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DOMINATED OPERATORS FROM LATTICE-NORMED SPACES TO SEQUENCE BANACH LATTICES

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Abstract. We show that every dominated linear operator from a Banach–Kantorovich space over an atomless Dedekind-complete vector lattice to a sequence Banach lattice $\ell_p(\Gamma)$ or $c_0(\Gamma)$ is narrow. As a consequence, we obtain that an atomless Banach lattice cannot have a finite-dimensional decomposition of a certain kind. Finally, we show that the order-narrowness of a linear dominated operator T from a lattice-normed space V to the Banach space with a mixed norm (W, F) over an order-continuous Banach lattice F implies the order-narrowness of its exact dominant $|T|$.

1. Introduction and preliminaries

Narrow operators generalize compact operators defined on function spaces (see [11] for the first systematic study; see also the recent monograph [12]). Different classes of narrow operators in framework of vector lattices and lattice-normed spaces were considered in [9], [10]. In the present article, we continue the investigation of narrow operators in lattice-normed spaces and show that every dominated linear operator from a Banach–Kantorovich space over an atomless Dedekind-complete vector lattice to a sequence Banach lattice is narrow. As a consequence, we obtain that an atomless Banach lattice cannot have a finite-dimensional decomposition of a certain kind.

We also consider a domination problem for the exact dominant of a dominated linear operator. In the classical sense, the domination problem can be stated as

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follows. Let E, F be vector lattices, and let $S, T : E \to F$ be linear operators with $0 \le S \le T$. Let $\mathcal{P}$ be some property of linear operators $R : E \to F$ so that $\mathcal{P}(R)$ means that R possesses $\mathcal{P}$. Does $\mathcal{P}(T)$ imply $\mathcal{P}(S)$? Another version: if $|S| \le T$, then does $\mathcal{P}(T)$ imply $\mathcal{P}(S)$? (See [3] for a survey on the domination problem for "small" operators.)

In the rest of this section we detail some basic definitions and facts. (For general information on vector lattices, Banach spaces, and lattice-normed spaces, see [1], [2], [4]–[6].)

Consider a vector space V and a real archimedean vector lattice E. A map $\lvert\!\cdot\!\rvert : V \to E$ is called a *vector norm* if it satisfies the following axioms:

(1) $\lvert v\rvert \ge 0$; $\lvert v\rvert = 0 \Leftrightarrow v = 0$, $\forall v \in V$,
(2) $\lvert v_1 + v_2\rvert \le \lvert v_1\rvert + \lvert v_2\rvert$, $v_1, v_2 \in V$,
(3) $\lvert \lambda v\rvert = |\lambda|\lvert v\rvert$, $\lambda \in \mathbb{R}$, $v \in V$.

A vector norm is called *decomposable* if

(4) for all $e_1, e_2 \in E_+$ and $x \in V$ from $\lvert x\rvert = e_1 + e_2$, it follows that there exist $x_1, x_2 \in V$ such that $x = x_1 + x_2$ and $\lvert x_k\rvert = e_k$, $k := 1, 2$.

A triple $(V, \lvert\cdot\rvert, E)$—also designated (V, E), or V for short—is called a *lattice-normed space* if $\lvert\cdot\rvert$ is a E-valued vector norm in the vector space V. If the norm $\lvert\cdot\rvert$ is decomposable, then the space V itself is called *decomposable*. We say that a net $(v_\alpha)_{\alpha\in\Delta}$ (bo)*-converges* to an element $v \in V$, and we write $v = (bo)\text{-}\lim_\alpha v_\alpha$ if there exists a decreasing net $(e_\xi)_{\xi\in\Xi}$ in E_+ such that $\inf_{\xi\in\Xi}(e_\xi) = 0$ and if for every $\xi \in \Xi$ there is an index $\alpha(\xi) \in \Delta$ such that $\lvert v - v_{\alpha(\xi)}\rvert \le e_\xi$ for all $\alpha \ge \alpha(\xi)$. A net $(v_\alpha)_{\alpha\in\Delta}$ is called (bo)*-fundamental* if the net $(v_\alpha - v_\beta)_{(\alpha,\beta)\in\Delta\times\Delta}$ (bo)-converges to zero. A lattice-normed space is called (bo)*-complete* if every (bo)-fundamental net (bo)-converges to an element of this space. Let e be a positive element of a vector lattice E. By $[0, e]$, we denote the set $\{v \in V : \lvert v\rvert \le e\}$. A set $D \subset V$ is called (bo)*-bounded* if there exists $e \in E_+$ such that $D \subset [0, e]$. Every decomposable (bo)-complete lattice-normed space is called a *Banach–Kantorovich space. All lattice-normed spaces we consider below are decomposable.*

