On the one-dimensional cubic nonlinear Schrödinger equation below L^2

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Abstract In this paper, we review several recent results concerning well-posedness of the one-dimensional, cubic nonlinear Schrödinger equation (NLS) on the real line $\mathbb{R}$ and on the circle $\mathbb{T}$ for solutions below the L^2-threshold. We point out common results for NLS on $\mathbb{R}$ and the so-called *Wick-ordered NLS* (WNLS) on $\mathbb{T}$, suggesting that WNLS may be an appropriate model for the study of solutions below $L^2(\mathbb{T})$. In particular, in contrast with a recent result of Molinet, who proved that the solution map for the periodic cubic NLS equation is not weakly continuous from $L^2(\mathbb{T})$ to the space of distributions, we show that this is not the case for WNLS.

1. Introduction

In this paper, we consider the one-dimensional cubic nonlinear Schrödinger equation (NLS)

$$\begin{cases} iu_t - u_{xx} \pm |u|^2 u = 0, \\ u|_{t=0} = u_0, \quad (x,t) \in \mathbb{T}\times\mathbb{R} \text{ or } \mathbb{R}\times\mathbb{R}, \end{cases} \tag{1.1}$$

where u is a complex-valued function and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$; (1.1) arises in various physical settings for the description of wave propagation in nonlinear optics, fluids, and plasmas (see [36] for a general review). It also arises in quantum field theory as a mean field equation for many-body boson systems. It is known to be one of the simplest partial differential equations (PDEs) with complete integrability (see [1], [2], [20]).

As a completely integrable PDE, (1.1) enjoys infinitely many conservation laws, starting with conservation of mass, momentum, and Hamiltonian:

$$\begin{aligned} &N(u) = \int |u|^2\,dx, \qquad P(u) = \operatorname{Im}\int \overline{u}u_x\,dx, \\ &H(u) = \frac{1}{2}\int |u_x|^2\,dx \pm \frac{1}{4}\int |u|^4\,dx. \end{aligned} \tag{1.2}$$

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In the focusing case (with the $-$ sign), (1.1) admits soliton and multisoliton solutions. Moreover, (1.1) is globally well posed in L^2 thanks to the conservation of the L^2-norm (see [37] on $\mathbb{R}$ and [3] on $\mathbb{T}$).

It is also well known that (1.1) is invariant under several symmetries. In the following, we concentrate on the dilation symmetry and the Galilean symmetry. The dilation symmetry states that if $u(x,t)$ is a solution to (1.1) on $\mathbb{R}$ with initial condition u_0, then $u^\lambda(x,t) = \lambda^{-1}u(\lambda^{-1}x, \lambda^{-2}t)$ is also a solution to (1.1) with the λ-scaled initial condition $u_0^\lambda(x) = \lambda^{-1}u_0(\lambda^{-1}x)$. Associated to the dilation symmetry, there is a scaling-critical Sobolev index s_c such that the homogeneous $\dot{H}^{s_c}$-norm is invariant under the dilation symmetry. In the case of the one-dimensional cubic NLS, the scaling-critical Sobolev index is $s_c = -1/2$. It is commonly conjectured that a PDE is ill posed in H^s for $s < s_c$. Indeed, on the real line, Christ, Colliander, and Tao [12] showed that the data-to-solution map is unbounded from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ for $s < -1/2$. The Galilean invariance states that if $u(x,t)$ is a solution to (1.1) on $\mathbb{R}$ with initial condition u_0, then $u^\beta(x,t) = e^{i(\beta/2)x}e^{i(\beta^2/4)t}u(x+\beta t, t)$ is also a solution to (1.1) with the initial condition $u_0^\beta(x) = e^{i(\beta/2)x}u_0(x)$. Note that the L^2-norm is invariant under the Galilean symmetry.* It turns out that this symmetry also leads to a kind of ill-posedness in the sense that the solution map cannot be *smooth* in H^s for $s < s_c^\infty = 0$. Indeed, a simple application of Bourgain's idea in [5] shows that the solution map of (1.1) cannot be C^3 in H^s for $s < s_c^\infty = 0$ (see Section 2 for more results in this direction).

Recently, Molinet [34] showed that the solution map for (1.1) on $\mathbb{T}$ cannot be continuous in $H^s(\mathbb{T})$ for $s < 0$ (see [14] and [9] for related results). His argument is based on showing that the solution map† is not continuous from $L^2(\mathbb{T})$ endowed with weak topology to the space of distributions $(C^\infty(\mathbb{T}))^*$. Several remarks are in order. First, on the real line, there is no corresponding result (i.e., failure of continuity of the solution map for $s < 0$). Also, the discontinuity in [34] is precisely caused by $2\mu(u)u$, where $\mu(u) := \fint |u|^2\,dx = (1/2\pi)\int_0^{2\pi} |u|^2\,dx$.

Our main goal in this paper is to propose an alternative formulation of the periodic cubic NLS below $L^2(\mathbb{T})$ to avoid this undesirable behavior. In particular, we show that this model has properties similar to those of (1.1) on the real line even below L^2. We consider the *Wick-ordered cubic NLS* (WNLS)

$$\begin{cases} iu_t - u_{xx} \pm (|u|^2 - 2\fint |u|^2)u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.3}$$

for $(x,t) \in \mathbb{T} \times \mathbb{R}$. Clearly, (1.1) and (1.3) are equivalent for $u_0 \in L^2(\mathbb{T})$. If u satisfies (1.1) with $u_0 \in L^2(\mathbb{T})$, then $v(t) = e^{\mp 2i\mu(u_0)t}u(t)$ satisfies (1.3). However, for $u_0 \notin L^2(\mathbb{T})$, we cannot freely convert solutions of (1.3) into solutions of (1.1).

*The Galilean symmetry does not preserve the momentum. Indeed, $P(u^\beta) = (\beta/2)N(u) + P(u)$.

†Strictly speaking, Molinet's result applies to the flow map; that is, for each nonzero $u_0 \in L^2(\mathbb{T})$, the map: $u_0 \to u(t)$ is not continuous.

The effect of this modification can be seen more clearly on the Fourier side. By writing the cubic nonlinearity as $\widehat{|u|^2u}(n) = \sum_{n=n_1-n_2+n_3} \widehat{u}(n_1)\overline{\widehat{u}}(n_2)\widehat{u}(n_3)$, we see that the additional term in (1.3) precisely removes resonant interactions caused by $n_2 = n_1$ or n_3 (see Section 4). Such a modification does not seem to have a significant effect on $\mathbb{R}$, since $\xi_2 = \xi_1$ or ξ_3 is a set of measure zero in the hyperplane $\xi = \xi_1 - \xi_2 + \xi_3$ (for fixed ξ).

