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PRODUCTS OF LAURENT OPERATORS AND FIELDS OF VALUES

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Abstract. One of the most fundamental properties of the field of values of an operator is the inclusion of the spectrum within its closure. Obtaining information on the spectrum of products of operators in terms of this spectral inclusion region is a demanding issue. Stating general results seems difficult; however, conclusions can be derived in some special instances. In this paper, we show that the field of values of products of Laurent operators is easily related to the product of their fields of values, and the same occurs for certain classes of Laurent operators with matrix symbols. The results also apply to the class of infinite upper (lower) triangular Toeplitz matrices.

1. Introduction

Let A be a bounded operator on a Hilbert space H equipped with an inner product $\langle\,,\,\rangle$. Denote by $B(H)$ the algebra of bounded linear operators over H. In our discussion, we identify H with $\mathbb{C}^n$ whenever H has dimension n. The *field of values* of A is the set of the complex plane defined as

$$W(A)=\big\{\langle Af,f\rangle/\langle f,f\rangle : f\in H, \langle f,f\rangle\neq 0\big\}.$$

This concept is a useful tool in studying linear operators, and it has been extensively investigated (see, e.g., [4] and the references therein).

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The Toeplitz–Hausdorff theorem (see [4]) states that $W(A)$ is a convex set whose closure contains the convex hull of the *spectrum* $\sigma(A)$ of A:

$$\overline{W(A)} \supseteq \operatorname{conv} \sigma(A), \tag{1.1}$$

where conv stands for convex hull. We recall that

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$$

with I the identity operator. When $A \in B(H)$ is normal, that is, $AA^* = A^*A$, equality holds in (1.1), $\operatorname{conv} \sigma(A) = \overline{W(A)}$. Proofs of these well-known facts may be found, for example, in [4].

Obtaining information on $W(AB)$ from the fields of values $W(A)$ and $W(B)$ is a challenging task, but answers in full generality seem difficult. Here, we investigate particular situations under which the field of values of a product is simply related with the product of the fields of values of the factors. Specifically, we shall be concerned with the fields of values of products of Laurent and Toeplitz operators, and we shall also focus on $W(A^k)$ for integers k.

This paper is organized as follows. In Section 2, some preliminaries on the state of the research are presented. In Section 3, we introduce pertinent notation and background on Laurent operators, and we extend Klein's theorem for these operators. As a consequence, inclusion regions for fields of values of products and powers of Laurent operators are obtained. Triangular Toeplitz operators are also considered in this framework. Related inequalities for the numerical radius and the Crawford number are easily derived. In Section 4, Laurent operators with matrix symbols are studied in the same context.

2. Preliminaries

Some authors have investigated connections between $W(AB)$ and $W(A)$ and $W(B)$ (see, e.g., [2]–[5], [10]). For instance, if A and B are $n \times n$ normal matrices and commute, then

$$W(AB) \subseteq \operatorname{conv} W(A)W(B).$$

In multiplicative perturbation theory, the product of operators AB is considered for B close to the identity, and it is of interest to relate in some way the spectra of products with the products of spectra (see [7], [9]).

The field of values is a spectral inclusion region in the sense that (1.1) holds. It can be easily verified that, for any $A, B \in B(H)$,

$$\sigma(A + B) \subset W(A + B) \subset W(A) + W(B).$$

Investigating the corresponding multiplicative version of this inclusion chain might be a demanding goal.

If B is *self-adjoint positive definite* (SPD) and $A \in B(H)$, then we have

$$\overline{W(B^{1/2}AB^{1/2})} \subseteq \overline{W(A)W(B)}$$

because

$$W(B^{1/2}AB^{1/2}) = \{x^* B^{1/2}AB^{1/2}x : x^*x = 1\} = \left\{ \frac{x^*Ax}{x^*B^{-1}x} : 0 \neq x \in \mathbb{C}^n \right\},$$

and so

$$\overline{W(B^{1/2}AB^{1/2})} \subseteq \frac{\overline{W(A)}}{\overline{W(B^{-1})}} = \overline{W(A)W(B)}.$$

Moreover,

$$\overline{W(B^{-1/2}AB^{-1/2})} \subseteq \frac{\overline{W(A)}}{\overline{W(B)}}. \tag{2.1}$$

Thus, if B is SPD, and commutes with A, then

$$\overline{W(AB)} \subseteq \overline{W(A)W(B)}$$

(see [4, Theorem 2.5(1)]); however, this inclusion may hold even if A and B do not commute, as the following example shows.