Let (V, E) be a lattice-normed space. A subspace V_0 of V is called a (bo)*-ideal* of V if, for $v \in V$ and $u \in V_0$, from $\lvert v\rvert \le \lvert u\rvert$, it follows that $v \in V_0$. A subspace V_0 of a decomposable lattice-normed space V is a (bo)-ideal if and only if $V_0 = h(L) := \{v \in V : \lvert v\rvert \in L\}$, where L is an order ideal in E (see [4, Proposition 2.1.6.1]). Let V be a lattice-normed space and let $y, x \in V$. If $\lvert x\rvert \perp \lvert y\rvert = 0$, then we call the elements x, y *disjoint* and write $x \perp y$. The equality $x = \coprod_{i=1}^n x_i$ means that $x = \sum_{i=1}^n x_i$ and $x_i \perp x_j$ if $i \ne j$. For $n = 2$, we write $x = x_1 \sqcup x_2$ if $x_1 \perp x_2$. An element $z \in V$ is called a *fragment* of $x \in V$ if $z \perp (x - z)$. Two fragments x_1, x_2 of x are called *mutually complemented* if $x = x_1 + x_2$. The notation $z \sqsubseteq x$ means that z is a fragment of x. The set of all fragments of an element $v \in V$ is denoted by $\mathcal{F}_v$.

Consider some important examples of lattice-normed spaces. We begin with simple extreme cases, namely, vector lattices and normed spaces. If $V = E$, then the absolute value of an element can be taken as its lattice norm: $\lvert v\rvert := |v| = v \vee (-v)$, $v \in E$. The decomposability of this norm follows from the *Riesz* *decomposition property* holding in every vector lattice (see [2, Theorem 1.13]). If $E=\mathbb{R}$, then V is a normed space.

Let Q be a compact topological space and let X be a Banach space. Let $V := C(Q,X)$ be the space of continuous vector-valued functions from Q to X. Assign $E := C(Q,\mathbb{R})$. Given $f \in V$, we define its lattice norm by the relation $\lvert f\rvert : t \mapsto \|f(t)\|_X$ $(t \in Q)$. Then $\lvert\cdot\rvert$ is a decomposable norm (see [4, Lemma 2.3.2]).

Let (Ω,Σ,μ) be a σ-finite measure space, let E be an order-dense ideal in $L_0(\Omega)$, and let X be a Banach space. We use $L_0(\Omega,X)$ to denote the space of (equivalence classes of) Bochner μ-measurable vector functions acting from Ω to X. As usual, vector-functions are equivalent if they have equal values at almost all points of the set Ω. For a measurable vector-function $f : \Omega \to X$, the map $t \mapsto \|f(t)\|$, $t \in \Omega$, is a scalar measurable function which is denoted by the symbol $\lvert f\rvert \in L_0(\mu)$. Assign by the definition

$$E(X) := \big\{f \in L_0(\mu,X) : \lvert f\rvert \in E\big\}.$$

Then $(E(X),E)$ is a lattice-normed space with a decomposable norm (see [4, Lemma 2.3.7]). If E is a Banach lattice, then the lattice-normed space $E(X)$ is a Banach space with respect to the norm $|||f||| := \|\|f(\cdot)\|_X\|_E$.

Let E be a Banach lattice and let (V,E) be a lattice-normed space. By definition, $\lvert x\rvert \in E_+$ for every $x \in V$, and we can introduce some *mixed norm* in V by the formula

$$\big\||x|\big\| := \big\|\lvert x\rvert\big\|, \quad \forall x \in V.$$

The normed space $(V,|||\cdot|||)$ is called a *space with a mixed norm*. In view of the inequality $|\lvert x\rvert - \lvert y\rvert| \le \lvert x-y\rvert$ and monotonicity of the norm in E, we have

$$\big\|\lvert x\rvert - \lvert y\rvert\big\| \le \big\||x-y|\big\|, \quad \forall x,y \in V,$$

so a vector norm is a norm-continuous operator from $(V,|||\cdot|||)$ to E. A lattice-normed space (V,E) is called a *Banach space with a mixed norm* if the normed space $(V,|||\cdot|||)$ is complete with respect to the norm convergence.