It turns out that (1.3) on $\mathbb{T}$ shares many common features with (1.1) on $\mathbb{R}$ (see Section 2). Equation (1.3) (in the defocusing case on $\mathbb{T}^2$) first appeared in the work of Bourgain [4] and [7], in the study of the invariance of the Gibbs measure, as an equivalent formulation of the Wick-ordered Hamiltonian equation, related to renormalization in the Euclidean φ^4_2 quantum field theory (see Section 3).

There are several results on (1.3). Using a power series method, Christ [10] proved the local-in-time existence of solutions in $\mathcal{F}L^p(\mathbb{T})$ for $p < \infty$, where the Fourier–Lebesgue space $\mathcal{F}L^p(\mathbb{T})$ is defined by the norm $\|f\|_{\mathcal{F}L^p(\mathbb{T})} = \|\widehat{f}(n)\|_{\ell^{p'}_n(\mathbb{Z})}$. In the periodic case, we have $\mathcal{F}L^p(\mathbb{T}) \supsetneq L^2(\mathbb{T})$ for $p > 2$. Grünrock and Herr [22] established the same result (with uniqueness) via the fixed point argument.

On the one hand, Molinet's ill-posedness result does not apply to (1.3) since we have removed the part responsible for the discontinuity. On the other hand, by a slight modification of the argument in [8], we see that the solution map for (1.3) is not uniformly continuous below $L^2(\mathbb{T})$ (see [16]). This, in particular, implies that one cannot expect well-posedness of (1.3) in $H^s(\mathbb{T})$ for $s < 0$ via the standard fixed point argument.

There are, however, positive results for (1.3) in $H^s(\mathbb{T})$ for $s < 0$. Christ, Holmer, and Tataru [15] established an a priori bound on the growth of (smooth) solutions in the H^s-topology for $s \geq -1/6$. In Section 4, we show that the solution map for (1.3) is continuous in $L^2(\mathbb{T})$ endowed with weak topology. These results have counterparts for (1.1) on $\mathbb{R}$.

In [16], Colliander and Oh considered the well-posedness question of (1.3) below $L^2(\mathbb{T})$ with randomized initial data of the form

$$u_0(x;\omega) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+|n|^{2\alpha}}} e^{inx}, \tag{1.4}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a family of independent standard complex-valued Gaussian random variables. It is known (see [40]) that $u_0(\omega) \in H^{\alpha-1/2-\varepsilon} \setminus H^{\alpha-1/2}$ almost surely in ω for any $\varepsilon > 0$ and that u_0 of the form (1.4) is a typical element in the support of the Gaussian measure

$$d\rho_\alpha = Z_\alpha^{-1} \exp\Big(-\frac{1}{2}\int |u|^2 - \frac{1}{2}|D^\alpha u|^2\,dx\Big) \prod_{x\in\mathbb{T}} du(x), \tag{1.5}$$

where $D = \sqrt{-\partial_x^2}$. In [16], local-in-time solutions were constructed for (1.3) almost surely (with respect to ρ_α) in $H^s(\mathbb{T})$ for each $s > -1/3$ ($s = \alpha - 1/2 - \varepsilon$ for small $\varepsilon > 0$), and global-in-time solutions almost surely in $H^s(\mathbb{T})$ for all $s > -1/12$. The argument is based on the fixed point argument around the linear solution, exploiting nonlinear smoothing under randomization on initial data.

The same technique can be applied to study the well-posedness issue of (1.3) with initial data of the form

$$u_0(x;\omega) = v_0(x) + \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\sqrt{1+|n|^{2\alpha}}} e^{inx}, \tag{1.6}$$

where v_0 is in $L^2(\mathbb{T})$. The initial data of the form (1.6) may be of physical importance since smooth data may be perturbed by a rough random noise; that is, initial data, which are smooth in an ideal situation, may be of low regularity in practice due to a noise. This is one of the reasons that we are interested in having a formulation of NLS below L^2.

Another physically relevant issue is the study of (1.3) with initial data of the form (1.4) when $\alpha = 0$. The Gaussian measure ρ_α then corresponds to the white noise on $\mathbb{T}$ (up to a multiplicative constant). It is conjectured (see [40]) that the white noise is invariant under the flow of the cubic NLS (1.1). In [35], Oh, Quastel, and Valkó proved that the white noise is a weak limit of probability measures that are invariant under the flow of (1.1) and (1.3). Note that the white noise ρ_0 is supported on $H^{-1/2-\varepsilon}(\mathbb{T}) \setminus H^{-1/2}(\mathbb{T})$ for $\varepsilon > 0$ (more precisely, on $B^{-1/2}_{2,\infty}$). Such a low regularity seems to be out of reach at this point. Hence, the result in [35] implies only a version of "formal" invariance of the white noise due to lack of well-defined flow of NLS on the support of the white noise. In view of Molinet's ill-posedness below $L^2(\mathbb{T})$, we need to pursue this issue with (1.3) in place of (1.1). In this respect, the result in [16] can be regarded as a partial progress toward this goal.

Note that the white noise (i.e., u_0 in (1.4) with $\alpha = 0$ up to multiplicative constant) can be regarded as a Gaussian randomization (on the Fourier coefficients) of the delta function $\delta(x) = \sum_n e^{inx}$. It is known (see [28]) that in considering the Cauchy problem (1.1) on $\mathbb{R}$ with the delta function as initial condition, we have either nonexistence or nonuniqueness in $C([-T,T];\mathcal{S}'(\mathbb{R}))$. Moreover, on $\mathbb{T}$, Christ [11] proved a nonuniqueness result of (1.3) in the class $C([-T,T];H^s(\mathbb{T}))$ for $s < 0$. Christ's result states that one cannot have unconditional uniqueness* in $H^s(\mathbb{T})$, $s < 0$. However, this is not an issue since, in discussing well-posedness, we usually construct a unique solution in $C([-T,T];H^s) \cap X_T$, where X_T is an auxiliary function space (such as Strichartz spaces or $X^{s,b}$-spaces).

