Leading terms of Thom polynomials and J- images

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Abstract We give two types of singularities of maps between $4q$-manifolds whose Thom polynomials with integer coefficients have nonvanishing coefficient of Pontrjagin class P_q. We show that an element of the J-image of dimension $4q-1$ has a fold map between S^{4q-1} and can be detected by the leading terms of Thom polynomials of those singularities of an extended map between D^{4q} of the fold map.

1. Introduction

The calculation of Thom polynomials of smooth maps in the real category began in [24], and has been developed mainly with $\mathbb{Z}_2$-coefficients by many authors (see, e.g., [17], [20], [21], [2], [16], [6], [18]). However, there have been known only a small number of orientable real singularities of codimension $4q$ of smooth maps between equi-dimensional manifolds whose Thom polynomials with $\mathbb{Z}$-coefficients have the nonvanishing leading term, namely, the term of the qth Pontrjagin class. This is a very different situation from the complex case in the calculation of Thom polynomials. The examples, as far as the author knows, are the singularities of type Σ^2 of codimension 4 in [20] and the singularities, which have been studied in [6], of codimension 8. In this paper we present two types of real singularities with such a property under a certain restrictive assumption on maps and apply the result to show a relationship between those singularities and the J-images of the stable homotopy groups of spheres.

Let $\mathcal{K}^{(k)}$ denote the contact group defined in [10] on the jet space $J^k(n,n)$. For an integer n with $n \geqq 8$, we consider an unfolding $f_\eta : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ of a genotype $\eta = \langle \eta_1(u,v), \eta_2(u,v) \rangle$ and the $\mathcal{K}^{(k)}$-orbit $\mathcal{K}^{(k)}(j^k f_\eta)$, which we denote, for simplicity, by $\mathcal{K}\eta$ in this paper. We deal with the genotypes $\langle u^2+v^2, u^m \rangle$ (S1) and $\langle u^2+v^3, uv^{m-2} \rangle$ (S2) for $m \geqq 4$. Note that S1 is of type $\Sigma^{2,0}$, called IV_m by [12], and S2 is of type $\Sigma^{2,1}$. They are orientable if m is an even integer $2q$. If a smooth map $f : X \to Y$ between smooth manifolds of dimension n with $n \geqq 4q$ such that $j^{2q-1}f(X)$ does not intersect with $\mathrm{cl}(\mathcal{K}\eta) \backslash \mathcal{K}\eta$ and $j^{2q-1}f(X)$ is transverse to $\mathcal{K}\eta$, then $(j^{2q-1})f^{-1}(\mathcal{K}\eta)$ is a manifold and we can define its

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Thom polynomial as proved in Corollary 5.6, which we denote by $\mathrm{tp}(\mathcal{K}\eta; f)$. We calculate the leading term of the Thom polynomials for these genotypes.

THEOREM 1.1

Let m be an even integer $2q$ ($q \geqq 2$). Let X and Y be orientable smooth manifolds of dimension n with $n \geqq 4q$, and let $f : X \to Y$ be a smooth map such that $j^{2q-1}f(X)$ does not intersect with $\mathrm{cl}(\mathcal{K}\eta)\backslash\mathcal{K}\eta$ and that $j^{2q-1}f(X)$ is transverse to $\mathcal{K}\eta$. Then the leading term of the Thom polynomial $\mathrm{tp}(\mathcal{K}\eta; f)$ with $\mathbb{Z}$-coefficients is equal to, up to sign,

$$(1) \quad (2q-1)!P_q \quad \text{for } \langle u^2+v^2, u^{2q}\rangle,$$

$$(2) \quad \begin{cases} 3P_2 & \text{if } q=2, \\ 3\{\prod_{i=3}^{2q-2} i\}P_q & \text{if } q \geqq 3 \end{cases} \qquad \text{for } \langle u^2+v^3, uv^{2q-2}\rangle,$$

where P_i denotes the Pontrjagin class $P_i(f^*(TY) - TX)$. In particular, these terms depend only on the homotopy class of f.

In the process of the calculation using the Gysin homomorphisms, the structures of the normal bundles of Boardman–Thom manifolds in [5] and [9] and of the normal bundle of $\mathcal{K}\eta$ in [10] play important roles. Note that we do not assert the existence of the Thom polynomials in the sense of [6]. Although it is better for the complete forms of Thom polynomials to apply the method in [6], [18], and [19] using the Vassiliev complexes and the structure groups of normal bundles of $\mathcal{K}$-orbits, it is rather hard to adopt it in our case. Fehér and Rimányi [6] have proved that $\mathcal{K}\langle u^2+v^3, uv^2\rangle - 2\mathcal{K}\langle u^2+v^2, u^4\rangle$ constitutes a cycle in a Vassiliev complex and have determined its precise Thom polynomial. Its leading term coincides with our leading term $\pm 9P_2(f^*(TY) - TX)$ in Theorem 1.1.

We next explain that the above Thom polynomial of the singularities $\mathcal{K}\eta$ detects elements of the J-images of the stable homotopy groups of spheres. In [3] we have studied the group of oriented cobordism classes of fold maps to S^n of degree zero. Two fold maps $f_i : N_i \to S^n$ $(i = 0, 1)$ of degree zero are called cobordant if there exists a fold map, say, $\widetilde{f} : (W, \partial W) \to (S^n \times [0,1], S^n \times 0 \cup S^n \times 1)$ of degree zero, where $\widetilde{f}|N_0 \times 0 = f_0$ and $\widetilde{f}|N_1 \times 1 = f_1$ together with the usually required properties, where N_i and W are oriented.

Let $\Omega_{\mathrm{fold},0}(S^n)$ denote the group of all oriented cobordism classes of fold maps to S^n of degree zero. Let π_n^s denote the nth stable homotopy group of spheres. Then we have proved that there exists an isomorphism $\omega_0 : \Omega_{\mathrm{fold},0}(S^n) \to \pi_n^s$ for $n \geq 1$. Consequently, an element in the J-image has a fold map $f : N \to S^n$ of degree zero via ω_0 and its extension $E^f : (V, N) \to (D^{n+1}, S^n)$ of degree zero, where V is a parallelizable manifold with $\partial V = N$ and $E^f|N = f$. We will apply a method introduced in [4] to detect an element of the J-image by the algebraic numbers of above singularities of E^f of codimension $n+1 = 4q$ and will describe the details in dimensions $4q \geqq 8$.

In Section 2 we explain the notation currently used in this paper. In Section 3 we briefly review the fundamental properties of Boardman–Thom manifolds. In

Section 4 we briefly review the results concerning $\mathcal{K}$-orbits in [10] and give preliminary lemmas and properties of the singularities of $\mathcal{K}\eta$. In Section 5 we give a number of results concerning the normal bundles of $\mathcal{K}\eta$. In Section 6 we give a proof of Theorem 1.1 in a general form. In Section 7 we apply Theorem 1.1 to show a relationship between the singularities of $\mathcal{K}\eta$ and the J-images of the stable homotopy groups of spheres in Theorem 7.2.

2. Notation

Throughout the paper all manifolds are Hausdorff, paracompact, and smooth of class C^∞.

Let $\pi^E : E \to W$ and $\pi^F : F \to W$ be smooth n-vector bundles over a smooth manifold W. Let $\mathrm{Hom}(E,F)$ denote the smooth vector bundle over W with fiber $\mathrm{Hom}(E_x,F_x)$, $x \in W$, which consists of all homomorphisms $E_x \to F_x$.

We set

$$J^k(E,F) = \mathrm{Hom}\Big(\bigoplus_{i=1}^{k} S^i(E), F\Big) \tag{2.1}$$

over W with projections π^J onto W. Here, $S^i(E)$ denote the vector bundle $\bigcup_{x\in W} S^i(E_x)$ over W, where $S^i(E_x)$ denotes the i-fold symmetric product of E_x. An element z of $J^k(E,F)$ with $\pi^J(z) = x$ gives the homomorphisms $h_{i,z} : S^i(E_x) \to F_x$. Let $(\partial x_1, \partial x_2, \ldots, \partial x_n)$ or $(\partial y_1, \partial y_2, \ldots, \partial y_n)$ denote the basis of E_x or F_y, and let $(x_1, x_2, \ldots, x_n)$ or $(y, y_2, \ldots, y_n)$ denote the dual basis of E_x^* and F_x^*. Then $\{h_{i,z}\}$ yields a map germ $f : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$, where $y_i \circ f(x_1,x_2,\ldots,x_n)$ is a polynomial of degree k for $i = 1,\ldots,n$. We identify z with $j_0^k f$.

Let $J^k(X,Y)$ denote the k-jet space of n-manifolds X and Y. Let p_X and p_Y be the projections of $X \times Y$, onto X and Y, respectively. If we provide X and Y with Riemannian metrics, then the Levi–Civita connections induce the exponential maps $\exp_{X,x} : T_xX \to X$ and $\exp_{Y,y} : T_yY \to Y$. In dealing with exponential maps we always consider convex neighborhoods (see [8]). We define the smooth bundle map

$$J^k(X,Y) \to J^k\big(p_X^*(TX), p_Y^*(TY)\big) \quad \text{over } X \times Y \tag{2.2}$$

by sending $z = j_x^k f \in J_{x,y}^k(X,Y)$ to the k-jet of $(\exp_{P,y})^{-1} \circ f \circ \exp_{X,x}$ at $0 \in T_xX$, which is regarded as an element of $J^k(T_xX, T_yY)$ (i.e., $J_{x,y}^k(TX,TY)$). Let $L^k(n)$ denote the group of all k-jets of local diffeomorphisms of $(\mathbb{R}^n,0)$. Then the smooth equivalence of the fiber bundles under the structure group $L^k(n) \times L^k(n)$ in (2.2) gives a smooth reduction of the structure group $L^k(n) \times L^k(n)$ of $J^k(X,Y)$ to the structure group $O(n) \times O(n)$ of $J^k(p_X^*(TX), p_Y^*(TY))$. Therefore, we will work in the jet spaces of types in (2.1).

3. Boardman–Thom singularities

Let us recall the fundamental properties of the intrinsic derivatives on Boardman–Thom manifolds in $J^k(E,F)$ following [5] and [9]. Let $\mathbf{D}$ denote the total tangent bundle which is isomorphic to $(\pi^J)^*E$. There have been defined the first, second, and third intrinsic derivatives.

(1) Let $\mathbf{d}_1 : \mathbf{D} \longrightarrow (\pi^J)^*F$ denote the first intrinsic derivative defined over $J^k(E,F)$. Let $\mathbf{K}$ and $\mathbf{Q}$ denote the 2-dimensional kernel and cokernel bundle of $\mathbf{d}_1$ defined over $\Sigma^2(E,F)$, respectively.

