Ann. Funct. Anal. 8 (2017), no. 2, 190–198
http://dx.doi.org/10.1215/20088752-3784361
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

A NOTE ON WEYL-TYPE THEOREMS AND RESTRICTIONS

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Communicated by J. Esterle

Abstract. Let X be a Banach space, let $T \in L(X)$ be a bounded linear operator, and let T_n be a restriction of T on $R(T^n)$. This article should be viewed as a note on the research work of Carpintero et al. We give here several different proofs for completeness, and we show the relations of T and T_n to a much greater extent. Moreover, we give sufficient conditions for which Weyl-type theorems for T are equivalent to Weyl-type theorems for T_n.

1. Introduction and preliminaries

Throughout this article, let $L(X)$ denote the Banach algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For $T \in L(X)$, let T^* denote its dual, $N(T)$ its kernel, $\alpha(T)$ its nullity, $R(T)$ its range, $\beta(T)$ its defect, $\sigma(T)$ its spectrum, and $\sigma_a(T)$ its approximate point spectrum. Two other classical quantities in operator theory are the *ascent* and *descent* of an operator T, defined as $p(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and $q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, respectively (the infimum of an empty set is defined to be ∞). Note that if $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$ (see [1, Theorem 3.3]). If $R(T)$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$), then T is said to be *upper semi-Fredholm* (resp., *lower semi-Fredholm*). If T is both upper and lower semi-Fredholm, then T is said to be *Fredholm*. If T is either upper or lower semi-Fredholm, then T is said to be *semi-Fredholm*, and its index is defined by $\operatorname{ind} T = \alpha(T) - \beta(T)$. An operator T is called *Browder* (resp., *upper semi-Browder*) if T is Fredholm and $p(T) = q(T) < \infty$ (resp., T is upper semi-Fredholm and $p(T) < \infty$). The operator T is called *Weyl* (resp., *upper*

Received Apr. 19, 2016; Accepted Aug. 14, 2016.
2010 *Mathematics Subject Classification.* Primary 47A10; Secondary 47A11, 47A53, 47A55.
Keywords. restriction, Weyl-type theorems, semi-Fredholm, pole of the resolvent.

semi-Weyl) if T is Fredholm and $\operatorname{ind} T = 0$ (resp., T is upper semi-Fredholm and $\operatorname{ind} T \leq 0$). For $T \in L(X)$, let us define the *Weyl spectrum* and the *upper semi-Weyl spectrum* of T, respectively, as follows:

$$\sigma_{\mathrm{w}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}$$

and

$$\sigma_{\mathrm{uw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\}.$$

For each $n \in \mathbb{N}$, define T_n to be the restriction of T on $R(T^n)$, viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_0 = T$). If there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n is Fredholm (resp., upper semi-Fredholm), then T is called *B-Fredholm* (resp., *upper semi-B-Fredholm*). Analogously, if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n is Browder (resp. upper semi-Browder), then T is called *B-Browder* (resp., *upper semi-B-Browder*). Furthermore, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent of T (see [11, Proposition 50.2]). An operator $T \in L(X)$ is said to be *left Drazin invertible* if $p(T) < \infty$ and $R(T^{p(T)+1})$ is closed; $T \in L(X)$ is called *Drazin invertible* if the ascent and the descent of T are both finite.

Other spectra related to semi-B-Fredholm operators are defined as follows. The *left Drazin invertible spectrum* is defined by

$$\sigma_{\mathrm{ld}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\}.$$

The *upper semi-B-Browder spectrum* is defined by

$$\sigma_{\mathrm{usbb}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-B-Browder}\}.$$

Clearly, by [2, Theorem 4.91], $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{usbb}}(T)$.

Now, we introduce an important property in local spectral theory. The localized version of this property was introduced by Finch [9], and in the framework of Fredholm theory this property has been characterized in several ways (see Chapter 3 of [1]). An operator $T \in L(X)$ is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for brevity) if for every open disk $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at λ_0, the only analytic function $f\colon \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T) f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0}$$

is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator T is said to have SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$. Evidently, T (or T^*) has SVEP at $\lambda \in \operatorname{iso} \sigma(T)$, and T has SVEP at $\lambda \in \operatorname{iso} \sigma_a(T)$. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \quad \Rightarrow \quad T \text{ has SVEP at } \lambda, \tag{1.1}$$

$$q(\lambda I - T) < \infty \quad \Rightarrow \quad T^* \text{ has SVEP at } \lambda. \tag{1.2}$$

By Chapter 3 of [1], the implications (1.1) and (1.2) are actually equivalences whenever $T \in L(X)$ is semi-Fredholm.