Consider lattice-normed spaces (V,E) and (W,F), a linear operator $T : V \to W$, and a positive operator $S \in L_+(E,F)$. If the condition

$$\lvert Tv\rvert \le S\lvert v\rvert, \quad \forall v \in V$$

is satisfied, then we say that S *dominates* or *majorizes* T or that S is *dominant* or *majorant* for T. In this case, T is called a *dominated* or *majorizable* operator. The set of all dominants of the operator T is denoted by $\mathrm{maj}(T)$. If there is the least element in $\mathrm{maj}(T)$ with respect to the order induced by $L_+(E,F)$, then it is called the *least* or the *exact dominant* of T, and it is denoted by $\lvert T\rvert$. The set of all dominated operators from V to W is denoted by $M(V,W)$. (Narrow operators in vector lattices were first introduced in [7]. Later in the setting of lattice-normed spaces, linear order-narrow operators were investigated in [9]. Recently in [8], the connection between narrow operators and the theory of vector measures was established.)

According to [2, p. 111], an element $e > 0$ of a vector lattice E is called an *atom* whenever $0 \le f_1 \le e$, $0 \le f_2 \le e$, and $f_1 \perp f_2$ imply that either $f_1 = 0$ or $f_2 = 0$.

Definition 1.1. A vector lattice E is said to be *atomless* if it has no atom. We say that a vector lattice E is *purely atomic* if there is a collection $(f_i)_{i\in I}$ of atoms in E_+, called a *generating collection of atoms*, such that $f_i \perp f_j$ for $i \neq j$ and such that, for every $e \in E$, if $|e| \wedge f_i = 0$ for each $i \in I$, then $e = 0$.

Lemma 1.2 ([8, Proposition 1.6])**.** *Any vector lattice E with the principal projection property has a decomposition $E = E_0 \oplus E_1$ into mutually complemented bands, where E_0 is a purely atomic vector lattice and E_1 is an atomless vector lattice.*

Lemma 1.3. *Let (V, E) be a lattice-normed space over vector lattice E with the principal projection property, and let $E = E_0 \oplus E_1$, where E_0, E_1 are mutually complemented bands in E. Then V has a decomposition $V = V_0 \oplus V_1$, where (V_i, E_i) are lattice-normed spaces over E_i, $i \in \{0, 1\}$.*

Proof. Take an arbitrary element $x \in V$. Then $\lvert x\rvert = e$ has the unique decomposition $e = e_0 \sqcup e_1$, $e_i \in E_i$, $i \in \{0, 1\}$. By the decomposability of the vector norm of the space V there exists the unique decomposition of the element $x = x_0 + x_1$, $\lvert x_i\rvert = e_i$, $i \in \{0, 1\}$ (see [4, Proposition 2.1.2.3]). Let $V_0 = \{x_0 : x = x_0 + x_1, x \in V\}$ and let $V_1 = \{x_1 : x = x_0 + x_1, x \in V\}$. It is clear that V_0, V_1 are vector spaces and that vector norm $\lvert \cdot \rvert : V_i \to E_i$, $i \in \{0, 1\}$ is well defined. □

Now we are ready to give some definitions.

Definition 1.4. Let (V, E) be a lattice-normed space over a vector lattice E and X be a Banach space. A linear operator $T : V \to X$ is called *order-to-norm-continuous* if T sends (bo)-convergent nets in V to norm-convergent nets in X.

Definition 1.5. Let (V, E) be a lattice-normed space over an atomless vector lattice E, and let X be a Banach space. A linear operator $T : V \to X$ is called *narrow* if for every $v \in V$ and $\varepsilon > 0$ there exist mutually complemented fragments v_1, v_2 of v such that $\|Tv_1 - Tv_2\| < \varepsilon$.

Note that if a vector lattice E is atomless then, for every nonzero element $x \in V$, the set $\mathcal{F}_x$ has infinite cardinality. Nevertheless, the following two lemmas show that if a vector lattice E has a principal projection property, then there is no need to restrict to the atomless vector lattice in this definition.