Lastly, another physical motivation for the study of NLS in the low-regularity setting is the localized induction approximation model for the flow of a vortex filament. The filament at time t is given by a curve $X(x,t)$ in $\mathbb{R}^3$ satisfying

$$X_t = X_x \times X_{xx}, \tag{1.7}$$

where x is the arc length. Then, under the Hasimoto transform [24],

$$u(x,t) = c(x,t)\exp\Big(i\int^x \tau(y,t)\,dy\Big), \tag{1.8}$$

*We say that a solution u is unconditionally unique if it is unique in $C([0,T];H^s)$ *without* intersecting with any auxiliary function space. Unconditional uniqueness is a concept of uniqueness which does not depend on how solutions are constructed (see [26]).

where $c(x,t)$ and $\tau(x,t)$ are the curvature and the torsion of $X(x,t)$, and the transformed function u satisfies the focusing cubic (1.1) on $\mathbb{R}$. Gutiérrez, Rivas, and Vega [23] showed that a smooth filament can develop a sharp corner in finite time, which corresponds, under (1.8), to a Dirac delta singularity for u in (1.1). This necessitates the study of NLS in the low-regularity setting.

This paper is organized as follows. In Section 2, we compare the results for NLS (1.1) on $\mathbb{R}$ and WNLS (1.3) on $\mathbb{T}$. In Section 3, we recall basic aspects of the Wick ordering and the derivation of (1.3) on $\mathbb{T}^2$ following [4]. In Section 4, we present the proof of the weak continuity of the solution map for (1.3) in $L^2(\mathbb{T})$.

2. NLS on $\mathbb{R}$ and Wick-ordered NLS on $\mathbb{T}$

In this section, we present several results that are common to (1.1) on $\mathbb{R}$ and (1.3) on $\mathbb{T}$. We show a summary of these results in Table 1 below. This analogy suggests that WNLS (1.3) on $\mathbb{T}$ is an appropriate model to study when interested in solutions below $L^2(\mathbb{T})$.

2.1. Well-posedness in L^2

On the real line, Tsutsumi [37] proved global well-posedness of (1.1) in $L^2(\mathbb{R})$. His argument is based on the smoothing properties of the linear Schrödinger operator expressed by the Strichartz estimates and the conservation of the L^2-norm. For the problem on the circle, Bourgain [3] introduced the $X^{s,b}$-space and proved global well-posedness of (1.1) in $L^2(\mathbb{T})$. His argument is based on the periodic L^4-Strichartz and the conservation of the L^2-norm. The same argument can be applied to establish global well-posedness of (1.3) in $L^2(\mathbb{T})$.

2.2. Ill-posedness in H^s for $s < 0$

An application of Bourgain's argument in [5] shows that the solution maps for (1.1) on $\mathbb{R}$ and (1.3) on $\mathbb{T}$ are not C^3 in H^s for $s < 0$. The method consists of examining the differentiability at $\delta = 0$ of the solution map with initial condition $u_0 = \delta\phi$ for some suitable ϕ, that is, differentiability at the zero solution in a certain direction.

On $\mathbb{R}$, Kenig, Ponce, and Vega [28] proved the failure of uniform continuity of the solution map for (1.1) in $H^s(\mathbb{R})$ for $s < 0$ in the focusing case, by constructing a family of smooth soliton solutions. In the defocusing case, Christ, Colliander, and Tao [12] established the same result by constructing a family of smooth

Table 1. Corresponding results for NLS on $\mathbb{R}$ and WNLS on $\mathbb{T}$ (and NLS on $\mathbb{T}$)

	NLS on $\mathbb{R}$	*WNLS on* $\mathbb{T}$	*NLS on* $\mathbb{T}$
GWP in L^2	[37]	[3]	[3]
Ill-posedness below L^2	[28], [12]	[8]	[8], [34]
Well-posedness in $\mathcal{F}L^p$, $p < \infty$	[21] (GWP for $p \in (2, 5/2)$)	[10], [22]	False [10]
A priori bound for $s \geq -1/6$	[29] ([13] for $s > -1/12$)	[15]	Not known
Weak continuity in L^2	[19]	Theorem 2.1	False [34]

approximate solutions. On $\mathbb{T}$, Burq, Gérard, and Tzvetkov [8] (also see [12]) constructed a family of explicit solutions supported on a single mode and showed the corresponding result for (1.1). By a slight modification of their argument, we can also establish the same result for (1.3). It is worthwhile to note that the momentum diverges to ∞ in these examples.

The above ill-posedness results state that the solution map is not smooth or uniformly continuous in H^s below $s < s_c^\infty = 0$. This does not say that (1.1) on $\mathbb{R}$ and (1.3) on $\mathbb{T}$ are ill-posed below L^2; that is, it is still possible to construct continuous flow below L^2. These results instead state that the fixed point argument cannot be used to show well-posedness of (1.1) on $\mathbb{R}$ and (1.3) on $\mathbb{T}$ below L^2, since such a method would make solution maps smooth. Compare the above results with the ill-posedness result by Molinet [34]—the discontinuity of the solution map below $L^2(\mathbb{T})$ for the periodic NLS (1.1).

2.3. Well-posedness in $\mathcal{F}L^p$

Define the Fourier–Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{R})$ equipped with the norm $\|f\|_{\mathcal{F}L^{s,p}(\mathbb{R})} = \|\langle\xi\rangle^s \widehat{f}(\xi)\|_{L^p(\mathbb{R})}$ with $\langle\cdot\rangle = 1 + |\cdot|$. When $s = 0$, we set $\mathcal{F}L^p = \mathcal{F}L^{0,p}$. The homogeneous $\mathcal{F}\dot{L}^{s,p}$-norm is invariant under the dilation scaling when $sp = -1$.

In [39], Vargas and Vega constructed (both local and global-in-time) solutions for initial data with infinite L^2-norm under certain conditions. This class of initial data, in particular, contains those satisfying

$$\left|\frac{d^j}{d\xi^j}\widehat{u}_0(\xi)\right| \lesssim \langle\xi\rangle^{-\alpha-j}, \quad j = 0, 1, \text{ for some } \alpha > \frac{1}{6}. \tag{2.1}$$

We point out that u_0 satisfying (2.1) is in $\mathcal{F}L^p(\mathbb{R})$ with $p > 1/\alpha$. Grünrock [21] considered (1.1) on $\mathbb{R}$ with initial data in $\mathcal{F}L^p(\mathbb{R})$ and proved local well-posedness for $p < \infty$ and global well-posedness for $2 < p < 5/2$. The method relies on the Fourier restriction norm method. For the global-in-time argument, he adapted Bourgain's [6] high-low method, where he separated a function in terms of the size of its Fourier coefficient instead of its frequency size as in [6].