Let $\Pi_{00}(T)$ denote the set of all poles of T. We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if $\lambda I - T$ is left Drazin inverse. Let $\Pi^a(T)$ denote the set of all left poles of T. We also set

$$p_{00}(T)=\{\lambda\in\Pi_{00}(T):\alpha(\lambda I-T)<\infty\},$$
$$p_{00}^a(T)=\{\lambda\in\Pi^a(T):\alpha(\lambda I-T)<\infty\},$$
$$E(T)=\{\lambda\in\operatorname{iso}\sigma(T):0<\alpha(\lambda I-T)\},$$
$$E^a(T)=\{\lambda\in\operatorname{iso}\sigma_a(T):0<\alpha(\lambda I-T)\},$$
$$\pi_{00}(T)=\{\lambda\in\operatorname{iso}\sigma(T):0<\alpha(\lambda I-T)<\infty\},$$

and

$$\pi_{00}^a(T)=\{\lambda\in\operatorname{iso}\sigma_a(T):0<\alpha(\lambda I-T)<\infty\},$$

where $\operatorname{iso}K$ denotes the set of all isolated points of $K\subseteq\mathbb{C}$.

Let $T\in L(X)$. Following Coburn [7], T is said to satisfy *Weyl's theorem*, in symbol (W), if $\sigma(T)\setminus\sigma_{\mathrm{w}}(T)=\pi_{00}(T)$, while, according to Rakočević [12], T is said to satisfy *a-Weyl's theorem*, in symbol (aW), if $\sigma_a(T)\setminus\sigma_{\mathrm{uw}}(T)=\pi_{00}^a(T)$. Following Harte and Lee [10], T is said to satisfy *Browder's theorem*, in symbol (B), if $\sigma(T)\setminus\sigma_{\mathrm{w}}(T)=p_{00}(T)$, while, according to Djordjević and Han [8], T is said to satisfy *a-Browder's theorem*, in symbol (aB), if $\sigma_a(T)\setminus\sigma_{\mathrm{uw}}(T)=p_{00}^a(T)$.

In Section 2, we will need the following basic results.

Lemma 1.1 ([3, Lemma 1.1])**.** *Let $T\in L(X)$ and $p=p(T)<\infty$. Then the following statements are equivalent.*

(1) *There exists $n\ge p+1$ such that $T^n(X)$ is closed,*
(2) *$T^m(X)$ is closed for all $m\ge p$.*

Lemma 1.2. *Let $T\in L(X)$, and let $T_n, n\in\mathbb{N}$ be the restriction of the operator T on the subspace $R(T^n)=T^n(X)$. Then, for all $\lambda\neq 0$, we have*

(1) *$N((\lambda I-T_n)^m)=N((\lambda I-T)^m)$, for any $m\in\mathbb{N}$,*
(2) *$R((\lambda I-T_n)^m)=R((\lambda I-T)^m)\cap R(T^n)$, for any $m\in\mathbb{N}$,*
(3) *$\alpha(\lambda I-T_n)=\alpha(\lambda I-T)$,*
(4) *$\beta(\lambda I-T_n)=\beta(\lambda I-T)$,*
(5) *$p(\lambda I-T_n)=p(\lambda I-T)$,*
(6) *$q(\lambda I-T_n)<\infty\Leftrightarrow q(\lambda I-T)<\infty$.*

Proof. The proofs of (1), (2), (3), (4), and (5) may be found in [3, Lemma 2.1].

(6) Suppose that $q(\lambda I-T)<\infty$. From (2) we show that $R((\lambda I-T_n)^{q(\lambda I-T)})=R((\lambda I-T)^{q(\lambda I-T)})\cap R(T^n)=R((\lambda I-T)^{q(\lambda I-T)+1})\cap R(T^n)=R((\lambda I-T_n)^{q(\lambda I-T)+1})$, so $q(\lambda I-T_n)<\infty$. On the other hand, if $q(\lambda I-T_n)<\infty$, from [4, Lemma 3] it follows that $q(\lambda I-T)<\infty$. □

Lemma 1.3 ([3, Lemma 2.2])**.** *If $R(T^n)$ is closed in X and $R((\lambda I-T_n)^m)$ is closed in $R(T^n)$, then there exists $k\in\mathbb{N}$ such that $R((\lambda I-T)^k)$ is closed in X.*

Lemma 1.4 ([6, Lemma 1.5])**.** *If $R(T^n)$ is closed in X and $R((\lambda I-T_n)^m)$ is closed in $R(T^n)$ for $\lambda\neq 0$, then $R((\lambda I-T)^m)$ is closed in X.*

Lemma 1.5(4) below was first established in [6, Lemma 2.2]. We give here a different proof for completeness.

