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SIMILARITY ORBITS OF COMPLEX SYMMETRIC OPERATORS

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Abstract. An operator T on a complex Hilbert space $\mathcal{H}$ is said to be *complex symmetric* if T can be represented as a symmetric matrix relative to some orthonormal basis for $\mathcal{H}$. In this article we explore the stability of complex symmetry under the condition of similarity. It is proved that the similarity orbit of an operator T is included in the class of complex symmetric operators if and only if T is an algebraic operator of degree at most 2.

1. Introduction and preliminaries

Throughout this paper, $\mathcal{H}$ will always denote a complex separable Hilbert space. We let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $CTC = T^*$; in this case, T is said to be C-*symmetric.* Recall that a conjugate-linear map C on $\mathcal{H}$ is called a *conjugation* if C is invertible, $C^{-1} = C$, and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. Note that T is complex symmetric if and only if there is an orthonormal basis of $\mathcal{H}$ with respect to which T has a complex symmetric (i.e., self-transpose) matrix representation (see [4, Lemma 2.16]).

The general study of complex symmetric operators was initiated by Garcia, Putinar and Wogen in [5]–[8], and has recently received much attention (see [3], [9], [11]). The class of complex symmetric operators is surprisingly large and

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includes many important special operators such as normal operators, Hankel operators, binormal operators, truncated Toeplitz operators, and many integral operators. The reader is referred to [4] for more historical comments about complex symmetric operators.

The aim of this paper is to study the stability of complex symmetry under the condition of *similarity*. Recall that two operators $A_1, A_2 \in \mathcal{B}(\mathcal{H})$ are said to be *similar*, denoted by $A_1 \sim A_2$, if there exists some invertible $X \in \mathcal{B}(\mathcal{H})$ such that $A_1X = XA_2$. In general, the complex symmetry is not invariant under similarity; that is, an operator similar to some complex symmetric operator is possibly not complex symmetric. In fact, each operator of finite rank is similar to a complex symmetric operator, and there exist many finite-rank operators that are not complex symmetric (see [5, Example 4]). The reader is also referred to [4, Example 2.21] for more concrete examples.

Then it is natural to ask the following question.

Question 1.1. When does an operator T satisfy that every operator similar to T is complex symmetric?

Given $T \in \mathcal{B}(\mathcal{H})$, denote by $\mathcal{S}(T)$ the *similarity orbit* of T; that is,

$$\mathcal{S}(T) = \{A \in \mathcal{B}(\mathcal{H}) : A \sim T\}.$$

A subset $\mathcal{E}$ of $\mathcal{B}(\mathcal{H})$ is said to be *similarity-invariant* if $\mathcal{S}(T) \subset \mathcal{E}$ for all $T \in \mathcal{E}$. In the following, we let CSO denote the set of all complex symmetric operators on $\mathcal{H}$. Thus Question 1.1 is equivalent to the question of when T satisfies $\mathcal{S}(T) \subset CSO$.

The main result of this paper is the following theorem which gives a complete answer to Question 1.1.

Theorem 1.2. *For $T \in \mathcal{B}(\mathcal{H})$, the following are equivalent:*

(i) $\mathcal{S}(T) \subset CSO$,
(ii) $\mathcal{S}(T) \subset \overline{CSO}$,
(iii) $\overline{\mathcal{S}(T)} \subset CSO$,
(iv) *T is an algebraic operator of degree at most 2.*

Recall that an operator T is *algebraic* if $p(T) = 0$ for some nonzero polynomial $p(z)$. The *degree* of an algebraic operator is defined to be the degree of the nonzero polynomial $p(z)$ of least degree for which $p(T) = 0$. By the spectral mapping theorem, the spectrum of an algebraic operator is finite. By Theorem 1.2, if T does not satisfy any polynomial of degree 2, then T is always similar to an operator that is not complex symmetric.

Statement (iv) of Theorem 1.2 means that T satisfies a polynomial of degree 2. Garcia and Wogen proved that such operators are always complex symmetric (see [8, Theorem 2]). Since these operators constitute a norm-closed, similarity-invariant subset of $\mathcal{B}(\mathcal{H})$, the following two results are immediate consequences of Theorem 1.2.

Corollary 1.3. *If $\mathcal{E}$ is a nonempty subset of CSO, then $\mathcal{E}$ is similarity-invariant if and only if $\mathcal{E}$ consists of the similarity orbits of some algebraic operators of degree at most 2.*

Corollary 1.4. *The set of all algebraic operators of degree at most* 2 *is the largest similarity-invariant subset of CSO.*

2. Proof of the main result

First, we offer some preparation.