Example 2.1. Let $A = (a_{i-j}), B = (b_{i-j})$ be 10×10 tridiagonal and pentadiagonal Toeplitz matrices, respectively, such that

$$\begin{aligned} &b_{-9} = \cdots = b_{-2} = 0, \qquad b_{-1} = 1, \qquad b_0 = 3,\\ &b_1 = 1, \qquad b_2 = \cdots = b_9 = 0,\\ &a_{-9} = \cdots = a_{-2} = 0, \qquad a_{-1} = -1,\\ &a_0 = a_1 = a_2 = a_3 = 1, \qquad a_4 = \cdots = a_9 = 0. \end{aligned}$$

These matrices do not commute. Nevertheless, not only do we have

$$W(B^{-1/2}AB^{-1/2}) \subset W(A)/W(B),$$

the following inclusion also occurs:

$$W(B^{-1}A) \subset W(A)/W(B).$$

The boundaries of these sets are represented in Figure 1. There is no simple inclusion relation between $W(B^{-1}A)$ and $W(B^{-1/2}AB^{-1/2})$.

Nevertheless, simple examples in the 2×2 case show that the inclusion

$$W(AB) \subseteq W(A)W(B)$$

does not hold in general. Even $W(A^2)$ and $W(A)^2$ are not easily related. For instance, let $A = \operatorname{diag}(1, i)$ so that $W(A) = [1, i]$ and $W(A^2) = [-1, 1]$. A simple computation shows that $W(A)^2 = \{z_1z_2 : z_1, z_2 \in W(A)\}$ is the region bounded by the line segments $y = 1 - x, 0 \le x \le 1$, $y = 1 + x, -1 \le x \le 0$, and the arc of parabola $y = (1 - x^2)/2, -1 \le x \le 1$. Thus $W(A^2) \nsubseteq W(A)^2$; however, $W(A^2) \subset \operatorname{conv}(W(A))^2$, and so $\sigma(A^2) \subseteq W(A)^2$.

If $A, B \in B(H)$, and $0 \notin \overline{W(B)}$, then B is invertible, and by (1.1) and (2.1) we clearly have (see [4, Theorem 2.4(1)])

$$\sigma(B^{-1}A) \subseteq \overline{W(A)}/\overline{W(B)}.$$

As a consequence, if B is positive definite, then $0 \notin \overline{W(B)}$ and

$$\sigma(BA) = \sigma\big((B^{-1})^{-1}A\big) \subseteq \overline{W(B)W(A)}$$

for any A.

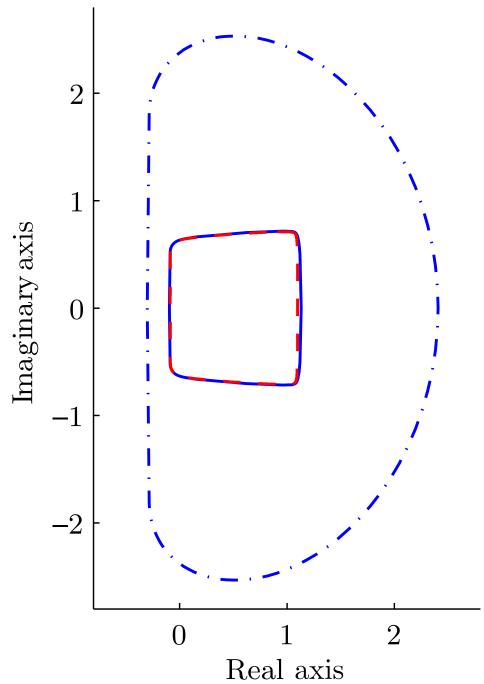

FIGURE 1. Boundaries of $W(B^{-1}A)$ (full line), $W(B^{-1/2}AB^{-1/2})$ (dashed line), and $W(A)/W(B)$ (dot-dashed line) for Example 2.1.