Lemma 1.6. *Let (V, E) be a lattice-normed space over a vector lattice E, $x \in V$, let $\mathcal{F}_x$ be a finite set, and let $T : V \to X$ be a narrow operator. Then $Tx = 0$.*

Proof. Assume that $Tx \neq 0$. Since the operator T is narrow, there exists $y \in \mathcal{F}_x$, such that $Ty \neq 0$. Note that $\mathcal{F}_y$ is the proper subset of the $\mathcal{F}_x$. Using the same arguments, we find a fragment $z \in \mathcal{F}_x$ such that $\mathcal{F}_z = \{0, z\}$ and $Tz \neq 0$. Hence by the narrowness, $\|Tz\| < \varepsilon$ for every $\varepsilon > 0$, and we have a contradiction. □

Lemma 1.7. *Let (V, E) be a lattice-normed space over a vector lattice E with a principal projection property; let $E = E_0 \oplus E_1$ be the decomposition into a purely atomic band E_0 and an atomless band E_1; let V be the decomposition $V = V_0 \oplus V_1$ where (V_i, E_i) are lattice-normed spaces over E_i, $i \in \{0, 1\}$; let X be a Banach space; and let $T : V \to X$ be a narrow operator. Then $Tx = 0$ for every $x \in V_0$.*

Proof. Take an nonzero element $x \in V_0$. Then $|x| = e \in E_{0+} > 0$, and there exists a finite number $f_1, \ldots, f_n$ of mutually disjoint positive atoms in E_0 and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_+$ such that $0 < e \leq \sum_{i=1}^{n} \lambda_i f_i$. Taking into account that $f_1, \ldots, f_n$ are atoms, we deduce that $\mathcal{F}_e$ and therefore $\mathcal{F}_x$ are finite sets. Thus by Lemma 1.6, we have $Tx = 0$. □

Next is the definition of an order-narrow operator.

Definition 1.8. Let (V, E) and (W, F) be lattice-normed spaces with E atomless. A linear operator $T : V \to W$ is called *order-narrow* if for every $v \in V$ there exists a net of decompositions $v = v_\alpha^1 \sqcup v_\alpha^2$ such that $(Tv_\alpha^1 - Tv_\alpha^2) \xrightarrow{(bo)} 0$.

2. Main results

In this section, we investigate narrow operators from a Banach–Kantorovich space to sequence Banach lattices. The first result here is the following theorem.

Theorem 2.1. *Let (V, E) be a Banach–Kantorovich space over an atomless Dedekind-complete vector lattice E, and let Γ be any set. Let $X = X(\Gamma)$ denote one of the Banach lattices $c_0(\Gamma)$ or $\ell_p(\Gamma)$ with $1 \leq p < \infty$. Then every order-to-norm-continuous linear dominated operator $T : V \to X$ is narrow.*

For the proof, we need two auxiliary lemmas.

Lemma 2.2 ([9, Lemma 4.11])**.** *Let (V, E) be a Banach–Kantorovich space over an atomless Dedekind-complete vector lattice E, and let F be a finite-dimensional Banach space. Then every order-to-norm-continuous dominated linear operator $T : V \to F$ is narrow.*

Lemma 2.3 ([4, Proposition 4.1.2])**.** *Let (V, E), (W, F) be lattice-normed spaces with V decomposable, and let F be Dedekind-complete. Then every dominated linear operator $T : V \to W$ has the exact dominant $|T|$.*

Proof of Theorem 2.1. Let $T : V \to X$ be a dominated operator, let $v \in V$, and let $\varepsilon > 0$. Note that by Lemma 2.3, the operator T has the exact dominant $|T|$. Take an arbitrary $u \in \mathcal{F}_v$. Since $u \perp (v - u)$, we have the estimation,

$$\begin{aligned} |Tu| &\leq |Tu| + |T(v-u)| \\ &\leq |T||u| + |T||v-u| \\ &= |T||v| = f \in F_+. \end{aligned}$$

Then we choose a finite subset $\Gamma_0 \subset \Gamma$ such that

(1) $|f(\gamma)| \leq \varepsilon/4$ for all $\gamma \in \Gamma \setminus \Gamma_0$, if $X = c_0(\Gamma)$, and
(2) $\sum_{\gamma \in \Gamma \setminus \Gamma_0} (f(\gamma))^p \leq (\varepsilon/4)^p$ if $X = \ell_p(\Gamma)$.