(2) Let $\mathbf{d}_2 : \mathbf{K} \longrightarrow \mathrm{Hom}(\mathbf{K},\mathbf{Q})$ denote the second intrinsic derivative defined over $\Sigma^2(E,F)$. The manifold $\Sigma^{2,1}(E,F)$ consists of all jets $z \in \Sigma^2(E,F)$ with $\mathbf{d}_{2,z}$ being of rank 1. Let $\mathbf{K}_2$ denote the kernel bundle of $\mathbf{d}_2$ over $\Sigma^{2,1}(E,F)$. Let $\widetilde{\mathbf{d}}_2 : S^2\mathbf{K} \to \mathbf{Q}$ over $\Sigma^2(E,F)$ denote the bundle homomorphisms, which are canonically induced from $\mathbf{d}_2$. This implies that $\widetilde{\mathbf{d}}_2$ is a smooth section of $\mathrm{Hom}(S^2\mathbf{K},\mathbf{Q})$ over $\Sigma^2(E,F)$. Let $\mathbf{K}_2^\perp$ denote the orthogonal complement of $\mathbf{K}_2$ in $\mathbf{K}$ such that $\widetilde{\mathbf{d}}_2 : \mathbf{K}_2^\perp \bigcirc \mathbf{K}_2^\perp \to \mathbf{Q}$ is injective. Let $\mathbf{I}_2$ denote the trivial line subbundle as the image $\widetilde{\mathbf{d}}_2(\mathbf{K}_2^\perp \bigcirc \mathbf{K}_2^\perp)$.

(3) Let $\mathbf{d}_3 : \mathbf{K}_2 \longrightarrow \mathrm{Cok}(\mathbf{d}_2)$ denote the third intrinsic derivative defined over $\Sigma^{2,1}(E,F)$. The manifold $\Sigma^{2,1,0}(E,F)$ consists of all jets $z \in \Sigma^{2,1}(E,F)$ such that $\mathbf{d}_{3,z}$ is injective.

In the paper we usually abbreviate (E,F) as Σ^2, $\Sigma^{2,1}$, and $\Sigma^{2,1,0}$.

PROPOSITION 3.1

(1) *The normal bundle of* Σ^2 *in* $J^k(E,F)$ *is isomorphic to* $\mathrm{Hom}(\mathbf{K},\mathbf{Q})$.

(2) *The normal bundle of* $\Sigma^{2,1}$ *in* Σ^2 *is isomorphic to*

$$\mathrm{Hom}(\mathbf{K}_2 \bigcirc \mathbf{K}_2^\perp, \mathbf{Q}/\mathbf{I}_2) \oplus \mathrm{Hom}(\mathbf{K}_2 \bigcirc \mathbf{K}_2, \mathbf{Q})$$

restricted to $\Sigma^{2,1}$.

Proof

(1) This is well known.

(2) Since $\widetilde{\mathbf{d}}_2$ vanishes exactly on $\mathbf{K}_2 \bigcirc \mathbf{K}$, it is a monomorphism of $\mathbf{K}_2^\perp \bigcirc \mathbf{K}_2^\perp$ to $\mathbf{Q}$. Therefore, the cokernel of $\mathbf{d}_2$ is isomorphic to $\mathrm{Hom}(\mathbf{K}_2^\perp, \mathbf{Q}/\mathbf{I}_2) \oplus \mathrm{Hom}(\mathbf{K}_2, \mathbf{Q})$. By [5], the normal bundle of $\Sigma^{2,1}$ in Σ^2 is isomorphic to

$$\mathrm{Hom}\big(\mathbf{K}_2, \mathrm{Hom}(\mathbf{K}_2^\perp, \mathbf{Q}/\mathbf{I}_2) \oplus \mathrm{Hom}(\mathbf{K}_2, \mathbf{Q})\big).$$

This shows the assertion. □

4. Local properties of singularities

In this section we study the singularities of unfoldings of the genotypes introduced in the introduction. In this section let k denote $m-1$.

Let us recall the tangent bundle and the normal bundle of the $\mathcal{K}^{(k)}$-orbit of the k-jet $z = j_0^k f$ for a C^∞-map germ $f : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ in $J^k(n,n)$ described in [10, Proposition 7.4]. Let $\theta(f)$ denote the vector space of germs of vector fields along f. Let $\mathrm{id}_{\mathbb{R}^n}$ be the identity map germs of $(\mathbb{R}^n,0)$. Then we have the homomorphisms

$$tf : \theta(\mathrm{id}_{\mathbb{R}^n}) \longrightarrow \theta(f) \qquad \text{and} \qquad wf : \theta(\mathrm{id}_{\mathbb{R}^n}) \longrightarrow \theta(f)$$

defined by $tf(s) = df \circ s$ and $wf(s) = s \circ f$ for sections $s \in \theta(\mathrm{id}_{\mathbb{R}^n})$. It has been proved that there exists a canonical isomorphism of the tangent bundle of $J^k(n,n)$ at z with $\mathfrak{m}_x\theta(f)/\mathfrak{m}_x^{k+1}\theta(f)$. Then the tangent bundle and the normal bundle of $\mathcal{K}^{(k)}z$ are expressed as

$$(4.1)\quad \begin{aligned} &T_z(\mathcal{K}^{(k)}(z)) = \{tf(\mathfrak{m}_x\theta(\mathrm{id}_{\mathbb{R}^n})) + f^*(\mathfrak{m}_y)\theta(f)\}/\mathfrak{m}_x^{k+1}\theta(f), \\ &\nu_z(\mathcal{K}^{(k)}(z)) = \mathfrak{m}_x\theta(f)/\big(tf(\mathfrak{m}_x\theta(\mathrm{id}_{\mathbb{R}^n})) + f^*(\mathfrak{m}_y)\theta(f) + \mathfrak{m}_x^{k+1}\theta(f)\big), \end{aligned}$$

respectively. Here, $\mathfrak{m}_x$ and $\mathfrak{m}_y$ denote the maximal ideals of C^∞-map germs on $(\mathbb{R}^n, 0)$ under coordinates $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$, respectively.

Let $\eta = \langle \eta_1, \eta_2\rangle$ denote a C^∞-map germ $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ with rank zero at the origin. An unfolding $f : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ of the genotype η implies a map germ $(u,v,t_1,\dots,t_{n-2}) \mapsto (f_1,\dots,f_n)$, where

$$(4.2)\quad \begin{aligned} &f_1 = \eta_1(u,v) + g_1(u,v,t_1,\dots,t_{n-2}), \\ &f_2 = \eta_2(u,v) + g_2(u,v,t_1,\dots,t_{n-2}), \\ &f_j = t_{j-2} \quad \text{for } 3 \leqq j \leqq n, \end{aligned}$$

such that $g_1(u,v,0,\dots,0) = g_2(u,v,0,\dots,0) = 0$.

The following lemma is an elementary consequence.

LEMMA 4.1

The tangent bundle and the normal bundle of $\mathcal{K}^{(k)}(j_0^k f)$ are isomorphic to those of $\mathcal{K}^{(k)}(j_0^k\eta)$ under the canonical isomorphism

$$\mathfrak{m}_x\theta(f)/\mathfrak{m}_x^{k+1}\theta(f) \approx \mathfrak{m}_{u,v}\theta(\eta)/\mathfrak{m}_{u,v}^{k+1}\theta(\eta).$$

Since the orbit $\mathcal{K}^{(k)}(j_0^k f)$ is determined by the genotype η, we denote the orbit, simply, by $\mathcal{K}_0\eta$ in what follows.

In this paper $\partial/\partial x_i, \partial/\partial y_j, \partial/\partial u$, and $\partial/\partial v$ are denoted by $\partial x_i, \partial y_j, \partial u$, and ∂v for simplicity. Let $(\partial u, \partial v)$ or $(\partial y_1, \partial y_2)$ be a basis of the source $\mathbb{R}^2$ or the target $\mathbb{R}^2$, respectively. The following proposition is a consequence of a direct calculation and is useful to study the normal bundle of $\mathcal{K}_0\eta$.

PROPOSITION 4.2

Let $m \geqq 4$. In the respective cases S1 and S2, we have the following.

(1) *The tangent space $T(\mathcal{K}_0\eta)$ is, respectively, generated by*

(S1) *$u^2\partial y_1$, $uv\partial y_1$, $v^2\partial y_1$, $uv^{m-1}\partial y_2$, and $(u^2+v^2)\partial/\partial y_i$, $v^m\partial/\partial y_i$ for $i = 1,2$ over $\mathfrak{m}_{u,v}$,*

(S2) *$2u^2\partial y_1 + uv^{m-2}\partial y_2$, $2uv\partial y_1 + v^{m-1}\partial y_2$, $3uv^2\partial y_1 + (m-2)u^2v^{m-3}\partial y_2$, $3v^3\partial y_1 + (m-2)uv^{m-2}\partial y_2$, and $(u^2+v^3)\partial y_i$, $uv^{m-2}\partial y_i$ for $i = 1,2$ over $\mathfrak{m}_{u,v}$.*

(2) *The normal space $\nu(\mathcal{K}_0\eta)$ is, respectively, generated by the vectors*

(S1) *$u\partial y_i$, $v\partial y_i$ for $i = 1,2$, and $u^j\partial y_2$, $u^{j-1}v\partial y_2$, where j varies over 2 to $m-1$,* (S2) *$u\partial y_i$, $v\partial y_i$, for $i=1,2$, and $uv\partial y_1$, $v^2\partial y_1$, $uv^{j-1}\partial y_2$, $v^j\partial/\partial y_2$, where j varies over 2 to $m-2$.*

We have the following lemma.

LEMMA 4.3
The orbit $\mathcal{K}_0\eta$ is a submanifold of codimension $2m$.

REMARK 4.4
In Proposition 3.1(2), u, v, uv, ∂y_1, and ∂y_2 correspond to $(\mathbf{K}_2^\perp)_z^*$, $(\mathbf{K}_2^*)_z$, $(\mathbf{K}_2^\perp)_z^* \bigcirc (\mathbf{K}_2^*)_z$, $(\mathbf{I}_2)_z$, and $(\mathbf{Q}/\mathbf{I}_2)_z$, respectively.

LEMMA 4.5
The topological closure of $\mathcal{K}_0\eta$ is an algebraic set of $J^k(n,n)$.