When $A \in B(H)$ is normal and k is a positive integer, then $\overline{W(A^k)}$ is the convex hull of the spectrum of A^k,

$$\overline{W(A^k)} = \operatorname{conv}\bigl(\sigma(A^k)\bigr),$$

which is a subset of the convex hull of the set

$$\overline{W(A)}^k = \bigl\{\eta_1 \cdots \eta_k : \eta_1, \ldots, \eta_k \in \overline{W(A)}\bigr\};$$

that is, the following inclusion holds:

$$\overline{W(A^k)} \subseteq \operatorname{conv} \overline{W(A)}^k, \quad k \in \mathbb{Z}^+.$$

This result may not be valid for a nonnormal operator even in the 2×2 case, as the following example shows.

Example 2.2. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

We find that

$$\partial W(A) = \bigl\{2\cos(\theta)\mathrm{e}^{i\theta} : 0 \le \theta < \pi\bigr\}$$

so that

$$\bigl(W(A)\bigr)^2 = \bigcup_{\theta=0}^{\pi} 2\cos\theta \mathrm{e}^{i\theta} W(A).$$

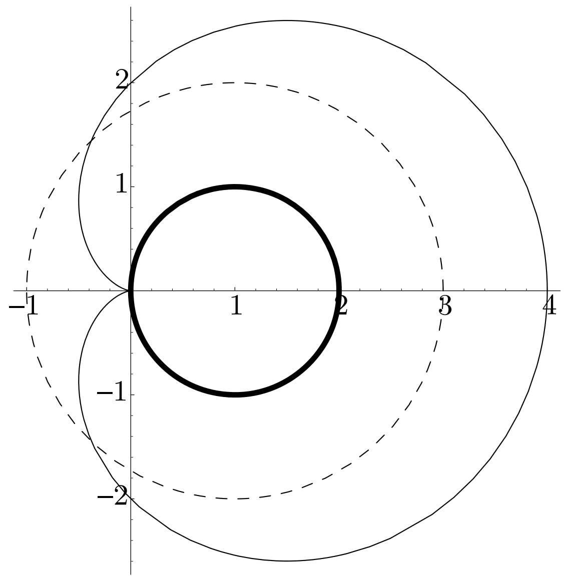

FIGURE 2. Boundaries of $W(A)$ (full line), $W(A^2)$ (dashed line), and $(W(A))^2$ (thin line), the cardioid, for Example 2.2.

We are led to consider the family of circles

$$\left\{4\cos\theta\cos\theta' \mathrm{e}^{i(\theta+\theta')}, \theta, \theta' \in [0,\pi]\right\}$$

whose envelope is the curve

$$x^2 + y^2 - 2x - 2\sqrt{x^2+y^2} = 0.$$

In Figure 2, the boundaries of $W(A)$ and $W(A^2)$ are represented, while $(W(A))^2$ is the cardioid. It can be easily confirmed that $W(A^2) \subsetneqq \operatorname{conv}(W(A))^2$.

3. Results for Laurent operators

Let ϕ be a bounded measurable function on the unit circle Γ. The multiplication induced by ϕ on the Lebesgue space L^2 (with respect to the normalized Lebesgue measure),

$$L_\phi f = \phi f, \quad \forall f \in L^2,$$

is called the *Laurent operator* induced by ϕ or the Laurent operator with *symbol* ϕ (for further information, see, e.g., [5]). The matrix of L_ϕ with respect to the standard orthonormal basis in L^2, $e_n(z) = z^n, n = 0, \pm1, \pm2, \ldots,$ is a Laurent matrix, that is, a bilaterally infinite matrix $[a_{ij}]_{-\infty}^{+\infty}$, all of whose diagonals parallel to the main diagonal are constant and

$$a_{ij} = \alpha_{i-j}, \quad i,j = 0, \pm1, \pm2, \ldots.$$

Further, $\phi = \sum_{n=-\infty}^{+\infty} \alpha_n e_n$ is the Fourier expansion of ϕ.

