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THE *BD* PROPERTY IN SPACES OF COMPACT OPERATORS

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ABSTRACT. For Banach spaces X and Y, let $K_{w^*}(X^*, Y)$ denote the space of all $w^* - w$ continuous compact operators from X^* to Y endowed with the operator norm. A Banach space X has the *BD* property if every limited subset of X is relatively weakly compact. We prove that if X has the Gelfand–Phillips property and Y has the *BD* property, then $K_{w^*}(X^*, Y)$ has the *BD* property.

1. Introduction and preliminaries

A bounded subset A of X is called a *limited* subset of X if every w^*-null sequence (x_n^*) in X^* tends to 0 uniformly on A; that is,

$$\lim_n \big(\sup\big\{\big|x_n^*(x)\big| : x \in A\big\}\big) = 0.$$

If A is a limited subset of X, then $T(A)$ is relatively compact for any operator $T : X \to c_0$ (see [1], [14]). A Banach space X has the *BD property* (see [1]) if every limited subset of X is relatively weakly compact. The space X has property *BD* whenever X is weakly sequentially complete or X does not contain ℓ_1 (see [1], [14]).

If X has the *BD* property, then $L^p(\mu, X)$, $1 \leq p < \infty$, also has the *BD* property (see [5], [11]). If $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$ and both X and Y have the *BD* property, then $K_{w^*}(X^*, Y)$ has the *BD* property (see [8]).

In this note, we study whether the space $K_{w^*}(X^*, Y)$ has the *BD* property when X and Y have the *BD* property. We give some applications to the spaces $(N_1(X, Y))^*$ and we prove that in some cases, if $L(X, Y)$ has the *BD* property, then $L(X, Y) = K(X, Y)$.

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Throughout this paper, X and Y will denote Banach spaces. The unit ball of X will be denoted by B_X, and X^* will denote the topological dual of X. The canonical unit vector basis of c_0 will be denoted by (e_n). An operator $T : X \to Y$ will be a continuous and linear function. The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X, Y)$, $W(X, Y)$, and $K(X, Y)$. The $w^* - w$ continuous (resp., compact) operators from X^* to Y will be denoted by $L_{w^*}(X^*, Y)$ (resp., $K_{w^*}(X^*, Y)$). The injective tensor product of two Banach spaces X and Y will be denoted by $X \otimes_\epsilon Y$. The space $X \otimes_\epsilon Y$ can be embedded into the space $K_{w^*}(X^*, Y)$ by identifying $x \otimes y$ with the rank 1 operator $x^* \to \langle x^*, x\rangle y$.

A bounded subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. Every limited set is weakly precompact (e.g., see [1]).

The space X has the *Gelfand–Phillips (GP) property* if every limited subset of X is relatively compact. The following spaces have the Gelfand–Phillips property: Schur spaces; spaces with w^*-sequential compact dual unit balls; separable spaces; reflexive spaces; spaces whose duals do not contain ℓ_1; subspaces of weakly compactly generated spaces; spaces whose duals have the Radon–Nikodym property (see [1], [14, p. 31]).

A topological space S is called *dispersed (or scattered)* if every nonempty closed subset of S has an isolated point. A compact Hausdorff space K is dispersed if and only if $\ell_1 \not\hookrightarrow C(K)$.

2. The *BD* property in $K_{w^*}(X^*, Y)$

Let wot denote the *weak operator topology* on $L(X, Y)$: $T_n \to T$ (wot) provided that $\langle T_n(x), y^*\rangle \to \langle T(x), y^*\rangle$ for all $x \in X$ and $y^* \in Y^*$ (see [10]). In [8, Lemma 4.7] it is shown that if (T_n) is a sequence in $K_{w^*}(X^*, Y)$ such that $T_n \to T$ (wot), where $T \in K_{w^*}(X^*, Y)$, then $T_n \to T$ weakly.

We note that if $K_{w^*}(X^*, Y)$ has the *BD* (resp., the Gelfand–Phillips) property, then X and Y have it too, since this property is inherited by closed subspaces.

We say that an operator $T : X \to Y$ is *limited weakly completely continuous* if it maps weakly Cauchy limited sequences to weakly convergent sequences.