Proof
By [11, Proposition 9.1], it is enough to prove the assertion in the case $n=2$. By [4] and [13], the topological closures of $\Sigma^{2,0}$ and $\Sigma^{2,1}$ are algebraic sets. A jet of a germ $(y_1\circ f, y_2\circ f)$ of $\Sigma^{2,0}$ lies in the topological closure of $\mathcal{K}_0\eta$ if and only if $y_1\circ f$ and $y_2\circ f$ vanish modulo $(u^2+v^2)+\mathfrak{m}_{u,v}^m$ by the arguments in the classification of simple singularities of type $\Sigma^{2,0}$ in [12]. If a jet of a germ $(y_1\circ f, y_2\circ f)$ lies in $\Sigma^{2,1}$, then the functions $\partial u(y_i\circ f), \partial v(y_i\circ f)$ for $i=1,2$ constitute a one-dimensional subspace of $\mathfrak{m}_{u,v}/\mathfrak{m}_{u,v}^2$. Let $w(u,v)$ denote such a nonsingular function in them. Then a jet of a germ $(y_1\circ f, y_2\circ f)$ of $\Sigma^{2,1}$ lies in the topological closure of $\mathcal{K}_0\eta$ if and only if $y_1\circ f$ and $y_2\circ f$ vanish modulo $(w^2)+\mathfrak{m}_{u,v}^{m-1}$ by the arguments in the classification of simple singularities of type $\Sigma^{2,1}$ in [12]. This shows the assertion. □

Let V be a 2-dimensional vector space with basis ∂u and ∂v, and let V^* be its dual space with basis u and v. Then S^iV^* is identified with the space of homogeneous polynomials of degree i with variables u and v. Since the element $(\partial u)^2+(\partial v)^2$ in S^2V is invariant with respect to the action of $O(2)$, it yields the 1-dimensional subspace L_V of S^2V. Hence, the subspaces $L_V \bigcirc S^{i-2}V$ in S^iV for $i \geqq 2$ yield the subspace $\Sigma_{i=2}^{t+1}(L_V \bigcirc S^{i-2}V)$ in $\Sigma_{i=2}^{t+1}S^iV$ of codimension $2t$.

REMARK 4.6
The quotient $S^iV/(L_V \bigcirc S^{i-2}V)$ has a basis $(\partial v)^i$ and $\partial u(\partial v)^{i-1}$. Let $z=\partial u+\sqrt{-1}\partial v$, and let $\mathcal{R}(z^i)$ and $\mathcal{I}(z^i)$ denote the real and imaginary part of z^i, respectively. Then $\mathcal{R}(z^i)$ and $\mathcal{I}(z^i)$ constitute a better basis. Indeed, for any homogeneous polynomial $g(u,v)$ of degree $i-2$, we have

$$\big(\mathcal{R}(z^i)+\sqrt{-1}\mathcal{I}(z^i)\big)(u^2+v^2)g(u,v)=0,$$

and so, $\mathcal{R}(z^i)$ and $\mathcal{I}(z^i)$ annihilate $(u^2+v^2)g(u,v)$.

We define $\mathcal{K}\eta(E_x, F_x)$ at $x \in X$ corresponding to $\mathcal{K}_0\eta$ in $J^k(n,n)$ applying the above argument similarly as in $J^k(E_x, F_x)$. Let $\mathcal{K}\eta(E,F)$ denote the subbundle of $J^k(E,F)$ over X with fiber $\mathcal{K}\eta(E_x, F_x)$. Let $T(\mathcal{K}\eta(E,F))$ and $\nu(\mathcal{K}\eta(E,F))$ denote the tangent bundle and the normal bundle of $\mathcal{K}\eta(E,F)$ in $J^k(E,F)$, respectively. If there is no confusion, then (E_x, F_x) and (E,F) may be abbreviated as $\mathcal{K}_x\eta$ and $\mathcal{K}\eta$.

We next determine the structure of the normal bundle of $\mathcal{K}\eta$ in $J^k(E,F)$ by using Propositions 3.1 and 4.2. Since $\mathcal{K}\eta$ lies in the Boardman–Thom manifold Σ^2 of codimension 4, it is enough for this purpose to determine the structure of the normal bundle of $\mathcal{K}\eta$ in Σ^2.

Let $\mathbf{L}$ denote the trivial line bundle in $S^2\mathbf{K}$, which is associated to the subspace $L_{\mathbf{K}_z}$ of $S^2\mathbf{K}_z$. Let $\mathfrak{K}$, $\mathfrak{L}$, $\mathfrak{Q}$, and $\mathfrak{K}_2^{\perp}$ in the case (S2) denote the restriction of $\mathbf{K}$, $\mathbf{L}$, $\mathbf{Q}$, and $\mathbf{K}_2^{\perp}$ in the case (S2) to $\mathcal{K}\eta$. For a jet $z \in \mathcal{K}\eta$, let q_z denote the oriented line of $\mathbf{Q}_z$ with the orthogonal projection $p(q_z): \mathbf{Q}_z \to q_z$.

We define two line bundles $\mathfrak{q}_i$ and their orthogonal complements $\mathfrak{q}_i^{\perp}$ for $i = 1,2$ in $\mathfrak{Q}$ over $\mathcal{K}\eta$. Namely, $\mathfrak{q}_{1,z}^{\perp}$ is generated by the image $\widetilde{\mathbf{d}}_{2,z}((\partial u)^2 + (\partial v)^2)$ in the case (S1), and $\mathfrak{q}_{2,z}^{\perp}$ is generated by the image $\widetilde{\mathbf{d}}_{2,z}((\partial u)^2)$ in the case (S2). We note that $\mathfrak{q}_i^{\perp}$ are trivial and $W_1(\mathfrak{q}_i) = W_1(\mathfrak{Q})$ over $\mathcal{K}\eta$.

Let $\nu(\mathcal{K}\eta)$ denote the following bundle over $\mathcal{K}\eta$ in the respective cases:

(S1) $\mathrm{Hom}\big(\bigoplus_{i=2}^{m-1} S^i\mathfrak{K}/(\mathfrak{L} \bigcirc S^{i-2}\mathfrak{K}), \mathfrak{q}_1\big)$,
(S2) $\mathrm{Hom}(S^2\mathfrak{K}/\mathfrak{K}_2^{\perp}, \mathfrak{Q}) \oplus \mathrm{Hom}\big(\big\{\bigoplus_{i=3}^{m-2} S^i\mathfrak{K}/(\mathfrak{K}_2^{\perp} \bigcirc S^{i-2}\mathfrak{K})\big\}, \mathfrak{q}_2\big)$.

The next proposition follows from Proposition 4.2.

PROPOSITION 4.7
We have the following:

(1) *the normal bundle of* $\mathcal{K}\eta$ *in* $J^k(E,F)$ *is isomorphic to* $\mathrm{Hom}(\mathfrak{K}, \mathfrak{Q}) \oplus \nu(\mathcal{K}\eta)$,

(2) *the normal bundle of* $\mathcal{K}\eta$ *is orientable if and only if* m *is even, respectively.*

Proof

(1) The assertion follows from Propositions 3.1 and 4.2.

(2) The first Stiefel–Whitney classes of $\mathrm{Hom}(\mathfrak{K}, \mathfrak{Q})$ and $\mathrm{Hom}(S^2\mathfrak{K}/\mathfrak{K}_2^{\perp}, \mathfrak{Q})$ are all equal to zero. Let $W(\mathfrak{K}) = (1+t_1)(1+t_2)$ and $W(\mathfrak{Q}) = (1+r_1)(1+r_2)$. Then we have $W_1(\mathfrak{K}) = W_1(\mathfrak{Q}) = t_1 + t_2$ and $W_1(S^i\mathfrak{K}) = (i(i+1)/2)W_1(\mathfrak{K})$. Since $\mathfrak{L}$ and $\mathfrak{K}_2^{\perp}$ are isomorphic to the trivial bundle ε, we have

$$W_1\big(S^i\mathfrak{K}/(\varepsilon \bigcirc S^{i-2}\mathfrak{K})\big) = W_1(S^i\mathfrak{K}) - W_1(S^{i-2}\mathfrak{K}) = W_1(\mathfrak{K}).$$

These identities show the assertions. □

5. Global properties of singularities

In this section we study the global structure of the normal bundle of $\mathcal{K}\eta$, which is necessary for the calculation of its Thom polynomial. In this section let k denote $m-1$.

Let X be orientable. Let $J^k(E,F)^\times = J^k(E,F)\backslash(\mathrm{cl}(\mathcal{K}\eta)\setminus\mathcal{K}\eta)$ with the projection π^J onto X. Let $G(E)$ denote the Grassmann bundle $G_{2,2w-2}((\pi^J)^*E)$ with canonical projection $\mathrm{pr}_E : G(E)\to X$. Let $G(E,F)$ denote the Grassmann bundle $G_{2,2w-2}((\mathrm{pr}_E)^*F)$ with projection $\mathrm{pr}_G : G(E,F)\to X$. Let K_G denote the canonical 2-plane bundle over $G(E,F)$, and let Q_G denote the canonical 2-plane bundle over $G(E,F)$ associated to $\mathrm{pr}_E^*(F)$. We always provide E, F, K_G, and Q_G with the structure groups $O(n)$ and $O(2)$, respectively. Let L_G denote the trivial line subbundle of S^2K_G. An element of $G(E,F)$ is expressed by (z,α,β), where $z\in J(E,F)^\times$ with $\pi^J(z)=x$, $\alpha\in G_{2,n-2}(E_x)$, $\beta\in G_{2,n-2}(F_x)$. Here, α and β are often written as K_z and Q_z, respectively. Let $\pi_G : G(E,F)\to J(E,F)^\times$ denote the map defined by $\pi_G(z,\alpha,\beta)=z$. Let s be a section of $J(E,F)^\times$ over X, which is transverse to $\mathcal{K}\eta$, and let $s_G : s^*G(E,F)\to G(E,F)$ denote the canonical bundle map covering s. Then we have the diagram with the given canonical maps:

$$
\begin{array}{ccc}
\big(S\eta_G \subset s^*G(E,F)\big) & \xrightarrow{\ s_G\ } & \big(\mathcal{K}\eta_G \subset G(E,F)\big) \\
\downarrow \qquad\quad \downarrow & & \downarrow \qquad \downarrow \pi_G \\
 & & \big(\mathcal{K}\eta \subset J^k(E,F)^\times\big) \\
 & \overset{s}{\nearrow} & \downarrow \\
(S\eta \quad \subset \quad X) & & X
\end{array}
\tag{5.1}
$$

The following notation is used at the end of this section. Let $S\eta$ denote the space $s^{-1}(\mathcal{K}\eta)$. The space S^2_G denotes the space that consists of all quadruples $(x,s(x),\alpha,\beta)$ with $s(x)\in\mathrm{cl}(\Sigma^2)$ such that $\alpha\subset\mathrm{Ker}(\mathbf{d}_{1,s(x)})$ and $\beta\perp\mathrm{Im}(\mathbf{d}_{1,s(x)})$. The space $S\eta_G$ denotes the subspace of S^2_G with $s(x)\in\mathcal{K}\eta$. Obviously, $S\eta_G$ is mapped onto $S\eta$ diffeomorphically by the canonical projection.

Let Σ^2_G or $\mathcal{K}\eta_G$ denote the space that consists of all triples $\tilde z=(z,K_z,Q_z)\in G(E,F)$ such that K_z and Q_z are 2-planes and that z lies in $\mathrm{cl}(\Sigma^2)$ or $\mathcal{K}\eta$, respectively. For an element $\tilde z\in\mathcal{K}\eta$, K_z, and Q_z are uniquely determined. Any jet $\tilde z$ in $G(E,F)$ induces an element of $J^k(K_z,Q_z)=\mathrm{Hom}\big(\bigoplus_{i=1}^k S^iK_z,Q_z\big)$ and a polynomial map $\zeta:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ of degree k. We use this notation ζ for $\tilde z$ without any comment.