A nonempty bounded open subset Ω of the complex plane $\mathbb{C}$ is a *Cauchy domain* if the following conditions are satisfied: (i) Ω has finitely many components, the closures of any two of which are disjoint, and (ii) the boundary of Ω is composed of a finite number of closed rectifiable Jordan curves, no two of which intersect. If, in addition, all curves of $\partial\Omega$ are regular analytic Jordan curves, we say that Ω is an *analytic Cauchy domain*.

Let $T \in \mathcal{B}(\mathcal{H})$. If σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$. We let $E(\sigma; T)$ denote the *Riesz idempotent* of T corresponding to σ; that is,

$$E(\sigma; T) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda - T)^{-1} \, \mathrm{d}\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we denote by $\mathcal{H}(\sigma; T)$ the range of $E(\sigma; T)$. Since $E(\sigma; T)T = TE(\sigma; T)$, one can see $T(\mathcal{H}(\sigma; T)) \subseteq \mathcal{H}(\sigma; T)$. Note that $E(\sigma; T)$ is the Riesz–Dunford functional calculus of T with respect to the following function:

$$f(z) = \begin{cases} 1, & z \in \Omega, \\ 0, & z \in \mathbb{C} \setminus \overline{\Omega}, \end{cases}$$

which is analytic on a neighborhood of $\sigma(T)$. If λ_0 is an isolated point of $\sigma(T)$ and $\dim \mathcal{H}(\{\lambda_0\}; T) < \infty$, then λ_0 is called a *normal eigenvalue* of T. Denote by $\sigma_0(T)$ the set of all normal eigenvalues of T. The reader is referred to Chapter 1 of [10] for more details about Riesz idempotents.

Lemma 2.1 ([14, Lemma 2.2])**.** *Let $T \in \mathcal{B}(\mathcal{H})$ and σ be a clopen subset of $\sigma(T)$. Then T can be written as*

$$T = \begin{bmatrix} A & F \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}(\sigma; T) \\ \mathcal{H}(\sigma; T)^\perp \end{matrix},$$

where $\sigma(A) = \sigma$ and $\sigma(B) = \sigma(T) \setminus \sigma$.

Lemma 2.2 ([10, Corollary 3.22])**.** *Let $T \in \mathcal{B}(\mathcal{H})$, and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. If T can be written as*

$$T = \begin{bmatrix} A & F \\ 0 & B \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$$

and $\sigma(A) \cap \sigma(B) = \emptyset$, then $T \sim A \oplus B$.

Given a subset Δ of $\mathbb{C}$, we let Δ^* denote the set $\{z \in \mathbb{C} : \overline{z} \in \Delta\}$.

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and σ be a clopen subset of $\sigma(T)$. If C is a conjugation on $\mathcal{H}$ and $CTC = T^*$, then $CE(\sigma; T)C = E(\sigma; T)^*$.*

Proof. Since T is C-symmetric, one can easily verify that $r(T)$ is C-symmetric for any rational function $r(z)$ with poles off $\sigma(T)$. If f is a function analytic on a neighborhood Ω of $\sigma(T)$, then, by Runge's theorem, there exist rational functions $\{r_n(z)\}_{n=1}^{\infty}$ with poles off $\sigma(T)$ such that $f(T) = \lim_n r_n(T)$. It follows that $Cf(T)C = f(T)^*$. Therefore, $CE(\sigma; T)C = E(\sigma; T)^*$. □

Let $T \in \mathcal{B}(\mathcal{H})$. Denote by $\ker T$ and $\operatorname{ran} T$ the kernel of T and the range of T, respectively. T is called a *semi-Fredholm* operator if $\operatorname{ran} T$ is closed and either $\dim\ker T$ or $\dim\ker T^*$ is finite; in this case, $\operatorname{ind} T := \dim\ker T - \dim\ker T^*$ is called the *index* of T. In particular, if $-\infty < \operatorname{ind} T < \infty$, then T is called a *Fredholm operator*. The *Wolf spectrum* $\sigma_{lre}(T)$ and the *essential spectrum* $\sigma_e(T)$ are defined by

$$\sigma_{lre}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}$$

and

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

respectively.

Lemma 2.4 ([2, Chapter XI, Theorem 6.8, Proposition 6.9])**.** *If $T \in \mathcal{B}(\mathcal{H})$, then $\partial\sigma(T) \subset \sigma_{lre}(T) \cup \sigma_0(T)$.*

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *polynomially compact* if $p(T)$ is compact for some nonzero polynomial $p(z)$. Olsen [12] proved that each polynomially compact operator is the sum of an algebraic operator and a compact one, and so, if T is polynomially compact, then $\sigma_e(T)$ is finite. Given a subset Δ of $\mathbb{C}$, we let $\operatorname{iso}\Delta$ denote the set of all isolated points of Δ.