Suppose that X and Y are Banach spaces and that M is a closed subspace of $L_{w^*}(X^*, Y)$. If $x^* \in X^*$ and $y^* \in Y^*$, the evaluation operators $\phi_{x^*} : M \to Y$ and $\psi_{y^*} : M \to X$ are defined by

$$\phi_{x^*}(T) = T(x^*), \qquad \psi_{y^*}(T) = T^*(y^*), \quad T \in M.$$

Theorem 2.1. *Suppose that X has the Gelfand–Phillips property. If the evaluation operator $\phi_{x^*} : K_{w^*}(X^*, Y) \to Y$ is limited weakly completely continuous for each $x^* \in X^*$, then $K_{w^*}(X^*, Y)$ has the BD property.*

Proof. Let H be a limited subset of $M = K_{w^*}(X^*, Y)$. For fixed $y^* \in Y^*$, the map $\psi_{y^*} : M \to X$ is a bounded operator. Then $H^*(y^*)$ is a limited subset of X, and thus is relatively compact.

Let (T_n) be a sequence in H. Since limited sets are weakly precompact (see [1]), without loss of generality we can assume that (T_n) is weakly Cauchy. For each $y^* \in Y^*$, $(T_n^*(y^*))$ is weakly Cauchy and relatively compact, and hence convergent. Let $x^* \in X^*$. Since $\phi_{x^*} : M \to Y$ is limited weakly completely continuous, $(T_n(x^*))$ is weakly convergent.

Define $T : X^* \to Y$ by $T(x^*) = w - \lim T_n(x^*)$, $x^* \in X^*$. Since $T_n(x^*) \xrightarrow{w} T(x^*)$, $T_n^*(y^*) \xrightarrow{w} T^*(y^*)$ for each $y^* \in Y^*$. Then $T^*(y^*) \in X$ for each $y^* \in Y^*$, and thus T is $w^* - w$ continuous.

We will show that $T^*(B_{Y^*})$ is a limited subset of X. Let (y_i^*) be a sequence in B_{Y^*} and (x_i^*) be a w^*-null sequence in X^*. Define $L : K_{w^*}(X^*, Y) \to c_0$ by $L(S) = (\langle x_i^*, S^*(y_i^*)\rangle)_i$, for $S \in K_{w^*}(X^*, Y)$. If $S \in K_{w^*}(X^*, Y)$, then

$$\left\langle x_i^*, S^*(y_i^*)\right\rangle = \left\langle S(x_i^*), y_i^*\right\rangle \le \left\|S(x_i^*)\right\| \to 0.$$

Thus S is a well-defined operator.

Since $(L(T_n))$ is a limited subset of c_0, it is relatively compact [1]. Note that $\lim_n \langle x_i^*, T_n^*(y_i^*)\rangle = \langle x_i^*, T^*(y_i^*)\rangle$ for all i. Therefore $\lim_i \langle x_i^*, T^*(y_i^*)\rangle = 0$. Then $T^*(B_{Y^*})$ is a limited subset of X, thus relatively compact. Then T^*, thus T, is compact. Hence $(T_n) \to T$ weakly by [8, Lemma 4.7]. Thus H is relatively weakly compact. □

Corollary 2.2. *If X has the Gelfand–Phillips property and Y has the BD property, then $K_{w^*}(X^*, Y)$ has the BD property.*

Proof. Since Y has the *BD* property, $\phi_{x^*} : K_{w^*}(X^*, Y) \to Y$ is limited weakly completely continuous for each $x^* \in X^*$. Apply Theorem 2.1. □

We recall the following well-known isometries:

(1) $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$, $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$ $(T \to T^*)$,
(2) $W(X, Y) \simeq L_{w^*}(X^{**}, Y)$ and $K(X, Y) \simeq K_{w^*}(X^{**}, Y)$ $(T \to T^{**})$.

Corollary 2.3. *If X has the BD property and Y has the Gelfand–Phillips property, then $K_{w^*}(X^*, Y)$ has the BD property.*

Corollary 2.4. *Suppose that X has the Gelfand–Phillips property and Y has the BD property (or X has the BD property and Y has the Gelfand–Phillips property). Then $X \otimes_\epsilon Y$ has the BD property.*

Proof. By Corollary 2.2 (or Corollary 2.3), $K_{w^*}(X^*, Y)$ has the *BD* property. Hence $X \otimes_\epsilon Y$ has the *BD* property, since property *BD* is inherited by closed subspaces. □

Example 2.5. The space $L_1(\mu)$, where μ is a finite measure, has the Gelfand–Phillips property. Suppose that X has the *BD* property. It is known that $L_1(\mu) \otimes_\epsilon X \simeq K_{w^*}(X^*, L_1(\mu))$ (see [3, Theorem 5]). By Corollary 2.3, this space has the *BD* property.