On $\mathbb{T}$, Christ [10] applied the power series method to construct local-in-time solutions (without uniqueness) for (1.3) in $\mathcal{F}L^p(\mathbb{T})$ for $p < \infty$. Grünrock and Herr [22] proved the same result (with uniqueness in a suitable $X^{s,b}$-space) via the fixed point argument. A subtraction of $2(\fint |u|^2\, dx)u$ in the nonlinearity in (1.3) is essential for continuous dependence. In [10], it was also stated (without proof) that (1.3) is globally well-posed in $\mathcal{F}L^p$ for sufficiently small (smooth) initial data.

2.4. A priori bound

Koch and Tataru [29] established an a priori bound on (smooth) solutions for (1.1) in $H^s(\mathbb{R})$ for $s \geq -1/6$ in the following form: given any $M > 0$, there exist $T, C > 0$ such that for any initial $u_0 \in L^2$ with $\|u_0\|_{H^s} \leq M$, we have $\sup_{t\in[0,T]} \|u(t)\|_{H^s} \leq C\|u_0\|_{H^s}$, where u is a solution of (1.1) with initial condition u_0 (see [13] for a related result). This result yields the existence of weak solutions (without uniqueness). In the periodic setting, Christ, Holmer, and Tataru [15] proved the same result for (1.3) when $s \geq -1/6$. In [29], relating mKdV and NLS through modulated wave train solutions, Koch and Tataru indicated how the regularity $s = -1/6$ arises by associating mKdV with initial data in L^2 to (1.1) with initial data in $H^{-1/6}$.

2.5. Weak continuity in L^2

The Galilean invariance for (1.1) yields the critical regularity $s_c^\infty = 0$. That is, the solution map is not uniformly continuous in H^s for $s < s_c^\infty = 0$. However, it does not imply that the solution map is not continuous in H^s for $s < 0$ (at least on $\mathbb{R}$). Heuristically speaking, given $s_0 \in \mathbb{R}$, one can consider the weak continuity of the solution map in H^{s_0} as an intermediate step between establishing the continuity in (the strong topology of) H^{s_0} and proving the continuity in H^s for $s < s_0$. For example, recall that if f_n converges weakly in H^{s_0}, then it converges strongly in H^s for $s < s_0$ (at least in bounded domains). Indeed if there is sufficient regularity for the solution map in H^s for some $s < s_0$, then its weak continuity in H^{s_0} can be treated by the approach used in the works of Martel and Merle [32], [33] and Kenig and Martel [27] related to the asymptotic stability of solitary waves. In these works, weak continuity of the flow map plays a central role in the study of the linearized operator around the solitary wave and in rigidity theorems (see [18] for a nice discussion on this issue).

There are several recent results in this direction. On $\mathbb{R}$, Goubet and Molinet [19] proved the weak continuity of the solution map for (1.1) in $L^2(\mathbb{R})$. Cui and Kenig [17] and [18] proved the weak continuity in the s_c^∞-critical Sobolev spaces for other dispersive PDEs. However, on $\mathbb{T}$, Molinet [34] showed that the solution map for (1.1) is not continuous from $L^2(\mathbb{T})$ endowed with weak topology to the space of distributions $(C^\infty(\mathbb{T}))^*$. This, in particular, implies that the solution map for (1.1) is not weakly continuous in $L^2(\mathbb{T})$.

When considering the cubic WNLS (1.3), we remove one of the resonant interactions. Indeed, we have the following result on the weak continuity of the solution map for (1.3).

THEOREM 2.1 (WEAK CONTINUITY OF WNLS ON $L^2(\mathbb{T})$)

Suppose that $u_{0,n}$ converges weakly to u_0 in $L^2(\mathbb{T})$. Let u_n and u denote the unique global solutions of (1.3) with initial data $u_{0,n}$ and u_0, respectively. Then, given $T > 0$, we have the following:

(a) *u_n converges weakly to u in $L^4_{T,x} := L^4([-T,T]; L^4(\mathbb{T}))$.*

(b) *For any $|t| \leq T$, $u_n(t)$ converges weakly to $u(t)$ in $L^2(\mathbb{T})$. Moreover, this weak convergence is uniform for $|t| \leq T$; that is, for any $\phi \in L^2(\mathbb{T})$,*

$$\lim_{n\to\infty} \sup_{|t|\leq T} |\langle u_n(t) - u(t), \phi\rangle_{L^2}| = 0.$$

We do not expect the weak continuity in the Strichartz space, that is, in $L^6_{T,x}$ (with $|t| \le T$). This is due to the failure of the $L^6_{x,t}$-Strichartz estimate in the periodic setting (see [3]). Although the proof of Theorem 2.1 is essentially contained in [34], we present it in Section 4 for the completeness of our presentation.

3. Wick ordering

3.1. Gaussian measures and Hermite polynomials

In this subsection, we briefly go over the basic relation between Gaussian measures and Hermite polynomials. For the following discussion, we refer to the works of Kuo [30], Ledoux and Talagrand [31], and Janson [25]. A nice summary is given by Tzvetkov in [38, Section 3] for the hypercontractivity of the Ornstein–Uhlenbeck semigroup related to products of Gaussian random variables.

Let ν be the Gaussian measure with mean zero and variance σ, and let $H_n(x;\sigma)$ be the Hermite polynomial of degree n with parameter σ. They are defined by

$$e^{tx-(1/2)\sigma t^2} = \sum_{n=0}^{\infty} \frac{H_n(x;\sigma)}{n!} t^n.$$

The first three Hermite polynomials are $H_0(x;\sigma) = 1$, $H_1(x;\sigma) = x$, and $H_2(x;\sigma) = x^2 - \sigma$. It is well known that every function $f \in L^2(\nu)$ has a unique series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{H_n(x;\sigma)}{\sqrt{n!\sigma^n}},$$

where $a_n = (n!\sigma^n)^{-1/2} \int_{-\infty}^{\infty} f(x) H_n(x;\sigma)\, d\nu(x)$, $n \ge 0$. Moreover, we have $\|f\|^2_{L^2(\nu)} = \sum_{n=0}^{\infty} a_n^2$. In the following, we set $H_n(x) := H_n(x;1)$.