Lemma 2.5 ([1, Theorem 9.2])**.** *If $T \in \mathcal{B}(\mathcal{H})$ is not polynomially compact, then $\overline{\mathcal{S}(T)}$ contains all operators $R \in \mathcal{B}(\mathcal{H})$ satisfying*

$$\operatorname{iso}\sigma(R) = \emptyset \qquad \text{and} \qquad \sigma(T) \subset \sigma(R) = \sigma_{lre}(R).$$

For $e, f \in \mathcal{H}$, define $e \otimes f \in \mathcal{B}(\mathcal{H})$ as $(e \otimes f)(x) = \langle x, f\rangle e$ for $x \in \mathcal{H}$.

Lemma 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be quasinilpotent; that is, $\sigma(T) = \{0\}$. If $T^2 \neq 0$, then $\overline{\mathcal{S}(T)}$ contains $0, e_1 \otimes e_2$ and $e_1 \otimes e_2 + e_2 \otimes e_3$ for any orthonormal triple $\{e_1, e_2, e_3\}$ in $\mathcal{H}$.*

Proof. If $T^n \neq 0$ for all $n \geq 1$, then, by Lemma 8.1 in [10], $\overline{\mathcal{S}(T)}$ contains all compact nilpotent operators.

Now we assume that $T^k = 0$ and $T^{k-1} \neq 0$ for some integer k with $2 < k < \infty$. Then $\dim\mathcal{H} \geq k$. For any orthonormal subset $\{e_i\}_{1=1}^{k}$ of $\mathcal{H}$, by the discussion on page 221 of [10], we have $\sum_{i=1}^{k-1} e_i \otimes e_{i+1} \in \overline{\mathcal{S}(T)}$. In view of Theorem 2.1 in [10], we deduce that $\overline{\mathcal{S}(T)}$ contains $0, e_1 \otimes e_2$ and $e_1 \otimes e_2 + e_2 \otimes e_3$. □

Using a similar argument as in Lemma 2.6, one can prove the following result.

Lemma 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be quasinilpotent. Then*

(i) *If $T \neq 0$, then $\overline{\mathcal{S}(T)}$ contains 0 and $e_1 \otimes e_2$ for any orthonormal pair $\{e_1, e_2\}$ in $\mathcal{H}$;*
(ii) *$0 \in \overline{\mathcal{S}(T)}$.*

Recall that an operator T is said to be *essentially normal* if $T^*T - TT^*$ is compact.

Lemma 2.8. *If $T \in \mathcal{B}(\mathcal{H})$ is polynomially compact, then $\overline{\mathcal{S}(T)}$ contains an essentially normal operator $R \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) = \sigma(R)$.*

Proof. By Lemma 7.1 in [10], we need only prove the case that T is essentially nilpotent; that is, there exists $1 \leq k < \infty$ such that T^k is compact. Then $\sigma(T) = \{0\} \cup \sigma_0(T)$ and $\sigma_e(T) = \sigma_{lre}(T) = \{0\}$. Without loss of generality, we may assume that T^{k-1} is not compact.

We need only consider the case that $T^{k-1} + T^*$ is not a Fredholm operator. In fact, if $T^{k-1}+T^*$ is a Fredholm operator, then, by the discussion at the beginning of Section 8.3.1 of [10], $\overline{\mathcal{S}(T)}$ contains an operator $A \in \mathcal{B}(\mathcal{H})$ similar to $T \oplus 0$ on $\mathcal{H} \oplus \mathcal{H}$. Thus A is essentially nilpotent of order k, and $A^{k-1} + A^*$ is not a Fredholm operator. It suffices to prove that $\overline{\mathcal{S}(A)}$ contains an essentially normal operator $R \in \mathcal{B}(\mathcal{H})$ with $\sigma(A) = \sigma(R)$.

Now we assume that $T^{k-1} + T^*$ is not a Fredholm operator.

Case 1. $\sigma_0(T) = \emptyset$.

This implies that T is quasinilpotent. By Proposition 8.5 in [10], we obtain $0 \in \overline{\mathcal{S}(T)}$.

Case 2. $\sigma_0(T)$ is nonempty and finite.

Assume that $\sigma_0(T) = \{\lambda_n : 1 \leq n \leq m\}$. Then, by Lemmas 2.1 and 2.2, $T \sim T_0 \oplus F$, where F acts on a finite-dimensional space and T_0 is quasinilpotent. One can check that T_0 is essentially nilpotent of order k and $T_0^{k-1} + T_0^*$ is not a Fredholm operator. By the proof in Case 1, $\overline{\mathcal{S}(T_0)}$ contains an essentially normal operator R_0 with $\sigma(R_0) = \{0\} = \sigma(T_0)$. Then $R := R_0 \oplus F$ is essentially normal lying in $\overline{\mathcal{S}(T)}$ and $\sigma(R) = \sigma(T)$.

Case 3. $\sigma_0(T)$ is infinite.