Example 2.6. The space c_0 has the Gelfand–Phillips property (see [1]). Suppose that X has the *BD* property. It is known that $c_0 \otimes_\epsilon X \simeq c_0(X)$, the Banach space of sequences in X that converge to zero, with the norm $\|(x_n)\| = \sup_n \|x_n\|$ (see [13, p. 47]). Then $c_0 \otimes_\epsilon X$ has the *BD* property by Corollary 2.4.

Definition 2.7. A subset S of a topological space $T = (T, \rho)$ is *ρ-conditionally sequentially compact* (shortly, (ρ-) CSC) if every sequence in S has a subsequence converging to a limit in S (see [4]). A topological space T *satisfies condition (DCSC)* if it has a *dense conditionally sequentially compact* subset S (see [4]).

Corollary 2.8.

(i) *If K is a compact Hausdorff topological space satisfying (DCSC) and Y has the BD property, then $C(K, Y)$ has the BD property.*
(ii) *If X contains no copy of ℓ_1 and Y has the Gelfand–Phillips property, then $K_{w^*}(X^*, Y)$ and $X \otimes_\epsilon Y$ have the BD property.*
(iii) *If K is dispersed and Y has the Gelfand–Phillips property, then $C(K, Y)$ has the BD property.*

Proof.

(i) If K is (DCSC), then $C(K)$ has the Gelfand–Phillips property (see [4]). Hence $C(K) \otimes_\epsilon Y \simeq C(K, Y)$ has the *BD* property, by Corollary 2.4.
(ii) Since X contains no copy of ℓ_1, X has the *BD* property (see [1], [14]). Apply Corollaries 2.3 and 2.4.
(iii) Since K is dispersed, $X = C(K)$ contains no copy of ℓ_1. By (ii), $C(K) \otimes_\epsilon Y \simeq C(K, Y)$ has the *BD* property. □

Remark 2.9. There is a Banach space Y such that Y contains no copies of ℓ_1 and Y does not have the Gelfand–Phillips property (see [14, Theorem 5.2.4]). If K is (DCSC), then $C(K, Y)$ has the *BD* property by Corollary 2.8(i), and it does not have the Gelfand–Phillips property. Schlumprecht constructed a $C(K)$ space which has the *BD* property, but does not have the Gelfand–Phillips property (see [14, Proposition 5.1.7]). If Y has the Gelfand–Phillips property, then $C(K, Y)$ has the *BD* property by Corollary 2.4, and it does not have the Gelfand–Phillips property.

Corollary 2.10. *Suppose that X^* has the Gelfand–Phillips property and Y has the BD property (or that X^* has the BD property and Y has the Gelfand–Phillips property). Then $K(X, Y)$ and $X^* \otimes_\epsilon Y$ have the BD property.*

Proof. Apply Corollaries 2.2, 2.3, and 2.4 (with X^* instead of X) and the isometry $K(X, Y) \simeq K_{w^*}(X^{**}, Y)$. □

Definition 2.11. A Banach space X has the *Grothendieck property* if every w^*-convergent sequence in X^* is weakly convergent.

If $X = C(K)$ has the Grothendieck property, then a bounded subset of X is weakly precompact if and only if it is limited (see [1], [14]). It is known that the space ℓ_∞ does not have property *BD* (see [14, Example 1.1.8]). For instance, let $(s_n) = (\sum_{i=1}^n e_i)$. Note that (s_n) is bounded, $(s_n) \subseteq c_0$, and (s_n) is weakly precompact. Further, (s_n) is not relatively weakly compact (since $(1, 1, 1, \ldots)$ is not in c_0). Thus, if X has property *BD*, then $\ell_\infty \not\hookrightarrow X$.

The next three results continue to concentrate on conditions which ensure that spaces of operators have the *BD* property.

