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DISJOINTNESS-PRESERVING LINEAR MAPS ON BANACH FUNCTION ALGEBRAS ASSOCIATED WITH A LOCALLY COMPACT GROUP

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Abstract. We introduce a certain property of commutative Banach algebras which we call property O$\mathbb{B}$. We prove that every bounded disjointness-preserving linear map from a commutative Banach algebra with the aforesaid property to any semisimple, commutative Banach algebra is a weighted composition map. Further, it is shown that a variety of important Banach algebras in harmonic analysis have the property O$\mathbb{B}$.

1. Introduction

The basic aim of this article is to bring together the theory of operator hyper-Tauberian Banach algebras developed by Samei in [15] and the pattern established in [1] with the purpose of analyzing the so-called *disjointness-preserving linear maps*. This class of maps has been extensively studied in different contexts: Banach lattices, function algebras, and general Banach algebras. A linear map $\Phi\colon A \to B$ between Banach function algebras A and B is said to be *disjointness-preserving* if $\Phi(a)\Phi(b) = 0$ whenever $a, b \in A$ are such that $ab = 0$. The question of whether such a map for certain algebras is a weighted composition map has been widely studied. This paper focuses on a variety of significant Banach function algebras associated with a locally compact group G such as the Figà–Talamanca–Herz algebra $A_p(G)$ and the Figà–Talamanca–Herz–Lebesgue

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algebra $A_p^q(G)$ for $p \in \,]1,\infty[$ and $q \in [1,\infty[$. Accordingly, it seems appropriate to refer the reader to the papers [1], [3]–[5], [7], [9], [12], [13], and [14]. The paper [4] is concerned with bijective disjointness-preserving linear maps $\Phi\colon A(G) \to A(H)$ between the Fourier algebras $A(G)$ and $A(H)$ of amenable locally compact groups G and H. In [14], the author removes the amenability assumption and studies the continuous bijective disjointness-preserving linear maps $\Phi\colon A_p(G) \to A_p(H)$ for arbitrary locally compact groups G and H. The article [12] deals with bijective disjointness-preserving linear maps between Fourier algebras, and, further, it is shown that such a map gives rise to a topological group isomorphism between the corresponding groups in the case where additionally it preserves a kind of orthogonality. In [3] and [13], the operator space structure of the Fourier algebra $A(G)$ of a locally compact group G is involved, and the authors are concerned with completely bounded surjective disjointness-preserving linear maps from $A(G)$ in the case where G is amenable. Further, articles [1] and [5] are devoted to group algebras ([5] is restricted to locally compact abelian groups).

In Section 2 we introduce a certain property of commutative Banach algebras which we call *property* $\mathrm{O}\mathbb{B}$, and we show that a variety of important Banach algebras in harmonic analysis have the aforementioned property, namely $A_p(G)$ together with its quotient $A_p(E)$ for any locally compact group G and $E \subset G$ closed, and $A_p^q(G)$ together with its quotient $A_p^q(E)$ whenever the group G is such that $A_p(G)$ has a certain approximate identity (which holds for G amenable) and $E \subset G$ closed.

In Section 3 we show that every bounded disjointness-preserving linear map $\Phi\colon A \to B$ from a commutative Banach algebra A with the property $\mathrm{O}\mathbb{B}$ into any semisimple, commutative Banach algebra B is a weighted composition map. We should stress that we do not require that the map Φ be either bijective or surjective, and we do not impose the amenability on the locally compact group G in the case where we deal with a Banach function algebra associated with G. Furthermore, we emphasize that although our approach makes use of the operator space theory, we do not require at all for the map Φ to be completely bounded (but this time-boundedness on Φ is enough).

2. Orthosymmetric bilinear maps: properties $\mathbb{B}$ and $\mathrm{O}\mathbb{B}$

Throughout this article, let A be a commutative Banach algebra. Let $\Omega(A)$ denote the character space of A. For a subset B of A and a subset E of $\Omega(A)$, let $h(B) = \{\gamma \in \Omega(A) : \gamma(a) = 0 \text{ for each } a \in B\}$ and $I(E) = \{a \in A : \gamma(a) = 0 \text{ for each } \gamma \in E\}$ denote the hull of B and the kernel of E, respectively (taking $h(\varnothing) = \Omega(A)$ and $I(\varnothing) = A$). The annihilator of $a \in A$ is $\mathrm{Ann}(a) = \{b \in A : ab = 0\}$. Consider the dual space A^* of A with the usual A-module action $(\phi \cdot a)(b) = \phi(ab)$ for all $\phi \in A^*$ and $a, b \in A$. The annihilator of $\phi \in A^*$ is $\mathrm{Ann}(\phi) = \{a \in A : \phi \cdot a = 0\}$.

Let X be a Banach space, and let $\varphi\colon A \times A \to X$ be a bilinear map. Then φ is said to be *orthosymmetric* if $\varphi(a, b) = 0$ whenever $a, b \in A$ are such that $ab = 0$.

It should be pointed out that the orthosymmetry has been widely studied in the context of Banach lattices and that it has been shown in [1] to be useful in studying disjointness-preserving linear maps on Banach algebras.