Assume that $\sigma_0(T) = \{\lambda_n : n \geq 1\}$, where $\lambda_n \neq \lambda_m$ whenever $n \neq m$. Since T is essentially nilpotent and $\sigma_e(T) = \{0\}$, it follows that $\lambda_n \to 0$ as $n \to \infty$. Denote by J_n the Jordan form of $(T - \lambda_n)|_{\mathcal{H}(\{\lambda_n\};T)}$ for $n \geq 1$. Set $R = \bigoplus_{n=1}^{\infty}(\lambda_n + \lambda_n J_n)$. Then R is compact, and $\sigma(R) = \sigma(T)$. By Proposition 8.6 in [10], we obtain $R \in \overline{\mathcal{S}(T)}$. □

Corollary 2.9. *If $T \in \mathcal{B}(\mathcal{H})$ and $\sigma(T)$ is contained in a Cauchy domain Ω, then $\overline{\mathcal{S}(T)}$ contains an essentially normal operator R with $\sigma(R) \subset \Omega$.*

Proof. Assume that $\{\Omega_i\}_{i=1}^m$ are all components of Ω. Then $\{\Omega_i\}_{i=1}^m$ is an open cover of $\sigma(T)$. Since $\sigma(T)$ is compact, there exists a finite subcover $\{\Omega_{n_i}\}_{i=1}^k$ such that $\sigma(T) \subset \bigcup_{i=1}^k \Omega_{n_i}$ and $\sigma(T) \cap \Omega_{n_i} \neq \emptyset$ for each i with $1 \leq i \leq k$. Set $\sigma_i = \sigma(T) \cap \Omega_{n_i}$ for $1 \leq i \leq k$. Then σ_i's are pairwise disjoint clopen subsets of $\sigma(T)$ and $\sigma(T) = \bigcup_{i=1}^k \sigma_i$. Then, by Lemmas 2.1 and 2.2, T is similar to an operator A of the form $A = \bigoplus_{i=1}^k T_i$, where T_i satisfies $\sigma(T_i) = \sigma_i$ for $1 \leq i \leq k$.

Fix an i with $1 \leq i \leq k$. It suffices to show that there exists an essentially normal R_i in $\overline{\mathcal{S}(T_i)}$ with $\sigma(R_i) \subset \Omega_{n_i}$. If T_i is polynomially compact, then, by Lemma 2.8, there exists an essentially normal operator R_i with $\sigma(R_i) = \sigma(T_i) \subset \Omega_{n_i}$. Now assume that T_i is not polynomially compact. Since Ω_i is connected and

$\sigma_i \subset \Omega_i$, we can choose a connected Cauchy domain G satisfying $\sigma_i \subset G \subset \overline{G} \subset \Omega_i$. Obviously we can construct a normal operator R_i on the underlying space of T_i with $\sigma(R_i) = \overline{G}$. Since G is a domain, we deduce that $\sigma(R_i) = \sigma_{lre}(R_i)$ and $\operatorname{iso}\sigma(R_i) = \emptyset$. Noting that $\sigma(T_i) \subset \sigma(R_i)$, it follows from Lemma 2.5 that $R_i \in \overline{\mathcal{S}(T_i)}$. This ends the proof. □

Let $\mathcal{H}_1, \mathcal{H}_2$ be two complex Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Assume that $A_i \in \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$. The Rosenblum operator τ_{A_2,A_1} induced by A_2 and A_1 is defined as

$$\begin{aligned} \tau_{A_2,A_1} : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) &\longrightarrow \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), \\ X &\longmapsto A_2 X - X A_1. \end{aligned}$$

Then it is easy to see that τ_{A_2,A_1} is a bounded linear operator on $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

Lemma 2.10 ([10, Corollary 3.20])**.** *Let A_1, A_2 be as above. If $\sigma(A_1) \cap \sigma(A_2) = \emptyset$, then τ_{A_2,A_1} is invertible.*

Lemma 2.11. *Let $R \in \mathcal{B}(\mathcal{H})$. Assume that $\mathcal{H} = \bigoplus_{i=1}^{3} \mathcal{H}_i$ with respect to which R admits the following matrix representation:*

$$R = \begin{bmatrix} A_1 & X & Y \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}.$$

If $\sigma_i = \sigma(A_i)$ for $1 \le i \le 3$ and $\sigma_i \cap \sigma_j = \emptyset$ whenever $i \ne j$, then

$$E(\sigma_2; R) = \begin{bmatrix} 0 & -\tau^{-1}_{A_1,A_2}(X) & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad E(\sigma_3; R) = \begin{bmatrix} 0 & 0 & -\tau^{-1}_{A_1,A_3}(Y) \\ 0 & 0 & 0 \\ 0 & 0 & I_3 \end{bmatrix},$$

where I_i is the identity on $\mathcal{H}_i$, $i = 2, 3$.