Lemma 1.5. *Let $T \in L(X)$, and let $T_n, n \in \mathbb{N}$ be the restriction of the operator T on the subspace $R(T^n)$. If $R(T^n)$ is closed, we have*

(1) $\sigma(T_n) \subseteq \sigma(T)$ *and* $\sigma_a(T_n) \subseteq \sigma_a(T)$,
(2) $\sigma_{\mathrm{w}}(T_n) \subseteq \sigma_{\mathrm{w}}(T)$ *and* $\sigma_{\mathrm{uw}}(T_n) \subseteq \sigma_{\mathrm{uw}}(T)$,
(3) *If* $0 \notin \Pi_{00}(T)$, *then* $\sigma(T) = \sigma(T_n)$,
(4) *If* $0 \notin \Pi^a(T)$, *then* $\sigma_a(T) = \sigma_a(T_n)$.

Proof. The proofs of (1) and (2) may be found in [6, Lemma 2.1].

(3) By (1), we need only to show that $\sigma(T) \subseteq \sigma(T_n)$. Let $\lambda \notin \sigma(T_n)$. Here we consider the two difference cases $\lambda \neq 0$ and $\lambda = 0$. If $\lambda \neq 0$, Lemma 1.2 implies $\alpha(\lambda I - T) = \alpha(\lambda I - T_n) = \beta(\lambda I - T_n) = \beta(\lambda I - T) = 0$, so $\lambda \notin \sigma(T)$. If $\lambda = 0$, then $p(T_n) = q(T_n) = 0$. By [4, Lemmas 2 and 3], $p(T) = q(T) < \infty$. Suppose that $0 < p(T) = q(T) < \infty$. Then $0 \in \Pi_{00}(T)$, which is a contradiction. Thus $p(T) = q(T) = 0$, which implies $\lambda \notin \sigma(T)$. Consequently, $\sigma(T) \subseteq \sigma(T_n)$.

(4) By (1), we need only to show that $\sigma_a(T) \subseteq \sigma_a(T_n)$. Let $\lambda \notin \sigma_a(T_n)$. Here we consider the two difference cases $\lambda \neq 0$ and $\lambda = 0$. If $\lambda \neq 0$, then $N(\lambda I - T_n) = \{0\}$ and $R(\lambda I - T_n)$ is closed. By Lemmas 1.2 and 1.4, $N(\lambda I - T) = \{0\}$ and $R(\lambda I - T)$ is closed, which implies that $\lambda I - T$ is bounded below, so $\lambda \notin \sigma_a(T)$. If $\lambda = 0$, then $p(T_n) = 0$ and $R(T_n)$ is closed. By [4, Lemma 2], $p(T) < \infty$. Moreover, by [4, Remark 1], $p(T) = \inf\{k \in \mathbb{N} : T_k \text{ is injective}\} \leq n$. By Lemma 1.1, $R(T^m)$ is closed for any $m \geq p(T)$, since $R(T^{n+1}) = R(T_n)$ is closed. Thus, if $0 \in \sigma_a(T)$, then $0 \in \Pi^a(T)$, which is a contradiction. Hence, $0 \notin \sigma_a(T)$. Consequently, $\sigma_a(T) \subseteq \sigma_a(T_n)$. □

From [3], [5], and [6], it follows that several spectral properties are the same for T and T_n. First we give a number of different proofs for completeness, and then we show the relations of T and T_n to a much greater extent. Moreover, we give sufficient conditions for which Weyl-type theorems for T are equivalent to Weyl-type theorems for T_n.

2. Relations of T and T_n

Theorems 2.1 and 2.2 below were first established in [3, Lemmas 2.3 and 2.4], respectively. We give some different proofs for completeness.

