2.2).*

REMARK 2.2

It should be noted that the above coercive estimates are more precise than those provided by [10, Theorem 1.2] and more adaptable to prove the regularity of solutions. In fact, the coercive estimate (2.2) is uniform with respect to $g$, which is a crucial point in the proof of Theorem 1.2. Though the estimate (2.3) is not used in the present paper, such a coercive estimate plays an important role in the study of the regularity of classical solutions for the spatially inhomogeneous Boltzmann equation given in [4] and [5], where solutions are constructed in $H^m_\ell$ with respect to the velocity variables $v$. More precisely, thanks to the coercive estimates (2.2) and (2.3), [7, Theorem 1.1] shows that the bounded classical solutions to the Cauchy problem given in [5] possess $C^\infty$-regularity under the suitable nonvanishing initial condition (see [4, Theorem 1.2]). Furthermore, we notice that if $\gamma + 4s > 0$ and $D(g,g) < \infty$, then $g$ belongs to $L^{3/(3+\gamma+2s')}_{|\gamma|}$ for $s' \in ]0, s[$ with $0 > \gamma + 2s' > -2s$, provided that $g \in L^1_\ell$ for a sufficiently large $\ell$. In fact, it follows from the proof of Corollary 2.4 below that $D(g,g) < \infty$ implies $\sqrt{g} \in H^s_{\gamma/2}$ and hence $\langle v\rangle^\gamma g \in L^{3/(3-2s)}$ by means of the Sobolev embedding theorem, which together with Lemma 3.10 below leads us to this conclusion.

*Proof*
Put

$$\mathcal{C}_\gamma(g,f) = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} b(\cdot)|v-v_*|^\gamma g_*(f'-f)^2\,dv\,dv_*\,d\sigma,$$

and note that

$$\big(Q(g,f),f\big) = -\frac{1}{2}\mathcal{C}_\gamma(g,f) + \frac{1}{2}\iiint \Phi b g_*(f'^2 - f^2)\,dv\,dv_*\,d\sigma.$$

It follows from the cancellation lemma and [1, Remark 6] that

$$\begin{aligned}\Big|\iiint b|v-v_*|^\gamma g_*(f^2-f'^2)\,dv\,dv_*\,d\sigma\Big| &\lesssim \Big|\iint |v-v_*|^\gamma g_* f^2\Big|\,dv\,dv_* \\ &\lesssim \|g\|_{L^1_{|\gamma|}}\|f\|^2_{H^{(-\gamma/2)^+}_{\gamma/2}},\end{aligned}\tag{2.4}$$

where the last inequality in the case $\gamma \geq 0$ is trivial, while for $\gamma < 0$, this follows from the fact that

$$|v-v_*|^\gamma \lesssim \langle v\rangle^\gamma\big\{\mathbf{1}_{|v-v_*|\geq\langle v\rangle/2} + \mathbf{1}_{|v-v_*|<\langle v\rangle/2}\langle v_*\rangle^{-\gamma}|v-v_*|^\gamma\big\},\tag{2.5}$$

and the Hardy inequality $\sup_{v_*}\int |v-v_*|^\gamma|F(v)|^2\,dv \lesssim \|F\|^2_{H^{-\gamma/2}}$ for $F = \langle v\rangle^{\gamma/2}f$. Furthermore, if $\gamma + 2s' < 0$ for some $s' \in (0,s)$, then it follows from the Hardy–Littlewood–Sobolev inequality that

$$\begin{aligned}\Big|\iint |v-v_*|^\gamma g_* f^2\Big|\,dv\,dv_* &\lesssim \|g\|_{L^1}\|F\|^2_{L^2} + \iint \frac{\langle v_*\rangle^{|\gamma|}g(v_*)F(v)^2}{|v-v_*|^{-\gamma}}\,dv\,dv_* \\ &\lesssim \|g\|_{L^1}\|F\|^2_{L^2} + \|\langle v\rangle^{|\gamma|}g\|_{L^{3/(3+\gamma+2s')}}\|F^2\|_{L^{3/(3-2s')}} \\ &\lesssim (\|g\|_{L^1} + \|g\|_{L^{3/(3+\gamma+2s')}_{|\gamma|}})\|f\|^2_{H^{s'}_{\gamma/2}},\end{aligned}$$

where we have used the Sobolev embedding inequality in the last inequality. Since we can replace the factor $\langle v_*\rangle^{-\gamma}|v-v_*|^\gamma$ in (2.5) by $\langle v_*\rangle^{-\gamma+\varepsilon}|v-v_*|^{\gamma-\varepsilon}$ for any $\varepsilon > 0$ we have

$$\Big|\iint |v-v_*|^\gamma g_* f^2\Big|\,dv\,dv_* \lesssim (\|g\|_{L^1} + \|g\|_{L^{3/(3+\gamma-\varepsilon+2s')}_{|\gamma-\varepsilon|}})\|f\|^2_{H^{s'}_{\gamma/2}}$$

if $\gamma - \varepsilon + 2s' < 0$.

For the proof of both (1) and (2) of the proposition, it now suffices to consider only the quantity $\mathcal{C}_\gamma(g,f)$ because one can apply the interpolation inequality

$$\|f\|^2_{H^{(-\gamma/2)+}_{\gamma/2}} \le \varepsilon \|f\|^2_{H^s_{\gamma/2}} + C_\varepsilon \|f\|^2_{L^2_{\gamma/2}}$$

to (2.4) when $\gamma+2s>0$. The case $\gamma=0$ is obvious. In fact, by [1, Corollary 3, Proposition 2], there exists a $c_0=c_0(D_0,E_0)>0$ depending only on $D_0,E_0>0$ such that

$$\mathcal{C}_0(g,f) \ge c_0 \int_{\{|\xi|\ge 1\}} \left| |\xi|^s \hat f(\xi)\right|^2 d\xi, \quad \forall f \in \mathcal{S}(\mathbb{R}^3), \tag{2.6}$$

where $\hat f(\xi)$ is the Fourier transform of $f$ with respect to the variable $v\in\mathbb{R}^3$. From the proof in [1], it should be noticed that (2.6) holds for any $f\in L^2$ such that the left-hand side is finite.

We now consider the case $\gamma\neq 0$, following the argument used in the proof of [1, Lemma 2]. Choose $R, r_0$ such that (2.1) holds. Let $\varphi_R$ be a nonnegative smooth function not greater than one, which is 1 for $|v|\ge 4R$ and 0 for $|v|\le 2R$. In view of

$$\frac{\langle v\rangle}{4} \le |v-v_*| \le 2\langle v\rangle \quad \text{on } \mathrm{supp}(\chi_{B(R)})_*\varphi_R,$$

we have

$$\begin{aligned}
4^{|\gamma|}\Phi(|v-v_*|)g_*(f'-f)^2 &\ge (g\chi_{B(R)})_*(\langle v\rangle^{\gamma/2}\varphi_R)^2(f'-f)^2\\
&\ge (g\chi_{B(R)})_*\Big[\frac12\big((\langle v\rangle^{\gamma/2}\varphi_R f)'-\langle v\rangle^{\gamma/2}\varphi_R f\big)^2\\
&\qquad -\big((\langle v\rangle^{\gamma/2}\varphi_R)'-\langle v\rangle^{\gamma/2}\varphi_R\big)^2 f'^2\Big].
\end{aligned}$$

It follows from the mean value theorem that for a $\tau\in(0,1)$,

$$\begin{aligned}
\left|(\langle v\rangle^{\gamma/2}\varphi_R)'-\langle v\rangle^{\gamma/2}\varphi_R\right| &\lesssim \langle v+\tau(v'-v)\rangle^{\gamma/2-1}|v-v_*|\sin\frac{\theta}{2}\\
&\lesssim \langle v_*\rangle^{|\gamma/2-1|}\langle v'-v_*\rangle^{\gamma/2}\sin\frac{\theta}{2}\\
&\lesssim \langle v_*\rangle^{|\gamma/2|+|\gamma/2-1|}\langle v'\rangle^{\gamma/2}\sin\frac{\theta}{2},
\end{aligned}$$

because $|v-v_*|/\sqrt2 \le |v'-v_*| \le |v+\tau(v'-v)-v_*| \le |v-v_*|$ for $\theta\in[0,\pi/2]$. Therefore, we have

$$\mathcal{C}_\gamma(g,f) \ge 2^{-1-2|\gamma|}\mathcal{C}_0(g\chi_{B(R)},\varphi_R\langle v\rangle^{\gamma/2}f) - C_R\|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}}, \tag{2.7}$$

for a positive constant $C_R\sim R^{|\gamma|+|\gamma-2|}$. For a set $B(4R)$ we take a finite covering

$$B(4R)\subset \bigcup_{v_j\in B(4R)} A_j, \quad A_j=\Big\{v\in\mathbb{R}^3; |v-v_j|\le \frac{r_0}{4}\Big\}.$$

For each $A_j$ we choose a nonnegative smooth function $\varphi_{A_j}$ which is 1 on $A_j$ and 0 on $\{|v-v_j|\geq r_0/2\}$. Note that

$$\frac{r_0}{2}\leq |v-v_*|\leq 6R \quad \text{on } \operatorname{supp}(\chi_{B_j(R,r_0)})_*\varphi_{A_j}.$$

Then we have

$$\begin{aligned}
&\Phi(|v-v_*|)g_*(f'-f)^2\\
&\qquad\gtrsim \min\{r_0^{\gamma^+},R^{-(-\gamma)^+}\}(g\chi_{B_j(R,r_0)})_*\varphi_{A_j}^2(f'-f)^2\\
&\qquad\gtrsim R^{-\gamma^+}\min\{r_0^{\gamma^+},R^{-(-\gamma)^+}\}(g\chi_{B_j(R,r_0)})_*\\
&\qquad\quad\times\Big[\frac{1}{2}\big((\langle v\rangle^{\gamma/2}\varphi_{A_j}f)'-\langle v\rangle^{\gamma/2}\varphi_{A_j}f\big)^2-\big((\langle v\rangle^{\gamma/2}\varphi_{A_j})'-\langle v\rangle^{\gamma/2}\varphi_{A_j}\big)^2 f'^2\Big].
\end{aligned}$$

Since $|(\langle v\rangle^{\gamma/2}\varphi_{A_j})'-\langle v\rangle^{\gamma/2}\varphi_{A_j}|\lesssim R^{|\gamma|+1}\langle v'\rangle^{\gamma}\sin\theta/2$ if $|v_*|\leq R$, we obtain

$$\begin{aligned}
\mathcal{C}_\gamma(g,f)\gtrsim{}& \min\big\{(r_0/R)^{\gamma^+},R^{-(-\gamma)^+}\big\}\mathcal{C}_0(g\chi_{B_j(R,r_0)},\varphi_{A_j}\langle v\rangle^{\gamma/2}f)\\
&-C'_{R,r_0}\|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}},
\end{aligned}\tag{2.8}$$

for a positive constant $C'_{R,r_0}\sim R^{2+2|\gamma|}$. It follows from (2.6)–(2.8) that there exist $c_0',C,C'>0$ depending only on $D_0,E_0$ such that

$$\begin{aligned}
\mathcal{C}_\gamma(g,f)&\geq c_0'\Big(\|\langle D\rangle^s\varphi_R\langle v\rangle^{\gamma/2}f\|^2+\sum_j\|\langle D\rangle^s\varphi_{A_j}\langle v\rangle^{\gamma/2}f\|^2\Big)-C\|f\|^2_{L^2_{\gamma/2}}\\
&\geq c_0'\|\langle v\rangle^{\gamma/2}f\|^2_{H^s}-C'\|f\|^2_{L^2_{\gamma/2}},
\end{aligned}\tag{2.9}$$

because $\varphi_R^2+\sum_j\varphi_{A_j}^2\geq 1$ and commutators $[\langle D\rangle^s,\varphi_R]$, $[\langle D\rangle^s,\varphi_{A_j}]$ are $L^2$-bounded operators. □

REMARK 2.3

The estimate (2.9) holds for any $f\in L^2_{\gamma/2}$ such that $\mathcal{C}_\gamma(g,f)$ is finite, because of the remark just after (2.6). Similarly, (2.2) holds for any $f\in L^2_{\gamma/2}$ if $\gamma\geq 0$ and if its left-hand side is finite.