The product of two Laurent operators, L_ϕ, L_ψ, with symbols ϕ, ψ, is still a Laurent operator with symbol $\phi\psi$ denoted by $L_{\phi\psi}$. Thus, Laurent operators always commute. The Laurent operator compressed to H^2 (the *Hardy subspace* of L^2) is the *Toeplitz operator* with symbol ϕ,

$$T_\phi f = P(\phi f), \quad \forall f \in H^2,$$

where P is the projection operator from L^2 onto H^2. The linear map $\phi \to T_\phi$ of functions such that the nth Fourier coefficient $\hat{f}(n) = 0$, for every $n < 0$, is not in general multiplicative. The Brown–Halmos theorem (see [5]) states that $T_f T_g = T_{fg}$ if and only if f^* (or g) is in H^∞, or, equivalently, the matrix of T_g (T_f) in the standard orthonormal basis is an infinite lower (upper) triangular matrix.

The field of values of a Toeplitz operator T_ϕ was characterized in [6]. Namely, it was shown that $W(T_\phi)$ is the relative interior of the convex hull of $\sigma(T_\phi)$ (for an extension of this result, see [1]). On the other hand, by the Brown–Halmos theorem, the latter set coincides with the convex hull of the *essential range* $\mathcal{R}(\phi)$ of ϕ, which consists of every z such that the preimage of any neighborhood of z under ϕ has a positive measure.

As shown in the following result, $W(L_\phi)$ cannot be completely characterized in terms of $\mathcal{R}(\phi)$, but its closure still can. The proof is inspired by one of Klein's theorems (see [6]).

Theorem 3.1. *Let L_ϕ be a Laurent operator. The closure of the set $W(L_\phi)$ coincides with* $\operatorname{conv}\sigma(L_\phi)$ *and with the convex hull of*

$$\mathcal{R}(\phi). \tag{3.1}$$

Proof. For L_ϕ, L_ψ Laurent operators, we observe that $L_\phi + L_\psi$ is also a Laurent operator. For any $\Phi \in \mathbb{C}$, it is clear that $L_\phi - \Phi I = L_{(\phi-\Phi)}$. It follows that $L_\phi - \Phi I$ is not invertible if and only if $\Phi \in \mathcal{R}(\phi)$; that is, $\sigma(L_\phi) = \mathcal{R}(\phi)$. Next, we compare the closure of $W(L_\phi)$ with the convex hull of (3.1). Any point of $W(L_\phi)$ is by definition of the form

$$\int_\Gamma x^*(t)\phi(t)x(t)\,dt, \tag{3.2}$$

where x is a unit vector in L^2. Approximating x and ϕ by functions with finitely many values and keeping the values Φ_j of the approximation of ϕ in the essential range of ϕ, we conclude that this approximation is a convex combination of Φ_j. Since

$$\Phi_j \in \mathcal{R}(\phi),$$

their convex combinations are in the convex hull of $\mathcal{R}(\phi)$. Considering that convex hulls of compact sets in $\mathbb{R}$ are compact, the integral (3.2) itself lies there. Thus

$$\overline{W(L_\phi)} \subseteq \operatorname{conv}\big\{\mathcal{R}(\phi)\big\}.$$

To prove the converse inclusion, we just need to show that any $\Phi \in \mathcal{R}(\phi)$ lies in the closure of $W(L_\phi)$ since the latter is convex. To this end, let

$$x_s(t) = \begin{cases} 1 & \text{if } |\Phi - \phi(t)| < s, \\ 0 & \text{otherwise.} \end{cases}$$

Normalizing this function in L^2 (due to the definition of the essential range we can do so because it differs from zero on a set with positive measure for any $s > 0$), and letting $s \to 0$, we conclude that the corresponding points in $W(L_\phi)$ converge to Φ. □

If L_ϕ is *self-adjoint*, it is a direct consequence of Theorem 3.1 that $\overline{W(L_\phi)}$ is a line segment whose endpoints are

$$\sup \mathcal{R}(\phi) = \sup\{z : z \in \mathcal{R}(\phi)\}$$

and

$$\inf \mathcal{R}(\phi) = \inf\{z : z \in \mathcal{R}(\phi)\}.$$

Further, $\overline{W(L_\phi)} = \operatorname{conv} \sigma(L_\phi)$.