Let $d_1|K_G : K_G\to\mathrm{pr}_G^*(F)$ denote the homomorphism induced from $\mathbf{d}_1$. Let $\mathrm{pr}(Q_G)$ denote the canonical projection of $\mathrm{pr}_G^*(F)$ onto Q_G. Then we have the bundles $\mathrm{Hom}(K_G,\mathrm{pr}_G^*(F))\oplus\mathrm{Hom}(K_G^\perp,Q)$ over $G(E,F)$ with a section σ_G defined by

$$\sigma_G(z)=d_{1,z}|(K_G)_z\oplus\mathrm{pr}(Q_G)\circ d_{1,z}|(K_G^\perp)_z.$$

The next proposition follows from [17] (see also [2]).

PROPOSITION 5.1

(i) *The section σ_G is transverse to the zero section, and its inverse images of the zero sections by σ_G coincide with Σ^2_G. Consequently, Σ^2_G is a submanifold.*

(ii) *The projection π_G maps Σ^2_G onto* $\mathrm{cl}(\Sigma^2)$ *so that $(\pi_G|\Sigma^2_G)^{-1}(\Sigma^2)$ is mapped onto Σ^2 diffeomorphically.*

We take a very small tubular neighborhood $U(\mathcal{K}\eta_G)$ of $\mathcal{K}\eta_G$ in the cases (S1) and (S2). Let $d_{2,\tilde{z}} : S^2K_z \to Q_z$ denote the composite of the homomorphism $h_{2,z}|S^2K_z$ and $\mathrm{pr}(Q_z)$. Note that $d_{2,\tilde{z}}$ coincides with the homomorphism induced from $\mathbf{d}_{2,z}$ at least for $z \in \Sigma^2$. We define two line bundles $\mathbf{q}_i$ and their orthogonal complements $\mathbf{q}_i^\perp$ for $i = 1, 2$ in Q_G. If $d_{2,\tilde{z}}(L_z)$ does not vanish, then $\mathbf{q}^\perp_{1,\tilde{z}}$ is defined to be the image $d_{2,\tilde{z}}(L_z)$ and $\mathbf{q}_{1,\tilde{z}}$ is its orthogonal line in Q_z, where L_z is generated by $\partial u^2 + \partial v^2$.

Let $\Sigma^{2,1}_G$ denote $(\pi_G|\Sigma^2_G)^{-1}(\Sigma^{2,1})$. We define $K_2^\perp$ to be a line subbundle of K_G over $\Sigma^{2,1}_G$ induced from $\mathbf{K}_2^\perp$. Let ∂u denote a unit basis of $K^\perp_{2,\tilde{z}}$, and let ∂v denote a basis of $K_{2,\tilde{z}}$. Then $(\partial u, \partial v)$ is an orthogonal basis of $K_{G,\tilde{z}}$. We take a small tubular neighborhood $U(\Sigma^{2,1}_G)$ of $\Sigma^{2,1}_G$ with radius ϵ within a tubular neighborhood with radius 2ϵ in Σ^2_G, where ϵ is a positive function on $\Sigma^{2,1}_G$. Let $U(\Sigma^{2,1}_G)$ contain $U(\mathcal{K}\eta_G)$ in the case (S2). We can extend $K_2^\perp$ to a trivial bundle over the tubular neighborhood denoted by the same symbol $K_2^\perp$. Let $d(\widetilde{z}, \Sigma^{2,1}_G)$ denote the distance of z and $\Sigma^{2,1}$, and let $w(t)$ be a smooth-increasing function such that $w(t) = 0$ for $t \leqq \epsilon$ and $w(t) = \epsilon$ for $t \geqq 2\epsilon$. We define a trivial line subbundle Θ_G of S^2K_G over Σ^2_G so that Θ_G coincides with $S^2K_2^\perp$ on $\Sigma^{2,1}_G$ and that $(\Theta_G)_{\tilde{z}}$ is generated by a vector

$$\big(1 - (1/\epsilon)w(d(\tilde{z}, \Sigma^{2,1}_G))\big)\partial u^2 + (1/\epsilon)w\big(d(\tilde{z}, \Sigma^{2,1}_G)\big)(\partial u^2 + \partial v^2).$$

If $d_{2,\tilde{z}}((\Theta_G)_{\tilde{z}})$ does not vanish, then $\mathbf{q}^\perp_{2,\tilde{z}}$ is defined to be the image $d_{2,\tilde{z}}((\Theta_G)_{\tilde{z}})$, and $\mathbf{q}_{2,\tilde{z}}$ is its orthogonal line in Q_z.

REMARK 5.2

In the case (S2) we choose a basis of $S^iK_G/(\Theta_G \bigcirc S^{i-2}K_G)$ denoted by R^i_Θ and I^i_Θ, which are equal to $\partial u(\partial v)^{i-1}$ and $(\partial v)^i$ over $U(\Sigma^{2,1}_G)$.

Let (u, v) and (y_1, y_2) denote orthogonal coordinates determined as above. We define a section r of

$$\mathrm{Hom}\big(S^iK_G/(L_G \bigcirc S^{i-2}K_G), Q_G\big) \qquad \text{and}$$
$$\mathrm{Hom}\big(S^iK_G/(\Theta_G \bigcirc S^{i-2}K_G), Q_G\big)$$

by

$$(5.2) \quad \begin{aligned} &(S1)\ \ r^i(z) = \begin{bmatrix} \mathcal{R}(\partial u + \sqrt{-1}\partial v)^i(y_1 \circ \zeta)|_{\mathbf{0}}, & \mathcal{I}(\partial u + \sqrt{-1}\partial v)^i(y_1 \circ \zeta)|_{\mathbf{0}} \\ \mathcal{R}(\partial u + \sqrt{-1}\partial v)^i(y_2 \circ \zeta)|_{\mathbf{0}}, & \mathcal{I}(\partial u + \sqrt{-1}\partial v)^i(y_2 \circ \zeta)|_{\mathbf{0}} \end{bmatrix}, \\ &(S2)\ \ r^i_\Theta(z) = \begin{bmatrix} R^i_\Theta(y_1 \circ \zeta)|_{\mathbf{0}}, & I^i_\Theta(y_1 \circ \zeta)|_{\mathbf{0}} \\ R^i_\Theta(y_2 \circ \zeta)|_{\mathbf{0}}, & I^i_\Theta(y_2 \circ \zeta)|_{\mathbf{0}} \end{bmatrix}. \end{aligned}$$

We show how $r^i(z)$ changes by the coordinate changes. We express them as $z=u+\sqrt{-1}v$, $z'=u'+\sqrt{-1}v'$ with $z=e^{\sqrt{-1}\theta}z'$, and $(y_1,y_2)=A(y_1',y_2')$, where A is an orthogonal 2-matrix. Let $T(\theta)$ denote the counterclockwise rotation by the angle θ. Then the following lemma is easy to prove.

LEMMA 5.3

(i) *If* $z=e^{\sqrt{-1}\theta}z'$, *then* $r^i(z)=Ar^i(z')^tT(\theta)$.
(ii) *If* $u=u'$ *and* $v=-v$, *then* $r^i(z)=Ar^i(z')\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$.

Let $o(Q_G)$ denote the line bundle determined by the first Stiefel–Whitney class $W_1(Q_G)$, which is isomorphic to the wedge product $Q_G\wedge Q_G$. Let ε^1_G denote the trivial line bundle over $G(E,F)$.

Since $\pi_3(S^1)=\{0\}$, the following lemma is easy to prove.

LEMMA 5.4

If Q_G *is the Whitney sum* $\mathbf{q}^\perp\oplus\mathbf{q}$ *over a subcomplex* W *of* Σ^2_G *with* $W_1(\mathbf{q})=W_1(Q_G|_W)$, *then there exists a fiberwise map*

$$\mu:\mathrm{Hom}\big(S^iK_G/(\varepsilon^1_G\bigcirc S^{i-2}K_G),Q_G\big)\to\mathrm{Hom}\big(S^iK_G/(\varepsilon^1_G\bigcirc S^{i-2}K_G),o(Q_G)\big)$$

over Σ^2_G *such that if* $\{0\}$ *lies in the image of* μ *on a point of* Σ^2_G, *then* $\mu^{-1}\{0\}=\{0\}$ *there and that* $\mu|\,\mathrm{Hom}(S^iK_G/(\varepsilon^1_G\bigcirc S^{i-2}K_G),\mathbf{q})|_W$ *is the identity on* W.

Proof

The restriction of the identity of $\mathrm{Hom}(S^iK_G/(\varepsilon^1_G\bigcirc S^{i-2}K_G),Q_G)$ to

$$\mathrm{Hom}\big(S^iK_G/(\varepsilon^1_G\bigcirc S^{i-2}K_G),Q_G\big)\backslash\{\text{zero section}\}$$

over W yields a fiberwise map to

$$\mathrm{Hom}\big(S^iK_G/(\varepsilon^1_G\bigcirc S^{i-2}K_G),\mathbf{q}\big)\backslash\{\text{zero section}\}$$

by using $\pi_3(S^1)=\{0\}$ so that $\mu|\,\mathrm{Hom}(S^iK_G/(\varepsilon^1_G\bigcirc S^{i-2}K_G),\mathbf{q})|_W$ is the identity on W. Then extend this fiberwise map to

$$\mathrm{Hom}\big(S^iK_G/(\varepsilon^1_G\bigcirc S^{i-2}K_G),o(Q_G)\big)\backslash\{\text{zero-section}\}.$$

Then construct the required map μ by extending this map by the conewise construction. □

Let $\mathcal{N}(\eta)_G$ denote the following vector bundles:

(S1) $\mathrm{Hom}\big(\bigoplus_{i=2}^{m-1}(S^iK_G/(L_G\otimes S^{i-2}K_G),o(Q_G))\big)$,
(S2) $\mathrm{Hom}(S^2K_G/\Theta_G,Q_G)\ \oplus\ \mathrm{Hom}\big(\{\big(\bigoplus_{i=3}^{m-2}S^iK_G/(\Theta_G\ \bigcirc\ S^{i-2}K_G)\big)\},o(Q_G)\big)$

over $G(E,F)$ in the cases (S1) and (S2), respectively. Let $\mathbf{n}(\eta)_{\Sigma^2}$ denote their restriction to Σ^2_G, respectively.

In the following proposition we apply the fiberwise map μ with $W=U(\mathcal{K}\eta_G)$ in the case (S1) and with $W=U(\Sigma^{2,1}_G)$ in the case (S2) together with r^i and r^i_Θ.

PROPOSITION 5.5

In the case (S1) or (S2), we have the following.