COROLLARY 2.4

*Let $f(t)\in L^1_{\max\{2,\gamma\}}\cap L\log L$ be a weak solution. Suppose that the cross section $B$ is the same as in Propostion 2.1. Assume that for a fixed $T>0$ we have*

$$\int_0^T D\big(f(\tau),f(\tau)\big)\,d\tau<\infty.\tag{2.10}$$

*Then there exist positive constants $c_f$ and $C_f>0$ such that*

$$c_f\int_0^T\|\sqrt{f(\tau)}\|^2_{H^s_{\gamma/2}}\,d\tau\leq\int_0^T D\big(f(\tau),f(\tau)\big)\,d\tau+C_f\int_0^T\|f(\tau)\|_{L^1_{\gamma^+}}\,d\tau.\tag{2.11}$$

*Proof*
We first consider the case $\gamma < 0$. Note that

$$\begin{aligned}
D(f,f) &= -\iiint B(f'_*f' - f_*f)\log f\, dv\, dv_*\, d\sigma\\
&= \frac{1}{4}\iiint B(f'f'_* - ff_*)(\log f'f'_* - \log ff_*)\, dv\, dv_*\, d\sigma\\
&\geq \frac{1}{4}\iiint b(\cdot)\langle v - v_*\rangle^\gamma (f'f'_* - ff_*)(\log f'f'_* - \log ff_*)\, dv\, dv_*\, d\sigma,
\end{aligned}$$

because $(x-y)(\log x - \log y) \geq 0$ and $\Phi(|v-v_*|) \geq \langle v - v_*\rangle^\gamma$. Then we have

$$\begin{aligned}
D(f,f) \geq &-\iiint b(\cdot)\langle v - v_*\rangle^\gamma (f'_*f' - f_*f)\log f\, dv\, dv_*\, d\sigma\\
&= \iiint b(\cdot)\langle v - v_*\rangle^\gamma f_*\Big(f\log\frac{f}{f'} - f + f'\Big)\, dv\, dv_*\, d\sigma\\
&\quad + \iiint b(\cdot)\langle v - v_*\rangle^\gamma f_*(f - f')\, dv\, dv_*\, d\sigma\\
&\geq \iiint b(\cdot)\langle v - v_*\rangle^\gamma f_*(\sqrt{f'} - \sqrt{f})^2\, dv\, dv_*\, d\sigma - C\|f\|^2_{L^1},
\end{aligned}$$

where we have used $x\log(x/y) - x + y \geq (\sqrt{x} - \sqrt{y})^2$ and the cancellation lemma in the last inequality, which was used similarly in the proof of [1, Theorem 1]. Since the proof of Proposition 2.1 still works with $\Phi$ replaced by $\langle v - v_*\rangle^\gamma$, we obtain the desired estimate in view of Remark 2.3. The case $\gamma \geq 0$ is easier because we do not need to replace $\Phi$ by $\langle v - v_*\rangle^\gamma$ when the cancellation lemma is applied. $\square$

## 3. Mollifier and commutator estimates

Since a weak solution is only in $L^1$, we cannot use it directly as a test function in the definition of weak solution to get the energy estimate. To overcome this difficulty, we need to mollify with some suitable mollifiers so that considering the commutators between the mollifiers and the collision operator becomes necessary.

Let $\lambda, N_0 \in \mathbb{R}$, let $\delta > 0$, and put

$$M^\delta_\lambda(\xi) = \frac{\langle\xi\rangle^\lambda}{(1+\delta\langle\xi\rangle)^{N_0}}, \quad \langle\xi\rangle = (1+|\xi|^2)^{1/2}. \tag{3.1}$$

Then $M^\delta_\lambda(\xi)$ belongs to the symbol class $S^{\lambda - N_0}_{1,0}$ of pseudodifferential operators and belongs to $S^\lambda_{1,0}$ uniformly with respect to $\delta \in ]0,1]$. The associated pseudodifferential operated is denoted by $M^\delta_\lambda(D_v)$. By direct calculation, we see that for any $\alpha$ there exists a $C_\alpha > 0$ independent of $\delta$ such that

$$|\partial^\alpha_\xi M^\delta_\lambda(\xi)| \leq C_\alpha M^\delta_\lambda(\xi)\langle\xi\rangle^{-|\alpha|}. \tag{3.2}$$

LEMMA 3.1

*There exists a constant $C > 0$ independent of $\delta$ such that*

$$
\begin{aligned}
&|M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*)| \\
&\le C\langle\xi\rangle^\lambda \mathbf{1}_{\langle\xi_*\rangle\ge\sqrt{2}|\xi|} + CM_\lambda^\delta(\xi-\xi_*)\Big\{\mathbf{1}_{\langle\xi_*\rangle\ge|\xi|/2} + \frac{\langle\xi_*\rangle}{\langle\xi\rangle}\mathbf{1}_{|\xi|/2>\langle\xi_*\rangle}\Big\} \\
&\quad + CM_\lambda^\delta(\xi-\xi_*)\Big(\frac{M_\lambda^\delta(\xi_*)(1+\delta\langle\xi-\xi_*\rangle)^{N_0}}{\langle\xi-\xi_*\rangle^\lambda}\Big)\mathbf{1}_{\sqrt{2}|\xi|>\langle\xi_*\rangle\ge|\xi|/2}.
\end{aligned}
\tag{3.3}
$$

*Moreover, if $p \ge N_0 - \lambda$,*

$$
\begin{aligned}
&|M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi - \xi_*)| \\
&\le CM_\lambda^\delta(\xi-\xi_*)\Big\{\Big(\frac{\langle\xi_*\rangle}{\langle\xi\rangle}\Big)^p\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt{2}|\xi|} \\
&\quad + \Big(\frac{M_\lambda^\delta(\xi_*)(1+\delta\langle\xi-\xi_*\rangle)^{N_0}}{\langle\xi-\xi_*\rangle^\lambda} + 1\Big)\mathbf{1}_{\sqrt{2}|\xi|>\langle\xi_*\rangle\ge|\xi|/2} + \frac{\langle\xi_*\rangle}{\langle\xi\rangle}\mathbf{1}_{|\xi|/2>\langle\xi_*\rangle}\Big\}.
\end{aligned}
\tag{3.4}
$$

*Proof*

We first note that

$$
\begin{cases}
\langle\xi\rangle \lesssim \langle\xi_*\rangle \sim \langle\xi-\xi_*\rangle, & \text{on supp}\,\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt{2}|\xi|}, \\
\langle\xi\rangle \sim \langle\xi-\xi_*\rangle, & \text{on supp}\,\mathbf{1}_{\langle\xi_*\rangle\le|\xi|/2}, \\
\langle\xi\rangle \sim \langle\xi_*\rangle \gtrsim \langle\xi-\xi_*\rangle, & \text{on supp}\,\mathbf{1}_{\sqrt{2}|\xi|\ge\langle\xi_*\rangle\ge|\xi|/2}\ .
\end{cases}
\tag{3.5}
$$

Since $\langle\xi\rangle^p M_\lambda^\delta(\xi)$ is increasing with respect to $\langle\xi\rangle$, we have

$$
\langle\xi\rangle^p M_\lambda^\delta(\xi) \lesssim \langle\xi_*\rangle^p M_\lambda^\delta(\xi_*) \sim \langle\xi_*\rangle^p M_\lambda^\delta(\xi-\xi_*) \quad \text{on supp}\,\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt{2}|\xi|},
$$

and also trivially,

$$
M_\lambda^\delta(\xi) \le \langle\xi\rangle^\lambda.
$$

Note that

$$
\begin{aligned}
&M_\lambda^\delta(\xi) \sim M_\lambda^\delta(\xi_*) \\
&\quad \sim M_\lambda^\delta(\xi-\xi_*)\frac{M_\lambda^\delta(\xi_*)(1+\delta\langle\xi-\xi_*\rangle)^{N_0}}{\langle\xi-\xi_*\rangle^\lambda} \quad \text{on supp}\,\mathbf{1}_{\sqrt{2}|\xi|\ge\langle\xi_*\rangle\ge|\xi|/2}.
\end{aligned}
$$

By the mean value theorem, we have

$$
\begin{aligned}
|M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi-\xi_*)| &\le \int_0^1 \big|(\nabla_\xi M_\lambda^\delta)\big(\xi+\tau(\xi-\xi_*)\big)\big|\, d\tau |\xi_*| \\
&\lesssim M_\lambda^\delta(\xi-\xi_*)\frac{\langle\xi_*\rangle}{\langle\xi\rangle} \quad \text{on supp}\,\mathbf{1}_{\langle\xi_*\rangle\le|\xi|/2}.
\end{aligned}
$$

Here we have used (3.2) and the second formula from (3.5). The above estimates imply (3.4) and (3.3). □

As regards the kinetic factor $|v-v_*|^\gamma$, we need to take into account its singular behavior close to $|v-v_*| = 0$ except $\gamma = 0$. Therefore, we decompose the kinetic factor in two parts. Let $0 \le \phi(z) \le 1$ be a smooth radial function with value 1 for $z$ close to 0, and 0 for large values of $z$. Set

$$\Phi_\gamma(z) = \Phi_\gamma(z)\phi(z) + \Phi_\gamma(z)\big(1 - \phi(z)\big) = \Phi_c(z) + \Phi_{\bar{c}}(z).$$

Then correspondingly we can write

$$Q(f,g) = Q_c(f,g) + Q_{\bar{c}}(f,g),$$

where the kinetic factors in these collision operators are defined according to the previous decomposition. Note that $\Phi_{\bar{c}}(z)$ is smooth, and $\Phi_{\bar{c}}(z) \lesssim \tilde{\Phi}_\gamma(z)$, where $\tilde{\Phi}_\gamma(|z|) = (1 + |z|^2)^{\gamma/2}$ is the regular kinetic factor studied in [4]. Then $Q_{\bar{c}}(f,g)$ has similar properties as $Q_{\tilde{\Phi}_\gamma}(f,g)$ does with regard to the upper bound and commutator estimations. In particular, let us recall [4, Proposition 2.9].

PROPOSITION 3.2

*Let $\lambda \in \mathbb{R}$, and let $M(\xi)$ be a positive symbol in $S^\lambda_{1,0}$ in the form of $M(\xi) = \tilde{M}(|\xi|^2)$. Assume that there exist constants $c, C > 0$ such that for any $s, \tau > 0$,*

$$c^{-1} \le \frac{s}{\tau} \le c \quad \textit{implies} \quad C^{-1} \le \frac{\tilde{M}(s)}{\tilde{M}(\tau)} \le C,$$

*and $M(\xi)$ satisfies*

$$|M^{(\alpha)}(\xi)| = |\partial^\alpha_\xi M(\xi)| \le C_\alpha M(\xi)\langle\xi\rangle^{-|\alpha|},$$

*for any $\alpha \in \mathbb{N}^3$. Then, if $0 < s < 1/2$, for any $N > 0$ there exists a $C_N > 0$ such that*

$$\begin{aligned} &\big|\big(M(D_v)Q_{\bar{c}}(f,g) - Q_{\bar{c}}(f, M(D_v)g), h\big)_{L^2}\big| \\ &\qquad \le C_N \|f\|_{L^1_{\gamma+}} \big(\|M(D_v)g\|_{L^2_{\gamma+}} + \|g\|_{H^{\lambda-N}_{\gamma+}}\big)\|h\|_{L^2}. \end{aligned} \tag{3.6}$$

*Furthermore, if $1/2 < s < 1$, for any $N > 0$ and any $\varepsilon > 0$ , there exists a $C_{N,\varepsilon} > 0$ such that*

$$\begin{aligned} &\big|\big(M(D_v)Q_{\bar{c}}(f,g) - Q_{\bar{c}}(f, M(D_v)g), h\big)_{L^2}\big| \\ &\qquad \le C_{N,\varepsilon} \|f\|_{L^1_{(2s+\gamma-1)^+}} \big(\|M(D_v)g\|_{H^{2s-1+\varepsilon}_{(2s+\gamma-1)^+}} + \|g\|_{H^{\lambda-N}_{\gamma+}}\big)\|h\|_{L^2}. \end{aligned} \tag{3.7}$$

*When $s = 1/2$ we have the same estimate as (3.7) with $(2s + \gamma - 1)$ replaced by $(\gamma + \kappa)$ for any small $\kappa > 0$.*

REMARK 3.3

In the case $\gamma > 0$ and $0 < s < 1/2$, it follows from [13, Lemma 3.1] and its proof that (3.6) can be replaced by

$$\begin{aligned} &\big|\big(M(D_v)Q_{\bar{c}}(f,g) - Q_{\bar{c}}(f, M(D_v)g), h\big)_{L^2}\big| \\ &\qquad \le C_N \|f\|_{L^1_\gamma} \big(\|M(D_v)g\|_{L^2_{\gamma/2}} + \|g\|_{H^{\lambda-N}_{\gamma/2}}\big)\|h\|_{L^2_{\gamma/2}}. \end{aligned}$$

From now on, we concentrate on the study for the singular part $Q_c(f,g)$.