Corollary 3.2. *For the Laurent operators L_ϕ, L_ψ,*

$$\operatorname{conv} \sigma(L_\phi L_\psi) = \overline{W(L_\phi L_\psi)} \subseteq \operatorname{conv} \overline{W(L_\phi) W(L_\psi)}.$$

Proof. We have

$$\operatorname{conv} \sigma(L_\phi L_\psi) = \operatorname{conv} \sigma(L_{\phi\psi}) = \operatorname{conv} \mathcal{R}(\phi\psi).$$

Clearly,

$$\mathcal{R}(\phi\psi) \subseteq \mathcal{R}(\phi)\mathcal{R}(\psi).$$

Moreover,

$$\mathcal{R}(\phi) \subseteq \overline{W(L_\phi)}, \qquad \mathcal{R}(\psi) \subseteq \overline{W(L_\psi)}$$

so that

$$\mathcal{R}(\phi\psi) \subseteq \operatorname{conv} \overline{W(L_\phi) W(L_\psi)}.$$ □

Corollary 3.3. *For L_ϕ, L_ψ, both self-adjoint Laurent operators,*

$$\operatorname{conv} \sigma(L_\phi L_\psi) = \overline{W(L_\phi L_\psi)} \subseteq \overline{W(L_\phi) W(L_\psi)}.$$

Proof. Since for $S \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$ the convex set ST is clearly convex, the result follows from Corollary 3.2 due to the convexity of the field of values. □

The next corollary is related to [7, Proposition 2.1]. Its proof requires the following lemma.

Lemma 3.4. *Let $S \subseteq \mathbb{R}^+$ and T be convex sets. Then ST is convex.*

Proof. For $x_S, y_S \in S$ and $x_T, y_T \in T$, we show that $r x_S x_T + (1-r) y_S y_T \in ST, 0 \leq r \leq 1$. Indeed, let $q = r x_S / (r x_S + (1-r) y_S)$, and so $(1-q) = (1-r) y_S / (r x_S + (1-r) y_S)$. Clearly, $((r x_S + (1-r) y_S))(q x_T + (1-q) y_T) = r x_S x_T + (1-r) x_S y_T \in ST$. □

Corollary 3.5. *For Laurent operators L_ϕ, L_ψ, with L_ϕ positive definite,*

$$\operatorname{conv} \sigma(L_\phi L_\psi) = \overline{W(L_\phi L_\psi)} \subseteq \overline{W(L_\phi) W(L_\psi)}.$$

Proof. The result follows from Theorem 3.1 and Lemma 3.4. □

The following example illustrates Corollary 3.5.

Example 3.6. Consider the bilaterally infinite matrices $A = [a_{ij}]_{-\infty}^{+\infty}$ and $B = [b_{ij}]_{-\infty}^{+\infty}$ such that $a_{ij} = a_{i-j}$, $b_{ij} = b_{i-j}$ with

$$\cdots = b_{-3} = b_{-2} = 0, \qquad b_{-1} = 1, \qquad b_0 = 3,$$
$$b_1 = 1, \qquad b_2 = b_3 = \cdots = 0,$$
$$\cdots = a_{-3} = a_{-2} = 0, \qquad a_{-1} = -1,$$
$$a_0 = a_1 = a_2 = a_3 = 1, \qquad a_4 = a_5 = \cdots = 0.$$

These matrices represent Laurent operators in the standard orthonormal basis in L^2. The symbols of A and B are

$$\phi_A(\mathrm{e}^{i\theta}) = 1 + 2i\sin\theta + \mathrm{e}^{5i\theta/2}\big(2\cos(\theta/2)\big), \qquad \phi_B(\mathrm{e}^{i\theta}) = 3 + 2\cos\theta, \quad 0 \le \theta \le 2\pi.$$

In Figure 3, the ranges of the symbols of $B^{-1}A$ (full line) and of A (dashed line) are represented. Since $W(B^{-1}) = [1/5, 1]$ and, in this case, $W(B^{-1}) \subset W(A)$, we have $W(B^{-1})W(A) = W(A)$. Thus $\operatorname{conv}\sigma(B^{-1}A) = W(B^{-1}A) \subset W(B^{-1})W(A) = W(A)$.