Now, consider the Hilbert space $L^2(\mathbb{R}^d, \mu_d)$ with $d\mu_d = (2\pi)^{-d/2} \exp(-|x|^2/2)\, dx$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. We define a *homogeneous Wiener chaos of order* n to be an element of the form $\prod_{j=1}^d H_{n_j}(x_j)$, $n = n_1 + \cdots + n_d$. Denote by $\mathcal{K}_n$ the collection of the homogeneous chaoses of order n. Given a homogeneous polynomial $P_n(x) = P_n(x_1, \dots, x_d)$ of degree n, we define *the Wick-ordered monomial* $: P_n(x) :$ to be its projection onto $\mathcal{K}_n$. In particular, we have $: x_j^n : = H_n(x_j)$ and $: \prod_{j=1}^d x_j^{n_j} : = \prod_{j=1}^d H_{n_j}(x_j)$ with $n = n_1 + \cdots + n_d$.

In the following, we discuss the key estimate for the well-posedness results of the cubic WNLS of [4] and [16]. Consider the Hartree–Fock operator $L = \Delta - x \cdot \nabla$, which is the generator for the Ornstein–Uhlenbeck semigroup. Then, by the hypercontractivity of the Ornstein–Uhlenbeck semigroup $U(t) = e^{Lt}$, we have the following proposition.

PROPOSITION 3.1

Fix $q \ge 2$. For every $f \in L^2(\mathbb{R}^d, \mu_d)$ and $t \ge (1/2)\log(q-1)$, we have

$$\|U(t) f\|_{L^q(\mathbb{R}^d,\mu_d)} \le \|f\|_{L^2(\mathbb{R}^d,\mu_d)}. \tag{3.1}$$

It is known that the eigenfunction of L with eigenvalue $-n$ is precisely the homogeneous Wiener chaos of order n. Thus, we have the following dimension-independent estimate.

PROPOSITION 3.2

Let $F(x)$ be a linear combination of homogeneous chaoses of order n. Then, for $q \geq 2$, we have

$$\|F(x)\|_{L^q(\mathbb{R}^d,\mu_d)} \leq (q-1)^{n/2}\|F(x)\|_{L^2(\mathbb{R}^d,\mu_d)}. \tag{3.2}$$

The proof is basically the same as that in [38, Propositions 3.3–3.5]. We only have to note that $F(x)$ is an eigenfunction of $U(t)$ with eigenvalue e^{-nt}. The estimate (3.2) follows from (3.1) by evaluating (3.1) at time $t = 1/2\log(q-1)$. In [4], [16], and [38], Proposition 3.2 was used in a crucial manner to estimate random elements in the nonlinearity after dyadic decompositions.

In order to motivate $:|u|^4:$, the Wick-ordered $|u|^4$, for a complex-valued function u, we consider the Wick ordering on complex Gaussian random variables. Let g denote a standard complex-valued Gaussian random variable. Then, g can be written as $g = x + iy$, where x and y are independent standard real-valued Gaussian random variables. Note that the variance of g is $\text{Var}(g) = 2$.

Next, we investigate the Wick-ordering on $|g|^{2n}$ for $n \in \mathbb{N}$, that is, the projection of $|g|^{2n}$ onto $\mathcal{K}_{2n}$. When $n = 1$, $|g|^2 = x^2 + y^2$ is Wick-ordered into

$$:|g|^2: = (x^2-1) + (y^2-1) = |g|^2 - \text{Var}(g).$$

When $n = 2$, $|g|^4 = (x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4$ is Wick-ordered into

$$\begin{aligned} :|g|^4: &= (x^4 - 6x^2 + 3) + 2(x^2-1)(y^2-1) + (y^4 - 6y^2 + 3) \\ &= x^4 + 2x^2y^2 + y^4 - 8(x^2+y^2) + 8 \\ &= |g|^4 - 4\,\text{Var}(g)|g|^2 + 2\,\text{Var}(g)^2, \end{aligned} \tag{3.3}$$

where we use $H_4(x) = x^4 - 6x^2 + 3$. In general, we have $:|g|^{2n}: \in \mathcal{K}_{2n}$. Moreover, we have

$$:|g|^{2n}: = |g|^{2n} + \sum_{j=0}^{n-1} a_j|g|^{2j} = |g|^{2n} + \sum_{j=0}^{n-1} b_j :|g|^{2j}: . \tag{3.4}$$

This follows from the fact that $|g|^{2n}$, as a polynomial in x and y only with even powers, is orthogonal to any homogeneous chaos of odd order, and it is radial; that is, it depends only on $|g|^2 = x^2 + y^2$. Note that $:|g|^{2n}:$ can also be obtained from the Gram–Schmidt process applied to $|g|^{2k}$, $k = 0, \dots, n$, with $\mu_2 = (2\pi)^{-1}\exp(-(x^2+y^2)/2)\,dx\,dy$.

3.2. Wick-ordered cubic NLS

In [4], Bourgain considered the issue of the invariant Gibbs measure for (1.1) on $\mathbb{T}^2$ in the defocusing case. In this subsection, we present his argument to derive

(1.3) on $\mathbb{T}^2$. First, consider the finite-dimensional approximation to (1.1):

$$\begin{cases} iu_t^N - \Delta u^N + \mathbb{P}_N(|u^N|^2 u^N) = 0, \\ u|_{t=0} = \mathbb{P}_N u_0, \quad (x,t) \in \mathbb{T}^2 \times \mathbb{R}, \end{cases} \tag{3.5}$$

where $u^N = \mathbb{P}_N u$ and $\mathbb{P}_N$ is the Dirichlet projection onto the frequencies $|n| \leq N$. This is a Hamiltonian equation with Hamiltonian $H(u^N)$, where H is as in (1.2) with the + sign. On $\mathbb{T}^2$, the Gaussian part $d\rho = Z^{-1} \exp(-1/2 \int |\nabla u|^2 \, dx) \times \prod_{x \in \mathbb{T}^2} du(x)$ of the Gibbs measure is supported on $\bigcap_{s<0} H^s(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$. However, the nonlinear part $\int |\mathbb{P}_N u|^4 \, dx$ of the Hamiltonian diverges to ∞ as $N \to \infty$ almost surely on the support of the Wiener measure ρ. Hence, we need to *renormalize* the nonlinearity.