PROPOSITION 3.4

*Assume that* $0 < s < 1, \gamma + 2s > 0$. *Let* $0 < s' < s$ *satisfy* $\gamma + 2s' > 0$ *and* $2s' \geq (2s-1)^+$. *If*

$$5 + \gamma \geq 2(N_0 - \lambda), \tag{3.8}$$

*then we have the following.*

(1) *If* $s' + \lambda < 3/2$, *then*

$\big|\big(M^\delta_\lambda(D_v)Q_c(f,g) - Q_c(f, M^\delta_\lambda(D_v)g), h\big)\big| \lesssim \|f\|_{L^1}\|M^\delta_\lambda(D_v)g\|_{H^{s'}}\|h\|_{H^{s'}}$.

(2) *If* $s' + \lambda \geq 3/2$, *then*

$$\begin{aligned}&\big|\big(M^\delta_\lambda(D_v)Q_c(f,g) - Q_c(f, M^\delta_\lambda(D_v)g), h\big)\big| \\ &\qquad \lesssim (\|f\|_{L^1} + \|f\|_{H^{(\lambda+s'-3)^+}})\|M^\delta_\lambda(D_v)g\|_{H^{s'}}\|h\|_{H^{s'}}.\end{aligned}$$

*Furthermore, if* $s > 1/2$ *and* $\gamma > -1$, *then the assumption (3.8) can be relaxed to*

$$4 + \gamma + 2s > 2(N_0 - \lambda). \tag{3.9}$$

*Proof*

We shall follow some of the arguments from [8]. By using the formula from [1, Appendix], we have

$$\begin{aligned}\big(Q_c(f,g),h\big) = \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} & b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)[\hat\Phi_c(\xi_* - \xi^-) - \hat\Phi_c(\xi_*)] \\ & \times \hat f(\xi_*)\hat g(\xi - \xi_*)\overline{\hat h(\xi)}\,d\xi\,d\xi_*\,d\sigma,\end{aligned}$$

where $\xi^- = (1/2)(\xi - |\xi|\sigma)$. Therefore

$$\begin{aligned}&\big(M^\delta_\lambda(D)Q_c(f,g) - Q_c(f, M^\delta_\lambda(D)g), h\big) \\ &\quad = \iiint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)[\hat\Phi_c(\xi_* - \xi^-) - \hat\Phi_c(\xi_*)] \\ &\quad\quad \times \big(M^\delta_\lambda(\xi) - M^\delta_\lambda(\xi - \xi_*)\big)\hat f(\xi_*)\hat g(\xi - \xi_*)\overline{\hat h(\xi)}\,d\xi\,d\xi_*\,d\sigma \\ &\quad = \iiint_{|\xi^-|\leq(1/2)\langle\xi_*\rangle}\cdots d\xi\,d\xi_*\,d\sigma + \iiint_{|\xi^-|\geq(1/2)\langle\xi_*\rangle}\cdots d\xi\,d\xi_*\,d\sigma \\ &\quad = \mathcal{A}_1(f,g,h) + \mathcal{A}_2(f,g,h).\end{aligned}$$

Then, we write $\mathcal{A}_2(f,g,h)$ as

$$\begin{aligned}\mathcal{A}_2 = &\iiint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbf{1}_{|\xi^-|\geq(1/2)\langle\xi_*\rangle}\hat\Phi_c(\xi_* - \xi^-)\cdots d\xi\,d\xi_*\,d\sigma \\ &- \iiint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\mathbf{1}_{|\xi^-|\geq(1/2)\langle\xi_*\rangle}\hat\Phi_c(\xi_*)\cdots d\xi\,d\xi_*\,d\sigma \\ = &\,\mathcal{A}_{2,1}(f,g,h) - \mathcal{A}_{2,2}(f,g,h).\end{aligned}$$

On the other hand, for $\mathcal{A}_1$ we use the Taylor expansion of $\hat\Phi_c$ of order 2 to have

$$\mathcal{A}_1 = \mathcal{A}_{1,1}(f,g,h) + \mathcal{A}_{1,2}(f,g,h),$$

where

$$\mathcal{A}_{1,1} = \iiint b\xi^- \cdot (\nabla\hat{\Phi}_c)(\xi_*)\mathbf{1}_{|\xi^-|\le(1/2)\langle\xi_*\rangle}\big(M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi-\xi_*)\big)$$
$$\times \hat{f}(\xi_*)\hat{g}(\xi-\xi_*)\bar{\hat{h}}(\xi)\,d\xi\,d\xi_*\,d\sigma,$$

and $\mathcal{A}_{1,2}(f,g,h)$ is the remaining term corresponding to the second-order term in the Taylor expansion of $\hat{\Phi}_c$.

We first consider $\mathcal{A}_{1,1}$. By writing

$$\xi^- = \frac{|\xi|}{2}\Big(\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\frac{\xi}{|\xi|} - \sigma\Big) + \Big(1 - \Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big)\frac{\xi}{2},$$

we see that the integral corresponding to the first term on the right-hand side vanishes because of the symmetry on $\mathbb{S}^2$. Hence, we have

$$\mathcal{A}_{1,1} = \iint_{\mathbb{R}^6} K(\xi,\xi_*)\big(M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi-\xi_*)\big)\hat{f}(\xi_*)\hat{g}(\xi-\xi_*)\bar{\hat{h}}(\xi)\,d\xi\,d\xi_*,$$

where

$$K(\xi,\xi_*) = \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big(1 - \Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big)\frac{\xi}{2}\cdot(\nabla\hat{\Phi}_c)(\xi_*)\mathbf{1}_{|\xi^-|\le(1/2)\langle\xi_*\rangle}\,d\sigma.$$

Note that $|\nabla\hat{\Phi}_c(\xi_*)| \lesssim 1/\langle\xi_*\rangle^{3+\gamma+1}$, from [8, Appendix]. If $\sqrt{2}|\xi| \le \langle\xi_*\rangle$, then $\sin(\theta/2)|\xi| = |\xi^-| \le \langle\xi_*\rangle/2$ because $0 \le \theta \le \pi/2$, and we have

$$|K(\xi,\xi_*)| \lesssim \int_0^{\pi/2} \theta^{1-2s}\,d\theta\frac{\langle\xi\rangle}{\langle\xi_*\rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma}}\Big(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\Big).$$

On the other hand, if $\sqrt{2}|\xi| \ge \langle\xi_*\rangle$, then

$$|K(\xi,\xi_*)| \lesssim \int_0^{\pi\langle\xi_*\rangle/(2|\xi|)} \theta^{1-2s}\,d\theta\frac{\langle\xi\rangle}{\langle\xi_*\rangle^{3+\gamma+1}} \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma}}\Big(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\Big)^{2s-1}.$$

Hence we obtain

$$\begin{aligned}|K(\xi,\xi_*)| \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma}}\Big\{&\Big(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\Big)\mathbf{1}_{\langle\xi_*\rangle\ge\sqrt{2}|\xi|} \\ &+ \mathbf{1}_{\sqrt{2}|\xi|\ge\langle\xi_*\rangle\ge|\xi|/2} + \Big(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\Big)^{2s-1}\mathbf{1}_{|\xi|/2\ge\langle\xi_*\rangle}\Big\}.\end{aligned} \tag{3.10}$$

Similarly to $\mathcal{A}_{1,1}$, we can also write

$$\mathcal{A}_{1,2} = \iint_{\mathbb{R}^6} \tilde{K}(\xi,\xi_*)\big(M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi-\xi_*)\big)\hat{f}(\xi_*)\hat{g}(\xi-\xi_*)\bar{\hat{h}}(\xi)\,d\xi\,d\xi_*,$$

where

$$\tilde{K}(\xi,\xi_*) = \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\int_0^1 (1-\tau)(\nabla^2\hat{\Phi}_c)(\xi_* - \tau\xi^-)\cdot\xi^-\cdot\xi^-\mathbf{1}_{|\xi^-|\le\frac{1}{2}\langle\xi_*\rangle}\,d\tau\,d\sigma.$$

Again from [8, Appendix], we have

$$|(\nabla^2\hat{\Phi}_c)(\xi_* - \tau\xi^-)| \lesssim \frac{1}{\langle\xi_* - \tau\xi^-\rangle^{3+\gamma+2}} \lesssim \frac{1}{\langle\xi_*\rangle^{3+\gamma+2}},$$

because $|\xi^-| \le \langle \xi_* \rangle/2$, which leads to

$$
\begin{aligned}
|\tilde K(\xi,\xi_*)| \lesssim \frac{1}{\langle \xi_*\rangle^{3+\gamma}} \Big\{ \Big( \frac{\langle \xi \rangle}{\langle \xi_* \rangle}\Big)^2 \mathbf{1}_{\langle \xi_*\rangle \ge \sqrt{2}|\xi|} \\
+ \mathbf{1}_{\sqrt{2}|\xi| \ge \langle \xi_* \rangle \ge |\xi|/2} + \Big( \frac{\langle \xi \rangle}{\langle \xi_* \rangle}\Big)^{2s} \mathbf{1}_{|\xi|/2 \ge \langle \xi_*\rangle} \Big\}.
\end{aligned} \tag{3.11}
$$

It follows from (3.4) of Lemma 3.1, (3.10), and (3.11) that if $p = N_0 - \lambda$, then

$$
|\mathcal{A}_1| \lesssim |\mathcal{A}_{1,1}| + |\mathcal{A}_{1,2}| \lesssim A_1 + A_2 + A_3,
$$

where

$$
\begin{aligned}
A_1 = \iint_{\mathbb{R}^6} \Big| \frac{\hat f(\xi_*)}{\langle \xi_*\rangle^{3+\gamma}} \Big| |M^\delta_\lambda(\xi - \xi_*) \hat g(\xi - \xi_*)| |\hat h(\xi)| \\
\times \Big( \frac{\langle \xi_* \rangle}{\langle \xi \rangle}\Big)^{p-1} \mathbf{1}_{\langle \xi_*\rangle \ge \sqrt{2}|\xi|} \, d\xi_* d\xi,
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
A_2 = \iint_{\mathbb{R}^6} \Big| \frac{\hat f(\xi_*)}{\langle \xi_*\rangle^{3+\gamma}} \Big| |M^\delta_\lambda(\xi - \xi_*) \hat g(\xi - \xi_*)| |\hat h(\xi)| \\
\times \Big( \frac{M^\delta_\lambda(\xi_*)(1 + (\delta \langle \xi - \xi_* \rangle)^{N_0})}{\langle \xi - \xi_* \rangle^\lambda} + 1 \Big) \mathbf{1}_{\sqrt{2}|\xi| > \langle \xi_* \rangle \ge |\xi|/2} \, d\xi_* d\xi,
\end{aligned}
$$

$$
A_3 = \iint_{\mathbb{R}^6} \Big| \frac{\hat f(\xi_*)}{\langle \xi_*\rangle^{3+\gamma}} \Big| |M^\delta_\lambda(\xi - \xi_*) \hat g(\xi - \xi_*)| |\hat h(\xi)| \Big( \frac{\langle \xi \rangle}{\langle \xi_* \rangle}\Big)^{2s-1} \mathbf{1}_{|\xi|/2 > \langle \xi_*\rangle} \, d\xi_* d\xi.
$$

Setting $\hat G(\xi) = \langle \xi \rangle^{s'} M^\delta_\lambda(\xi) \hat g(\xi)$ and $\hat H(\xi) = \langle \xi \rangle^{s'} \hat h(\xi)$, we get

$$
\begin{aligned}
|A_1|^2 &\lesssim \|\hat f\|^2_{L^\infty} \Big( \int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3_\xi} |\hat H(\xi)|^2 \, d\xi \Big) \\
&\quad \times \Big( \int_{\mathbb{R}^3} \frac{d\xi}{\langle \xi \rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3} \Big( \frac{\langle \xi \rangle}{\langle \xi_* \rangle}\Big)^{3+\gamma-2(p-1)} \mathbf{1}_{\langle \xi_*\rangle \ge \sqrt{2}|\xi|} |\hat G(\xi - \xi_*)|^2 \, d\xi_* \Big) \\
&\lesssim \|f\|^2_{L^1} \|M^\delta_\lambda g\|^2_{H^{s'}} \|h\|^2_{H^{s'}},
\end{aligned}
$$

because $\gamma + 2s' > 0$, and $3 + \gamma - 2(p-1) \ge 0$ from (3.8). Here we have used the fact that $\langle \xi_* \rangle \sim \langle \xi - \xi_* \rangle$ if $\langle \xi_* \rangle \ge \sqrt{2}|\xi|$.