2.1. **Property $\mathbb{B}$ and hyper-Tauberian Banach algebras.** The paper [1] makes heavy use of the orthosymmetric bilinear maps even though the orthosymmetric bilinear functionals would suffice. We say that the Banach algebra A *has the property* $\mathbb{B}$ if every bounded orthosymmetric bilinear functional $\varphi\colon A\times A\to\mathbb{C}$ satisfies $\varphi(ab,c)=\varphi(a,bc)$ for all $a,b,c\in A$.

Remark 2.1. Let A have the property $\mathbb{B}$, and let $\varphi\colon A\times A\to X$ be a bounded orthosymmetric bilinear map for some Banach space X. For every continuous linear functional ξ on X, the composition $\xi\circ\varphi$ is a bounded orthosymmetric bilinear functional, and therefore $\xi(\varphi(ab,c))=\xi(\varphi(a,bc))$ for all $a,b,c\in A$. We thus get $\varphi(ab,c)=\varphi(a,bc)$ for all $a,b,c\in A$. This shows that our definition agrees with that of [1].

In [15] Samei confines himself to regular, semisimple, commutative Banach algebras and develops the theory of hyper-Tauberian Banach algebras through the local maps from the algebra to its dual space. Suppose that A is a regular semisimple, commutative Banach algebra. We can think of A as a Banach function algebra on $\Omega(A)$. Then $\operatorname{supp}(a)=h(\operatorname{Ann}(a))$ for each $a\in A$. The support of $\phi\in A^*$ is defined to be the set $\operatorname{supp}(\phi)=h(\operatorname{Ann}(\phi))$. A linear map $\Phi\colon A\to A^*$ is said to be *local* if $\operatorname{supp}(\Phi(a))\subset\operatorname{supp}(a)$ for each $a\in A$. The algebra A is defined to be *hyper-Tauberian* if every bounded local map $\Phi\colon A\to A^*$ is an A-module homomorphism.

Lemma 2.2. *Let A be a regular, semisimple, commutative Banach algebra, and let $\varphi\colon A\times A\to\mathbb{C}$ be a bounded orthosymmetric bilinear functional. Then the map $\Phi\colon A\to A^*$ defined by $\Phi(a)(b)=\varphi(a,b)$ for all $a,b\in A$ is local.*

Proof. It suffices to check that $\operatorname{Ann}(a)\subset\operatorname{Ann}(\Phi(a))$ for each $a\in A$. Let $a\in A$, and let $b\in\operatorname{Ann}(a)$. If $c\in A$, then $a(bc)=0$ and the orthosymmetry of φ yields $(\Phi(a)\cdot b)(c)=\Phi(a)(bc)=\varphi(a,bc)=0$, which shows that $b\in\operatorname{Ann}(\Phi(a))$ as required. □

Remark 2.3. It is worth noting that it is easy to construct a bounded local linear map $\Phi\colon A\to A^*$ for some commutative Banach algebra A such that the corresponding bilinear functional is not orthosymmetric. Let A be the space consisting of all sequences a of complex numbers with $(na(n))$ convergent. Then A is a regular, semisimple, commutative Banach algebra with respect to pointwise multiplication and the norm given by $\|a\|=\sup_{n\in\mathbb{N}} n|a(n)|$. We define the bounded bilinear functional $\varphi\colon A\times A\to\mathbb{C}$ by $\varphi(a,b)=a(1)\lim nb(n)$ for all $a,b\in A$. Define $a,b\in A$ by $a(1)=1$ and $a(n)=0$ for each $n>1$, and $b(1)=0$ and $b(n)=1/n$ for each $n>1$. Then $ab=0$ and $\varphi(a,b)=1$, which shows that φ is not orthosymmetric. Nevertheless, it is immediate to check that $\operatorname{Ann}(\varphi(a,\cdot))=A$ for each $a\in A$, which shows that the map $a\mapsto\varphi(a,\cdot)$ from A to A^* is local.

Proposition 2.4. *Let A be a hyper-Tauberian Banach algebra. Then A has the property $\mathbb{B}$.*

Proof. Let $\varphi\colon A \times A \to \mathbb{C}$ be a bounded orthosymmetric bilinear functional, and let $\Phi\colon A \to A^*$ be the local map defined in Lemma 2.2. Since A is hyper-Tauberian, it follows that Φ is an A-module homomorphism, which gives

$$\varphi(ab, c) = \Phi(ab)(c) = \big(\Phi(a) \cdot b\big)(c) = \Phi(a)(bc) = \varphi(a, bc)$$

for all $a, b, c \in A$. □

2.2. **Property $\mathrm{O}\mathbb{B}$ and operator hyper-Tauberian algebras.** Throughout this section, A is a commutative quantized Banach algebra. For the convenience of the reader, we recall that a *quantized Banach algebra* is an algebra A which is also an operator space such that the multiplication $A \times A \to A$ is a completely bounded bilinear map. This is the same as asserting that it determines a completely bounded linear map on the operator space projective tensor product. It is worth noting that a bilinear functional $\varphi\colon A \times A \to \mathbb{C}$ is completely bounded if and only if it determines a completely bounded linear map from A to A^*. We refer the reader to [2] for the necessary background from operator space theory. It seems appropriate to mention that we always assume the operator spaces to be complete.