Theorem 2.1. *Let $T \in L(X)$. If $0 \notin \Pi_{00}(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $\pi_{00}(T) \subseteq \pi_{00}(T_n)$.*

Proof. Let $\lambda \in \pi_{00}(T)$. Then $\lambda \in \operatorname{iso}\sigma(T)$ and $0 < \alpha(\lambda I - T) < \infty$; Lemma 1.5 implies $\lambda \in \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T_n)$. Now, here we consider two difference cases. Case I: if $\lambda \neq 0$, by Lemma 1.2, $0 < \alpha(\lambda I - T_n) = \alpha(\lambda I - T) < \infty$, so $\lambda \in \pi_{00}(T_n)$. Case II: if $\lambda = 0$, we claim that $\alpha(T_n) > 0$. Suppose that $\alpha(T_n) = 0$. We have $p(T_n) = 0$ and, by [4, Lemma 2], $p(T) < \infty$. Also $\alpha(T^n) < \infty$, because $\alpha(T) < \infty$. We have that T^n is upper semi-Fredholm, since $R(T^n)$ is closed. Thus T is upper semi-Fredholm by [2, Theorem 1.46], and so T^{n+1} is upper semi-Fredholm by [2, Theorem 1.42]. Hence $R(T_n) = R(T^{n+1})$ is closed, which implies that T_n is bounded below, and so T_n is semi-Fredholm. Also T_n^* has SVEP at 0, because $0 \in \operatorname{iso}\sigma(T_n)$. Then by [2, Theorem 2.46], $q(T_n) < \infty$, which implies $q(T) < \infty$

by [4, Lemma 3]. Thus $0 < p(T) = q(T) < \infty$, which is a contradiction since $0 \notin \Pi_{00}(T)$. Hence, $\alpha(T_n) > 0$. On the other hand, since $N(T_n) \subseteq N(T)$ and $\alpha(T) < \infty$, we have $\alpha(T_n) < \infty$. Thus $0 < \alpha(T_n) < \infty$, and so $0 \in \pi_{00}(T_n)$. Consequently, $\pi_{00}(T) \subseteq \pi_{00}(T_n)$. □

Theorem 2.2. *Let $T \in L(X)$. If $0 \notin \Pi^a(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $\pi^a_{00}(T) \subseteq \pi^a_{00}(T_n)$.*

Proof. Let $\lambda \in \pi^a_{00}(T)$. Then $\lambda \in \operatorname{iso}\sigma_a(T)$ and $0 < \alpha(\lambda I - T) < \infty$. Since $0 \notin \Pi^a(T)$, by Lemma 1.5, $\lambda \in \operatorname{iso}\sigma_a(T) = \operatorname{iso}\sigma_a(T_n)$. For $\lambda \neq 0$, by Lemma 1.2, $0 < \alpha(\lambda I - T_n) = \alpha(\lambda I - T) < \infty$, so $\lambda \in \pi^a_{00}(T_n)$. For $\lambda = 0$, we claim that $\alpha(T_n) > 0$. If $\alpha(T_n) = 0$, then in similar way as in the proof of Theorem 2.1, we can prove that T_n is bounded below, so $0 \notin \sigma_a(T_n)$, which is a contradiction. Hence $\alpha(T_n) > 0$. On the other hand, since $N(T_n) \subseteq N(T)$ and $\alpha(T) < \infty$, we have $\alpha(T_n) < \infty$. Thus $0 < \alpha(T_n) < \infty$, and so $\lambda = 0 \in \pi^a_{00}(T_n)$. Consequently, $\pi^a_{00}(T) \subseteq \pi^a_{00}(T_n)$. □

Theorems 2.1 and 2.2 were first extended in [5, Lemmas 2.3 and 2.4]. Theorems 2.3 and 2.4 below provide some different conclusions.

Theorem 2.3. *Let $T \in L(X)$. If $0 \notin \Pi_{00}(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $E(T_n) \subseteq E(T)$.*

Proof. Let $\lambda \in E(T_n)$. Then $\lambda \in \operatorname{iso}\sigma(T_n)$ and $0 < \alpha(\lambda I - T_n)$. By Lemma 1.5, $\lambda \in \operatorname{iso}\sigma(T_n) = \operatorname{iso}\sigma(T)$. Also, since $N(\lambda I - T_n) \subseteq N(\lambda I - T)$, we have $0 < \alpha(\lambda I - T_n) \leq \alpha(\lambda I - T)$. So $\lambda \in E(T)$. Hence, $E(T_n) \subseteq E(T)$. □