Theorem 2.12. *Suppose that $L_{w^*}(X^*,Y) = K_{w^*}(X^*,Y)$. The following statements are equivalent:*

(i) *X and Y have the BD property and either $\ell_2 \not\hookrightarrow X$ or $\ell_2 \not\hookrightarrow Y$, and if, moreover, Y is a dual space Z^*, then the condition $\ell_2 \not\hookrightarrow Y$ implies that $\ell_1 \not\hookrightarrow Z$;*

(ii) *$L_{w^*}(X^*,Y)$ has the BD property.*

Proof. (i) $\Rightarrow$ (ii) This is proved by [8, Corollary 4.11].

(ii) $\Rightarrow$ (i) Suppose that $L_{w^*}(X^*,Y)$ has *BD* property. Then X and Y have the *BD* property, since the *BD* property is inherited by closed subspaces. Suppose $\ell_2 \hookrightarrow X$ and $\ell_2 \hookrightarrow Y$. Then c_0 embeds in $K_{w^*}(X^*,Y)$ (by [9, Theorem 20]). Since $c_0 \hookrightarrow L_{w^*}(X^*,Y)$ and X and Y do not have the Schur property, $\ell_\infty \hookrightarrow L_{w^*}(X^*,Y)$ (by [9, Corollary 2]). This contradiction proves the first assertion.

Now suppose $Y = Z^*$ and $\ell_1 \hookrightarrow Z$. Then $L_1 \hookrightarrow Z^*$ ([2, p. 212]). Also, the Rademacher functions span ℓ_2 inside of L_1, hence $\ell_2 \hookrightarrow Z^*$. □

A similar argument shows that if $L_{w^*}(X^*,Y)$ has the *BD* property, then X and Y have the *BD* property and either $\ell_p \not\hookrightarrow X$ or $\ell_q \not\hookrightarrow Y$, for $1 < p' \leq q < \infty$ (where p and p' are conjugate).

Corollary 2.13. *Suppose that $W(X,Y) = K(X,Y)$. The following statements are equivalent:*

(i) *X^* and Y have the BD property and either $\ell_1 \not\hookrightarrow X$ or $\ell_2 \not\hookrightarrow Y$, and if, moreover, Y is a dual space Z^*, then the condition $\ell_2 \not\hookrightarrow Y$ implies that $\ell_1 \not\hookrightarrow Z$;*

(ii) *$W(X,Y)$ has the BD property.*

Proof. Apply Theorem 2.12 and the isometries in (2) in Corollary 2.2. □

Corollary 2.14. *Suppose that $L(X,Y^*) = K(X,Y^*)$ and that both X^* and Y^* have the BD property. Then $K(X,Y^*)$ has the BD property and $\ell_1 \overset{c}{\not\hookrightarrow} X \otimes_\pi Y$.*

Proof. Note that $L(X,Y^*) \simeq (X \otimes_\pi Y)^*$ (see [13, p. 24]) and that $L(X,Y^*) = K(X,Y^*)$ has the *BD* property by [8, Corollary 4.12]. Hence $\ell_\infty \not\hookrightarrow L(X,Y^*)$. By a result of Bessaga and Pełczyński (see [2, Theorem 8]), $\ell_1 \overset{c}{\not\hookrightarrow} X \otimes_\pi Y$. □

Corollary 2.15. *Suppose that $L(X^*,Y^*) = K(X^*,Y^*)$ and that both X^{**} and Y^* have the BD property. Then the dual of the space of all nuclear operators $N_1(X,Y)$ has the BD property, and hence $\ell_1 \overset{c}{\not\hookrightarrow} N_1(X,Y)$.*

Proof. It is known that $N_1(X,Y)$ is a quotient of $X^* \otimes_\pi Y$ (see [13, p. 41]). By [8, Corollary 4.12], $(X^* \otimes_\pi Y)^* \simeq L(X^*,Y^*)$ has the *BD* property. Hence the dual of $N_1(X,Y)$ is a closed subspace of $(X^* \otimes_\pi Y)^*$, so it inherits the *BD* property of $(X^* \otimes_\pi Y)^* \simeq L(X^*,Y^*)$. Thus $\ell_\infty \not\hookrightarrow (N_1(X,Y))^*$. By Bessaga and Pełczyński's result mentioned above, $\ell_1 \overset{c}{\not\hookrightarrow} N_1(X,Y)$. □

Next we present some results about the necessity of the conditions $L_{w^*}(X^*,Y) = K_{w^*}(X^*,Y)$ and $L(X,Y) = K(X,Y)$.