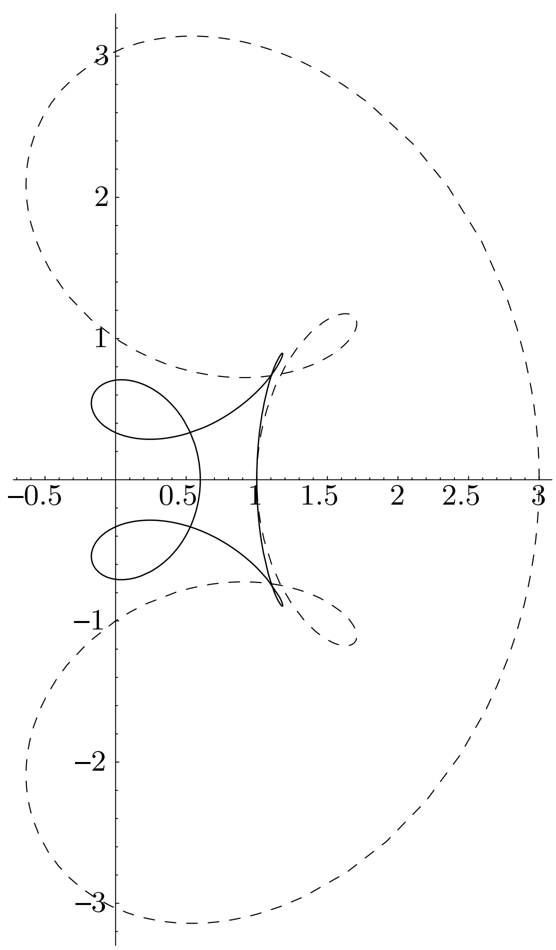

FIGURE 3. The ranges of the symbols of $B^{-1}A$ (full line) and of A (dashed line) (Example 3.6).

Corollary 3.5 may not hold if L_ϕ is not positive definite self-adjoint. Consider as an example the Laurent operator such that, for $t \in \Gamma$, $\phi(t) = (1+i)/2 + (1-i)(t+\bar{t})/4$ so that $\mathcal{R}(\phi) = \overline{W(L_\phi)} = [1, i]$. We have $\mathcal{R}(\phi^2) = \{x + i(1-x^2)/2 : -1 \le x \le 1\}$, and so

$$\overline{W(L_\phi^2)} = \overline{W(L_{\phi^2})} = \operatorname{conv}\bigl\{x + i(1-x^2)/2 : -1 \le x \le 1\bigr\}.$$

Since $(\overline{W(L_\phi)})^2$ is the region bounded by the line segments $y = 1-x, 0 \le x \le 1$, $y = 1+x, -1 \le x \le 0$, and the arc of parabola $y = (1-x^2)/2, -1 \le x \le 1$, it follows that $\overline{W(L_\phi^2)} \nsubseteq (\overline{W(L_\phi)})^2$. Nevertheless, $\overline{W(L_\phi^2)} \subseteq \operatorname{conv}(\overline{W(L_\phi)})^2$, and this inclusion occurs in general, as stated below.

Corollary 3.7. *For a Laurent operator L_ϕ,*

$$\overline{W(L_\phi^k)} = \operatorname{conv}\sigma(L_\phi^k) = \operatorname{conv}\mathcal{R}(\phi^k) \subseteq \operatorname{conv}\overline{W(L_\phi)}^k, \quad k \in \mathbb{Z}^+. \tag{3.3}$$

Proof. It is a simple consequence of Corollary 3.2. □

Corollary 3.8. *If L_ϕ is a positive definite self-adjoint operator, then $\overline{W(L_\phi^{-1})} = \overline{W(L_\phi)}^{-1} = \operatorname{conv}\{z^{-1} : z \in W(L_\phi)\}$.*

Proof. If ϕ is a bounded measurable function on Γ, then so is ϕ^{-1}, and henceforth $L_\phi^{-1} = L_{\phi^{-1}}$. Now, the result easily follows from Theorem 3.1. □

We observe that if L_ϕ is not positive definite, then the equality $\overline{W(L_\phi^{-1})} = \overline{W(L_\phi)}^{-1}$ may not hold. As an example, consider the Laurent operator such that $\mathcal{R}(\phi) = [-1, 1]$. Then $\overline{W(L_\phi^{-1})} = [-1, 1]$, while $\overline{W(L_\phi)}^{-1} =]-\infty, -1] \cup [1, +\infty[$.