Proof. For $\lambda \notin \sigma(R)$, since

$$\begin{aligned} (\lambda - R)^{-1} &= \begin{bmatrix} \lambda - A_1 & -X & -Y \\ 0 & \lambda - A_2 & 0 \\ 0 & 0 & \lambda - A_3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (\lambda - A_1)^{-1} & (\lambda - A_1)^{-1} X (\lambda - A_2)^{-1} & (\lambda - A_1)^{-1} Y (\lambda - A_3)^{-1} \\ 0 & (\lambda - A_2)^{-1} & 0 \\ 0 & 0 & (\lambda - A_3)^{-1} \end{bmatrix}, \end{aligned}$$

it follows that $E(\sigma_2; R)$ and $E(\sigma_3; R)$ can be written respectively as

$$E(\sigma_2; R) = \begin{bmatrix} 0 & U & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad E(\sigma_3; R) = \begin{bmatrix} 0 & 0 & V \\ 0 & 0 & 0 \\ 0 & 0 & I_3 \end{bmatrix}.$$

Noting that $E(\sigma_i; R) R = R E(\sigma_i; R)$ for $i = 2, 3$, a direct matrical calculation shows that

$$A_1 U + X = U A_2 \qquad \text{and} \qquad A_1 V + Y = V A_3.$$

Since $\sigma(A_1) \cap \sigma(A_2) = \emptyset = \sigma(A_1) \cap \sigma(A_3)$, it follows from Lemma 2.10 that $\tau_{A_1,A_2}, \tau_{A_1,A_3}$ are both invertible. Then $U = -\tau^{-1}_{A_1,A_2}(X)$ and $V = -\tau^{-1}_{A_1,A_3}(Y)$. □

Lemma 2.12. *Let $A_i \in \mathcal{B}(\mathcal{H}_i), i = 1,2,3$. If $\sigma(A_1)$, $\sigma(A_2)$, and $\sigma(A_3)$ are pairwise disjoint, then there exist rank-one operators $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $Y \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$ such that the operator R defined by*

$$R = \begin{bmatrix} A_1 & X & Y \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{matrix}$$

is not complex symmetric.

Proof. For $1 \le i \le 3$, take a unit vector $e_i \in \mathcal{H}_i$. Define $F_1 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $F_2 \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1)$ as

$$F_1 = e_1 \otimes e_2, \qquad F_2 = e_1 \otimes e_3.$$

One can see that $\operatorname{ran} F_1 = \operatorname{ran} F_2 = \mathbb{C}e_1$. Define $X = F_1A_2 - A_1F_1$ and $Y = F_2A_3 - A_1F_2$. Since $\sigma(A_1) \cap \sigma(A_2) = \emptyset = \sigma(A_1) \cap \sigma(A_3)$, it follows from Lemma 2.10 that τ_{A_1,A_2} and τ_{A_1,A_3} are both invertible,

$$F_1 = -\tau^{-1}_{A_1,A_2}(X) \qquad \text{and} \qquad F_2 = -\tau^{-1}_{A_1,A_3}(Y). \tag{2.1}$$

Set

$$R = \begin{bmatrix} A_1 & X & Y \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}.$$

In view of Lemma 2.11 and (2.1), we have

$$E(\sigma_2; R) = \begin{bmatrix} 0 & F_1 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad E(\sigma_3; R) = \begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ 0 & 0 & I_3 \end{bmatrix}.$$

Then we obtain

$$E(\sigma_2; R)^* = \begin{bmatrix} 0 & 0 & 0 \\ F_1^* & I_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad E(\sigma_3; R)^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F_2^* & 0 & I_3 \end{bmatrix}.$$

Thus

$$\left\langle E(\sigma_2; R)^*x, E(\sigma_3; R)^*y \right\rangle = 0, \quad \forall x, y \in \mathcal{H}. \tag{2.2}$$

Now it suffices to prove that R is not complex symmetric. In fact, if not, then there exists a conjugation C such that $CRC = R^*$. By Lemma 2.3, $CE(\sigma_i; R)C = E(\sigma_i; R)^*$ for $1 \le i \le 3$. Then, by (2.2), we have

$$\begin{aligned} 1 = \langle e_1, e_1 \rangle &= \left\langle E(\sigma_2; R)e_2, E(\sigma_3; R)e_3 \right\rangle \\ &= \left\langle CE(\sigma_2; R)^*Ce_2, CE(\sigma_3; R)^*Ce_3 \right\rangle \\ &= \left\langle E(\sigma_3; R)^*Ce_3, E(\sigma_2; R)^*Ce_2 \right\rangle = 0, \end{aligned}$$

which is absurd. This ends the proof. □

Corollary 2.13. *If $T \in \mathcal{B}(\mathcal{H})$ and $\sigma(T)$ consists of at least three components, then T is similar to an operator that is not complex symmetric.*