(i) *The normal bundle of* $\mathcal{K}\eta_G$ *in* Σ^2_G *is induced from* $\mathbf{n}(\eta)_{\Sigma^2}$ *by the inclusion* $\mathcal{K}\eta_G$ *in* Σ^2_G.

(ii) *There exists a section* ψ *of* $\mathbf{n}(\eta)_{\Sigma^2}$ *over* Σ^2_G, *which is transverse to the zero-section on* $\mathcal{K}\eta_G$, *whose inverse image of the zero section coincides with* $\mathcal{K}\eta_G$.

Proof

For an element $\tilde{z} = (z, K_z, Q_z)$ of Σ^2_G with $z \in \mathrm{cl}(\Sigma^2)$, we take local orthogonal coordinate systems (u, v) and (y_1, y_2) for K_z and Q_z with the associated polynomial map $\zeta : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$.

(1) We have defined the lines $\mathbf{q}_1^\perp$ and $\mathbf{q}_1$ such that $Q_G \approx \mathbf{q}_1^\perp \oplus \mathbf{q}_1$ over a very small tubular neighborhood $U(\mathcal{K}\eta_G)$ of $\mathcal{K}\eta_G$. Let (y_1, y_2) be the coordinates associated to $(\mathbf{q}_1^\perp, \mathbf{q}_1)$. Then it follows from (5.2) and Lemma 5.4 that we have a section

$$\mu \circ r^i : U(\mathcal{K}\eta_G) \to \mathrm{Hom}\big(S^i K_G/(L_G \otimes S^{i-2} K_G), \mathbf{q}_1\big)$$

defined over $U(\mathcal{K}\eta_G)$. We set the section ψ_U on $U(\mathcal{K}\eta_G)$ as

$$\psi_U(\tilde{z}) = \Big(\bigoplus_{i=2}^{m-1} \mu \circ r^i(\tilde{z})\Big).$$

By definition, $\overline{\psi}_U$ is transverse to the zero section on $\mathcal{K}\eta$. Furthermore, ψ_U vanishes on $\mathcal{K}\eta_G$ and never vanishes on $U(\mathcal{K}\eta_G)\backslash\mathcal{K}\eta_G$. Suppose that $\psi_U(\tilde{z})$ vanishes. Then there exists a nonzero real number c such that

$$y_1 \circ \zeta(u, v) = (u^2 + v^2)\big(c + g_1(u, v)\big),$$
$$y_2 \circ \zeta(u, v) = (u^2 + v^2)\big(g_2(u, v)\big),$$

modulo $\mathfrak{m}^m_{u,v}$, where $\deg g_i$ is greater than zero. By the Morse theorem we may assume under a suitable choice of coordinates (u, v) that $y_1 \circ \zeta(u, v) = c(u^2 + v^2)$. Hence, we may assume under a suitable choice of coordinates (y_1, y_2) that $y_2 \circ \zeta(u, v) = 0$ modulo $\mathfrak{m}^m_{u,v}$. This implies that $\tilde{z}$ lies in $\mathcal{K}\eta_G$.

We next extend ψ_U to a section of

$$\mathrm{Hom}\Big(\bigoplus_{i=2}^{m-1}(S^i K_G/(L_G \otimes S^{i-2} K_G), o(Q_G)\Big)\backslash(\text{zero section})$$

over Σ^2_G. By using $\mu \circ r^i$ in Lemma 5.4, we can extend the section r^i on $\partial U(\mathcal{K}\eta_G)$ to a section

$$\psi^i : \Sigma^2_G \backslash \mathrm{Int}\, U\big(\mathcal{K}\eta_G\big) \to \mathrm{Hom}\big(S^i K_G/(L_G \bigcirc S^{i-2} K_G), Q_G\big)$$

over $\Sigma^2_G \backslash \mathrm{Int}\, U(\mathcal{K}\eta_G)$ such that $\psi^i(\tilde{z})$ vanishes if and only if $\mu \circ \psi^i(\tilde{z})$ vanishes. Now we define the section ψ' over $\Sigma^2_G \backslash \mathrm{Int}\, U(\mathcal{K}\eta_G)$ by

$$\psi'(\tilde{z}) = \Big(\bigoplus_{i=2}^{m-1} \mu \circ \psi^i(\tilde{z})\Big).$$

We have to show that ψ' never vanish on $\Sigma^2_G \backslash \operatorname{Int} U(\mathcal{K}\eta_G)$. Suppose that $\psi'(\tilde{z})$ vanishes. This implies that $y_i \circ \zeta(u,v)$ lies in the ideal (u^2+v^2) modulo $\mathfrak{m}^m_{u,v}$ for $i=1,2$. First let z lie in Σ^2. If one of $y_i \circ \zeta(u,v)$ is equal to $c(u^2+v^2)$ modulo $\mathfrak{m}^3_{u,v}$ with $c \neq 0$, then we may again suppose that $y_i \circ \zeta(u,v) = c(u^2+v^2)$, and hence, z lies in $\mathcal{K}\eta_G$. This is impossible. Hence, $c=0$ and z lies in $\Sigma^{2,2}$. Since the normal bundle $\operatorname{Hom}(S^2\mathbf{K}, \mathbf{Q})$ of $\Sigma^{2,2}$ cannot be a subbundle of $\mathcal{N}(\eta)_G$ by considering the structure group of $\mathcal{N}(\eta)_G$, this is also impossible. If z lies in the closure of Σ^3, then the normal bundle of Σ^i for $i>2$ cannot be a subbundle of $\mathcal{N}(\eta)_G$ by the same reason. Therefore, $\psi'(\tilde{z})$ never vanish.

By the definition of ψ_U and ψ', they coincide on $\partial U(\mathcal{K}\eta_G)$ with each other. Thus we have obtained the required section ψ defined on Σ^2_G such that it vanishes only on $\mathcal{K}\eta_G$ and is transverse to the zero section on $\mathcal{K}\eta_G$.

(2) In the case (S2), we have defined the lines $\mathbf{q}_2^{\perp}$ and $\mathbf{q}_2$ such that $Q_G \approx \mathbf{q}_2^{\perp} \oplus \mathbf{q}_2$ over a very small tubular neighborhood $U(\Sigma^{2,1}_G)$ of $\Sigma^{2,1}_G$. Let (y_1,y_2) be the corresponding coordinates. By (5.2) and Lemma 5.4 we have the sections

$$\mu \circ r^i_\Theta : U(\Sigma^{2,1}_G) \to \operatorname{Hom}\big(S^iK_G/(\Theta_G \otimes S^{i-2}K_G), \mathbf{q}_2\big) \quad \text{for } i \geqq 3.$$

We set the section $\overline{\psi}_U$ on $U(\Sigma^{2,1}_G)$ as

$$\overline{\psi}_U(\tilde{z}) = r^2_\Theta(\tilde{z}) \oplus \Big(\bigoplus_{i=3}^{m-2} \mu \circ r^i_\Theta(\tilde{z})\Big).$$

By definition, $\overline{\psi}_U$ is transverse to the zero section on $\mathcal{K}\eta$. Furthermore, $\overline{\psi}_U$ vanishes on $\mathcal{K}\eta_G$ and never vanishes on $U(\Sigma^{2,1}_G)\backslash \mathcal{K}\eta_G$. In fact, suppose that $\overline{\psi}_U(\tilde{z})$ vanishes. Since $r^2_\Theta(\tilde{z})$ vanishes, we may write $y_1 \circ \zeta(u,v) = au^2$ with $a \neq 0$ and $y_2 \circ \zeta(u,v) = 0$ modulo $\mathfrak{m}^3_{u,v}$ under a suitable choice of coordinates (y_1,y_2), and z lies in $\Sigma^{2,1}$. By the splitting theorem, we may assume under a suitable choice of coordinates (u,v) that $y_1 \circ \zeta(u,v) = a_1u^2 + h(v)$ modulo $\mathfrak{m}^{m-1}_{u,v}$, where $\deg h > 2$. If $\deg h = 3$, then we may assume that $y_1 \circ \zeta(u,v) = u^2+v^3$ and $y_2 \circ \zeta(u,v) = 0$ modulo $(u^2+v^3) + \mathfrak{m}^{m-1}_{u,v}$. Hence, we can prove under a suitable choice of coordinates (u,v) and (y_1,y_2) that $y_1 \circ \zeta(u,v) = u^2+v^3$ and $y_2 \circ \zeta(u,v) = 0$ modulo $\mathfrak{m}^{m-1}_{u,v}$. This implies by the result concerning a classification of simple singularities in [12, Section 8] that z lies in $\mathcal{K}\eta$. If $a=0$ or $\deg h > 3$, then we first have $h(v) = 0$ modulo $\mathfrak{m}^{m-1}_{u,v}$. Next we take the germs ζ_λ such that

$$y_1 \circ \zeta_\lambda(u,v) = \lambda(u^2+v^3),$$
$$y_2 \circ \zeta_\lambda(u,v) = (u^2+v^3)g(u,v)$$

modulo $\mathfrak{m}^{m-1}_{u,v}$, which yields the jets $z_\lambda = z + j^{m-2}\zeta_\lambda$. If $\lambda \neq 0$, then z_λ similarly lies in $\mathcal{K}\eta$, and so z lies in $\operatorname{cl}(\mathcal{K}\eta)\backslash\mathcal{K}\eta$. This is impossible. Hence, $\overline{\psi}_U$ never vanish.