We consider the case $s > 1/2, \gamma > -1$. For $s > s' > 1/2$ we have

$$
\begin{aligned}
|A_1|^2 &\lesssim \|\hat f\|^2_{L^\infty} \Big( \int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_* \rangle^{3+\gamma+1}} \int_{\mathbb{R}^3_\xi} |\hat H(\xi)|^2 \, d\xi \Big) \\
&\quad \times \Big( \int_{\mathbb{R}^3} \frac{d\xi}{\langle \xi \rangle^{3+\gamma+1}} \int_{\mathbb{R}^3} \Big( \frac{\langle \xi \rangle}{\langle \xi_* \rangle}\Big)^{3+\gamma+(2s'-1)-2(p-1)} \\
&\quad \times \frac{\mathbf{1}_{\langle \xi_*\rangle \ge \sqrt{2}|\xi|}}{\langle \xi \rangle^{2(2s'-1)}} |\hat G(\xi - \xi_*)|^2 \, d\xi_* \Big) \\
&\lesssim \|f\|^2_{L^1} \|M^\delta_\lambda g\|^2_{H^{s'}} \|h\|^2_{H^{s'}},
\end{aligned}
$$

if $3+\gamma+(2s'-1)-2(p-1)>0$. Thus (3.8) can be relaxed to (3.9) to get the desired estimate for $A_1$. Here we remark that (3.8) and (3.9) are only required to estimate the part $A_1$.

Noticing the third formula of (3.5), we get

$$
\begin{aligned}
|A_2|^2 \lesssim \Big\{ \int_{\mathbb{R}^3} \frac{|\hat f(\xi_*)|^2\, d\xi_*}{\langle \xi_*\rangle^{6+2\gamma+2s'}} \int_{\langle \xi-\xi_*\rangle \lesssim \langle \xi_*\rangle} \Big( \frac{\langle \xi_*\rangle^{2\lambda}}{\langle \xi-\xi_*\rangle^{2(\lambda+s')}} \\
+ \frac{\langle \xi_*\rangle^{2(\lambda-N_0)}}{\langle \xi-\xi_*\rangle^{2(\lambda-N_0+s')}} + \frac{1}{\langle \xi-\xi_*\rangle^{2s'}} \Big) d\xi \Big\} \\
\times \Big( \iint_{\mathbb{R}^6} |\hat G(\xi-\xi_*)|^2 |\hat H(\xi)|^2\, d\xi\, d\xi_* \Big).
\end{aligned}
$$

If $\lambda+s'<3/2$, then

$$
\begin{aligned}
|A_2|^2 &\lesssim \int_{\mathbb{R}^3} \frac{|\hat f(\xi_*)|^2}{\langle \xi_*\rangle^{3+2(\gamma+2s')}}\, d\xi_* \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 \\
&\lesssim \|f\|_{L^1}^2 \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 .
\end{aligned}
$$

If $\lambda+s'\ge 3/2$, then

$$
\begin{aligned}
|A_2|^2 &\lesssim \int_{\mathbb{R}^3} \frac{|\hat f(\xi_*)|^2 \langle \xi_*\rangle^{2(\lambda+s'+\varepsilon)}}{\langle \xi_*\rangle^{6+2(\gamma+2s')}}\, d\xi_* \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 \\
&\lesssim \|f\|_{H^{\lambda+s'-3}}^2 \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 .
\end{aligned}
$$

Since $2s'\ge 2s-1$ and $\gamma+2s'>0$, we have

$$
\begin{aligned}
|A_3|^2 &\lesssim \|\hat f\|_{L^\infty}^2 \Big( \int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_*\rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3} |\hat H(\xi)|^2\, d\xi \Big) \\
&\quad \times \Big( \int_{\mathbb{R}^3} \frac{d\xi_*}{\langle \xi_*\rangle^{3+\gamma+2s'}} \int_{\mathbb{R}^3} \Big( \frac{\langle \xi_*\rangle}{\langle \xi\rangle} \Big)^{2\{2s'-(2s-1)\}} \mathbf{1}_{|\xi|/2\ge \langle \xi_*\rangle} |\hat G(\xi-\xi_*)|^2\, d\xi \Big) \\
&\lesssim \|f\|_{L^1}^2 \|M_\lambda^\delta g\|_{H^{s'}}^2 \|h\|_{H^{s'}}^2 .
\end{aligned}
$$

The above four estimates yield the desired estimate for $\mathcal{A}_1(f,g,h)$.

Next consider $\mathcal{A}_2(f,g,h) = \mathcal{A}_{2,1}(f,g,h) - \mathcal{A}_{2,2}(f,g,h)$. Since $|\xi^-| = |\xi| \times \sin(\theta/2) \ge \langle \xi_*\rangle/2$ and $\theta\in[0,\pi/2]$, we have $\sqrt{2}|\xi|\ge \langle \xi_*\rangle$. Write

$$
\mathcal{A}_{2,j} = \iint_{\mathbb{R}^6} K_j(\xi,\xi_*) \big( M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi-\xi_*) \big) \hat f(\xi_*) \hat g(\xi-\xi_*) \bar{\hat h}(\xi)\, d\xi\, d\xi_* .
$$

Then we have

$$
\begin{aligned}
|K_2(\xi,\xi_*)| &= \Big| \int b\Big( \frac{\xi}{|\xi|}\cdot\sigma \Big) \hat\Phi_c(\xi_*) \mathbf{1}_{|\xi^-|\ge \frac12 \langle \xi_*\rangle}\, d\sigma \Big| \\
&\lesssim \frac{1}{\langle \xi_*\rangle^{3+\gamma}} \frac{\langle \xi\rangle^{2s}}{\langle \xi_*\rangle^{2s}} \mathbf{1}_{\sqrt{2}|\xi|\ge \langle \xi_*\rangle} \\
&\lesssim \frac{1}{\langle \xi_*\rangle^{3+\gamma}} \Big\{ \mathbf{1}_{\sqrt{2}|\xi|\ge \langle \xi_*\rangle \ge |\xi|/2} + \Big( \frac{\langle \xi\rangle}{\langle \xi_*\rangle} \Big)^{2s} \mathbf{1}_{|\xi|/2\ge \langle \xi_*\rangle} \Big\},
\end{aligned}
$$

which shows the desired estimate for $\mathcal{A}_{2,2}$, in exactly the same way as the estimation on $A_2$ and $A_3$.

As for $\mathcal{A}_{2,1}$, it suffices to work under the condition $|\xi_* \cdot \xi^-| \geq (1/2)|\xi^-|^2$. In fact, on the complement of this set, we have $|\xi_* - \xi^-| > |\xi_*|$, and $\hat{\Phi}_c(\xi_* - \xi^-)$ is the same as $\hat{\Phi}_c(\xi_*)$. Therefore, we consider $\mathcal{A}_{2,1,p}$, which is defined by replacing $K_1(\xi, \xi_*)$ by

$$K_{1,p}(\xi,\xi_*) = \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\hat{\Phi}_c(\xi_* - \xi^-)\mathbf{1}_{|\xi^-|\geq(1/2)\langle\xi_*\rangle}\mathbf{1}_{|\xi_*\cdot\xi^-|\geq(1/2)|\xi^-|^2}\,d\sigma.$$

By writing

$$\mathbf{1} = \mathbf{1}_{\langle\xi_*\rangle\geq|\xi|/2}\mathbf{1}_{\langle\xi-\xi_*\rangle\leq2\langle\xi_*-\xi^-\rangle} + \mathbf{1}_{\langle\xi_*\rangle\geq|\xi|/2}\mathbf{1}_{\langle\xi-\xi_*\rangle>2\langle\xi_*-\xi^-\rangle} + \mathbf{1}_{\langle\xi_*\rangle<|\xi|/2},$$

we decompose, respectively,

$$\mathcal{A}_{2,1,p} = B_1 + B_2 + B_3.$$

On the sets corresponding to the above integrals, we have $\langle\xi_* - \xi^-\rangle \lesssim \langle\xi_*\rangle$, because of $|\xi^-| \lesssim |\xi_*|$, which follows from $|\xi^-|^2 \leq 2|\xi_*\cdot\xi^-| \lesssim |\xi^-||\xi_*|$. Furthermore, on the sets for $B_1$ and $B_2$ we have $\langle\xi\rangle \sim \langle\xi_*\rangle$, so that $\langle\xi_* - \xi^-\rangle \lesssim \langle\xi\rangle$ and $b\mathbf{1}_{|\xi^-|\geq(1/2)\langle\xi_*\rangle}\mathbf{1}_{\langle\xi_*\rangle\geq|\xi|/2}$ is bounded. Putting again $\hat{G}(\xi) = \langle\xi\rangle^{s'}M^\delta_\lambda(\xi)\hat{g}(\xi)$ and $\hat{H}(\xi) = \langle\xi\rangle^{s'}\hat{h}(\xi)$, by Lemma 3.1 we have

$$\begin{aligned}|B_1|^2 \lesssim \Big[\iiint\Big|\frac{\hat{\Phi}_c(\xi_*-\xi^-)}{\langle\xi_*-\xi^-\rangle^{s'}}\Big|^2|\hat{f}(\xi_*)|^2 \\ \times\Big\{M^\delta_\lambda(\xi_*)^2\Big(\frac{\mathbf{1}_{\langle\xi-\xi_*\rangle\lesssim\langle\xi_*-\xi^-\rangle}}{\langle\xi-\xi_*\rangle^{2(s'+\lambda)}} + \frac{\delta^{2N_0}\mathbf{1}_{\langle\xi-\xi_*\rangle\lesssim\langle\xi_*-\xi^-\rangle}}{\langle\xi-\xi_*\rangle^{2(s'+\lambda-N_0)}}\Big) \\ + \frac{\mathbf{1}_{\langle\xi-\xi_*\rangle\lesssim\langle\xi_*-\xi^-\rangle}}{\langle\xi-\xi_*\rangle^{2s'}}\Big\}\,d\xi\,d\xi_*\,d\sigma\Big]\Big(\iiint|\hat{G}(\xi-\xi_*)|^2|\hat{H}(\xi)|^2\,d\sigma\,d\xi\,d\xi_*\Big).\end{aligned}$$

Noticing that $\langle\xi_*\rangle \sim \langle\xi\rangle \sim \langle\xi^+\rangle \lesssim \langle\xi^+ - u\rangle + \langle u\rangle$ with $u = \xi_* - \xi^-$, and moreover, $\langle u\rangle \lesssim \langle\xi_*\rangle$, we see that if $\lambda \geq 0$, then

$$M^\delta_\lambda(\xi_*)^2 \lesssim \frac{\langle\xi^+-u\rangle^{2\lambda} + \langle u\rangle^{2\lambda}}{(1+\delta\langle u\rangle)^{2N_0}}.$$

This is true even if $\lambda < 0$. Therefore, if $s' + \lambda < 3/2$ we have

$$\begin{aligned}|B_1|^2 &\lesssim \|f\|^2_{L^1}\int\langle u\rangle^{-(6+2\gamma+2s')} \\ &\quad\times\Big\{\int_{\langle\xi^+-u\rangle\leq\langle u\rangle}\frac{(\langle\xi^+-u\rangle^{2s'} + \langle u\rangle^{2\lambda})}{(1+\delta\langle u\rangle)^{2N_0}} \\ &\quad\times\Big(\frac{1}{\langle\xi^+-u\rangle^{2(s'+\lambda)}} + \frac{\delta^{2N_0}}{\langle\xi^+-u\rangle^{2(s'+\lambda-N_0)}}\Big)\,d\xi^+ \\ &\quad+\int_{\langle\xi^+-u\rangle\leq\langle u\rangle}\frac{d\xi^+}{\langle\xi^+-u\rangle^{2s'}}\Big\}\,du\,\|M^\delta_\lambda(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}} \\ &\lesssim \|f\|^2_{L^1}\|M^\delta_\lambda(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}\int\frac{du}{\langle u\rangle^{3+2(\gamma+2s')}}.\end{aligned}$$

Here we have used the change of variables $(\xi,\xi_*)\to(\xi^+,u)$ whose Jacobian is

$$\Big|\frac{\partial(\xi^+,u)}{\partial(\xi,\xi_*)}\Big|=\Big|\frac{\partial\xi^+}{\partial\xi}\Big|=\frac{|I+(\xi/|\xi|)\otimes\sigma|}{8}$$
$$=\frac{|1+(\xi/|\xi|)\cdot\sigma|}{8}=\frac{\cos^2(\theta/2)}{4}\geq\frac{1}{8},\quad \theta\in\Big[0,\frac{\pi}{2}\Big].$$

If $s'+\lambda\geq 3/2$, in view of $\gamma+2s'>0$ we have

$$|B_1|^2\lesssim\int|\hat f(\xi_*)|^2\big\{\langle u\rangle^{2\lambda-(6+2\gamma+2s')}\log\langle u\rangle\big\}\,d\xi_*\,\|M_\lambda^\delta(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}}$$
$$\lesssim\|f\|^2_{H^{(\lambda+s'-3)^+}}\|M_\lambda^\delta(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}},$$

because $\langle u\rangle\lesssim\langle\xi_*\rangle$ on the set of the integral.