Proof. Since $\sigma(T)$ consists of at least three components, we can find nonempty clopen subsets $\sigma_1, \sigma_2, \sigma_3$, which are pairwise disjoint so that $\sigma(T) = \bigcup_{i=1}^{3} \sigma_i$. Using Lemma 2.1 twice, we can write T as

$$T = \begin{bmatrix} A_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & A_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2, \\ \mathcal{H}_3 \end{matrix}$$

where $\mathcal{H} = \bigoplus_{i=1}^{3} \mathcal{H}_i$ and $\sigma(A_i) = \sigma_i$, $i = 1, 2, 3$. By Lemma 2.12, there exist rank-one operators $X : \mathcal{H}_2 \to \mathcal{H}_1$ and $Y : \mathcal{H}_3 \to \mathcal{H}_1$ such that the operator R defined by

$$R = \begin{bmatrix} A_1 & X & Y \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{matrix}$$

is not complex symmetric. By Lemma 2.2, one can see that

$$T \sim \bigoplus_{i=1}^{3} A_i \sim R.$$

This ends the proof. ∎

Lemma 2.14 ([9, Theorem 7.3])**.** *If $T \in \mathcal{B}(\mathcal{H})$ is essentially normal, then $T \in \overline{CSO}$ if and only if $T \in CSO$.*

Proposition 2.15. *If $T \in \mathcal{B}(\mathcal{H})$ and $\sigma(T)$ consists of at least three components, then $\mathcal{S}(T) \nsubseteq \overline{CSO}$.*

Proof. Since $\sigma(T)$ consists of at least three components, we can find nonempty clopen subsets $\sigma_1, \sigma_2, \sigma_3$, which are pairwise disjoint so that $\sigma(T) = \bigcup_{i=1}^{3} \sigma_i$. Using Lemma 2.1 twice, we can write T as

$$T = \begin{bmatrix} A_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & A_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2, \\ \mathcal{H}_3 \end{matrix}$$

where $\mathcal{H} = \bigoplus_{i=1}^{3} \mathcal{H}_i$ and $\sigma(A_i) = \sigma_i$, $i = 1, 2, 3$.

Choose analytic Cauchy domains Ω_1, Ω_2, and Ω_3 such that $\sigma_i \subset \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$. By Corollary 2.9, we can choose essentially normal $B_i \in \overline{\mathcal{S}(A_i)}$ with $\sigma(B_i) \subset \Omega_i$ for $1 \leq i \leq 3$, and so $\bigoplus_{i=1}^{3} B_i$ lies in the closure of the similarity orbit of $\bigoplus_{i=1}^{3} A_i$.

By Lemma 2.12, there exists rank-one operators $X : \mathcal{H}_2 \to \mathcal{H}_1$ and $Y : \mathcal{H}_3 \to \mathcal{H}_1$ such that the operator R defined by

$$R = \begin{bmatrix} B_1 & X & Y \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{matrix}$$

is not complex symmetric. Since R is essentially normal, it follows from Lemma 2.14 that $R \notin \overline{CSO}$. Since $\sigma(A_i)$'s are pairwise disjoint, it follows from Lemma 2.2 that

$$R \sim \bigoplus_{i=1}^{3} B_i \qquad \text{and} \qquad T \sim \bigoplus_{i=1}^{3} A_i.$$

Thus $R \in \overline{\mathcal{S}(T)}$. This means that $\overline{\mathcal{S}(T)} \nsubseteq \overline{CSO}$ or, equivalently, $\mathcal{S}(T) \nsubseteq \overline{CSO}$. □

Lemma 2.16. *Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{C}^3$, and let $T = e_1 \otimes e_2 + \lambda e_3 \otimes e_3$, where $\lambda \in \mathbb{C}$ is nonzero. Then T is similar to an operator that is not complex symmetric.*

Proof. Obviously, T can be written as

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{matrix} e_1 \\ e_2. \\ e_3 \end{matrix}$$

By Lemma 2.2, T is similar to the following operator:

$$R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{matrix} e_1 \\ e_2. \\ e_3 \end{matrix}$$

Now it remains to show that R is not complex symmetric.