Suppose $0 \notin \mathcal{R}(\phi)$ so that L_ϕ is invertible. Since $\overline{W(L_\phi)} = \operatorname{conv}\mathcal{R}(\phi)$, we have

$$\overline{W(L_\phi^{-1})} = \operatorname{conv}\mathcal{R}(\phi^{-1}) = \operatorname{conv}\sigma(L_{\phi^{-1}}).$$

Thus Corollary 3.7 holds for negative integers k as well.

We recall that the *numerical radius* of an operator A is defined by

$$w(A) = \sup\bigl\{|z| : z \in W(A)\bigr\}.$$

The famous *power inequality* for the numerical radius states that $w(A^m) \le w(A)^m$ for any positive integer m (an elementary proof of this result is given in [8]). For Laurent operators, we have the following.

Corollary 3.9. *The numerical radius of a Laurent operator L_ϕ coincides with $\sup|\phi| = (\sup|\phi|^2)^{1/2} = (\sup\{z : z \in \mathcal{R}(|\phi|^2)\})^{1/2}$. Further, $w(L_\phi^m) = w(L_\phi)^m$ for any positive integer m.*

Proof. The first part of the corollary is a direct consequence of Theorem 3.1. The second part is a trivial consequence of the fact that $|\phi^m| = |\phi|^m, m \in \mathbb{Z}^+$. □

Suppose L_ϕ is invertible and p is a positive integer. The following inequality clearly holds:

$$w(L_\phi^{-1}) \ge \bigl(w(L_\phi)\bigr)^{-1},$$

and so

$$w(L_\phi^{-p}) \geq w(L_\phi)^{-p}.$$

Equality occurs if L_ϕ is a nonzero multiple of a unitary operator; the converse also holds (see [3, Theorem 3.9]).

Corollary 3.10. *If L_ϕ is a unitary Laurent operator, then $w(L_\phi) = w(L_\phi^{-1}) = 1$; the converse also holds.*

The *Crawford number* of a Laurent operator L_ϕ such that $0 \notin \operatorname{conv}\mathcal{R}(\phi)$ is by definition

$$c(L_\phi) = \inf\bigl\{|z| : z \in \operatorname{conv}\mathcal{R}(\phi)\bigr\}.$$

Suppose that L_ϕ is invertible and that $0 \notin \operatorname{conv}\mathcal{R}(\phi)$; then we have $c(L_\phi) \leq \inf\{z : z \in \mathcal{R}(|\phi|)\}$. The following question arises: Does the inequality

$$c(L_\phi^{-1}) \leq \bigl(c(L_\phi)\bigr)^{-1}$$

hold? In the case of an affirmative answer and for p a positive integer, we have

$$c(L_\phi^{-p}) \leq c(L_\phi)^{-p}.$$

Remark 3.11. As previously mentioned, a necessary and sufficient condition for the product $T_f T_g$ of two Toeplitz operators being a Toeplitz operator is that both operators are represented in the standard orthonormal basis by infinite upper (or lower) triangular Toeplitz matrices. Further, under this condition, $T_f T_g = T_{fg}$ (see [5, p. 138]), and Corollaries 3.3, 3.2, 3.7, and 3.9 are easily seen to be valid for the operators of this class.

Example 3.12. Let L_ϕ be the Laurent operator with symbol ϕ such that

$$\mathcal{R}(\phi) = \{1 + \mathrm{e}^{i\theta} : 0 \leq \theta \leq 2\pi\}. \tag{3.4}$$

Then $\sigma(L_\phi^2)$ is the region whose boundary is the cardioid

$$x^2 + y^2 - 2x - 2\sqrt{x^2+y^2} = 0.$$

Moreover, the convex hull of the cardioid is also the boundary of $\overline{W(L_\phi)}^2$. Thus $\overline{W(L_\phi^2)} = \operatorname{conv}\overline{W(L_\phi)}^2$.

The Toeplitz operator with symbol (3.4) behaves precisely in the same way.