Theorem 2.16. *Suppose that X and Y are infinite-dimensional Banach spaces satisfying the following assumption: if T is an operator in $L_{w^*}(X^*, Y)$, then there is a sequence of operators (T_n) in $K_{w^*}(X^*, Y)$ such that, for each $x^* \in X^*$, the series $\sum T_n(x^*)$ converges unconditionally to $T(x^*)$. If $L_{w^*}(X^*, Y)$ has the BD property, then $L_{w^*}(X^*, Y) = K_{w^*}(X^*, Y)$.*

Proof. Suppose that the assumption holds. If $L_{w^*}(X^*, Y) \neq K_{w^*}(X^*, Y)$, then [9, Theorem 14] implies that ℓ_∞ embeds in $L_{w^*}(X^*, Y)$. Hence, $L_{w^*}(X^*, Y)$ does not have the *BD* property. □

The assumption of the previous theorem is satisfied, for instance, in the following cases.

(1) X (or Y) has an *unconditional compact expansion of the identity* (UCEI), (i.e. there is a sequence (A_n) of compact operators from X to X such that $\sum A_n(x)$ converges unconditionally to x for all $x \in X$); in this case, (A_n) is called a UCEI of X.

(2) Y is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n) and $L(X^*, Z_n) = K(X^*, Z_n)$ for each n. (A sequence (X_n) of closed subspaces of a Banach space X is called an *unconditional Schauder decomposition of* X if every $x \in X$ has a unique representation of the form $x = \sum x_n$, with $x_n \in X_n$, for every n, and the series converges unconditionally.)

Corollary 2.17. *Suppose that X and Y are infinite-dimensional Banach spaces such that X^* or Y has UCEI. If $W(X, Y)$ has the BD property, then $W(X, Y) = K(X, Y)$.*

Proof. Apply Theorem 2.16 and the isometries in (2) in Corollary 2.2. □

Definition 2.18. A series $\sum x_n$ in X is said to be *weakly unconditionally convergent* (wuc) if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. Equivalently, $\sum x_n$ is wuc if $\{\sum_{n \in A} x_n : A \subseteq \mathbb{N}, A \text{ finite}\}$ is bounded.

Definition 2.19. A basis (x_i) of E is *shrinking* if the associated sequence of coordinate functionals (x_i^*) is a basis for E^*.

Definition 2.20. A separable Banach space X has the *bounded approximation property* (bap) if there is a sequence (A_n) of finite rank operators from X to X such that $\sum A_n(x)$ converges to x for all $x \in X$ (see [7]).

Definition 2.21. The space X has (Rademacher) *cotype* q for some $2 \leq q \leq \infty$ if there is a constant C such that for every n and every $x_1, x_2, \ldots, x_n$ in X,

$$\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \leq C\Big(\int_0^1 \big\|r_i(t)x_i\big\|^2 \, dt\Big)^{1/2},$$

where (r_n) are the Rademacher functions.

For the definition of $\mathcal{L}_\infty$-spaces and $\mathcal{L}_1$-spaces, we refer the reader to [2, p. 181] or [13, p. 31, 51]. The dual of an $\mathcal{L}_1$-space (resp., $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space (resp., $\mathcal{L}_1$-space).

Theorem 2.22. *Suppose that X, Y are infinite-dimensional Banach spaces satisfying one of the assumptions:*

(i) *if $T : X \to Y$ is an operator, then there is a sequence (T_n) in $K(X,Y)$ such that, for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally to $T(x)$;*
(ii) *c_0 embeds in either Y or X^*;*
(iii) *X is an $\mathcal{L}_\infty$-space and Y is a closed subspace of an $\mathcal{L}_1$-space;*
(iv) *$X = C(K)$, K a compact Hausdorff space, and Y is a space with cotype 2;*
(v) *X is weakly compactly generated, Y is a subspace of a space Z with a shrinking unconditional basis and X^* or Y^* has the bounded approximation property;*
(vi) *X has UCEI.*

If $L(X,Y)$ has the BD property, then $L(X,Y) = K(X,Y)$.