We next extend $\overline{\psi}_U$ to a section of $\operatorname{Hom}(S^iK_G/(\Theta_G \bigcirc S^{i-2}K_G), Q_G)\backslash$(zero section) over Σ^2_G. Over $\Sigma^2_G \backslash \operatorname{Int} U(\Sigma^{2,1}_G)$, we have the section

$$r^2_\Theta : \Sigma^2_G \backslash \operatorname{Int} U(\Sigma^{2,1}_G) \to \operatorname{Hom}(S^2K_G/\Theta_G, Q_G),$$
$$r^i_\Theta : \Sigma^2_G \backslash \operatorname{Int} U(\Sigma^{2,1}_G) \to \operatorname{Hom}\big(S^iK_G/(\Theta_G \bigcirc S^{i-2}K_G), Q_G\big)$$

in (5.2). Therefore, it follows from Lemma 5.4 that it induces a section

$$\mu\circ r^i_\Theta:\Sigma^2_G\backslash\operatorname{Int}U(\Sigma^{2,1}_G)\to\operatorname{Hom}\bigl(S^iK_G/(\Theta_G\bigcirc S^{i-2}K_G),o(Q_G)\bigr)$$

such that $\mu\circ r^i_\Theta(\tilde z)$ vanishes if and only if $r^i_\Theta(\tilde z)$ vanishes. Now we define the section $\overline{\psi}'$ of

$$\operatorname{Hom}(S^2K_G/\Theta_G,Q_G)\oplus\operatorname{Hom}\Bigl(\Bigl\{\Bigl(\bigoplus_{i=3}^{m-2}S^iK_G/(\Theta_G\bigcirc S^{i-2}K_G)\Bigr)\Bigr\},o(Q_G)\Bigr)$$

over $\Sigma^2_G\backslash\operatorname{Int}U(\Sigma^{2,1}_G)$ by

$$\overline{\psi}'(\tilde z)=r^2_\Theta(\tilde z)\oplus\Bigl(\bigoplus_{i=3}^{m-2}\mu\circ r^i_\Theta(\tilde z)\Bigr).$$

We have to show that $\overline{\psi}'$ never vanish on $\Sigma^2_G\backslash\operatorname{Int}U(\Sigma^{2,1}_G)$. Suppose that $\overline{\psi}'(\tilde z)$ vanishes. Then we may write $y_1\circ\zeta(u,v)=a(u^2+v^2)$ and $y_2\circ\zeta(u,v)=0$ modulo $\mathfrak{m}^3_{u,v}$. First let z lie in Σ^2. If $a\neq 0$, then we may assume by the Morse theorem that $y_1\circ\zeta(u,v)=a(u^2+v^2)$. This implies that z lies in $\operatorname{cl}(\mathcal{K}\langle x^2+y^2,x^{m-1}\rangle)$. By the transversality of $\overline{\psi}'$, the structure groups of the normal bundles at z and of $\mathcal{N}(\eta)_G$ are different. This is impossible. If $a=0$, namely, $y_i\circ\zeta(u,v)=0$ for $i=1,2$, then z lies in $\Sigma^{2,2}$, and hence, similarly as in (1), it is impossible. Therefore, it follows as in (1) that z lies in $\operatorname{cl}(\Sigma^3)$. Similarly as in (1), this is also impossible. Hence, $\psi'(\tilde z)$ never vanish.

From the definition of $\overline{\psi}_U$, it follows that $\overline{\psi}_U$ and $\overline{\psi}'$ coincide with each other on $\partial U(\Sigma^{2,1}_G)$. Thus we have obtained the required section $\overline{\psi}$ defined on Σ^2_G such that it vanishes only on $\mathcal{K}\eta_G$ and is transverse to the zero section on $\mathcal{K}\eta_G$. This completes the proof. □

COROLLARY 5.6

Let X be of dimension not less than $2m$. Let s be a section of $J^k(E,F)$ over X such that $s(X)\cap(\operatorname{cl}(\mathcal{K}\eta)\backslash\mathcal{K}\eta)$ is empty and s is transverse to Boardman–Thom manifolds. Then the section ψ_s over S^2_G of $(s_G|S^2_G)^*\mathbf{n}(\eta)_{\Sigma^2}$, which is induced by $\psi\circ s$, is transverse to the zero section on $S\eta_G$ and its inverse image of the zero section is exactly equal to $S\eta_G$ in the case (S1) or (S2), respectively.

6. Thom polynomials

In what follows let k denote $2q-1$. We calculate the Thom polynomial of $\mathcal{K}\eta$ under the condition that a section of $J^k(E,F)$ does not intersect with $\operatorname{cl}(\mathcal{K}\eta)\backslash\mathcal{K}\eta$ by properties of Gysin homomorphisms and characteristic classes (see [7], [15], [22]) and prove Theorem 1.1. We first prepare several lemmas.

Let H be a $2w$-vector bundle over a connected orientable manifold Z. Let $\pi^G:G_{2,2w-2}(H)\to Z$ denote the Grassmann bundle associated to H with fiber $G_{2,2w-2}$. Let $(\pi^G)_!:H^*(G_{2,2w-2}(H);\mathbb{Z})\to H^*(Z;\mathbb{Z})$ denote the Gysin homomorphism. Let K denote the canonical 2-plane bundle over $G_{2,2w-2}(H)$ and $H_G=$

$(\pi^G)^* H$. We express the total Pontrjagin class $P(K)$ of K as $P(K) = 1 + P_1(K)$. Let $P(H)^{-1} = 1 + \overline{P}_1(H) + \cdots + \overline{P}_i(H) + \cdots$.

We have the following lemma. This is well known (see [17]).

LEMMA 6.1

(1) $G_{2,2w-2}(H)$ *is orientable.*

(2) *We have* $(\pi^G)_!(P_i(H_G/K)) = \begin{cases} 1 & \textit{for } i = w-1, \\ 0 & \textit{for } i \neq w-1. \end{cases}$

(3) *We have* $(\pi^G)_!(P_1(K)^{w-1+\ell}) = (-1)^{w-1+\ell}\overline{P}_\ell(H)$.

Proof

(1) The tangent bundle of $G_{2,2w-2}(H)$ is isomorphic to $\mathrm{Hom}(K, H_G/K)$, and its first Stiefel–Whitney class is equal to $(2w-2)W_1(K) + 2W_1(H/K) = 0$.

(2) If $i \neq w-1$, then $P_i(H_G/K)$ vanishes by the dimensional reason. By regarding H_x with $\mathbb{C}^w$ for a point $x \in X$, we take a 1-dimensional complex subspace $\mathbb{C}$ of H_x. Let $i_x : x \to X$ and $\widetilde{i_x} : G_{2,2w-2}(H_x) \to G_{2,2w-2}(H)$ be the inclusions. Let $H_G^x = (\widetilde{i_x})^* H_G$ and $K^x = (\widetilde{i_x})^* K$. Then we have a vector bundle $\mathrm{Hom}(\mathbb{C}, H_G^x/K^x)$ over $G_{2,2w-2}(H_x)$ and its section $\varkappa$ such that $\varkappa(b)$, for $b \in G_{2,2w-2}(H_x)$, maps $\mathbb{C}$ to H_x/b by the orthogonal projection along b of H_x onto H_x/b. Obviously, $\varkappa(b)$ is a null homomorphism if and only if $b = \mathbb{C}$. Furthermore, it is elementary to show that $\varkappa$ is transverse to the zero section of $\mathrm{Hom}(\mathbb{C}, H_G^x/K^x)$. This implies that the fundamental cohomology class of $G_{2,2w-2}(H_x)$ is equal to the Euler class $\chi(\mathrm{Hom}(\mathbb{C}, H_G^x/K^x))$. Furthermore, we have

$$\chi\big(\mathrm{Hom}(\mathbb{C}, H_G^x/K^x)\big) = C_{2w-2}\big((H_G^x/K^x) \otimes \mathbb{C}\big) = P_{w-1}(H_G^x/K^x),$$

where C_{2w-2} denotes the $(2w-2)$-th Chern class. For the Gysin homomorphisms

$$\big(\pi^G | G_{2,2w-2}(H_x)\big)_! : H^*\big(G_{2,2w-2}(H_x); \mathbb{Z}\big) \longrightarrow H^*(x; \mathbb{Z}),$$

we have

$$\big(\pi^G | G_{2,2w-2}(H_x)\big)_! \big(P_{w-1}(H_G^x/K^x)\big) = 1.$$

In the commutative diagram

$$\begin{array}{ccc} G_{2,2w-2}(H_x) & \xrightarrow{\widetilde{i_x}} & G_{2,2w-2}(H) \\ \downarrow & & \downarrow \\ x & \xrightarrow{\widetilde{i_x}} & X \end{array}$$

it follows that

$$\begin{aligned} (i_x)^*\big((\pi^G)_!(P_i(H_G/K))\big) &= \big(\pi^G | G_{2,2w-2}(H_x)\big)_! \big\{(\widetilde{i_x})^*\big(P_i(H_G/K)\big)\big\} \\ &= \big(\pi^G | G_{2,2w-2}(H_x)\big)_! \big(P_i(H_G^x/K^x)\big). \end{aligned}$$

Since $(i_x)^*$ induces an isomorphism of $\mathbb{Z}$ in the zeroth dimension, this proves the assertion.

(3) Since $H_G = K \oplus H_G/K$, we have $P(H_G) = P(K)P(H_G/K)$, and so, $P(K)^{-1} = P(H_G)^{-1}P(H_G/K)$. By comparing the terms of degree $w-1+\ell$, we have

$$(-1)^{w-1+\ell}P_1(K)^{w-1+\ell} = \sum_{j=0}^{w-1}\overline{P}_{w-1+\ell-j}(H_G)P_j(H_G/K).$$

By (2) and the naturality of the Gysin homomorphism, we have

$$\begin{aligned}(-1)^{w-1+\ell}(\pi^G)_!\big(P_1(K)^{w-1+\ell}\big) &= \sum_{j=0}^{w-1}\overline{P}_{w-1+\ell-j}(H_G)(\pi^G)_!\big(P_j(H_G/K)\big)\\ &= \overline{P}_\ell(H).\end{aligned}$$

□

As is well known, we may reduce the calculation to the case where F is trivial. In fact, let $F^\perp$ denote a vector bundle such that $F \oplus F^\perp$ is trivial. Let

$$\mathcal{L} : J^k(E,F) \to J^k(E\oplus F^\perp, F\oplus F^\perp)$$

denote a bundle map defined by $\mathcal{L}(h) = h + \mathrm{id}_{F^\perp}$, where $h \in J^k(E,F)$ and $\mathrm{id}_{F^\perp}$ is the identity of $F^\perp$. Then the following lemma is elementary.

LEMMA 6.2

(1) *The inverse images of Boardman–Thom manifolds* $\Sigma^I(E\oplus F^\perp, F\oplus F^\perp)$ *with any symbol* I, $\mathcal{K}\eta$, *and* $\mathrm{cl}(\mathcal{K}\eta)\backslash\mathcal{K}\eta$ *in* $J^k(E\oplus F^\perp, F\oplus F^\perp)$ *by* $\mathcal{L}$ *coincide with those spaces in* $J^k(E,F)$, *respectively.*

(2) $\mathcal{L}$ *is transverse to each* $\Sigma^I(E\oplus F^\perp, F\oplus F^\perp)$ *and* $\mathcal{K}\eta(E\oplus F^\perp, F\oplus F^\perp)$.

In the following E and F imply $E\oplus F^\perp$ and the trivial bundle $F\oplus F^\perp$ of dimension $2w$, respectively. Let i_{S^2} denote the inclusion of S^2_G into $s^*G(E,F)$. Note that $(i_{\Sigma^2})^*(\chi(\mathcal{N}(\eta)_G) = \chi(\mathbf{n}(\eta)_{\Sigma^2})$.