Let P be the projection of X onto $X(\Gamma_0)$ along $X(\Gamma \setminus \Gamma_0)$, and let $Q = \mathrm{Id} - P$ be the orthogonal projection. Obviously, both P and Q are positive linear bounded operators. Since $S = P \circ T : V \to X(\Gamma_0)$ is a finite-rank order-to-norm-continuous dominated operator, by Lemma 2.2, S is narrow, and hence, there are mutually complemented fragments v_1, v_2 of v with $\|S(v_1) - S(v_2)\| < \varepsilon/2$. Since $|T(v_i)| \leq f$, by the positivity of Q we have $Q(Tv_i) \leq Qf$ and $\|Q(T(v_i))\| \leq \|Q(f)\|$ for $i = 1, 2$. Moreover, by (1) and (2), $\|Q(f)\| \leq \varepsilon/4$. Then

$$\begin{aligned}\big\|T(v_1) - T(v_2)\big\| &= \big\|S(v_1) + Q\big(T(v_1)\big) - S(v_2) - Q\big(T(v_2)\big)\big\| \\ &\leq \big\|S(v_1) - S(v_2)\big\| + \big\|Q\big(T(v_1)\big)\big\| + \big\|Q\big(T(v_2)\big)\big\| \\ &< \frac{\varepsilon}{2} + \big\|Q(f)\big\| + \big\|Q(f)\big\| < \varepsilon. \qquad \square\end{aligned}$$

For a space with a mixed norm, we obtain the following consequence of Theorem 2.1.

Lemma 2.4. *Let (V, E) be a Banach space with a mixed norm over an atomless order-continuous Banach lattice E, and let Γ be any set. Let $X = X(\Gamma)$ denote one of the Banach lattices $c_0(\Gamma)$ or $\ell_p(\Gamma)$ with $1 \leq p < \infty$. Then every continuous dominated linear operator $T : V \to X$ is narrow.*

Proof. It is enough to prove that every continuous operator $T : V \to X$ is order-to-norm-continuous. Indeed, take a net $(v_\alpha)_{\alpha\in\Lambda}$ which is (bo)-convergent to zero. This means that $(|v_\alpha|)_{\alpha\in\Lambda} \subset E_+$ is (o)-convergent to zero. Since E is an order-continuous Banach lattice, we have $|||v_\alpha||| = \||v_\alpha|\| \longrightarrow 0$. Taking into account the fact that T is a continuous operator, we have $\|Tv_\alpha\|_{\alpha\in\Lambda} \longrightarrow 0$. Hence, the operator T is order-to-norm-continuous, and by Theorem 2.1 we deduce that T is narrow. $\square$

The idea used in the proof of Theorem 2.1 could be generalized as follows.

Definition 2.5. Let E, F be ordered vector spaces. We say that a linear operator $T : E \to F$ is *quasimonotone with a constant* $M > 0$ if for each $x, y \in E^+$ the inequality $x \leq y$ implies that $Tx \leq MTy$. An operator $T : E \to F$ is said to be *quasimonotone* if it is quasimonotone with some constant $M > 0$.

If $T \neq 0$ in the above definition, we easily obtain $M \geq 1$. Observe also that the quasimonotone operators with constant $M = 1$ exactly are the positive operators.

Recall that a sequence of elements $(e_n)_{n=1}^\infty$ (resp., of finite-dimensional subspaces $(E_n)_{n=1}^\infty$) of a Banach space E is called a *basis* (resp., a *finite-dimensional decomposition*, or *FDD*, for short) if for every $e \in E$ there exists a unique sequence of scalars $(a_n)_{n=1}^\infty$ (resp., sequence $(u_n)_{n=1}^\infty$ of elements $u_n \in E_n$) such that $e = \sum_{n=1}^\infty a_n e_n$ (resp., $e = \sum_{n=1}^\infty u_n$). Every basis (e_n) generates the FDD $E_n = \{\lambda e_n : \lambda \in \mathbb{R}\}$. Any basis (e_n) (resp., any FDD(E_n)) of a Banach space generates the corresponding *basis projections* (P_n) defined by

$$P_n\Big(\sum_{k=1}^\infty a_k e_k\Big) = \sum_{k=1}^n a_k e_k \quad \Big(\text{resp., } P_n\Big(\sum_{k=1}^\infty u_k\Big) = \sum_{k=1}^n u_k\Big),$$

which are uniformly bounded. (For more details about these notions, we refer the reader to [5].) The orthogonal projections to P_n's defined by $Q_n = \mathrm{Id} - P_n$, where Id is the identity operator on E, we will call the *residual projections* associated with the basis $(e_n)_{n=1}^\infty$ (resp., to the FDD$(E_n)_{n=1}^\infty$).

Definition 2.6. A basis (e_n) (resp., an FDD(E_n)) of a Banach lattice E is called *residually quasimonotone* if there is a constant $M > 0$ such that all corresponding residual projections are quasimonotone with the constant M.