A typical element in the support of the Wiener measure ρ is given by

$$u(x;\omega) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x}, \tag{3.6}$$

where $\{g_n\}_{n \in \mathbb{Z}}$ is a family of independent standard complex-valued Gaussian random variables.* For simplicity, assume that $\mathrm{Var}(g_n) = 1$. For u of the form (3.6), define a_N by

$$a_N = \mathbb{E}\Big[\fint |u^N|^2 \, dx \Big] = \sum_{|n| \leq N} \frac{1}{1+|n|^2}.$$

We have $a_N \sim \log N$ for large N. We define the Wick-ordered truncated Hamiltonian H_N by

$$\begin{aligned} H_N(u^N) &= \frac{1}{2} \int_{\mathbb{T}^2} |\nabla u^N|^2 \, dx + \frac{1}{4} \int_{\mathbb{T}^2} :|u^N|^4: \, dx \\ &= \frac{1}{2} \int_{\mathbb{T}^2} |\nabla u^N|^2 \, dx + \frac{1}{4} \int_{\mathbb{T}^2} |u^N|^4 - 4a_N |u^N|^2 + 2a_N^2 \, dx \end{aligned} \tag{3.7}$$

(cf. (3.7) with (3.3)). From (3.7), we obtain a Hamiltonian equation that is the Wick-ordered version of (3.5):

$$iu_t^N - \Delta u^N + \mathbb{P}_N(|u^N|^2 u^N) - 2a_N u^N = 0. \tag{3.8}$$

Let $c_N = \fint |u^N|^2 - a_N$. We see that $c_\infty(\omega) = \lim_{N\to\infty} c_N(\omega) < \infty$ almost surely. Under the change of variables $v^N = e^{-2ic_N t} u^N$, (3.8) becomes

$$iv_t^N - \Delta v^N + \mathbb{P}_N \Big(|v^N|^2 - 2 \fint |v^N|^2 \Big) v^N = 0. \tag{3.9}$$

Finally, letting $N \to \infty$, we obtain the Wick-ordered NLS

$$iv_t - \Delta v + \Big(|v|^2 - 2 \fint |v|^2 \Big) v = 0. \tag{3.10}$$

*The expression (3.6) is a representation of elements in the support of $d\tilde{\rho} = \tilde{Z}^{-1} \exp(-1/2 \int |u|^2 - 1/2 \int |\nabla u|^2) \prod_{x \in \mathbb{T}^2} du(x)$ due to the problems at the zero Fourier mode for ρ. However, we do not worry about this issue.

On $\mathbb{T}$, one can repeat the same argument. Note the following issue. On the one hand, the assumption that $u(t)$ is of the form (1.4) and is natural for $\alpha \in \mathbb{N}\cup\{0\}$ in view of the conservation laws. On the other hand, $c_N = \fint |u^N|^2 - \mathbb{E}[\fint |u^N|^2] < \infty$ for $\alpha > 1/4$. That is, $\alpha = 1$ is the smallest integer value of such α. In this case, there is no need for the WNLS (1.3) since $u \in H^{1/2-} \subset L^2$ a.s. for $\alpha = 1$.

4. Weak continuity of the Wick-ordered cubic NLS in $L^2(\mathbb{T})$

In this section, we present the proof of Theorem 2.1. First, write (1.3) as an integral equation:

$$u(t) = S(t)u_0 \pm i \int_0^t S(t-t')\mathcal{N}(u)(t')\,dt', \tag{4.1}$$

where $\mathcal{N}(u) = (|u|^2 - 2\fint |u|^2)u$ and $S(t) = e^{-i\partial_x^2 t}$. Define $\mathcal{N}_1(u_1,u_2,u_3)$ and $\mathcal{N}_2(u_1,u_2,u_3)$ by

$$\mathcal{N}_1(u_1,u_2,u_3) = \sum_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} \widehat{u}_1(n_1)\overline{\widehat{u}}_2(n_2)\widehat{u}_3(n_3)e^{inx},$$

$$\mathcal{N}_2(u_1,u_2,u_3) = -\sum_n \widehat{u}_1(n)\overline{\widehat{u}}_2(n)\widehat{u}_3(n)e^{inx}.$$

Moreover, let $\mathcal{N}_j(u) := \mathcal{N}_j(u,u,u)$. Then, we have $\mathcal{N}(u) = \mathcal{N}_1(u) + \mathcal{N}_2(u)$.

In [3], Bourgain established global well-posedness of (1.1) (and (1.3)) by introducing a new weighted space-time Sobolev space $X^{s,b}$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^2\rangle^b \widehat{u}(n,\tau)\|_{L^2_{n,\tau}(\mathbb{Z}\times\mathbb{R})}$$

where $\langle\cdot\rangle = 1 + |\cdot|$. Define the local-in-time version $X^{s,b}_\delta$ on $[-\delta,\delta]$ by

$$\|u\|_{X^{s,b}_\delta} = \inf\big\{\|\widetilde{u}\|_{X^{s,b}};\, \widetilde{u}|_{[-\delta,\delta]} = u\big\}.$$

In the following, we list the estimates needed for local well-posedness of (1.3). Let $\eta(t)$ be a smooth cutoff function such that $\eta = 1$ on $[-1,1]$ and $\eta = 0$ on $[-2,2]$.

- Homogeneous linear estimate: for $s, b \in \mathbb{R}$, we have

$$\|\eta(t)S(t)f\|_{X^{s,b}} \le C_1\|f\|_{H^s}. \tag{4.2}$$

- Nonhomogeneous linear estimate: for $s \in \mathbb{R}$ and $b > 1/2$, we have

$$\Big\|\eta(t)\int_0^t S(t-t')F(t')\,dt'\Big\|_{X^{s,b}_\delta} \lesssim C(\delta)\|F\|_{X^{s,b-1}_\delta}. \tag{4.3}$$

- Periodic L^4-Strichartz estimate: Zygmund [41] proved

$$\|S(t)f\|_{L^4_{x,t}(\mathbb{T}\times[-1,1])} \lesssim \|f\|_{L^2}, \tag{4.4}$$

which was improved by Bourgain [3]:

$$\|u\|_{L^4_{x,t}(\mathbb{T}\times[-1,1])} \lesssim \|u\|_{X^{0,3/8}}. \tag{4.5}$$

These estimates allow us to prove local well-posedness of (1.3) via the fixed point argument such that a solution u exists on the time interval $[-\delta,\delta]$ with $\delta = \delta(\|u_0\|_{L^2})$. Moreover, we have $\|u\|_{X_\delta^{0,1/2+}} \lesssim \|u_0\|_{L^2}$. Such local-in-time solutions can be extended globally in time thanks to the L^2-conservation.