Theorem 2.4. *Let $T \in L(X)$. If $0 \notin \Pi^a(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $E^a(T_n) \subseteq E^a(T)$.*

Proof. By arguments similar to those in Theorem 2.3, we can prove that $E^a(T_n) \subseteq E^a(T)$. □

To get the inclusions $\pi_{00}(T_n) \subseteq \pi_{00}(T)$, $\pi^a_{00}(T_n) \subseteq \pi^a_{00}(T)$, $E(T) \subseteq E(T_n)$, and $E^a(T) \subseteq E^a(T_n)$, we need to resolve the following questions.

Question 2.5. Let $T \in L(X)$, and let $T_n, n \in \mathbb{N}$ be the restriction of the operator T on the subspace $R(T^n)$. If $R(T^n)$ is closed, we have the following.

(1) If $0 \notin \Pi_{00}(T)$, then $\alpha(T_n) < \infty \Rightarrow \alpha(T) < \infty$? $\alpha(T) > 0 \Rightarrow \alpha(T_n) > 0$?
(2) If $0 \notin \Pi^a(T)$, then $\alpha(T_n) < \infty \Rightarrow \alpha(T) < \infty$? $\alpha(T) > 0 \Rightarrow \alpha(T_n) > 0$?

Corollary 2.6. *Let $T \in L(X)$. If $0 \notin E(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $E(T) = E(T_n)$.*

Proof. Since $0 \notin E(T)$ and $\Pi_{00}(T) \subseteq E(T)$, we have that $0 \notin \Pi_{00}(T)$. By Theorem 2.3, we need only to show that $E(T) \subseteq E(T_n)$. Let $\lambda \in E(T)$. Then $\lambda \in \operatorname{iso}\sigma(T)$ and $\alpha(\lambda I - T) > 0$. By Lemma 1.5, $\lambda \in \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T_n)$. Observe that, $0 \notin E(T)$ implies $\lambda \neq 0$. By Lemma 2.1, $\alpha(\lambda I - T_n) = \alpha(\lambda I - T) > 0$. So $\lambda \in E(T_n)$. Hence, $E(T) \subseteq E(T_n)$. □

Corollary 2.7. *Let $T \in L(X)$. If $0 \notin E^a(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $E^a(T) = E^a(T_n)$.*

Proof. By arguments similar to those in Corollary 2.6, we can prove that $E^a(T) = E^a(T_n)$. □

Similarly as in the above theorems, we have the following relations.

Theorem 2.8. *Let $T \in L(X)$. If $0 \notin \Pi_{00}(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $\Pi_{00}(T) = \Pi_{00}(T_n)$.*

Proof. We show $\Pi_{00}(T) \subseteq \Pi_{00}(T_n)$. Let $\lambda \in \Pi_{00}(T)$. Then $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. Observe that $0 \notin \Pi_{00}(T)$ implies $\lambda \neq 0$. From Lemma 1.2, we have $0 < p(\lambda I - T_n) = p(\lambda I - T) < \infty$ and $q(\lambda I - T_n) < \infty$. Hence $0 < p(\lambda I - T_n) = q(\lambda I - T_n) < \infty$, which implies $\lambda \in \Pi_{00}(T_n)$. Consequently, $\Pi_{00}(T) \subseteq \Pi_{00}(T_n)$.

We show $\Pi_{00}(T_n) \subseteq \Pi_{00}(T)$. Let $\lambda \in \Pi_{00}(T_n)$. Then $\lambda \in \sigma(T_n)$ and $0 < p(\lambda I - T_n) = q(\lambda I - T_n) < \infty$. By [4, Lemmas 2 and 3], $p(\lambda I - T) = q(\lambda I - T) < \infty$. Suppose that $p(\lambda I - T) = q(\lambda I - T) = 0$. Then $\lambda \notin \sigma(T)$. By Lemma 1.5, $\lambda \notin \sigma(T_n)$, which leads to a contradiction. Hence $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$, which implies $\lambda \in \Pi_{00}(T)$. Consequently, $\Pi_{00}(T_n) \subseteq \Pi_{00}(T)$. □

Theorem 2.9. *Let $T \in L(X)$. If $0 \notin \Pi^a(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $\Pi^a(T) = \Pi^a(T_n)$.*