For a proof by contradiction, we assume that C is a conjugation on $\mathbb{C}^3$ and $CRC = R^*$. Compute to see that

$$\ker R = \{\alpha e_1 : \alpha \in \mathbb{C}\}, \qquad \ker R^* = \{\alpha e_2 : \alpha \in \mathbb{C}\},$$

and

$$\ker(R - \lambda) = \big\{\alpha(e_1 + \lambda e_3) : \alpha \in \mathbb{C}\big\}, \qquad \ker(R - \lambda)^* = \{\alpha e_3 : \alpha \in \mathbb{C}\}.$$

From $CRC = R^*$ one can see that $C(\ker R^*) = \ker R$. Then there exists nonzero α_1 such that $Ce_2 = \alpha_1 e_1$. On the other hand, since $CRC = R^*$, we deduce that $C(R - \lambda)C = (R - \lambda)^*$ and $C(\ker(R - \lambda)^*) = \ker(R - \lambda)$. Then there exists nonzero α_2 such that $Ce_3 = \alpha_2(e_1 + \lambda e_3)$. Hence

$$0 = \langle e_2, e_3 \rangle = \langle Ce_3, Ce_2 \rangle = \big\langle \alpha_2(e_1 + \lambda e_3), \alpha_1 e_1 \big\rangle = \overline{\alpha_1}\alpha_2 \neq 0,$$

a contradiction. Thus R is not complex symmetric. This ends the proof. □

Lemma 2.17 ([9, Lemma 3.2])**.** *If $T = A \oplus N$, where N is normal, then T is complex symmetric if and only if A is complex symmetric.*

Lemma 2.18. *If $T \in \mathcal{B}(\mathcal{H})$ is not polynomially compact, then $\mathcal{S}(T) \nsubseteq \overline{CSO}$.*

Proof. Choose a real number $\delta > \|T\|$. Choose two infinite-dimensional subspaces $\mathcal{H}_1, \mathcal{H}_2$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Choose a normal operator $N \in \mathcal{B}(\mathcal{H}_1)$ with $\sigma(N) = \{z \in \mathbb{C} : |z| \leq \delta\}$. Assume that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_2$. Define $A \in \mathcal{B}(\mathcal{H}_2)$ as

$$Ae_i = \delta e_{i+1}, \quad \forall i \geq 1.$$

Set $R = N \oplus A$. It is easy to see that R is essentially normal with $\sigma(R) = \sigma_{lre}(R) = \{z \in \mathbb{C} : |z| \le \delta\}$. By Lemma 2.5, we obtain $R \in \overline{\mathcal{S}(T)}$.

On the other hand, noting that $\dim\ker A = 0 \neq 1 = \dim\ker A^*$, we deduce that A is not complex symmetric. In view of Lemma 2.17, R is not complex symmetric. Since R is essentially normal, it follows from Lemma 2.14 $R \notin \overline{CSO}$. This shows that $\overline{\mathcal{S}(T)} \nsubseteq \overline{CSO}$ or, equivalently, $\mathcal{S}(T) \nsubseteq \overline{CSO}$. □

Proposition 2.19. *Let $T \in \mathcal{B}(\mathcal{H})$, and assume that $\sigma(T)$ consists of two components. If $\mathcal{S}(T) \subseteq \overline{CSO}$, then there exist distinct complex numbers λ_1, λ_2 such that $(T - \lambda_1)(T - \lambda_2) = 0$.*

Proof. Assume that $\sigma(T) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are components of $\sigma(T)$. Clearly, σ_1 and σ_2 are connected clopen subsets of $\sigma(T)$. Then T can be written as

$$T = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix},$$

where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\sigma(A_i) = \sigma_i, i = 1, 2$. Then, by Lemma 2.2, $T \sim A_1 \oplus A_2$.

We claim that each of σ_1, σ_2 is a singleton. In fact, if not, then we may assume that σ_1 is not a singleton. Since σ_1 is connected, it follows that $\partial\sigma_1$ is infinite and, by Lemma 2.4, $\sigma_e(T) \supset \sigma_e(A_1) \supset \partial\sigma_1$ is infinite. Then T is not polynomially compact and, by Lemma 2.18, we obtain $\mathcal{S}(T) \nsubseteq \overline{CSO}$, a contradiction.

Assume that $\sigma(A_i) = \{\lambda_i\}$ for $i = 1, 2$. Without loss of generality, we may assume that $\lambda_1 = 0$. Otherwise, noting that $\mathcal{S}(T) \subset \overline{CSO}$ implies $\mathcal{S}(T - \lambda_1) \subset \overline{CSO}$, we need only deal with $T - \lambda_1$.

Now it suffices to prove that $A_1 = 0$ and $A_2 = \lambda_2 I_2$, where I_2 is the identity operator on $\mathcal{H}_2$. For a proof by contradiction, we may directly assume that $A_1 \neq 0$. Choose two unit vectors $e_1, e_2 \in \mathcal{H}_1$ with $\langle e_1, e_2 \rangle = 0$. Then, by Lemma 2.7, $e_1 \otimes e_2 \in \overline{\mathcal{S}(A_1)}$ and $\lambda_2 I_2 \in \overline{\mathcal{S}(A_2)}$. Hence $R = (e_1 \otimes e_2) \oplus \lambda_2 I_2 \in \overline{\mathcal{S}(A_1 \oplus A_2)} = \overline{\mathcal{S}(T)}$.