In other words, an FDD(E_n) of E is residually quasimonotone if the corresponding approximation of smaller in-modulus elements is better, up to some constant multiple: if $x, y \in E$ with $|x| \le |y|$, then $\|x - P_n x\| \le M\|y - P_n y\|$ for all n (observe that $\|z - P_n z\| \to 0$ as $n \to \infty$ for all $z \in E$).

Theorem 2.7. *Let (V, E) be a Banach–Kantorovich space over an atomless Dedekind-complete vector lattice E, and let F be a Banach lattice with a residually quasimonotone basis or, more generally, a residually quasimonotone FDD. Then every continuous dominated linear operator $T : V \to F$ is narrow.*

Proof. Let (F_n) be an FDD of F with the corresponding projections (P_n), and let $M > 0$ be such that for every $n \in \mathbb{N}$ the operator $Q_n = \mathrm{Id} - P_n$ is quasimonotone with constant M. Let $T : V \to F$ be a dominated linear operator, let $v \in V$, and let $\varepsilon > 0$. Choose $f \in F_+$ so that $|Tx| \le f$ for all $x \sqsubseteq v$. Since $\lim_{n\to\infty} P_n f = f$, we have $\lim_{n\to\infty} Q_n f = 0$. Choose n so that

$$\|Q_n f\| \le \frac{\varepsilon}{4M}. \tag{2.1}$$

Since $S = P_n \circ T : V \to E_n$ is a finite-rank dominated linear operator by Lemma 2.2, and since S is narrow, there are mutually complemented fragments v_1, v_2 of v such that $\|Sv_1 - Sv_2\| < \varepsilon/2$. Since $|Tv_i| \le f$, by the quasimonotonicity of Q_n we have $\|Q_n(Tv_i)\| \le M\|Q_n f\|$ for $i = 1, 2$. Then by (2.1),

$$\begin{aligned}\|Tv_1 - Tv_2\| &= \big\|Sv_1 + Q(Tv_1) - Sv_2 - Q(Tv_2)\big\| \\ &\le \|Sv_1 - Sv_2\| + \big\|Q(Tv_1)\big\| + \big\|Q(Tv_2)\big\| \\ &< \frac{\varepsilon}{2} + M\|Qf\| + M\|Qf\| < \varepsilon. \end{aligned}$$ ☐

Remark 2.8. An atomless order-continuous Banach lattice E cannot admit a residually quasimonotone FDD.

Proof. The Banach lattice E is a lattice-normed space (E, E), where the vector norm coincides with the absolute value. Thus, it is enough to observe that the identity operator of such a Banach lattice is not narrow. ☐

Recall that a vector lattice E is said to possesses the *strong Freudenthal property*, if for $f, e \in E$ such that $|f| \le \lambda|e|$, $\lambda \in \mathbb{R}_+$ f can be e-uniformly approximated by linear combinations $\sum_{k=1}^n \lambda_k \pi_k e$, where $\pi_1, \dots, \pi_n$ are order projections in E.

Denote by E_{0+} the conic hull of the set $\lvert V\rvert = \{\lvert v\rvert : v \in V\}$ (i.e., the set of elements of the form $\sum_{k=1}^n \lvert v_k\rvert$, where $v_1, \dots, v_n \in V$, $n \in \mathbb{N}$).

Theorem 2.9 ([4, Theorem 4.1.8])**.** *Let (V, E), (W, F) be lattice-normed spaces. Suppose that E possesses the strong Freudenthal property, suppose that V is decomposable, and suppose that F is Dedekind-complete. Then the exact dominant of an arbitrary operator $T \in M(V, W)$ can be calculated by the following* *formulas:*

$$\begin{aligned}
&|T|(e) = \sup\Big\{\sum_{i=1}^{n} |Tv_i| : \sum_{i=1}^{n} |v_i| = e, |v_i| \perp |v_j|; i \neq j; n \in \mathbb{N}; e \in E_{0+}\Big\},\\
&|T|(e) = \sup\big\{|T|(e_0) : e_0 \in E_{0+}; e_0 \leq e\big\}, \quad e \in E_+,\\
&|T|(e) = |T|(e_+) - |T|(e_-), \quad e \in E.
\end{aligned}$$

The next theorem is the second main result of the article. Here, we generalized the first part of the Theorem 5.1 from [9].