As for $B_2$, we first note that, on the set of the integration, $\xi^+=\xi-\xi_*+u$ implies

$$\frac{\langle\xi-\xi_*\rangle}{2}\leq\langle\xi-\xi_*\rangle-|u|\leq\langle\xi^+\rangle\leq\langle\xi-\xi_*\rangle+|u|\lesssim\langle\xi-\xi_*\rangle,$$

so that

$$\big(M_\lambda^\delta(\xi)\sim\big)M_\lambda^\delta(\xi^+)\sim M_\lambda^\delta(\xi-\xi_*),$$

and hence we have by the Cauchy–Schwarz inequality

$$|B_2|^2\lesssim\|f\|^2_{L^1}\iiint\frac{|\hat\Phi_c(\xi_*-\xi^-)|}{\langle\xi_*-\xi^-\rangle^{2s'}}|\hat G(\xi-\xi_*)|^2\,d\sigma\,d\xi\,d\xi_*$$
$$\times\iiint\frac{|\hat\Phi_c(\xi_*-\xi^-)|}{\langle\xi_*-\xi^-\rangle^{2s'}}|\hat H(\xi)|^2\,d\sigma\,d\xi\,d\xi_*$$
$$\lesssim\|f\|^2_{L^1}\|M_\lambda^\delta(D)g\|^2_{H^{s'}}\|h\|^2_{H^{s'}},$$

because $\gamma+2s'>0$.

On the set of the integration for $B_3$ we recall $\langle\xi\rangle\sim\langle\xi-\xi_*\rangle$ and

$$|M_\lambda^\delta(\xi)-M_\lambda^\delta(\xi-\xi_*)|\lesssim\frac{\langle\xi_*\rangle}{\langle\xi\rangle}M_\lambda^\delta(\xi-\xi_*),$$

so that

$$|B_3|^2\lesssim\|f\|^2_{L^1}\iiint b\mathbf{1}_{|\xi^-|\geq(1/2)\langle\xi_*\rangle}\frac{|\hat\Phi_c(\xi_*-\xi^-)|\langle\xi_*\rangle}{\langle\xi\rangle^{2s'+1}}|\hat G(\xi-\xi_*)|^2\,d\sigma\,d\xi\,d\xi_*$$
$$\times\iiint b\mathbf{1}_{|\xi^-|\geq(1/2)\langle\xi_*\rangle}\frac{|\hat\Phi_c(\xi_*-\xi^-)|\langle\xi_*\rangle}{\langle\xi\rangle^{2s'+1}}|\hat H(\xi)|^2\,d\sigma\,d\xi\,d\xi_*.$$

We use the change of variables $\xi_*\to u=\xi_*-\xi^-$. Note that $|\xi^-|\geq(1/2)\langle u+\xi^-\rangle$ implies $|\xi^-|\geq\langle u\rangle/\sqrt{10}$, and that

$$\langle\xi_*\rangle\lesssim\langle\xi_*-\xi^-\rangle+|\xi|\sin\theta/2.$$

Then we have

$$\iint b\mathbf{1}_{|\xi^-|\geq(1/2)\langle\xi_*\rangle}\frac{|\hat\Phi_c(\xi_*-\xi^-)|\langle\xi_*\rangle}{\langle\xi\rangle^{2s'+1}}\,d\sigma\,d\xi_*$$

$$
\begin{aligned}
&\lesssim \int \frac{\mathbf{1}_{\langle u\rangle \lesssim |\xi|}}{\langle u\rangle^{3+\gamma+2s'}} \Big(\frac{\langle u\rangle}{\langle \xi\rangle}\Big)^{2s'} \\
&\quad \times \Big( \int b \mathbf{1}_{|\xi^-| \gtrsim \langle u\rangle} \frac{\langle u\rangle}{\langle \xi\rangle}\, d\sigma + \int b \sin(\theta/2) \mathbf{1}_{|\xi^-| \gtrsim \langle u\rangle}\, d\sigma \Big)\, du,
\end{aligned}
$$

from which we can also obtain the desired bound for $B_3$ if $\gamma + 2s' > 0$. In fact, the first integral on the sphere is bounded from above by $\langle u\rangle^{1-2s}/\langle \xi\rangle^{1-2s}$, and the second integral has the same bound when $s > 1/2$. On the other hand, the second integral is bounded by a constant when $s < 1/2$ and by $|\log(\langle \xi\rangle/\langle u\rangle)|$ when $s = 1/2$. The proof of (1) and (2) of the proposition is then completed. $\square$

REMARK 3.5

As seen from the above proof, the restrictions (3.8) and (3.9) on the pair $(N_0, \lambda)$ in the formula (3.1) are only required to estimate the part $A_1$. It follows from (3.3) of Lemma 3.1, (3.10), and (3.11) that $A_1$ can be replaced by

$$
\tilde{A}_{1,\lambda} = \iint_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_*\rangle^{3+\gamma}} |\hat{g}(\xi - \xi_*)||\hat{h}(\xi)| \langle \xi\rangle^{\lambda} \mathbf{1}_{\langle \xi_*\rangle \geq \sqrt{2}|\xi|} \frac{\langle \xi\rangle}{\langle \xi_*\rangle}\, d\xi\, d\xi_*. \tag{3.13}
$$

Consequently, for any pair $(N_0, \lambda)$, assertion (1) holds if $\tilde{A}_{1,\lambda}$ is added on the right-hand side of the estimate.

The combination of Proposition 3.4 and Proposition 3.2 together with its remark yields the following theorem.

THEOREM 3.6

*Assume that* $0 < s < 1, \gamma + 2s > 0$. *Let* $0 < s' < s$ *satisfy* $\gamma + 2s' > 0, 2s' \geq (2s-1)^+$. *Assume that the pair* $(N_0, \lambda)$ *satisfies (3.8). Then*

(1) *if* $s' + \lambda < 3/2$, *we have*

$$
\begin{aligned}
&\big|\big(M_\lambda^\delta(D_v) Q(f,g) - Q(f, M_\lambda^\delta(D_v) g), h\big)\big| \\
&\qquad \lesssim \|f\|_{L^1_{\gamma^+ + (2s-1)^+}} \|M_\lambda^\delta(D_v) g\|_{H^{s'}_{\gamma^+ + (2s-1)^+}} \|h\|_{H^{s'}};
\end{aligned} \tag{3.14}
$$

(2) *if* $s' + \lambda \geq 3/2$, *we have*

$$
\begin{aligned}
&\big|\big(M_\lambda^\delta(D_v) Q(f,g) - Q(f, M_\lambda^\delta(D_v) g), h\big)\big| \\
&\qquad \lesssim (\|f\|_{L^1_{\gamma^+ + (2s-1)^+}} + \|f\|_{H^{(\lambda+s'-3)^+}}) \|M_\lambda^\delta(D_v) g\|_{H^{s'}_{\gamma^+ + (2s-1)^+}} \|h\|_{H^{s'}}.
\end{aligned}
$$

*Furthermore, if* $s > 1/2$ *and* $\gamma > -1$, *then the same conclusion as above holds even when the condition (3.8) is replaced by (3.9). When* $0 < s < 1/2$ *and* $\gamma > 0$, *we can use* $\|M_\lambda^\delta(D_v) g\|_{H^{s'}_{\gamma/2}} \|h\|_{H^{s'}_{\gamma/2}}$ *for the corresponding terms in the above estimates with smaller weight in the variable* $v$.

REMARK 3.7

It follows from Remark 3.5 that for any pair $(N_0, \lambda)$ the commutator estimate

$$
\begin{aligned}
(3.15)\qquad &\big|\big(M_\lambda^\delta(D_v)Q(f,g)-Q(f,M_\lambda^\delta(D_v)g),h\big)\big| \\
&\qquad\lesssim \|f\|_{L^1_{\gamma^++(2s-1)^+}}\|M_\lambda^\delta(D_v)g\|_{H^{s'}_{\gamma^++(2s-1)^+}}\|h\|_{H^{s'}}+\tilde{A}_{1,\lambda}
\end{aligned}
$$

holds, instead of (3.14), where $\tilde{A}_{1,\lambda}$ is defined by (3.13).

We recall also the following upper-bound estimate, [7, Proposition 2.1], where the assumption $\gamma+2s>0$ is needed (see also [4, Theorem 2.1]).

PROPOSITION 3.8

*Let $\gamma+2s>0$, and let $0<s<1$. For any $r\in[2s-1,2s]$ and $\ell\in[0,\gamma+2s]$ we have*

$$
\big|\big(Q(f,g),h\big)_{L^2(\mathbb{R}^3)}\big|\lesssim\|f\|_{L^1_{\gamma+2s}}\|g\|_{H^r_{\gamma+2s-\ell}}\|h\|_{H^{2s-r}_\ell}.
$$

In the following analysis, we shall need an interpolation inequality concerning weighted-type Sobolev spaces with respect to variable $v$ (see, for instance, [12], [13]).

LEMMA 3.9

*For any $k\in\mathbb{R}$, $p\in\mathbb{R}_+$, $\delta>0$,*

$$
\|f\|^2_{H^k_p(\mathbb{R}^3_v)}\le C_\delta\|f\|_{H^{k-\delta}_{2p}(\mathbb{R}^3_v)}\|f\|_{H^{k+\delta}_0(\mathbb{R}^3_v)}.
$$

And we need also another interpolation inequality in $L^q$ given by the following lemma.

LEMMA 3.10

*Let $1<q<p$. Assume that $f\in L^p$, and assume that $\langle v\rangle^\ell f\in L^1$ for any $\ell$. Then $\langle v\rangle^\ell f\in L^q$ for any $\ell$. More precisely, we have*

$$
\|f\|_{L^q_\ell}\le 2\|f\|_{L^p}^{(p(q-1))/(q(p-1))}\|f\|^{(p-q)/(q(p-1))}_{L^1_{\ell q(p-1)/(p-q)}}.
$$

*Proof*

For any $\lambda>0$, we can write

$$
\begin{aligned}
\|f\|^q_{L^q_\ell}&=\int_{\langle v\rangle^{\ell q}|f(v)|^{q-p}\le\lambda}\langle v\rangle^{\ell q}|f(v)|^q\,dv+\int_{\langle v\rangle^{\ell q}|f(v)|^{q-p}>\lambda}\langle v\rangle^{\ell q}|f(v)|^q\,dv \\
&\le\lambda\|f\|^p_{L^p}+\lambda^{(q-1)/(q-p)}\|f\|_{L^1_{\ell q(p-1)/(p-q)}}.
\end{aligned}
$$

Taking

$$
\lambda=\|f\|^{(p-q)/(p-1)}_{L^1_{\ell q(p-1)/(p-q)}}\|f\|_{L^p}^{-(p(p-q))/(p-1)},
$$

we obtain the desired estimate. □

## 4. Smoothing effect of $L^2$-weak solutions

We start from a weak solution in $L^2$ having finite moments of all orders.