THEOREM 6.3

We assume that the coefficient group is $\mathbb{Z}$ *when* m *is even and is* $\mathbb{Z}/2\mathbb{Z}$ *when* m *is odd. Then we have the following in the cases (S1) and (S2):*

$$(\pi_G)_!\big\{s^*_G\big(\chi(\mathrm{Hom}(K,\varepsilon^{2w})\oplus\mathrm{Hom}(K^\perp,Q))\cup\chi(\mathcal{N}(\eta)_G)\big)\big\} = [S\eta]$$

Proof

We give a proof for the case (S2), and the proof for the case (S1) is similar. Indeed, we have

$$\begin{aligned}&s^*_G\big(\chi(\mathrm{Hom}(K,\varepsilon^{2w})\oplus\mathrm{Hom}(K^\perp,Q))\cup\chi(\mathcal{N}(\eta)_G)\big)\cap[s^*G(E,F)]\\ &\quad= s^*_G\big(\chi(\mathcal{N}(\eta)_G)\big)\cap\big\{s^*_G\big(\chi(\mathrm{Hom}(K,\varepsilon^{2w})\oplus\mathrm{Hom}(K^\perp,Q))\big)\cap[s^*G(E,F)]\big\}\\ &\quad= s^*_G\big(\chi(\mathcal{N}(\eta)_G)\big)\cap\big((i_{S^2})_*([S^2_G])\big)\end{aligned}$$

$$= (i_{S^2})^*\big(s_G^*(\chi(\mathcal{N}(\eta)_G))\big) \cap ([S_G^2])$$
$$= \chi\big((i_{S^2})^*(\mathbf{n}(\eta)_{\Sigma^2})\big) \cap ([S_G^2])$$
$$= [S\eta_G].$$

Furthermore, we have that $S\eta_G$ is mapped diffeomorphically onto $S\eta$. This shows the theorem. □

Now we calculate the Euler class of $\mathrm{Hom}(K, \varepsilon^{2w}) \oplus \mathrm{Hom}(K^\perp, Q) \oplus \mathcal{N}(\eta)_G$.

LEMMA 6.4

The following formulas hold up to sign, where ε is a trivial line bundle over $G(E,F)$.

(i) $\chi\{\mathrm{Hom}(\bigoplus_{i=t+1}^{t+2\ell} S^i K_G/(\Theta_G \bigcirc S^{i-2}K_G), o(Q_G))\} = \{\prod_{i=t+1}^{t+2\ell} i\} \times P_1(K_G)^\ell$.

(ii) $\chi\{\mathrm{Hom}(K_G^\perp, Q_G)\} = \sum_{i=0}^{w-1}(-1)^i P_i(K_G^\perp)P_1(Q_G)^{w-1-i}$ *over* $G(E,F)$.

(iii) $\chi(\mathrm{Hom}(S^2K_G/L_G, Q_G)) = 3P_1(K_G)$ *over* $G(E,F)$.

Proof

In this proof, $=$ will mean the equality modulo 2-torsion. In the proof we set $K = K_G$, $Q = Q_G$, and $\varepsilon = \Theta_G = L_G$. Let $E(i) \to \mathrm{BO}(i)$ denote the classifying vector bundle over a classifying space of i-dimensional vector bundles. Let $c_K : G(E,F) \to \mathrm{BO}(2)$, $c_{K^\perp} : G(E,F) \to \mathrm{BO}(2w-2)$, and $c_Q : G(E,F) \to \mathrm{BO}(2)$ denote the classifying maps of K, $K^\perp$, and Q, respectively. Then we note that

$$\chi\Big\{\mathrm{Hom}\Big(\bigoplus_{i=t+1}^{t+2\ell} S^iK/(\varepsilon \bigcirc S^{i-2}K), o(Q)\Big)\Big\}$$
$$= (c_K \times c_Q)^*\bigg(\chi\Big\{\mathrm{Hom}\Big(\bigoplus_{i=t+1}^{t+2\ell} S^iE(2)/(\varepsilon \bigcirc S^{i-2}E(2)), o(E(2))\Big)\Big\}\bigg),$$
$$\chi\big\{\mathrm{Hom}(K^\perp, Q)\big\} = (c_{K^\perp} \times c_Q)^*\big(\chi\big\{\mathrm{Hom}\big(E(2w-2), E(2)\big)\big\}\big),$$
$$\chi\big(\mathrm{Hom}(S^2K/\varepsilon, Q)\big) = (c_K \times c_Q)^*\big(\chi\big(\mathrm{Hom}(S^2E(2)/\varepsilon, E(2))\big)\big).$$

Let $C(K^{\mathbb{C}})$ and $C(E(2)^{\mathbb{C}})$, which are corresponded by $(c_K)^*$, be represented by the same symbol $(1+t_1)(1+t_2)$, let $C(Q^{\mathbb{C}})$ and $C(E(2)^{\mathbb{C}})$, which are corresponded by $(c_Q)^*$, be represented by the same symbol $(1+r_1)(1+r_2)$, and similarly, let

$$C\big((K^\perp)^{\mathbb{C}}\big) \text{ or } C\big(E(2w-2)\big) = \prod_{j=3}^{2w-2}(1+t_j).$$

(i) For $i \geqq 2$, we have

$$C\big(S^iE(2)^{\mathbb{C}}/(\varepsilon^{\mathbb{C}} \otimes S^{i-2}E(2)^{\mathbb{C}})\big)$$
$$= C\big(S^iE(2)^{\mathbb{C}}\big)C(\varepsilon^{\mathbb{C}} \otimes S^{i-2}E(2)^{\mathbb{C}})^{-1}$$

$$
\begin{aligned}
&= C\big(S^iE(2)^{\mathbb{C}}\big)C\big(S^{i-2}E(2)^{\mathbb{C}}\big)^{-1} \\
&= \prod_{s=0}^{i}\big(1+st_1+(i-s)t_2\big)\Big\{\prod_{j=0}^{i-2}\big(1+jt_1+(i-2-j)t_2\big)\Big\}^{-1} \\
&= \prod_{s=0}^{i}\big(1+st_1+(i-s)t_2\big)\Big\{\prod_{j=0}^{i-2}\big(1+jt_1+(i-2-j)t_2+t_1+t_2\big)\Big\}^{-1} \\
&= (1+it_1)(1+it_2)
\end{aligned}
$$

modulo 2-torsion. Since

$$
\begin{aligned}
\chi\big\{\big(S^iE(2)^{\mathbb{C}}/\big(\varepsilon^{\mathbb{C}}\otimes S^{i-2}E(2)^{\mathbb{C}}\big)\big)\big\}^2 &= C_2\big((S^iE(2)^{\mathbb{C}}/(\varepsilon^{\mathbb{C}}\otimes S^{i-2}E(2)^{\mathbb{C}}))\big) \\
&= i^2t_1t_2 \\
&= i^2C_2\big(E(2)^{\mathbb{C}}\big),
\end{aligned}
$$

we calculate as

$$
\begin{aligned}
&\chi\Big\{\mathrm{Hom}\Big(\bigoplus_{i=t+1}^{t+2\ell} S^iE(2)/\big(\varepsilon \bigcirc S^{i-2}E(2)\big), o\big(E(2)\big)\Big)\Big\}^2 \\
&\quad= C_{4\ell}\Big(\mathrm{Hom}\Big(\bigoplus_{i=t+1}^{t+2\ell}\big(S^iE(2)^{\mathbb{C}}/\varepsilon^{\mathbb{C}}\otimes S^{i-2}E(2)^{\mathbb{C}}\big)\Big), \mathbb{C}\Big) \\
&\quad= \prod_{i=t+1}^{t+2\ell} C_2\big(S^iE(2)^{\mathbb{C}}/\varepsilon^{\mathbb{C}}\otimes S^{i-2}E(2)^{\mathbb{C}}\big) \\
&\quad= \prod_{i=t+1}^{t+2\ell}(it_1)(it_2) \\
&\quad= \Big\{\prod_{i=t+1}^{t+2\ell} i^2\Big\}C_2\big(E(2)^{\mathbb{C}}\big)^{2\ell} \\
&\quad= \Big\{\prod_{i=t+1}^{t+2\ell} i^2\Big\}P_1\big(E(2)\big)^{2\ell}.
\end{aligned}
$$

By considering the cohomology ring of BO(2) modulo 2-torsion, we have

$$
\chi\Big\{\mathrm{Hom}\Big(\bigoplus_{i=t+1}^{t+2\ell} S^iE(2)/\big(\varepsilon \bigcirc S^{i-2}E(2)\big), o\big(E(2)\big)\Big)\Big\} = \Big\{\prod_{i=t+1}^{t+2\ell} i\Big\}P_1\big(E(2)\big)^{\ell}.
$$

Thus we obtain the assertion (i) by applying $(c_K \times c_Q)^*$.

The following proofs of (ii) and (iii) are similar, and so we only give outlines of calculations.

(ii) We have

$$
\begin{aligned}
&\chi\big\{\mathrm{Hom}(E(2w-2), E(2))\big\}^2 \\
&\quad= C_{4w-4}\big(\mathrm{Hom}(E(2w-2)^{\mathbb{C}}, E(2)^{\mathbb{C}})\big)
\end{aligned}
$$

$$= \prod_{i=3}^{2w} (r_1 - t_i)(r_2 - t_i)$$

$$= \prod_{i=3}^{2w} (r_1 r_2 + t_i^2)$$

$$= \prod_{i=3}^{2w} \bigl(C_2(E(2)^{\mathbb{C}}) + t_i^2\bigr)$$

modulo 2-torsion. Setting $x^2 = C_2(E(2)^{\mathbb{C}})$, this is equal, modulo 2-torsion, to

$$\begin{aligned}
&\prod_{i=3}^{2w} \bigl(x^2 - (\sqrt{-1}t_i)^2\bigr) \\
&\quad = \prod_{i=3}^{2w} (x + \sqrt{-1}t_i) \prod_{i=3}^{2w} (x - \sqrt{-1}t_i) \\
&\quad = \Bigl(\sum_{i=0}^{2w-2} (\sqrt{-1})^i C_i\bigl(E(2w-2)^{\mathbb{C}}\bigr) x^{2w-2-i} \Bigr) \\
&\qquad \times \Bigl(\sum_{i=0}^{2w-2} (-\sqrt{-1})^i C_i\bigl(E(2w-2)^{\mathbb{C}}\bigr) x^{2w-2-i} \Bigr) \\
&\quad = \Bigl(\sum_{i=0}^{w-1} (-1)^i C_{2i}\bigl(E(2w-2)^{\mathbb{C}}\bigr) x^{2w-2-2i} \Bigr)^2 \\
&\quad = \Bigl(\sum_{i=0}^{w-1} (-1)^i C_{2i}\bigl(E(2w-2)^{\mathbb{C}}\bigr) C_2\bigl(E(2)^{\mathbb{C}}\bigr)^{w-1-i} \Bigr)^2 \\
&\quad = \Bigl(\sum_{i=0}^{w-1} (-1)^i P_i\bigl(E(2w-2)\bigr) P_1\bigl(E(2)\bigr)^{w-1-i} \Bigr)^2.
\end{aligned}$$

Hence,

$$\chi\bigl\{\mathrm{Hom}\bigl(E(2w-2), E(2)\bigr)\bigr\} = \pm \sum_{i=0}^{w-1} (-1)^i P_i\bigl(E(2w-2)\bigr) P_1\bigl(E(2)\bigr)^{w-1-i}.$$

(iii) Similarly, we have that

$$\begin{aligned}
&\chi\bigl(\mathrm{Hom}(S^2K/\varepsilon, Q)\bigr)^2 \\
&\quad = C_4\bigl(\mathrm{Hom}(S^2K^{\mathbb{C}}/\varepsilon^{\mathbb{C}}, Q^{\mathbb{C}})\bigr) \\
&\quad = (r_1 - 2t_1)(r_1 - 2t_2)(r_2 - 2t_1)(r_2 - 2t_2) \\
&\quad = (r_1^2 + 4t_1t_2)(r_2^2 + 4t_1t_2) \\
&\quad = \bigl(4C_2(K)\bigr)^2 - 8r_1r_2C_2(K)C_2(Q) + C_2(Q)^2
\end{aligned}$$

$$= \big(4C_2(K) - C_2(Q)\big)^2$$
$$= \big(4P_1(K) - P_1(Q)\big)^2.$$

Hence, we obtain $\chi(\mathrm{Hom}(S^2K/\varepsilon, Q)) = \pm(4P_1(K) - P_1(Q))$. □

The next theorem follows from Theorem 6.3 and Lemma 6.4.