Choose a unit vector $e_3 \in \mathcal{H}_2$. Set $\mathcal{H}_3 = \vee\{e_1, e_2, e_3\}$, and set $\mathcal{H}_4 = \mathcal{H} \ominus \mathcal{H}_3$, where $\vee$ denotes closed linear span. Then $\mathcal{H}_3$ reduces R, $N := R|_{\mathcal{H}_4}$ is normal, and

$$R|_{\mathcal{H}_3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{matrix} e_1 \\ e_2. \\ e_3 \end{matrix}$$

By Lemma 2.16, $R|_{\mathcal{H}_3}$ is similar to an operator F on $\mathcal{H}_3$ that is not complex symmetric. By Lemma 2.17, $N \oplus F$ is not complex symmetric. Since $N \oplus F$ is essentially normal, it follows from Lemma 2.14 that $N \oplus F \notin \overline{CSO}$. Noting that $R \sim N \oplus F$ and $R \in \overline{\mathcal{S}(T)}$, we obtain $N \oplus F \in \overline{\mathcal{S}(T)}$. Hence $\overline{\mathcal{S}(T)} \nsubseteq \overline{CSO}$. Equivalently, we obtain $\mathcal{S}(T) \nsubseteq \overline{CSO}$, a contradiction. This ends the proof. □

Proposition 2.20. *Let $T \in \mathcal{B}(\mathcal{H})$, and assume that $\sigma(T)$ is connected. If $\mathcal{S}(T) \subset \overline{CSO}$, then there exists $\lambda \in \mathbb{C}$ such that $(T - \lambda)^2 = 0$.*

Proof. First we claim that $\sigma(T)$ is a singleton. In fact, if not, then $\sigma(T)$ is an infinite, connected set. It follows that $\partial\sigma(T)$ is infinite and, by Lemma 2.4, $\sigma_e(T) \supset$

$\partial\sigma(T)$ is infinite. Then T is not polynomially compact and, by Lemma 2.18, we have $\mathcal{S}(T) \nsubseteq \overline{CSO}$, a contradiction.

Assume that $\sigma(T) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$. It remains to prove that $(T-\lambda)^2 = 0$. For a proof by contradiction, we assume that $(T-\lambda)^2 \neq 0$. Choose an orthonormal triple $\{e_1, e_2, e_3\} \subset \mathcal{H}$. Then, by Lemma 2.6, $S = e_1 \otimes e_2 + \frac{e_2}{2} \otimes e_3 \sim e_1 \otimes e_2 + e_2 \otimes e_3 \in \overline{\mathcal{S}(T-\lambda)}$. Hence $S + \lambda \in \overline{\mathcal{S}(T)}$.

Denote $\mathcal{H}_1 = \vee\{e_1, e_2, e_3\}$, and denote $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$. Then $\mathcal{H}_1$ reduces S, $S|_{\mathcal{H}_2} = 0$, and

$$S|_{\mathcal{H}_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} e_1 \\ e_2 . \\ e_3 \end{matrix}$$

By Lemma 2.17 and [13, Proposition 5.6], S is not complex symmetric. Since S is essentially normal, it follows from Lemma 2.14 that $S \notin \overline{CSO}$. This implies $S + \lambda \notin \overline{CSO}$. Thus $\overline{\mathcal{S}(T)} \nsubseteq \overline{CSO}$ or, equivalently, $\mathcal{S}(T) \nsubseteq \overline{CSO}$, a contradiction. This ends the proof. □

Now we are going to give the proof of Theorem 1.2.

Proof of Theorem 1.2. The implications (iii)⟹(i)⟹(ii) are clear.

(iv)⟹(iii). Assume that $\lambda_1, \lambda_2 \in \mathbb{C}$ and $(T-\lambda_1)(T-\lambda_2) = 0$. If $R \in \overline{\mathcal{S}(T)}$, then there exist invertible operators $\{X_n : n \geq 1\}$ so that $X_n T X_n^{-1} \to R$. It follows that $(X_n T X_n^{-1} - \lambda_1)(X_n T X_n^{-1} - \lambda_2) \longrightarrow (R-\lambda_1)(R-\lambda_2)$. Noting that

$$(X_n T X_n^{-1} - \lambda_1)(X_n T X_n^{-1} - \lambda_2) = X_n (T-\lambda_1)(T-\lambda_2) X_n^{-1} = 0,$$

we obtain $(R-\lambda_1)(R-\lambda_2) = 0$. Then, in view of [8, Theorem 2], R is complex symmetric.

(ii)⟹(iv). Since $\mathcal{S}(T) \subset \overline{CSO}$, by Proposition 2.15, $\sigma(T)$ consists of at most two components. In view of Propositions 2.19 and 2.20, we deduce that T is an algebraic operator of degree at most 2. □

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