Now, fix $u_0 \in L^2(\mathbb{T})$, and let $u_{0,n}$ converge weakly to u_0 in $L^2(\mathbb{T})$. Denote by u_n and u the unique global solutions of (1.3) with initial data $u_{0,n}$ and u_0. By the uniform boundedness principle, we have $\|u_{0,n}\|_{L^2}, \|u_0\|_{L^2} \le C$ for some $C > 0$. Hence, the local well-posedness guarantees the existence of the solutions u_n, u on the time interval $[-\delta,\delta]$ with $\delta = \delta(C)$, uniformly in n. In the following, we assume $\delta = 1$. That is, we assume that all the estimates hold on $[-1,1]$. (Otherwise we can replace $[-1,1]$ by $[-\delta,\delta]$ for some $\delta > 0$ and iterate the argument in view of the L^2-conservation.)

4.1. Proof of Theorem 2.1(a)

First, we show that u_n converges to u as space-time distributions.

- Linear part: Since $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$, we have $\|u_{0,n} - u_0\|_{H^{-\varepsilon}(\mathbb{T})} \to 0$ for any $\varepsilon > 0$. Let $\phi \in C_c^\infty(\mathbb{T}\times\mathbb{R})$ be a test function. Then, by Hölder inequality and (4.2), we have

$$\begin{aligned}\iint \eta(t)S(t)(u_{0,n}-u_0)\phi(x,t)\,dx\,dt &\le \|\eta(t)S(t)(u_{0,n}-u_0)\|_{X^{-\varepsilon,1/2+}}\|\phi\|_{X^{\varepsilon,-1/2-}}\\ &\lesssim C_\phi \|u_{0,n}-u_0\|_{H^{-\varepsilon}} \to 0.\end{aligned}$$

Hence, $\eta(t)S(t)u_{0,n}$ converges to $\eta(t)S(t)u_0$ as space-time distributions.

- Nonlinear part: Let $\mathcal{M}(u)$ denote the Duhamel term; that is,

$$\mathcal{M}(u)(t) := \pm i\int_0^t S(t-t')\mathcal{N}(u)(t')\,dt'.$$

Similarly, define $\mathcal{M}_j(u_1,u_2,u_3)$ by

$$\mathcal{M}_j(u_1,u_2,u_3)(t) := \pm i\int_0^t S(t-t')\mathcal{N}_j(u_1,u_2,u_3)(t')\,dt'$$

for $j = 1,2$. Also, let $\mathcal{M}_j(u) := \mathcal{M}_j(u,u,u)$.

From the local theory, we have $\|u_n\|_{X_1^{0,1/2+}} \lesssim \|u_{0,n}\|_{L^2} \le C$ for all n. Thus, there exists a subsequence u_{n_k} converging weakly to some v in $X_1^{0,1/2+}$. It follows from [34, Lemmas 2.2, 2.3] that $\mathcal{N}_j$, $j = 1,2$, is weakly continuous from $X_1^{0,1/2+}$ into $X_1^{0,-7/16}$. Hence, $\mathcal{N}_j(u_{n_k}) \rightharpoonup \mathcal{N}_j(v)$ in $X_1^{0,-7/16}$.

Recall the following. Given Banach spaces X and Y with a continuous linear operator $T: X \to Y$, we have $T^*: Y^* \to X^*$. If $f_n \rightharpoonup f$ in X, then we have, for $\phi \in Y^*$, $\langle T(f_n - f), \phi\rangle = \langle f_n - f, T^*\phi\rangle \to 0$ since $T^*\phi \in X^*$. Hence, $Tf_n \rightharpoonup Tf$ in Y.

It follows from (4.3) that the map $F \longmapsto \int_0^t S(t-t')F(t')\,dt'$ is linear and continuous from $X_1^{0,-7/16}$ into $X_1^{0,1/2+}$. Hence, $\mathcal{M}(u_{n_k}) \rightharpoonup \mathcal{M}(v)$ in $X_1^{0,1/2+}$. In particular, $\mathcal{M}(u_{n_k})$ converges to $\mathcal{M}(v)$ as space-time distributions.

Since u_{n_k} is a solution to (1.3) with initial data u_{0,n_k}, we have

$$u_{n_k} = \eta S(t) u_{0,n_k} + \eta \mathcal{M}(u_{n_k}).$$

By taking the limits of both sides, we obtain

$$v = \eta S(t) u_0 + \eta \mathcal{M}(v),$$

where the equality holds in the sense of space-time distributions. From the uniqueness of solutions to (1.3) in $X_1^{0,1/2+}$, we have $v = u$ in $X_1^{0,1/2+}$.

In fact, we can show that uniqueness of solutions to (1.3) holds in $L^4_{x,t}(\mathbb{T} \times [-1,1])$ with little effort. For simplicity, we replace $\mathcal{N}(u)$ in (4.1) by $|u|^2 u$. Then, by (4.4) and (4.5), we have

$$\begin{aligned}\|\eta(t) u\|_{L^4_{x,t}} &\le \|\eta(t) S(t) u_0\|_{L^4_{x,t}} + \left\| \eta(t) \int_0^t S(t-t') |\eta u(t')|^2 \eta u(t')\, dt' \right\|_{L^4_{x,t}} \\ &\lesssim \|u_0\|_{L^2_x} + \left\| \eta(t) \int_0^t S(t-t') |\eta u(t')|^2 \eta u(t')\, dt' \right\|_{X^{0,3/8}}.\end{aligned}$$

Moreover, we can use (4.3), duality, $L^4_{x,t} L^4_{x,t} L^4_{x,t} L^4_{x,t}$-Hölder inequality, and (4.5) to estimate the second term by

$$\begin{aligned}&\lesssim \| |\eta u|^2 \eta u\|_{X^{0,-3/8}} \\ &= \sup_{\|v\|_{X^{0,3/8}}=1} \iint v |\eta u|^2 (\eta u)\, dx\, dt \le \sup_{\|v\|_{X^{0,3/8}}=1} \|v\|_{L^4_{x,t}} \|\eta u\|^3_{L^4_{x,t}} \le \|\eta u\|^3_{L^4_{x,t}}.\end{aligned}$$

This shows that u is indeed a unique solution in $L^4_{x,t}(\mathbb{T} \times [-1,1])$.