Theorem 2.10. *Let (V, E) be a lattice-normed space, let E be atomless and possessed of the strong Freudenthal property, let (W, F) be a Banach space with a mixed norm, let F be a Banach lattice with an order-continuous norm, and let T be a (bo)-continuous dominated linear operator from V to W. If T is an order-narrow operator, then the same is its exact dominant $|T| : E \to F$.*

Proof. Since the lattice-normed space V is decomposable and the Banach lattice F is Dedekind-complete, by the Lemma 2.3 every dominated operator $T : V \to W$ has the exact dominant $|T|$. By ([9, Lemma 3.4]), instead of order-narrowness, we will consider narrowness. Fix any $e \in E_{0+}$ and $\varepsilon > 0$. Since

$$\Big\{\sum_{i=1}^{n} |Tv_i| : \coprod_{i=1}^{n} |v_i| = e; n \in \mathbb{N}\Big\}$$

is an increasing net, there exists a net of finite collections $\{v_1^\alpha, \dots, v_{n_\alpha}^\alpha\} \subset V$, $\alpha \in \Lambda$ with $e = \bigsqcup_{i=1}^{n_\alpha} |v_i^\alpha|, \alpha \in \Lambda$, and $(|T|(e) - \sum_{i=1}^{n_\alpha} |Tv_i^\alpha|) \leq y_\alpha \xrightarrow{(o)} 0$, where $0 \leq y_\alpha$, $\alpha \in \Lambda$, is an decreasing net and $\inf(y_\alpha)_{\alpha\in\Lambda} = 0$. The norm in F is order-continuous, and therefore we may assume that $\||T|(e) - \sum_{i=1}^{n_\alpha} |Tv_i^\alpha|\| \leq \frac{\varepsilon}{3}$ for some $\{v_1^\alpha, \dots, v_{n_\alpha}^\alpha\}$, $\alpha \in \Lambda$. Since T is an order-narrow operator, we may assume that there exists a finite set of a nets of decompositions $v_i^\alpha = u_i^{\beta_\alpha} \sqcup w_i^{\beta_\alpha}$, $i \in \{1, \dots, n_\alpha\}$, which depends of α, indexed by the same set Δ, such that $|Tu_i^{\beta_\alpha} - Tw_i^{\beta_\alpha}| \xrightarrow{(bo)} 0$, $i \in \{1, \dots, n_\alpha\}$. By [9, Lemma 3.4] and the fact that the norm in F is order-continuous, we may assume that $|||Tu_i^{\beta_\alpha} - Tw_i^{\beta_\alpha}||| < \frac{\varepsilon}{3n_\alpha}$ for every $i \in \{1, \dots, n_\alpha\}$ and some β_α. Let $f^{\beta_\alpha} = \coprod_{i=1}^{n_\alpha} |u_i^{\beta_\alpha}|$ and $g^{\beta_\alpha} = \coprod_{i=1}^{n_\alpha} |w_i^{\beta_\alpha}|$. Then we have

$$\begin{aligned}
0 \leq \Big\||T|(f^{\beta_\alpha}) - \sum_{i=1}^{n_\alpha} |Tu_i^{\beta_\alpha}|\Big\| \leq \Big\||T|(e) - \sum_{i=1}^{n_\alpha} |Tv_i^\alpha|\Big\|,\\
0 \leq \Big\||T|(g^{\beta_\alpha}) - \sum_{i=1}^{n_\alpha} |Tw_i^{\beta_\alpha}|\Big\| \leq \Big\||T|(e) - \sum_{i=1}^{n_\alpha} |Tv_i^\alpha|\Big\|.
\end{aligned}$$

Now we may write

$$\big\||T|f^{\beta_\alpha} - |T|g^{\beta_\alpha}\big\| = \Big\||T|f^{\beta_\alpha} - \sum_{i=1}^{n_\alpha} |Tu_i^{\beta_\alpha}| + \sum_{i=1}^{n_\alpha} |Tu_i^{\beta_\alpha}|$$