Proof. We show $\Pi^a(T) \subseteq \Pi^a(T_n)$. Let $\lambda \in \Pi^a(T)$. Then $\lambda \in \operatorname{iso}\sigma_a(T), p(\lambda I - T) < \infty$ and $R((\lambda I - T)^{p(\lambda I - T)+1})$ is closed. From Lemma 1.5, we have $\lambda \in \operatorname{iso}\sigma_a(T) = \operatorname{iso}\sigma_a(T_n)$. Observe that $0 \notin \Pi^a(T)$ implies $\lambda \neq 0$. By Lemma 1.2, $p(\lambda I - T_n) = p(\lambda I - T) < \infty$ and $R((\lambda I - T_n)^{p(\lambda I - T_n)+1}) = R((\lambda I - T)^{p(\lambda I - T_n)+1}) \cap R(T^n) = R((\lambda I - T)^{p(\lambda I - T)+1}) \cap R(T^n)$ is closed. Hence $\lambda \in \Pi^a(T_n)$. Consequently, $\Pi^a(T) \subseteq \Pi^a(T_n)$.

We show $\Pi^a(T_n) \subseteq \Pi^a(T)$. Let $\lambda \in \Pi^a(T_n)$. Here we consider two difference cases. Case I: if $\lambda \neq 0$, then $\lambda \in \sigma_a(T_n), p(\lambda I - T_n) < \infty$ and $R((\lambda I - T_n)^{p(\lambda I - T_n)+1})$ is closed. By Lemma 1.5, $\lambda \in \sigma_a(T_n) = \sigma_a(T)$. Also, it follows from Lemmas 1.2 and 1.4 that $p(\lambda I - T) = p(\lambda I - T_n) < \infty$ and that $R((\lambda I - T)^{p(\lambda I - T)+1}) = R((\lambda I - T)^{p(\lambda I - T_n)+1})$ is closed. Hence $\lambda \in \Pi^a(T)$. Case II: if $\lambda = 0$, then $0 \in \Pi^a(T_n) = \sigma_a(T_n) \backslash \sigma_{\mathrm{ld}}(T_n)$. Since $\sigma_{\mathrm{ld}}(T_n) = \sigma_{\mathrm{usbb}}(T_n)$, T_n is upper semi-B-Browder, then there exits $m \in \mathbb{N}$ such that $R(T_n^m) = R(T^{n+m})$ is closed and $T_{n[m]}$ is upper semi-Browder, where $T_{n[m]}$ denotes the restriction of T on $R(T^{m+n})$. Thus T is upper semi-B-Browder, and so $0 \notin \sigma_{\mathrm{usbb}}(T) = \sigma_{\mathrm{ld}}(T)$. By Lemma 1.5, $0 \in \sigma_a(T_n) = \sigma_a(T)$, so $0 \in \sigma_a(T) \setminus \sigma_{\mathrm{ld}}(T) = \Pi^a(T)$. Hence, $\Pi^a(T_n) \subseteq \Pi^a(T)$. □

Theorems 2.10 and 2.11 were first established in [6, Lemma 2.3]. We give different proofs for completeness.

Theorem 2.10. *Let $T \in L(X)$. If $0 \notin \Pi_{00}(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $p_{00}(T) = p_{00}(T_n)$.*

Proof. Observe that $0 \notin \Pi_{00}(T)$ implies $0 \notin p_{00}(T)$. Suppose that $0 \in p_{00}(T_n)$. Then $0 \in \Pi_{00}(T_n)$. By Theorem 2.8, $\Pi_{00}(T) = \Pi_{00}(T_n)$. Then $0 \in \Pi_{00}(T)$, which leads to a contradiction. Hence, $0 \notin p_{00}(T_n)$.

Let $0 \neq \lambda \in \mathbb{C}$. By Lemma 1.2, $\alpha(\lambda I - T) = \alpha(\lambda I - T_n)$. Also, from Theorem 2.8 we know that $\Pi_{00}(T) = \Pi_{00}(T_n)$. Thus $p_{00}(T) \setminus \{0\} = p_{00}(T_n) \setminus \{0\}$. Consequently, $p_{00}(T) = p_{00}(T_n)$. □

Theorem 2.11. *Let $T \in L(X)$. If $0 \notin \Pi^a(T)$ and there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, then $p^a_{00}(T) = p^a_{00}(T_n)$.*

Proof. By arguments similar to those in Theorem 2.10, we can prove that $p^a_{00}(T) = p^a_{00}(T_n)$. □

3. Weyl-type theorems and restrictions

In this section, we give conditions for which Weyl's theorem (resp., a-Weyl's theorem, Browder's theorem, a-Browder's theorem) for an operator $T \in L(X)$ is equivalent to Weyl's theorem (resp., a-Weyl's theorem, Browder's theorem, a-Browder's theorem) for a certain restriction T_n of T.