THEOREM 6.5

Let m be an even integer $2q$. Let X be an orientable manifold. Then the leading term of the Thom polynomial $\mathrm{tp}(\mathcal{K}\eta; s)$ with $\mathbb{Z}$-coefficients is equal to the following.

(S1) *We have* $(2q-1)!P_q(F - E)$,

(S2) *We have* $\begin{cases} 3P_2(F-E) & \textit{if } q = 2, \\ 3\{\prod_{i=3}^{2q-2} i\}P_q(F-E) & \textit{if } q \geqq 3 \end{cases}$

up to sign. In particular, $\mathrm{tp}(\mathcal{K}\eta; s)$ *depends only on the homotopy class of* s.

Proof

In this proof, $=$ will mean the equality modulo 2-torsion. In the proof E implies $E \oplus F^\perp$. As is well known, we have

$$\chi\big\{\mathrm{How}(K, \varepsilon^{2w})\big\} = \chi(K^{\mathbb{C}})^w = C_2(K^{\mathbb{C}})^w = P_1(K)^w \tag{6.1}$$

over $G(E, F)$.

(S1) The coefficient of $P_1(Q)^{w-1}$ of $\chi(\mathrm{Hom}(K^\perp, Q))$ is equal to 1. Hence, we have

$$\chi\Big\{\mathrm{How}(K, \varepsilon^{2w}) \oplus \mathrm{Hom}\Big(\bigoplus_{i=2}^{2q-1} S^iK/(L \bigcirc S^{i-2}K), o(Q)\Big)\Big\}$$
$$= \Big\{\prod_{i=2}^{2q-1} i\Big\} P_1(K)^{q-1+w}.$$

By the commutativity of the diagram (5.1), $(\mathrm{pr}_E)_!$ maps the Euler class to

$$\Big\{\prod_{i=2}^{2q-1} i\Big\}\overline{P}_q(E - F) = \Big\{\prod_{i=2}^{2q-1} i\Big\}P_q(F - E).$$

(S2) We have

$$\mathrm{How}(S^2K/\Theta \oplus E/K, Q) = \big(4P_1(K) - P_1(Q)\big)\Big(\sum_{i=0}^{w-1}(-1)^i P_i(E/K)P_1(Q)^{w-1-i}\Big).$$

The coefficient of the term $P_1(Q)^{w-1}$ is

$$P_1(E/K) + 4P_1(K) = P_1(E) + 3P_1(K).$$

Ignoring $P_1(E)$, $(\mathrm{pr}_F)_!$ maps the Euler class of

$$\mathrm{How}(K,\varepsilon^{2w}) \oplus \mathrm{How}(S^2K/\Theta \oplus E/K, Q) \oplus \mathrm{Hom}\Big(\bigoplus_{i=3}^{2q-2} S^iK/(\Theta \bigcirc S^{i-2}K), o(Q)\Big)$$

to

$$(\mathrm{pr}_E)_!\Big(3\Big\{\prod_{i=3}^{2q-2} i\Big\}P_1(K)^{q-1+w}\Big) = 3\Big\{\prod_{i=3}^{2q-2} i\Big\}P_q(F-E).$$

This proves the theorem. □

Proof of Theorems 1.1
By setting $F = f^*(TY)$, $E = TX$, and $s = (\mathrm{id}_X \times f)^*(j^k f)$, the assertions follow from Theorem 6.5 by replacing P_i with $P_i(f^*(TY) - TX)$. □

7. J-images

In this section we show a relationship of the Thom polynomials in Theorem 1.1 and the J-images.

Let us recall the J-image of the J-homomorphism

$$J : \pi_n(\mathrm{SO}) \longrightarrow \pi_n^s$$

in [1] and [23]. Recall the cobordism group $\Omega_{\mathrm{fold},j}(S^n)$ of fold maps of closed oriented n-dimensional manifolds to S^n of degree j and an isomorphism $\omega_j : \Omega_{\mathrm{fold},j}(S^n) \to \pi_n^s$ from [3, Theorem 1]. We have proved in [3, Proposition 5.2] that an element $\alpha \in \pi_n^s$ lies in the J-image if and only if there exists a fold map $f : S^n \to S^n$ of degree 1 with $\omega_1([f]) = \alpha$. This assertion is also true in the case of degree zero by [3, Lemmas 2.5, 3.4]. In fact, a fold map $f : N \to S^n$ of degree j determines the homotopy class of the bundle map

$$\mathcal{T}(f) : TN \oplus \varepsilon_N \longrightarrow TS^n \oplus \varepsilon_{S^n}$$

covering f. If $N = S^n$ and f is of degree j, then $\mathcal{T}(f)$ determines an element of $\pi_n(\mathrm{SO}(n+1))$, whose image of J coincides with $\omega_j([f])$.

For a fold map f of degree zero, we take a parallelizable $(n+1)$-manifold V with $\partial V = S^n$ and an extended map $F : V \to D^{n+1}$ such that the restriction of F between the collars $S^n \times [0,\varepsilon]$ of V and D^{n+1} is equal to $f \times \mathrm{id}_{[0,\varepsilon]}$ for a sufficiently small ε. Let $\widehat{V}$ denote the manifold, which is the union of $V \cup_{S^n} D^{n+1}$, where V and D^{n+1} are pasted on S^n. For a sufficiently large integer k, let $\tau(f)$ denote

$$\mathcal{T}(f) \oplus (f \times \mathrm{id}_{\mathbb{R}^{k-n-1}}) : TS^n \oplus \varepsilon_{S^n}^{k-n} \longrightarrow TS^n \oplus \varepsilon_{S^{n-1}}^{k-n}.$$

Let $\tau(\widehat{V}, \tau(f))$ be the k-dimensional vector bundle over $\widehat{V}$, which is obtained by pasting $TV \oplus \varepsilon_{S^{n-1}}^{k-n-1}$ and $TD^{n+1} \oplus \varepsilon_{S^n}^{k-n-1}$ by $\tau(f)$ on S^n.

Now consider the jet space $J^k\big(\tau(\widehat{V}, \tau(f)), \varepsilon_{\widehat{V}}^k\big)$, whose restriction to V (resp., D^{n+1}) is equal to $J^k(V, D^{n+1})$ (resp., $J^k(D^{n+1}, D^{n+1})$). Let $s(f)$ denote its section defined by

$$s(f)(x) = \begin{cases} j^k(\mathrm{id}_{D^{n+1}}) \times \mathrm{id}_{\mathbb{R}^{k-n-1}} & \text{for } x \in D^{n+1}, \\ j^k_x F \times \mathrm{id}_{\mathbb{R}^{k-n-1}} & \text{for } x \in V. \end{cases}$$

Let $n = 4q - 1$ in the following. The J-image $\pi_{4q-1}(\mathrm{SO})$ is a cyclic group of order j_q. The next lemma follows from [14, Lemma 2].

LEMMA 7.1

Let $n = 4q - 1$. Let $\mathfrak{o}\big(\tau(\widehat{V}, \tau(f))\big)$ denote the element of $\pi_{4q-1}(\mathrm{SO})$, which is determined as the primary obstruction of $\tau(\widehat{V}, \tau(f))$ to being trivial. Then the Pontrjagin class $P_q\big(\tau(\widehat{V}, \tau(f))\big)$ is related by the identity

$$P_q\big(\tau(\widehat{V}, \tau(f))\big) = \pm a_q (4q-1)!\,\mathfrak{o}\big(\tau(\widehat{V}, \tau(f))\big),$$

where $a_q = 2$ for q odd and $a_q = 1$ for q even.

By definition, $s(f)$ is transverse to $\mathcal{K}\eta$ and $(s(f)|D^{8q})^{-1}(\mathcal{K}\eta)$ is empty. Then the Thom polynomials $\mathrm{tp}(\mathcal{K}\eta, s(f))$ are as given in Theorem 1.1, and they are nothing but the Poincaré duals of $\mathcal{K}\eta$ of E^f. Therefore, we have the following theorem.

THEOREM 7.2

Let α be an element of the J-image in π^s_{4q-1}, which has a fold map $f : S^{4q-1} \to S^{4q-1}$ of degree zero with $\alpha = \mathfrak{o}\big(\tau(\widehat{V}, \tau(f))\big)$. Then the algebraic number of singularities of type $\mathcal{K}\eta$ of the extension E^f is equal, modulo $(4q-1)!j_q$, to

$$\text{(S1)} \quad (2q-1)!(4q-1)!a_q\alpha,$$

$$\text{(S2)} \quad \begin{cases} 3 \cdot 7!\alpha & \text{if } q = 2, \\ 3\big\{\prod_{i=3}^{2q-2} i\big\}(4q-1)!a_q\alpha & \text{if } q \geqq 3 \end{cases}$$

up to sign.

In dimension 12, the J-image is of order $2^3 3^2 7$, and the algebraic number of singularities of type $\mathcal{K}\eta$ of the extension E^f is equal, modulo $2 \cdot 11! \cdot 2^3 3^2 7$, to $5!11! \cdot 2\alpha$ in the case (S1) and to $3^2 \cdot 2^2 \cdot 11!\alpha$ in the case (S2), where an integer α varies from 1 to $2^3 3^2 7$.

In the case where a fold map $f : N \to S^n$ of degree zero has a parallelizable manifold V and an extension E^f such that $\omega_0([f]) = \alpha$ does not lie in the J-image, we can define the Thom polynomial $\mathrm{tp}(\mathcal{K}\eta, s(f))$. However, the author does not know whether it is effective to detect α or not. The theorem implies that the singularities with nonvanishing leading terms of Thom polynomials detect elements of the J-image. Therefore, the classification of those singularities and the calculation of Thom polynomials will be important to clarify the relationship between singularities and the stable homotopy groups of spheres.

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