THEOREM 4.1

*Assume that $0 < s < 1$, $\gamma + 2s > 0$. Let $f$ be in $L^\infty([t_0, T]; L^2_\ell(\mathbb{R}^3))$ for any $\ell \in \mathbb{N}$ and a nonnegative weak solution of (1.1). Then, for any $t_0 < \tilde{t}_0 < T$, we have*

$$f \in L^\infty\big([\tilde{t}_0, T]; \mathcal{S}(\mathbb{R}^3)\big).$$

*Proof*

Without loss of generality, let $t_0 = 0$. Assume that, for some $a \geq 0$, we have

$$\sup_{[0,T]} \|f(t,\cdot)\|_{H^a_\ell} < \infty \quad \text{for any } \ell \in \mathbb{N}. \tag{4.1}$$

Take $\lambda(t) = Nt + a$ for $N > 0$. Choose $N_0 = a + (5+\gamma)/2$. Then the pair $(N_0, \lambda(t))$ satisfies (3.8). If we choose $N, T_1 > 0$ such that $NT_1 = (1-s)$, then

$$\lambda(T_1) - N_0 - a \leq \lambda(T_1) - N_0 < -3/2,$$

from which we have, for $t, t' \in [0, T_1]$,

$$M^\delta_{\lambda(t)} f(t') \in L^\infty\big([0, T_1] \times [0, T_1]; H^{3/2}_\ell(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\big), \tag{4.2}$$

because of (4.1). We show in Lemma 4.3 below that (see also [14]):

$$M^\delta_{\lambda(t)} f(t) \in C\big([0, T_1]; L^2(\mathbb{R}^3)\big), \tag{4.3}$$

and for any $t \in ]0, T_1]$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)^2\, dv - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} f(\tau)\big(\partial_\tau (M^\delta_{\lambda(\tau)})^2\big) f(\tau)\, dv\, d\tau \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^3} (M^\delta_{\lambda(0)} f_0)^2\, dv \\
&\qquad + \int_0^t \big(Q(f(\tau), M^\delta_{\lambda(\tau)} f(\tau)), M^\delta_{\lambda(\tau)} f(\tau)\big)_{L^2}\, d\tau \\
&\qquad + \int_0^t \big(M^\delta_{\lambda(\tau)} Q(f(\tau), f(\tau)) - Q(f(\tau), M^\delta_{\lambda(\tau)} f(\tau)), M^\delta_{\lambda(\tau)} f(\tau)\big)_{L^2}\, d\tau,
\end{aligned} \tag{4.4}$$

by taking $(M^\delta_{\lambda(t)})^2 f(t)$ as a test function in the definition of the weak solution, though it does not belong to $L^\infty([0, T_1]; W^{2,\infty}(\mathbb{R}^3))$.

Noticing that

$$\partial_t M^\delta_{\lambda(t)} = N(\log\langle\xi\rangle) M^\delta_{\lambda(t)},$$

it follows from Theorem 3.6 that we have

$$\begin{aligned}
\frac{1}{2}\|(M^\delta_\lambda f)(t)\|^2_{L^2} &\leq \frac{1}{2}\|f(0)\|^2_{H^a} + \int_0^t \big(Q(f(\tau), (M^\delta_\lambda f)(\tau)), (M^\delta_\lambda f)(\tau)\big)\, d\tau \\
&\quad + C_f \int_0^t \|(M^\delta_\lambda f)(\tau)\|_{H^{s'}_{\gamma^+ + (2s-1)^+}} \|(M^\delta_\lambda f)(\tau)\|_{H^{s'}}\, d\tau \\
&\quad + CN \int_0^t \|(\log\langle D\rangle)^{1/2} (M^\delta_\lambda f)(\tau)\|^2_{L^2}\, d\tau.
\end{aligned} \tag{4.5}$$

Since the uniform coercive estimate (2.2) together with interpolation in the Sobolev space yields

$$\big(Q(f(\tau),(M_\lambda^\delta f)(\tau)),(M_\lambda^\delta f)(\tau)\big) \le -c_f\|(M_\lambda^\delta f)(\tau)\|^2_{H^s_{\gamma/2}} + C_f\|f(\tau)\|^2_{H^{-2}_{\gamma/2}},$$

by means of Lemma 3.9 we have

$$\|(M_\lambda^\delta f)(t)\|^2_{L^2} + c_f\int_0^t \|(M_\lambda^\delta f)(\tau)\|^2_{H^s_{\gamma+/2}}\,d\tau \le \|f(0)\|^2_{H^a} + C_f\int_0^t \|f(\tau)\|^2_{H^a_\ell}\,d\tau. \tag{4.6}$$

Taking $\delta\to+0$ and $t=T_1$, we have $f(T_1)\in H^{\lambda(T_1)}=H^{NT_1+a}$. This is true for any $0<T_1\le T$. Choosing $N=(1-s)T_1^{-1}$, we have that for any $0<T_1\le T$,

$$f(T_1)\in H^{(1-s)+a}.$$

Fix $0<s_0<(1-s)$. Then, by using Lemma 3.9 and the assumption (4.1), we see that for any $0<t_1<\tilde t_0$ and any $\ell$,

$$\sup_{[t_1,T]}\|f(t,\cdot)\|_{H^{s_0+a}_\ell}<\infty.$$

We can restart by replacing $a$ by $a+s_0=a_1$ and $t_0$ by $t_1$. By induction, for $a_0=0, a_k=ks_0$, and $t_k=\tilde t_0-(2k)^{-1}(\tilde t_0-t_0)$, we have for any $k\in\mathbb{N}$ and any $\ell$,

$$f\in L^\infty\big([t_k,T];H^{a_k}_\ell(\mathbb{R}^3)\big),$$

which concludes the proof of Theorem 4.1. □

REMARK 4.2

When $0<s<1/2$ and $\gamma>0$ we can use $\int_0^t\|(M_\lambda^\delta f)(\tau)\|^2_{H^{s'}_{\gamma/2}}\,d\tau$ for the corresponding term in (4.5). Hence, instead of (4.6), we can obtain

$$\|(M_\lambda^\delta f)(t)\|^2_{L^2} \le \|f(0)\|^2_{H^a} + C_f\int_0^t\|f(\tau)\|^2_{H^{-2}_{\gamma/2}}\,d\tau,$$

which shows that $f(t)\in L^\infty([0,T];L^2\cap L^1_2(\mathbb{R}^3))$ implies $f(t)\in H^\infty(\mathbb{R}^3)$ for $t>0$.

LEMMA 4.3

*Let $T_1>0$, and let $M^\delta_{\lambda(t)}(\xi)$ be defined by (3.1) with $\lambda=\lambda(t)=Nt+a$ for $NT_1<1$ and $a\in\mathbb{R}$. Suppose that*

$$f\in L^1\big([0,T_1];L^1_{\max\{\gamma+2s,2\}}(\mathbb{R}^3)\big)\cap L^\infty\big([0,T_1];H^a(\mathbb{R}^3)\big).$$

*If there exists $s_1>s$ such that*

$$M^\delta_{\lambda(t)}f(t',v)\in L^\infty\big([0,T_1]_t\times[0,T_1]_{t'};H^{s_1}_{\ell_0}(\mathbb{R}^3_v)\big)$$

*for $\ell_0=\max\{\gamma/2+s,\gamma^++(2s-1)^+\}$, then we have (4.3), and (4.4) for any $t\in[0,T_1]$. Furthermore, if $0<s<1/2$ and $\gamma>0$ we can take $\ell_0=\gamma/2+s$.*

*Proof*

In Definition 1.1, taking $\varphi(t,v)=\psi(v)\in C_0^\infty(\mathbb{R}^3)$ as a test function independent of the $t$-variable, we get

$$\int_{\mathbb{R}^3} f(t)\psi\,dv-\int_{\mathbb{R}^3} f(t')\psi\,dv=\int_{t'}^{t}d\tau\int_{\mathbb{R}^3} Q\big(f(\tau),f(\tau)\big)\psi\,dv,\quad 0\le t'\le t\le T_1. \tag{4.7}$$

For any fixed $\bar t\in[0,T_1]$, we can take a sequence $\{\psi_j(v)\}_{j=1}^\infty\subset C_0^\infty(\mathbb{R}^3_v)$ such that $(M^\delta_{\lambda(\bar t)})^{-1}\psi_j\to M^\delta_{\lambda(\bar t)}f(\bar t)$ in $H^s_{\ell_0}$. Because all terms in (4.7) make sense for $\psi=(M^\delta_{\lambda(\bar t)})^2 f(\bar t)$, we can set it as a test function in (4.7). In fact, we have

$$\Big|\int_{\mathbb{R}^3} f(t)(M^\delta_{\lambda(\bar t)})^2 f(\bar t)\,dv\Big|\le \|M^\delta_{\lambda(\bar t)}f(t)\|_{L^2}\|M^\delta_{\lambda(\bar t)}f(\bar t)\|_{L^2}<+\infty,$$

and by noticing that

$$\big(Q(f,f),(M^\delta_\lambda)^2 f\big)=\big(Q(f,M^\delta_\lambda f),M^\delta_\lambda f\big)+\big(M^\delta_\lambda Q(f,f)-Q(f,M^\delta_\lambda f),M^\delta_\lambda f\big),$$

we have

$$\begin{aligned}
&\Big|\int_{t'}^{t}d\tau\int_{\mathbb{R}^3} Q\big(f(\tau),f(\tau)\big)(M^\delta_{\lambda(\bar t)})^2 f(\bar t)\,dv\Big|\\
&\quad\lesssim \int_{t'}^{t}\|f(\tau)\|_{L^1_{\gamma+2s}}\,d\tau\Big(\sup_{\tau,\bar t\in[0,T_1]}\|M^\delta_{\lambda(\bar t)}f(\tau)\|_{H^s_{\gamma/2+s}}\|M^\delta_{\lambda(\bar t)}f(\bar t)\|_{H^s_{\gamma/2+s}}\Big)\\
&\qquad+\Big(\int_{t'}^{t}\|f(\tau)\|_{L^1_{\gamma^++(2s-1)^+}}\,d\tau+|t-t'|\sup_{\tau\in[0,T_1]}\|f(\tau)\|_{H^a}\Big)\\
&\qquad\times\Big(\sup_{\tau,\bar t\in[0,T_1]}\|M^\delta_{\lambda(\bar t)}f(\tau)\|_{H^s_{\gamma^++(2s-1)^+}}\|M^\delta_{\lambda(\bar t)}f(\bar t)\|_{H^s}\Big),
\end{aligned}\tag{4.8}$$

thanks to Proposition 3.8 and Theorem 3.6. Setting $\psi=(M^\delta_{\lambda(\bar t)})^2 f(\bar t)$ with $\bar t=t,t'$, and taking the sum, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}\big(M^\delta_{\lambda(t)}f(t)\big)^2\,dv-\int_{\mathbb{R}^3}\big(M^\delta_{\lambda(t')}f(t')\big)^2\,dv\\
&\quad=\int_{\mathbb{R}^3} f(t)\big((M^\delta_{\lambda(t)})^2-(M^\delta_{\lambda(t')})^2\big)f(t')\,dv\\
&\qquad+\int_{t'}^{t}d\tau\int_{\mathbb{R}^3} Q\big(f(\tau),f(\tau)\big)\big((M^\delta_{\lambda(t)})^2 f(t)+(M^\delta_{\lambda(t')})^2 f(t')\big)\,dv.
\end{aligned}\tag{4.9}$$

The second term goes to zero if $t'\to t$ thanks to (4.8). By the mean value theorem, the first term on the right-hand side of (4.9) is estimated by

$$|t-t'|\sup_{0\le t'<\tilde\tau<t\le T_1}\|M^\delta_{\lambda(\tilde\tau)}f(t)\|_{L^2}\|(\log\langle D\rangle)M^\delta_{\lambda(\tilde\tau)}f(t')\|_{L^2}.$$

Hence, (4.9) gives

$$\lim_{t'\to t}\int_{\mathbb{R}^3}\big(M^\delta_{\lambda(t')}f(t')\big)^2\,dv=\int_{\mathbb{R}^3}\big(M^\delta_{\lambda(t)}f(t)\big)^2\,dv. \tag{4.10}$$

Taking the difference, instead of (4.9), we get

$$\begin{aligned}\int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)^2\, dv &+ \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t')} f(t')\big)^2\, dv \\ &= \int_{\mathbb{R}^3} f(t)\big((M^\delta_{\lambda(t)})^2 + (M^\delta_{\lambda(t')})^2\big) f(t')\, dv \\ &\quad + \int_{t'}^{t} d\tau \int_{\mathbb{R}^3} Q\big(f(\tau), f(\tau)\big)\big((M^\delta_{\lambda(t)})^2 f(t) - (M^\delta_{\lambda(t')})^2 f(t')\big)\, dv,\end{aligned}$$

which shows

$$\lim_{t'\to t} \int_{\mathbb{R}^3} f(t)\big((M^\delta_{\lambda(t)})^2 + (M^\delta_{\lambda(t')})^2\big) f(t')\, dv = 2\int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)^2\, dv,$$

and moreover,

$$\lim_{t'\to t} \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)\big(M^\delta_{\lambda(t')} f(t')\big)\, dv = \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f(t)\big)^2\, dv. \tag{4.11}$$