Proof. Suppose that $L(X,Y)$ has the *BD* property and that $L(X,Y) \neq K(X,Y)$.

(i) Let $T : X \to Y$ be a noncompact operator. Let (T_n) be a sequence as in the hypothesis. By the *uniform boundedness principle*, $\{\sum_{n\in A} T_n : A \subseteq \mathbb{N}, A \text{ finite}\}$ is bounded in $K(X,Y)$. Then $\sum T_n$ is wuc and not unconditionally convergent (since T is noncompact). By a result of Bessaga and Pełczyński (see [2, Theorem 8]), $c_0 \hookrightarrow K(X,Y)$.

(ii) If c_0 embeds in Y or X^*, then c_0 embeds in $K(X,Y)$.

Suppose that (iii) or (iv) holds. It is known that any operator $T : X \to Y$ is 2-absolutely summing (see [2, p. 189]), and hence it factorizes through a Hilbert space. Then $c_0 \hookrightarrow K(X,Y)$ (by [6, Remark 3]).

(v) By [7, Theorem 4], $c_0 \hookrightarrow K(X,Y)$.

(vi) By [10, Theorem 6], $c_0 \hookrightarrow K(X,Y)$.

By [9, Theorem 1], $\ell_\infty \hookrightarrow L(X,Y)$. Since the *BD* property is inherited by closed subspaces and ℓ_∞ does not have this property, we have a contradiction. □

Assumption (i) of the Theorem 2.22 is satisfied, for instance, in the following cases:

(1) X^* (or Y) has UCEI;
(2) Y is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n) and $L(X, Z_n) = K(X, Z_n)$ for each n.

Example 2.23. For $1 < p < q < \infty$, the natural inclusion map $i : \ell_p \to \ell_q$ is not compact. Then $c_0 \hookrightarrow K(\ell_p, \ell_q)$, $\ell_\infty \hookrightarrow L(\ell_p, \ell_q) = W(\ell_p, \ell_q)$ (by [9, Theorem 14]), and $L(\ell_p, \ell_q)$ does not have the *BD* property.

Definition 2.24. An operator $T : X \to Y$ is *completely continuous* (or *Dunford–Pettis*) if T maps weakly convergent sequences to norm convergent sequences. A Banach space X has the *Dunford–Pettis property* (DPP) if every weakly compact operator T with domain X is completely continuous.

Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP. The reader can check [2] and [3] for a guide to the extensive classical literature dealing with the DPP. The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the DPP.

Definition 2.25. An operator $T : X \to Y$ is called *limited* if $T(B_X)$ is a limited subset of Y.

The operator T is limited if and only if $T^* : Y^* \to X^*$ is w^*-norm sequentially continuous. Let $Li(X, Y)$ denote the space of limited operators $T : X \to Y$.

Theorem 2.26. *Assume that one of the following assumptions holds:*

(i) *X has the DPP and $\ell_1 \hookrightarrow Y$,*

(ii) *X and Y have the DPP.*

If $W(X, Y^*)$ has the BD property, then $Li(X, Y^*) = K(X, Y^*)$.

Proof. Since every compact operator $T : X \to Y^*$ is limited, we only need to show that every limited operator $T : X \to Y^*$ is compact.

Suppose that $W(X, Y^*)$ has property BD. Since Y^* has the BD property, every limited operator $T : X \to Y^*$ is weakly compact, by [8, Theorem 3.12].

(i) Since $W(X, Y^*)$ has the BD property, either $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$, by the second part of Corollary 2.13. By the assumption $\ell_1 \hookrightarrow Y$, we obtain $\ell_1 \not\hookrightarrow X$. Since moreover X has the DPP, X^* has the Schur property (see [2, p. 212]). Let $T : X \to Y^*$ be a weakly compact operator. Then $T^* : Y^{**} \to X^*$ is weakly compact, thus compact, since X^* has the Schur property. Therefore T is compact. Thus $W(X, Y^*) = K(X, Y^*)$, which proves the result.

(ii) Assume that X and Y have the DPP. Then $W(X, Y^*) = K(X, Y^*)$ either by (i) if $\ell_1 \hookrightarrow Y$, or because Y^* has the Schur property (see [2, p. 212]) if $\ell_1 \not\hookrightarrow Y$.