Theorem 3.1. *Let $T \in L(X)$. If $0 \notin \Pi_{00}(T)$, then T satisfies (W) if and only if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (W).*

Proof. (Sufficiency) Suppose that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (W). Let $\lambda \in \pi_{00}(T)$. By Theorem 2.1 and hypothesis, $\lambda \in \pi_{00}(T_n) = \sigma(T_n)\backslash\sigma_{\mathrm{w}}(T_n)$. Suppose that $\lambda = 0$. Since T_n is Fredholm and $0 \in \operatorname{iso}\sigma(T_n)$, by [2, Corollary 2.49], $0 < p(T_n) = q(T_n) < \infty$. From [4, Lemmas 2 and 3] we have $0 < p(T) = q(T) < \infty$, which is a contradiction because $0 \notin \Pi_{00}(T)$. Thus $\lambda \neq 0$. Since $\lambda I - T_n$ is Weyl and $\lambda \in \sigma(T_n)$, $0 < \alpha(\lambda I - T_n) = \beta(\lambda I - T_n) < \infty$. By Lemma 1.2, $0 < \alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, so $\lambda I - T$ is Weyl, and thus $\lambda \in \sigma(T)\backslash\sigma_{\mathrm{w}}(T)$. Hence, $\pi_{00}(T) \subseteq \sigma(T)\backslash\sigma_{\mathrm{w}}(T)$. On the other hand, let $\lambda \in \sigma(T)\backslash\sigma_{\mathrm{w}}(T)$. By Lemma 1.5 and hypothesis, $\lambda \in \sigma(T_n)\backslash\sigma_{\mathrm{w}}(T_n) = \pi_{00}(T_n)$. By arguments similar to those above, we have that $\lambda \neq 0$. Since $\lambda I - T_n$ is Weyl and $\lambda \in \sigma(T_n)$, $0 < \alpha(\lambda I - T_n) = \beta(\lambda I - T_n) < \infty$, by Lemma 1.2, $0 < \alpha(\lambda I - T) < \infty$. From the hypothesis and Lemma 1.5, $\lambda \in \pi_{00}(T_n) \subseteq \operatorname{iso}\sigma(T_n) = \operatorname{iso}\sigma(T)$, thus $\lambda \in \pi_{00}(T)$. Hence, $\sigma(T)\backslash\sigma_{\mathrm{w}}(T) \subseteq \pi_{00}(T)$. Consequently, $\sigma(T)\backslash\sigma_{\mathrm{w}}(T) = \pi_{00}(T)$, which implies that T satisfies (W).

(Necessary) Suppose that T satisfies (W). Then for $n = 0$, $R(T^0) = X$ is closed and $T_0 = T$ satisfies (W). □

By arguments similar to those in Theorem 3.1, we can prove the necessary conditions of the following theorems, thus we only need to proof the sufficiency of these theorems.

Theorem 3.2. *Let $T \in L(X)$. If $0 \notin \Pi^a(T)$, then T satisfies (aW) if and only if there exits $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (aW).*

Proof. (Sufficiency) Suppose that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (aW). Let $\lambda \in \pi^a_{00}(T)$. By Theorem 2.2 and hypothesis, $\lambda \in \pi^a_{00}(T_n) = \sigma_a(T_n)\backslash\sigma_{\mathrm{uw}}(T_n)$. Since $\lambda I - T_n$ is upper semi-Fredholm, $\alpha(\lambda I - T_n) < \infty$ and $R(\lambda I - T_n)$ is closed. By Lemma 1.3, there exists $k \in \mathbb{N}$ such that $R(\lambda I - T)^k$ is closed. Also, $\alpha(\lambda I - T) < \infty$ because $\lambda \in \pi^a_{00}(T)$. Then $\alpha((\lambda I - T)^k) < \infty$. Thus, $(\lambda I - T)^k$ is upper semi-Fredholm. By [2, Theorem 1.46], $\lambda I - T$ is upper semi-Fredholm. Moreover, T has SVEP at λ since $\lambda \in \operatorname{iso}\sigma_a(T)$. Then by [2, Corollary 2.48], $\operatorname{ind}(\lambda I - T) \leq \infty$. Thus $\lambda I - T$ is upper semi-Weyl, and so $\lambda \in \sigma_a(T)\backslash\sigma_{\mathrm{uw}}(T)$. Hence, $\pi^a_{00}(T) \subseteq \sigma_a(T)\backslash\sigma_{\mathrm{uw}}(T)$. On the other hand, let $\lambda \in \sigma_a(T)\backslash\sigma_{\mathrm{uw}}(T)$, we have that $\alpha(\lambda I - T) < \infty$. By Lemma 1.5 and hypothesis, $\lambda \in$