$$
\begin{aligned}
&- \sum_{i=1}^{n_\alpha} |Tw_i^{\beta_\alpha}| + \sum_{i=1}^{n_\alpha} |Tw_i^{\beta_\alpha}| - |T|g^{\beta_\alpha}\Big\| \\
&\leq \Big\| |T|f^{\beta_\alpha} - \sum_{i=1}^{n_\alpha} |Tu_i^{\beta_\alpha}| \Big\| \\
&\quad + \Big\| |T|g^{\beta_\alpha} - \sum_{i=1}^{n_\alpha} |Tw_i^{\alpha}| \Big\| + \Big\| \sum_{i=1}^{n_\alpha} |Tu_i^{\beta_\alpha}| - \sum_{i=1}^{n_\alpha} |Tw_i^{\beta_\alpha}| \Big\| \\
&\leq 2\Big(\Big\| |T|(e) - \sum_{i=1}^{n_\alpha} |Tv_i^{\alpha}| \Big\| \Big) + \sum_{i=1}^{n_\alpha} \Big\| |Tu_i^{\beta_\alpha}| - |Tw_i^{\beta_\alpha}| \Big\| \\
&\leq 2\Big(\Big\| |T|(e) - \sum_{i=1}^{n_\alpha} |Tv_i^{\alpha}| \Big\| \Big) + \sum_{i=1}^{n_\alpha} \| |Tu_i^{\beta_\alpha} - Tw_i^{\beta_\alpha}| \| \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

Therefore $e = f^{\beta_\alpha} \sqcup g^{\beta_\alpha}$, $\alpha \in \Lambda$, $\beta_\alpha \in \Delta$, is the desirable decomposition of the element e. Now, let $e \in E_+$. Note that $D = \{f \leq e : f \in E_{0+}\}$ is a directed set. Indeed, let $f_1 = \coprod_{i=1}^{k} |u_i|$, $f_1 \leq e$; $f_2 = \coprod_{j=1}^{n} |w_j|$, $f_2 \leq e$, $u_i, w_j \in V$, $1 \leq i \leq k$, $1 \leq j \leq n$. Then by the decomposability of the vector norm there exists the set of mutually disjoint elements (v_{ij}), $1 \leq i \leq k$, $1 \leq j \leq n$ such that $u_i = \coprod_{j=1}^{n} v_{ij}$ for every $1 \leq i \leq k$ and $w_j = \coprod_{i=1}^{k} v_{ij}$ for every $1 \leq j \leq n$. Let $f = \coprod |v_{ij}|$. It is clear that $|T|f_i \leq |T|f$, $i \in \{1, 2\}$. Let $(e_\alpha)_{\alpha \in \Lambda}$, $e_\alpha \in D$, be the net, where $|T|e = \sup_\alpha |T|e_\alpha$. Fix $\alpha \in \Lambda$ such that $\| |T|e - |T|e_\alpha \| < \frac{\varepsilon}{2}$. For $e_\alpha \in D$ there exists the net of decompositions $e_\alpha = f_\alpha^\beta \sqcup g_\alpha^\beta$, $\beta \in \Delta$, such that $\| |T|f_\alpha^\beta - |T|g_\alpha^\beta \| < \frac{\varepsilon}{2}$. Thus we have

$$
\begin{aligned}
\big\| |T|(e - e_\alpha + f_\alpha^\beta) - |T|g_\alpha^\beta \big\| &= \big\| |T|(e - e_\alpha) + |T|f_\alpha^\beta - |T|g_\alpha^\beta \big\| \\
&\leq \big(\big\| |T|e - |T|e_\alpha \big\| + \big\| |T|f_\alpha^\beta - |T|g_\alpha^\beta \big\| \big) \\
&< \varepsilon.
\end{aligned}
$$

Hence, $e = ((e - e_\alpha) \sqcup f_\alpha^\beta)) \sqcup g_\alpha^\beta$ is the desirable decomposition of the element e. Finally, for an arbitrary element $e \in E$ we have $e = e_+ - e_-$, and by Theorem 2.9 we have $|T|(e) = |T|(e_+) - |T|(e_-)$. Thus, if $e_+ = f_1^\alpha \sqcup f_2^\alpha$ and $e_- = g_1^\alpha \sqcup g_2^\alpha$ are necessary decompositions, then we have

$$
\begin{aligned}
\big\| |T|(f_1^\alpha + g_1^\alpha) - |T|(f_2^\alpha + g_2^\alpha) \big\| &= \big\| |T|f_1^\alpha - |T|f_2^\alpha + |T|g_1^\alpha - |T|g_2^\alpha) \big\| \\
&\leq \big(\big\| |T|(f_1^\alpha - f_2^\alpha) \big\| + \big\| |T|(g_1^\alpha - g_2^\alpha) \big\| \big) \\
&< \varepsilon,
\end{aligned}
$$

and $e = (f_1^\alpha + g_1^\alpha) \sqcup (f_2^\alpha + g_2^\alpha)$ is the desirable decomposition. □

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