4. Laurent operators with matrix symbol

In this section, we shall be concerned with Laurent operators with a matrix symbol. For this purpose, we introduce some additional notation. We denote by L_n^2 the linear space of column vectors f of length n (i.e., $f = (f_j(\mathrm{e}^{it}))_1^n$) for

$$f_j : \Gamma \to \mathbb{C}, \qquad \int_0^{2\pi} \bigl|f_j(\mathrm{e}^{it})\bigr|^2 \,\mathrm{dt} = \sum_{k=-\infty}^{+\infty} |f_{jk}|^2 < \infty;$$

that is,

$$f_{jk} = \frac{1}{2\pi}\int_0^{2\pi} f_j(e^{it}) e^{-ikt}\,\mathrm{dt}.$$

We also consider the usual Hardy space H_n^2 of all functions $f \in L_n^2$ whose Fourier transform vanishes on the negative integers

$$H_n^2 = \{f \in L_n^2 : (f_{jk})_1^n = 0, k \in \mathbb{Z}^-\}.$$

Let $L_{n\times n}^\infty$ be the algebra of the $n \times n$ matrices whose entries are measurable and essentially bounded functions on Γ. If $n = 1$, then this set is simply denoted by L^∞. Let us consider the matrix function a:

$$a = \big(a_{jk}(t)\big)_{j,k=1}^n \in L_{n\times n}^\infty, \quad a_{jk} : \Gamma \to L^\infty, t \to a_{jk}(t) = \sum_{l=-\infty}^{l=+\infty} e^{ilt} a_{jk}^{(l)},$$

where

$$a_{jk}^{(l)} := \frac{1}{2\pi} \int_0^{2\pi} a_{jk}(t) e^{-ilt}\, dt.$$

The *multiplication operator* by a on L_n^2 is given by

$$M(a) : L_n^2 \to L_n^2, \qquad (f_k)_{k=1}^n \to \Big(\sum_{j=1}^n a_{kj} f_j\Big)_{k=1}^n.$$

By definition, the Laurent operator with matrix symbol a coincides with $M(a)$,

$$L_a := M(a).$$

Denote by P the *projection operator* on the space L_n^2 defined as

$$P : L_n^2 \to H_n^2, \qquad P\Big(\sum_{k=-\infty}^{+\infty} g_k e^{ikt}\Big) = \sum_{k=0}^{+\infty} g_k e^{ikt},$$

and denote by T_a the respective Toeplitz operator on H_n^2,

$$T_a : H_n^2 \to H_n^2, \qquad T_a := PM(a)P.$$

In [1], the following result was proved.

Theorem 4.1. *The closure of the sets* $W(L_a)$ *and* $\operatorname{conv}\sigma(L_a)$ *are the same and coincide with the convex hull of*

$$\{W(A) : A \in \mathcal{R}(a)\}.$$

We notice that in Theorem 4.1, A and L_a are operators acting on different Hilbert spaces $\mathbb{C}^n$ and L^n, respectively. As an illustrative example, we consider $n = 2$ so that

$$a = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}, \quad 0 \le t < 2\pi,$$

and so

$$W(L_a) = \operatorname{conv} \bigcup_{t=0}^{2\pi} W\left(\begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}\right).$$

We say that the symbol a is *normal* if $[a(\mathrm{e}^{it}), (a(\mathrm{e}^{it}))^*] = 0$ for any $\mathrm{e}^{it} \in \Gamma$, and we say that the symbols a, b *commute* if $[a(\mathrm{e}^{it}), b(\mathrm{e}^{it})] = 0$ for any $\mathrm{e}^{it} \in \Gamma$. As a consequence of Theorem 4.1, it can be easily seen that the results in Section 2 are valid for Laurent operators with matrix symbols. Corollaries 3.2 and 3.9 are easily adapted in terms of the following formulations.

Corollary 4.2. *Let L_a and L_b be Laurent operators with matrix symbols a and b which are normal and commute. Then*

$$\overline{W(L_aL_b)} \subseteq \operatorname{conv} \overline{W(L_a)W(L_b)}.$$

Corollary 4.3. *Let L_a be a Laurent operator with matrix symbol a which is normal. Then the numerical radius of L_a coincides with* $\sup\{|z| : z \in \sigma(a(\mathrm{e}^{it})), \mathrm{e}^{it} \in \Gamma\}$.

Corollary 4.4. *Let L_a and L_b be Laurent operators such that the matrix symbol a is normal and commutes with b. Then*

$$w(L_aL_b) \leq w(L_a)w(L_b).$$

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