Therefore every limited operator $T : X \to Y^*$ is compact. □

The preceding proof shows that if X and Y satisfy one of the hypotheses (i) or (ii) and if $W(X, Y^*)$ does not contain ℓ_∞ (as a closed subspace), then $W(X, Y^*) = K(X, Y^*)$.

Definition 2.27. An operator $T : X \to Y$ is *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

Definition 2.28. A bounded subset A of X (resp., A of X^*) is called a V^*-*subset* of X (resp., a V-*subset* of X^*) provided that

$$\lim_n \big(\sup\{|x_n^*(x)| : x \in A\}\big) = 0$$
$$\big(\text{resp.}, \lim_n \big(\sup\{|x^*(x_n)| : x^* \in A\}\big) = 0\big)$$

for each wuc series $\sum x_n^*$ in X^* (resp., wuc series $\sum x_n$ in X). The Banach space X has *property* (V) (resp., (V^*)) if every V-subset of X^* (resp., V^*-subset of X) is relatively weakly compact.

The following results were established in [12]: $C(K)$ spaces have property (V); L_1-spaces have property (V^*); reflexive Banach spaces have both properties (V) and (V^*); the Banach space X has property (V) if and only if every unconditionally converging operator T from X to any Banach space Y is weakly compact; if X has property (V^*), then X is weakly sequentially complete.

Remark 2.29. If $T : Y \to X^*$ is an operator such that $T^*|_X$ is (weakly) compact, then T is (weakly) compact. To see this, let $T : Y \to X^*$ be an operator such that

$T^*|_X$ is (weakly) compact. Let $S = T^*|_X$. Suppose that $x^{**} \in B_{X^{**}}$ and choose a net (x_α) in B_X which is w^*-convergent to x^{**}. Then $(T^*(x_\alpha)) \overset{w^*}{\to} T^*(x^{**})$. Now, $(T^*(x_\alpha)) \subseteq S(B_X)$, which is a relatively (weakly) compact set. Then $(T^*(x_\alpha)) \to T^*(x^{**})$ (resp., $(T^*(x_\alpha)) \overset{w}{\to} T^*(x^{**})$). Hence $T^*(B_{X^{**}}) \subseteq \overline{S(B_X)}$, which is relatively (weakly) compact. Therefore, $T^*(B_{X^{**}})$ is relatively (weakly) compact, and thus T is (weakly) compact.

It follows that if $L(X,Y^*) = K(X,Y^*)$, then $L(Y,X^*) = K(Y,X^*)$ and if $L(X,Y^*) = W(X,Y^*)$, then $L(Y,X^*) = W(Y,X^*)$.

Corollary 2.30. *Assume that one of the following assumptions holds:*

(i) *X and Y have the DPP and Y^* (or X^*) has property (V^*),*
(ii) *X and Y have the DPP and Y (or X) has property (V),*
(iii) *X and Y are infinite-dimensional $\mathcal{L}_\infty$-spaces.*

If $W(X,Y^*)$ has the BD property, then $L(X,Y^*) = K(X,Y^*)$.

Proof. Suppose that $W(X,Y^*)$ has the BD property.

(i) Since Y^* has the BD property, $\ell_\infty \not\hookrightarrow Y^*$ and thus $c_0 \not\hookrightarrow Y^*$ (see [2, p. 48]). Similarly, $c_0 \not\hookrightarrow X^*$. Let $T : X \to Y^*$ be an operator. Then $T^* : Y^{**} \to X^*$ is unconditionally converging (since $c_0 \not\hookrightarrow X^*$). If Y^* has property (V^*), then T is weakly compact (see [8, Theorem 3.10]). If X^* has property (V^*), then a similar argument shows that $L(Y,X^*) = W(Y,X^*)$. Thus $L(X,Y^*) = W(X,Y^*)$. By the proof of Theorem 2.26, $W(X,Y^*) = K(X,Y^*)$, which proves the result.

(ii) If Y has property (V), then Y^* has property (V^*) (see [12]). Apply (i).

(iii) Since X and Y are infinite-dimensional $\mathcal{L}_\infty$-spaces, $L(X,Y^*) = W(X,Y^*) = CC(X,Y^*)$ (see [2, p. 189, 61], [13, p. 148, 155]). By the proof of Theorem 2.26, $W(X,Y^*) = K(X,Y^*)$, which proves the result. □

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