It follows from $(L^4_{x,t}(\mathbb{T} \times [-1,1]))^* \subset (X_1^{0,1/2+})^*$ that weak convergence in $X_1^{0,1/2+}$ implies weak convergence in $L^4_{x,t}(\mathbb{T} \times [-1,1])$. Hence, the subsequence u_{n_k} converges weakly to u in $X_1^{0,1/2+}$ and $L^4_{x,t}(\mathbb{T} \times [-1,1])$. The argument above also shows that u is the *only* weak limit point of u_n in $X_1^{0,1/2+}$ and $L^4_{x,t}(\mathbb{T} \times [-1,1])$. Then, it follows from the boundedness of u_n in $X_1^{0,1/2+}$ and $L^4_{x,t}(\mathbb{T} \times [-1,1])$ that the whole sequence u_n converges weakly to u. Indeed, suppose that the whole sequence u_n does not converge weakly to u. Then, there exists $\phi \in (X_1^{0,1/2+})^*$ such that $\langle u_n, \phi \rangle \nrightarrow \langle u, \phi \rangle$. This, in turn, implies that there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$, there exists $n \ge N$ such that $|\langle u_n - u, \phi \rangle| > \varepsilon$. Given $\varepsilon > 0$, we can construct a subsequence u_{n_k} with $|\langle u_{n_k} - u, \phi \rangle| > \varepsilon$ for each k. However, by repeating the previous argument (from the uniform boundedness of u_{n_k} in $X_1^{0,1/2+}$), u_{n_k} has a sub-subsequence converging to u, which is a contradiction. This establishes Theorem 2.1(a) on $[-1,1]$. □

4.2. Proof of Theorem 2.1(b)

Recall the following embedding. For $b > 1/2$, we have

$$\|u\|_{L^\infty([-1,1]; H^s)} \le C_2 \|u\|_{X_1^{s,b}}. \tag{4.6}$$

Fix $\phi \in L^2(\mathbb{T})$ in the following.

• Linear part: Given $\varepsilon > 0$, choose $\psi \in H^1(\mathbb{T})$ such that $\|\phi - \psi\|_{L^2} < \varepsilon/(2KC_1C_2)$, where $K = \sup_n \|u_{0,n} - u_0\|_{L^2} < \infty$ and C_1, C_2 are as in (4.2) and (4.6). Then, by (4.2) and (4.6), we have

$$\begin{aligned}
&\sup_{|t|\leq 1} |\langle S(t)(u_{0,n} - u_0), \phi\rangle_{L^2}| \\
&\quad\leq \sup_{|t|\leq 1} |\langle S(t)(u_{0,n} - u_0), \psi\rangle_{L^2}| + \sup_{|t|\leq 1} |\langle S(t)(u_{0,n} - u_0), \phi - \psi\rangle_{L^2}| \\
&\quad\leq \|S(t)(u_{0,n} - u_0)\|_{L^\infty([-1,1];H^{-1})}\|\psi\|_{H^1} \\
&\qquad + \|S(t)(u_{0,n} - u_0)\|_{L^\infty([-1,1];L^2)}\|\phi - \psi\|_{L^2} \\
&\quad\leq C_\psi \|S(t)(u_{0,n} - u_0)\|_{X_1^{-1,1/2+}} + \frac{\varepsilon}{(2KC_1)}\|S(t)(u_{0,n} - u_0)\|_{X_1^{0,1/2+}} \\
&\quad\leq C\|u_{0,n} - u_0\|_{H^{-1}} + \frac{\varepsilon}{(2K)}\|u_{0,n} - u_0\|_{L^2}.
\end{aligned}$$

Hence, there exists N_1 such that for $n \geq N_1$,

$$\sup_{|t|\leq 1} |\langle S(t)(u_{0,n} - u_0), \phi\rangle_{L^2}| < \varepsilon$$

since $u_{0,n}$ converges strongly to u_n in H^{-1}.

• Nonlinear part: Since $u_n \rightharpoonup u$ in $X_1^{0,1/2+}$, we see that $\mathcal{N}(u_n)$ converges strongly to $\mathcal{N}(u)$ in $X_1^{-1,-7/16}$ (see the proof of [34, Lemmas 2.2, 2.3]). Then, it follows from (4.3) that $\mathcal{M}(u_n)$ converges strongly to $\mathcal{M}(u)$ in $X_1^{-1,1/2+}$.

Given $\varepsilon > 0$, choose $\psi \in H^1(\mathbb{T})$ such that $\|\phi - \psi\|_{L^2} < \varepsilon/(2KC_2)$, where $K = \sup_n \|\mathcal{M}(u_n) - \mathcal{M}(u_n)\|_{X_1^{0,1/2+}} < \infty$ and C_2 is as in (4.6). Then, by (4.6), we have

$$\begin{aligned}
&\sup_{|t|\leq 1} |\langle \mathcal{M}(u_n) - \mathcal{M}(u), \phi\rangle| \\
&\quad\leq \sup_{|t|\leq 1} |\langle \mathcal{M}(u_n) - \mathcal{M}(u), \psi\rangle_{L^2}| + \sup_{|t|\leq 1} |\langle \mathcal{M}(u_n) - \mathcal{M}(u), \phi - \psi\rangle_{L^2}| \\
&\quad\leq \|\mathcal{M}(u_n) - \mathcal{M}(u)\|_{L^\infty([-1,1];H^{-1})}\|\psi\|_{H^1} \\
&\qquad + \|\mathcal{M}(u_n) - \mathcal{M}(u)\|_{L^\infty([-1,1];L^2)}\|\phi - \psi\|_{L^2} \\
&\quad\leq C_\psi \|\mathcal{M}(u_n) - \mathcal{M}(u)\|_{X_1^{-1,1/2+}} + \frac{\varepsilon}{2K}\|\mathcal{M}(u_n) - \mathcal{M}(u)\|_{X_1^{0,1/2+}}.
\end{aligned}$$

Hence, there exists N_2 such that for $n \geq N_2$,

$$\sup_{|t|\leq 1} |\langle \mathcal{M}(u_n) - \mathcal{M}(u), \phi\rangle| < \varepsilon.$$

Therefore, we have

$$\lim_{n\to\infty} \sup_{|t|\leq 1} |\langle u_n(t) - u(t), \phi\rangle_{L^2}| = 0.$$

Given $[-T, T]$, we can iterate the argument above on each $[j, j+1]$ and obtain Theorem 2.1. □

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