$\sigma_a(T_n)\backslash\sigma_{\mathrm{uw}}(T_n) = \pi_{00}^a(T_n)$. Then $\lambda \in \operatorname{iso}\sigma_a(T_n) = \operatorname{iso}\sigma_a(T)$ and $0 < \alpha(\lambda I - T_n) < \infty$. Since $N(\lambda I - T_n) \subseteq N(\lambda I - T)$ and $\alpha(\lambda I - T_n) > 0$, we have $\alpha(\lambda I - T) > 0$. Thus $0 < \alpha(\lambda I - T) < \infty$, and so $\lambda \in \pi_{00}^a(T)$. Hence, $\sigma_a(T)\backslash\sigma_{\mathrm{uw}}(T) \subseteq \pi_{00}^a(T)$. Consequently, $\sigma_a(T)\backslash\sigma_{\mathrm{uw}}(T) = \pi_{00}^a(T)$, which implies that T satisfies (aW). □

Theorem 3.3. *Let $T \in L(X)$. If $0 \notin \Pi_{00}(T)$, then T satisfies (B) if and only if there exits $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (B).*

Proof. (Sufficiency) Suppose that there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (B). From [2, Theorem 4.25] it follows that $\sigma(T_n) = \sigma_{\mathrm{w}}(T_n) \cup \operatorname{iso}\sigma(T_n)$. By the hypothesis and Lemma 1.5, $\sigma(T) = \sigma(T_n) = \sigma_{\mathrm{w}}(T_n) \cup \operatorname{iso}\sigma(T) \subseteq \sigma_{\mathrm{w}}(T) \cup \operatorname{iso}\sigma(T)$, so $\sigma(T) \subseteq \sigma_{\mathrm{w}}(T) \cup \operatorname{iso}\sigma(T)$. Observe that $\sigma_{\mathrm{w}}(T) \cup \operatorname{iso}\sigma(T) \subseteq \sigma(T)$ holds for every $T \in L(X)$. Hence $\sigma(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{iso}\sigma(T)$, and by [2, Theorem 4.25], T satisfies (B). □

Theorem 3.4. *Let $T \in L(X)$. If $0 \notin \Pi^a(T)$, then T satisfies (aB) if and only if there exits $n \in \mathbb{N}$ such that $R(T^n)$ is closed and T_n satisfies (aB).*

Proof. (Sufficiency) By [2, Theorem 4.35], we need only to show that $\sigma_a(T) = \sigma_{\mathrm{uw}}(T) \cup \operatorname{iso}\sigma_a(T)$. Observe first that $\sigma_{\mathrm{uw}}(T) \cup \operatorname{iso}\sigma_a(T) \subseteq \sigma_a(T)$ holds for every $T \in L(X)$. And by arguments similar to those in Theorem 3.3, we have that $\sigma_a(T) \subseteq \sigma_{\mathrm{uw}}(T) \cup \operatorname{iso}\sigma_a(T)$. So $\sigma_a(T) = \sigma_{\mathrm{uw}}(T) \cup \operatorname{iso}\sigma_a(T)$. Hence, T satisfies (aB). □

Remark 3.5. Obviously, if we replace the assumptions of Theorems 3.1 and 3.3 (resp., Theorems 3.2 and 3.4) by $0 \notin \operatorname{iso}\sigma(T)$, $p(T) = \infty$, or $q(T) = \infty$ (resp., $0 \notin \operatorname{iso}\sigma_a(T)$ or $p(T) = \infty$), then the results are true.

Acknowledgment. The authors are highly grateful to the referee for a careful reading and valuable comments on this article.

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