By (4.10) and (4.11) we have

$$\lim_{t'\to t} \|M^\delta_{\lambda(t')} f(t') - M^\delta_{\lambda(t)} f(t)\|^2_{L^2} = 0,$$

which is (4.3), namely, $M^\delta_{\lambda(t)} f(t) \in C([0, T_1]; L^2(\mathbb{R}^3))$. Taking

$$\psi = (\log\langle D\rangle)^2 (M^\delta_{\lambda(\bar t)})^2 f(\bar t)$$

with $\bar t = t, t'$, similarly we have

$$(\log\langle D\rangle) M^\delta_{\lambda(t)} f(t) \in C\big([0, T_1]; L^2(\mathbb{R}^3)\big).$$

To prove (4.4), we need to mollify the solution with respect to moment as well as regularity, so we introduce the following mollifier:

$$M^{\delta,\kappa}_{\lambda(t)}(D, v) = \frac{M^\delta_{\lambda(t)}(D)}{1 + \kappa\langle D\rangle} \cdot \frac{1}{(1 + \kappa\langle v\rangle)^{\gamma^+/2+s}},$$

with a new parameter $\kappa > 0$. Divide $[0, t]$ into $k$ subintervals with the same length, and put $t_j = jt/k$ for $j = 0, \ldots, k$. Similarly to (4.9), we have

$$\begin{aligned}\int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f_\kappa(t_j)\big)^2\, dv &- \int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t_{j-1})} f_\kappa(t_{j-1})\big)^2\, dv \\ &= \int_{\mathbb{R}^3} f_\kappa(t_j)\big((M^\delta_{\lambda(t_j)})^2 - (M^\delta_{\lambda(t_{j-1})})^2\big) f_\kappa(t_{j-1})\, dv \\ &\quad + \int_{t_{j-1}}^{t_j} d\tau \int_{\mathbb{R}^3} Q\big(f(\tau), f(\tau)\big)(1 + \kappa\langle v\rangle)^{-\gamma^+/2-s}(1 + \kappa\langle D\rangle)^{-1} \\ &\quad \times \big((M^\delta_{\lambda(t_j)})^2 f_\kappa(t_j) + (M^\delta_{\lambda(t_{j-1})})^2 f_\kappa(t_{j-1})\big)\, dv,\end{aligned} \tag{4.12}$$

where $f_\kappa = (1 + \kappa\langle D\rangle)^{-1}(1 + \kappa\langle v\rangle)^{-\gamma^+/2-s} f$. Since we have

$$\int f_\kappa(t_j)\big((M^\delta_{\lambda(t_j)})^2 - (M^\delta_{\lambda(t_{j-1})})^2\big) f_\kappa(t_{j-1})\, dv$$

$$
\begin{aligned}
&= \int 2N f_\kappa(t_j)(\log\langle D\rangle)(M^\delta_{\lambda(\tau_j)})^2 f_\kappa(t_{j-1})\,dv(t_j-t_{j-1}) \quad \tau_j \in ]t_{j-1},t_j[ \\
&= 2N\int \big(\sqrt{\log\langle D\rangle} M^\delta_{\lambda(t_j)} f_\kappa(t_j)\big)\big(\sqrt{\log\langle D\rangle} M^\delta_{\lambda(t_{j-1})} f_\kappa(t_{j-1})\big)\,dv(t_j-t_{j-1}) \\
&\quad + N^2\Big(\sup_{\tau',\tau''\in[0,T_1]} \|\log\langle D\rangle M^\delta_{\lambda(\tau')} f_\kappa(\tau'')\|_{L^2}\Big)^2 O(|t_j-t_{j-1}|^2),
\end{aligned}
$$

it follows from a formula similar to (4.11) that

$$
\begin{aligned}
&\lim_{k\to\infty}\sum_{j=1}^k \int f_\kappa(t_j)\big((M^\delta_{\lambda(t_j)})^2-(M^\delta_{\lambda(t_{j-1})})^2\big) f_\kappa(t_{j-1})\,dv \\
&\qquad = N\int_0^t \int_{\mathbb{R}^3} \big(\sqrt{\log\langle D\rangle} M^\delta_{\lambda(\tau)} f_\kappa(\tau)\big)^2\,dv\,d\tau.
\end{aligned}
$$

Summing up (4.12) with respect to $j=1,\dots,k$ and letting $k\to\infty$, we obtain

$$
\begin{aligned}
&\frac12\int_{\mathbb{R}^3} \big(M^\delta_{\lambda(t)} f_\kappa(t)\big)^2\,dv - \frac12\int_0^t\int_{\mathbb{R}^3} f(\tau)\big(\partial_\tau (M^\delta_{\lambda(\tau)})^2\big) f_\kappa(\tau)\,dv\,d\tau \\
&\quad = \frac12\int_{\mathbb{R}^3} (M^{\delta,\kappa}_{\lambda(0)} f_0)^2\,dv \\
&\qquad + \int_0^t \big(Q(f(\tau), M^\delta_{\lambda(\tau)} f(\tau)), (M^\delta_{\lambda(\tau)})^{-1}(M^{\delta,\kappa}_{\lambda(\tau)})^* M^{\delta,\kappa}_{\lambda(\tau)} f(\tau)\big)_{L^2}\,d\tau \\
&\qquad + \int_0^t \big(M^\delta_{\lambda(\tau)} Q(f(\tau), f(\tau)) \\
&\qquad - Q(f(\tau), M^\delta_{\lambda(\tau)} f(\tau)), (M^\delta_{\lambda(\tau)})^{-1}(M^{\delta,\kappa}_{\lambda(\tau)})^* M^{\delta,\kappa}_{\lambda(\tau)} f(\tau)\big)_{L^2}\,d\tau,
\end{aligned}
\tag{4.13}
$$

thanks to Proposition 3.8 and Theorem 3.6. In fact, for example, we have

$$
\begin{aligned}
&\int_0^t \big|\big(Q(f(\tau), M^\delta_{\lambda(\tau)} f(\tau)), (M^\delta_{\lambda(\tau)})^{-1}\big\{(M^{\delta,\kappa}_{\lambda(t_j)})^* M^{\delta,\kappa}_{\lambda(t_j)} f(t_j) \\
&\qquad - (M^{\delta,\kappa}_{\lambda(\tau)})^* M^{\delta,\kappa}_{\lambda(\tau)} f(\tau)\big\}\big)_{L^2}\big|\,d\tau \\
&\quad \lesssim \int_0^t \|f(\tau)\|_{L^1_{\gamma+2s}}\,d\tau \sup_{\tau\in[0,T_1]} \|M^\delta_{\lambda(\tau)} f(\tau)\|_{H^s_{\gamma/2+s}} \\
&\qquad \times \sup_{\tau,t_j\in[0,T_1]} \big\|(M^\delta_{\lambda(\tau)})^{-1}\big\{(M^{\delta,\kappa}_{\lambda(t_j)})^* M^{\delta,\kappa}_{\lambda(t_j)} f(t_j) \\
&\qquad - (M^{\delta,\kappa}_{\lambda(\tau)})^* M^{\delta,\kappa}_{\lambda(\tau)} f(\tau)\big\}\big\|_{H^s_{\gamma/2+s}},
\end{aligned}
$$

and hence the Lebesgue convergence theorem yields (4.13) because we have

$$
\begin{aligned}
&\big\|(M^\delta_{\lambda(\tau)})^{-1}\big\{(M^{\delta,\kappa}_{\lambda(t_j)})^* M^{\delta,\kappa}_{\lambda(t_j)} f(t_j) - (M^{\delta,\kappa}_{\lambda(\tau)})^* M^{\delta,\kappa}_{\lambda(\tau)} f(\tau)\big\}\big\|_{H^s_{\gamma/2+s}} \\
&\quad \lesssim |t_j-\tau|\|M^\delta_{\lambda(\tau)} f(\tau)\|_{H^s} + \|M^\delta_{\lambda(t_j)} f(t_j) - M^\delta_{\lambda(\tau)} f(\tau)\|_{L^2}.
\end{aligned}
$$

Letting $\kappa\to 0$ in (4.13) we obtain the desired formula.

The last assertion of the lemma in the case $0 < s < 1/2, \gamma > 0$, follows easily from Theorem 3.6. $\square$

## 5. Smoothing effect of $L^1$-weak solutions

We come back to the proof of Theorem 1.2 starting from an $L^1$-weak solution. The first part of the theorem is restated as follows.

THEOREM 5.1

*Assume that* $0 < s < 1$, $\gamma > \max\{-2s, -1\}$. *If* $f$ *belongs to* $L^\infty([t_0, T]; L^1_\ell(\mathbb{R}^3))$ *for any* $\ell \in \mathbb{N}$ *and is a weak solution of (1.1), then for any* $t_0 < \tilde{t}_0 < T$, *we have*

$$f \in L^\infty\big([\tilde{t}_0, T]; \mathcal{S}(\mathbb{R}^3)\big).$$

*Proof*

By Theorem 4.1, it is sufficient to prove, for any $0 < t_1 \le T$, (taking again $t_0 = 0$), that

$$f \in L^\infty\big([t_1, T]; L^2_\ell(\mathbb{R}^3)\big). \tag{5.1}$$

Since $L^1(\mathbb{R}^3) \subset H^{-3/2-\varepsilon}$, we may assume that for any $\ell$ and any $0 < \varepsilon \ll 1$,

$$\sup_{[0,T]} \|f(t,\cdot)\|_{H_\ell^{-3/2-\varepsilon}} < \infty. \tag{5.2}$$

As in the proof of Theorem 4.1, we shall prove the theorem by induction. Assume that for $0 > a \ge -3/2 - \varepsilon$, we have

$$\sup_{[0,T]} \|f(t,\cdot)\|_{H^a_\ell} < \infty.$$

Take also $\lambda(t) = Nt + a$ for $N > 0$.

We first consider the case $0 < s \le 1/2$. Choose $N_0 = a + (5+\gamma)/2 \ge 1 - \varepsilon + (\gamma/2) > 0$ such that (3.8) is fulfilled. Put $\varepsilon_0 = (1 - 2s')/8 > 0$, and consider $\varepsilon = \varepsilon_0$, where $0 < s' < s$ is chosen to satisfy $\gamma + 2s' > 0$. If we choose $N, T_1 > 0$ such that $NT_1 = \varepsilon_0$, then

$$s + \lambda(T_1) - N_0 - a = s + \varepsilon_0 - N_0 \le s - 1 + 2\varepsilon_0 - (\gamma/2) < (s' - 1/2) + 2\varepsilon_0 < 0,$$

which shows that

$$M^\delta_{\lambda(t)} f(t) \in L^\infty\big([0, T_1]; H^s_\ell(\mathbb{R}^3)\big). \tag{5.3}$$

This estimate and Lemma 4.3 lead to (4.4), and hence we obtain (4.5) using Theorem 3.6, and (4.6) by means of (2.2) and Lemma 3.9. The same procedure as in the proof of Theorem 4.1 shows (5.1) by induction.

When $s > 1/2$ we choose $1/2 < s' < s$ such that $\gamma + 2s' > 0, 2s' \ge (2s-1)$. Choose $N_0 = a + (5 + \gamma + 2s' - 1)/2$ such that (3.9) is satisfied. Put $\varepsilon_0 = (\gamma + 1)/10 > 0$, and consider $\varepsilon = \varepsilon_0$. Then, we have

$$s + \lambda(T_1) - N_0 - a = s + \varepsilon_0 - N_0 \le s - s' + 2\varepsilon_0 - (1+\gamma)/2 = s - s' - 3\varepsilon_0. \tag{5.4}$$

Since we may assume $s - s' \le \varepsilon_0$, (5.4) also shows (5.3), which completes the proof of the theorem in the same way as in the case $0 < s \le 1/2$. $\square$

In view of Remark 4.2 and the last assertion of Lemma 4.3, the proof of Theorem 5.1 in the case $0 < s < 1/2$ leads us easily to the following theorem where the assumption (1.5) can be removed.

THEOREM 5.2
*Suppose that the cross section $B$ of the form (1.2) satisfies (1.3) and (1.4) with $0 < s < 1/2$ and $\gamma > 0$. If*

$$f \in L^\infty\big([0,T]; L^1_{\max\{2,\gamma/2+s\}}(\mathbb{R}^3) \cap L\log L\big) \cap L^1\big([0,T]; L^1_{2+\gamma}(\mathbb{R}^3)\big)$$

*is a weak solution, then $f \in L^\infty([t_0,T]; H^\infty(\mathbb{R}^3))$ for any $t_0 \in ]0,T[$.*

We consider now the second part of Theorem 1.2, which is stated as follows.

THEOREM 5.3
*Assume that $-1 \geq \gamma > -2s$. Let $f \in L^\infty([t_0,T]; L^1_\ell(\mathbb{R}^3))$ for any $\ell \in \mathbb{N}$ be a weak solution of (1.1) satisfying the entropy dissipation estimate*

$$\int_{t_0}^T D\big(f(t), f(t)\big)\, dt < +\infty. \tag{5.5}$$

*Then for any $t_0 < \tilde t_0 < T$, we have*

$$f \in L^\infty\big([\tilde t_0, T]; \mathcal{S}(\mathbb{R}^3)\big).$$

Thanks to Corollary 2.4, the assumption (5.5) yields a certain a priori regularity estimate

$$\int_{t_0}^T \|\langle v\rangle^{\gamma/2}\sqrt{f(t)}\|^2_{H^s}\, dt < \infty, \tag{5.6}$$

directly without mollification of a weak solution. This extra regularity enables us to use the commutator estimate (3.15) stated in Remark 3.7. Note that we can now choose an arbitrarily large $N_0$ in (3.1). Hence $(M^\delta_{\lambda(t)})^2 f(t)$ belongs to $W^{2,\infty}$, which can be taken as a test function. However, $\lambda(t)$ cannot be taken as large as we want, because the regularity (5.6) gained from the dissipation estimate is too weak. This is why other arguments are needed hereafter.

By means of Theorem 4.1, it suffices to show that $f \in L^\infty([t_1,T]; L^2_\ell)$ for $0 < t_1 \leq T$ by induction, starting from (5.2) where we take again $t_0 = 0$.

We divide the proof into three steps.

*First step.* Noticing the hypothesis $-1 \geq \gamma > -2s$, we take $s' > 1/2$ such that $\gamma + 2s' > 0$ and $s' < s$. Put $s_0 = (1/4)(\gamma + 2s')$. For arbitrary $t > 0$ and $N > 0$ satisfying $Nt = s_0$, we set

$$\lambda_1(\tau) = N\tau - \frac{3}{2} - \varepsilon \quad \text{for } \tau \in [0,t],$$

where $\varepsilon > 0$ is arbitrarily small. If we substitute $\lambda = \lambda_1(\tau)$ into (3.13), then, in view of $N\tau \le s_0$, we have

$$\tilde{A}_{1.\lambda_1}(\tau) \lesssim \|\hat{g}\|_{L^\infty} \iint_{\mathbb{R}^6} \frac{|\hat{f}(\xi_*)|}{\langle \xi_* \rangle^{3+\gamma-s_0}} \frac{|\hat{h}(\xi)|}{\langle \xi \rangle^{3/2+\varepsilon}} \, d\xi \, d\xi_*$$

$$\lesssim \|\hat{f}\|_{L^{3/(2s')}} \|g\|_{L^1} \|h\|_{L^2} \lesssim \|f\|_{L^{3/(3-2s')}} \|g\|_{L^1} \|h\|_{L^2},$$

using Hölder, inequality and the fact that $(3+\gamma-s_0)\{3/(3-2s')\} > 3$. By means of Lemma 3.10, we have for some $\ell_0 > 0$,

$$\tilde{A}_{1,\lambda_1} \lesssim \big( \|\langle v \rangle^\gamma f\|_{L^{3/(3-2s)}} + \|f\|_{L^1_{\ell_0}} \big) \|g\|_{L^1} \|h\|_{L^2}$$

$$\lesssim \big( \|\langle v \rangle^{\gamma/2} \sqrt{f}\|^2_{H^s} + \|f\|_{L^1_{\ell_0}} \big) \|g\|_{L^1} \|h\|_{L^2}.$$

Putting $f = g = f(\tau, v)$ and $h = M^\delta_{\lambda_1(\tau)} f(\tau, v)$, we have a term coming from $\tilde{A}_{1,\lambda_1}$ in estimating

$$\int_0^t \big( M^\delta_{\lambda_1} Q(f(\tau), f(\tau)) - Q(f(\tau), M^\delta_{\lambda_1} f(\tau)), M^\delta_{\lambda_1} f(\tau) \big) \, d\tau$$

as follows:

$$\Big( \sup_{\tau \in [0,t]} \|f(\tau)\|_{L^1} \|M^\delta_{\lambda_1(\tau)} f(\tau)\|_{L^2} \Big) \int_0^t \|\langle v \rangle^{\gamma/2} \sqrt{f(\tau)}\|^2_{H^s} \, d\tau$$

$$+ \Big( \sup_{\tau \in [0,t]} \|f(\tau)\|_{L^1} \|f(\tau)\|_{L^1_{\ell_0}} \Big) \int_0^t \|M^\delta_{\lambda_1(\tau)} f(\tau)\|_{L^2} \, d\tau$$

$$\le \varepsilon' \sup_{\tau \in [0,t]} \|M^\delta_{\lambda_1(\tau)} f(\tau)\|^2_{L^2} + \frac{\|f_0\|^2_{L^1}}{4\varepsilon'} \Big( \int_0^T \|\langle v \rangle^{\gamma/2} \sqrt{f(\tau)}\|^2_{H^s} \, d\tau \Big)^2$$

$$+ \frac{\|f_0\|^2_{L^1}}{4} \sup_{\tau \in [0,t]} \|f(\tau)\|^2_{L^1_{\ell_0}} + T \int_0^t \|M^\delta_{\lambda_1(\tau)} f(\tau)\|^2_{L^2} \, d\tau$$

$$\le \varepsilon' \sup_{\tau \in [0,t]} \|M^\delta_{\lambda_1(\tau)} f(\tau)\|^2_{L^2} + C_{f,T,\varepsilon'} \Big( 1 + \int_0^t \|M^\delta_{\lambda_1(\tau)} f(\tau)\|^2_{L^2} \, d\tau \Big),$$

if $0 < t \le T$, because of (5.6) and (1.5). Using (3.15) we obtain, instead of (4.5),

$$\frac{1}{2} \|M^\delta_{\lambda_1(t)} f(t)\|^2_{L^2} - \varepsilon' \sup_{\tau \in [0,t]} \|M^\delta_{\lambda_1(\tau)} f(\tau)\|^2_{L^2}$$

$$\le \frac{1}{2} \|f(0)\|^2_{H^a} + \int_0^t \big( Q(f(\tau), M^\delta_{\lambda_1(\tau)} f(\tau)), M^\delta_{\lambda_1(\tau)} f(\tau) \big) \, d\tau$$

$$+ C_{f,T,\varepsilon'} \Big( 1 + \int_0^t \|M^\delta_{\lambda_1(\tau)} f(\tau)\|_{H^{s'}_{\gamma^+ + (2s-1)^+}} \|M^\delta_{\lambda_1(\tau)} f(\tau)\|_{H^{s'}} \, d\tau \Big)$$

$$+ CN \int_0^t \|(\log \langle D \rangle)^{1/2} M^\delta_{\lambda_1(\tau)} f(\tau)\|^2_{L^2} \, d\tau,$$

where $a = -3/2 - \varepsilon$. If we consider $\tau \in [0,t]$ instead of $t$, then the first term on the left-hand side can be replaced by $\sup_{\tau \in [0,t]} \|M^\delta_{\lambda_1(\tau)} f(\tau)\|^2_{L^2}$, which absorbs the second term on the left-hand side. Therefore, in the same way as in (4.6), we get

$$\|M^\delta_{\lambda(t)} f(t)\|^2_{L^2} + \int_0^t \|M^\delta_{\lambda_1(\tau)} f(\tau)\|^2_{H^s_{\gamma+/2}}\, d\tau$$
$$\lesssim \|f(0)\|^2_{H^a} + 1 + \int_0^t \|f(\tau)\|^2_{H^a_\ell}\, d\tau.$$

Letting $\delta \to 0$ we obtain, in view of $Nt = s_0$,

$$\|\langle D\rangle^{s_0-3/2-\varepsilon} f(t)\|_{L^2} < \infty \tag{5.7}$$

and

$$\int_0^t \|\langle D\rangle^{N\tau-3/2-\varepsilon} f(\tau)\|^2_{H^{s'}}\, d\tau < \infty. \tag{5.8}$$

*Second step.* Let $\kappa > 0$ be arbitrarily small. Considering $\tau \in [\kappa, t]$ instead of $t$ in (5.7), we may assume

$$\sup_{\tau\in[\kappa,t]} \|\langle D\rangle^{s_0-3/2-\varepsilon} f(\tau)\|_{L^2} < \infty.$$

For arbitrary $t > \kappa$ and $N > 0$ satisfying $N(t-\kappa) = s_0$ we set

$$\lambda_2(\tau) = s_0 + N(\tau-\kappa) - \frac{3}{2} - \varepsilon \quad \text{for } \tau \in [\kappa, t].$$

If we substitute $\lambda = \lambda_2(\tau)$ into (3.13), then we have

$$\begin{aligned}\tilde A_{1.\lambda_2}(\tau) &\lesssim \int_{\mathbb{R}^3} \frac{|\langle\xi\rangle^{s'}\hat h(\xi)|}{\langle\xi\rangle^{3/2+\varepsilon}} \\ &\quad\times \bigg(\int_{\mathbb{R}^3} \frac{\langle\xi_*\rangle^{s_0-3/2-\varepsilon}|\hat f(\xi_*)|\langle\xi_*-\xi\rangle^{N(\tau-\kappa)-3/2-\varepsilon+s'}|\hat g(\xi-\xi_*)|}{\langle\xi_*\rangle^{3+\gamma+2s'-3-2\varepsilon}} \\ &\quad\times \Big(\frac{\langle\xi\rangle}{\langle\xi_*\rangle}\Big)^{1-s'} d\xi_*\bigg)\, d\xi \\ &\lesssim \|f\|_{H^{s_0-3/2-\varepsilon}} \|\langle D\rangle^{s'+N(\tau-\kappa)-3/2-\varepsilon} g\|_{L^2} \|h\|_{H^{s'}},\end{aligned}$$

if $\gamma + 2s' > 2\varepsilon$. Putting $f = g = f(\tau, v)$ and $h = M^\delta_{\lambda_2(\tau)} f(\tau, v)$ we have a term coming from $\tilde A_{1,\lambda_2}$ in estimating

$$\Big|\int_\kappa^t \big(M^\delta_{\lambda_2} Q(f(\tau), f(\tau)) - Q(f(\tau), M^\delta_{\lambda_2} f(\tau)), M^\delta_{\lambda_2} f(\tau)\big)\, d\tau\Big|$$

as follows:

$$\Big(\sup_{\tau\in[\kappa,t]} \|f(\tau)\|_{H^{s_0-3/2-\varepsilon}}\Big)\Big\{\int_\kappa^t \|M^\delta_{\lambda_2(\tau)} f(\tau)\|^2_{H^{s'}}\, d\tau\Big\}$$
$$+ \int_\kappa^t \|\langle D\rangle^{s'+N(\tau-\kappa)-3/2-\varepsilon} f(\tau)\|^2_{L^2}\, d\tau\Big\}.$$

To avoid any confusion we write $N = N_2 = s_0/(t-\kappa)$ in this second step and $N = N_1 = s_0/t$ in (5.8). Then we have

$$N_2(\tau-\kappa) \le N_1\tau \quad \text{if } \tau \in [\kappa, t],$$

from which we can use (5.8) to estimate the term coming from $\tilde{A}_{1,\lambda_2}$. In this step we obtain, finally, in view of $N(t-\kappa) = s_0$,

$$\|\langle D\rangle^{2s_0-3/2-\varepsilon} f(t)\|_{L^2} < \infty$$

and

$$\int_\kappa^t \|\langle D\rangle^{s_0-3/2-\varepsilon} f(\tau)\|_{H^{s'}}^2 \, d\tau < \infty. \tag{5.9}$$

*Third step.* For $k \ge 2$, suppose that

$$\sup_{\tau\in[(k-1)\kappa,t]} \|\langle D\rangle^{(k-1)s_0-3/2-\varepsilon} f(\tau)\|_{L^2} < \infty.$$

For arbitrary $t > k\kappa$ and $N > 0$ satisfying $N(t-k\kappa) = s_0$ we set

$$\lambda_k(\tau) = (k-1)s_0 + N(\tau-\kappa) - \frac{3}{2} - \varepsilon \quad \text{for } \tau \in [\kappa, t].$$

Consider $M^\delta_{\lambda_k(\tau)}$. Then, using (5.9) instead of (5.8), we can proceed with the induction method in almost the same way as in the second step. Since $\kappa > 0$ is arbitrary, we obtain the desired conclusion.

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