We say that A *has the property* $\mathrm{O}\mathbb{B}$ (O for *operator*) if every completely bounded orthosymmetric bilinear functional $\varphi\colon A \times A \to \mathbb{C}$ satisfies $\varphi(ab, c) = \varphi(a, bc)$ for all $a, b, c \in A$.

Suppose that A is a regular, semisimple, commutative quantized Banach algebra. The algebra A is said to be *operator hyper-Tauberian* if every completely bounded local map $\Phi\colon A \to A^*$ is an A-module homomorphism.

The same argument as in Lemma 2.4 gives the following.

Proposition 2.5. *Let A be an operator hyper-Tauberian Banach algebra. Then A has the property* $\mathrm{O}\mathbb{B}$.

Remark 2.6. Let A be a quantized Banach algebra. It is clear that if A is hyper-Tauberian, then A is operator hyper-Tauberian. Nevertheless, the Fourier algebra $A(G)$ of the group G of rotations of $\mathbb{R}^3$ is operator hyper-Tauberian (see [15, Theorem 26(v)]), but it is not weakly amenable (see [10, Corollary 7.3]), which implies that it is not hyper-Tauberian (see [15, Theorem 5(iii)]). It is also clear that if A has the property $\mathbb{B}$, then A has the property $\mathrm{O}\mathbb{B}$. We don't know whether or not having the property $\mathrm{O}\mathbb{B}$ implies having the property $\mathbb{B}$. In Remark 2.15 we show that a regular, semisimple, commutative quantized Banach algebra with the property $\mathbb{B}$ need not be operator hyper-Tauberian.

2.3. **Hereditary properties and examples.** In [1], the authors give examples of (in general, noncommutative) Banach algebras with the property $\mathbb{B}$ such as the group algebra $L^1(G)$ of any locally compact group G, and it is shown that this property is stable under the usual methods of constructing Banach algebras. In [15], the author provides some examples of (operator) hyper-Tauberian Banach algebras and investigates the hereditary properties of this class. As a matter of fact, it is shown in [15] that the algebra $A_p(G)$ of a locally compact group G for $p \in \,]1, \infty[$ is operator hyper-Tauberian.

We gather here various facts concerning the behavior of properties $\mathbb{B}$ and $\mathrm{O}\mathbb{B}$ with respect to some basic constructions.

Proposition 2.7. *Let A be a (quantized) Banach algebra with the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$), let B be a (quantized) Banach algebra, and let $\Phi\colon A \to B$ be a (completely) bounded homomorphism with dense range. Then B has the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$).*

Proof. The nonquantized statement is given in [1, Proposition 2.6]. The proof of the quantized counterpart can be handled in much the same way. □

Corollary 2.8. *Let A be a (quantized) Banach algebra with the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$), and let I be a closed ideal of A. Then the quotient algebra A/I has the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$).*

Proof. It suffices to apply the preceding result to the quotient homomorphism $\Phi\colon A \to A/I$. □

Proposition 2.9. *Let A be a (quantized) Banach algebra with the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$), and let I be an ideal of A. Suppose that*

(1) *I is a (quantized) Banach algebra with respect to some norm (operator space structure),*
(2) *the inclusion map from I into A is (completely) bounded,*
(3) *the multiplication $A \times I \to I$ is (completely) bounded,*
(4) *the linear span of the set AI is dense in I.*

Then I has the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$).

Proof. This follows by the same method as in the proof of [1, Proposition 2.5(ii)]. □

Corollary 2.10. *Let A be a (quantized) Banach algebra with the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$), and let I be a closed ideal of A such that the linear span of the set AI is dense in I. Then I has the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$).*

Proof. We equip I with the (operator space) Banach space structure inherited from A. Then Proposition 2.9 applies. □

Proposition 2.9 applies equally well to (operator) abstract Segal algebras in A. We recall that a subalgebra B of A is an *abstract Segal algebra* in A if

(i) B is a dense ideal of A,
(ii) B is a Banach algebra with respect to a norm $\|\cdot\|_B$,
(iii) there exists $\alpha > 0$ such that $\|b\| \le \alpha \|b\|_B$ for each $b \in B$,
(iv) there exists $\beta > 0$ such that $\|ab\|_B \le \beta \|a\| \|b\|_B$ for all $a \in A$ and $b \in B$.

Abstract Segal algebras have been studied in [6] from an operator space perspective. The authors keep (i) and replace (ii), (iii), and (iv) by the quantized counterparts, namely:

(Oii) B is a quantized Banach algebra with respect to some operator space structure,
(Oiii) the inclusion map from B into A is completely bounded,
(Oiv) the multiplication $A \times B \to B$ is completely bounded.

Consequently, Proposition 2.9 clearly gives the following.

Corollary 2.11. *Let A be a (quantized) Banach algebra with the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$), and let B be an (operator) abstract Segal algebra with respect to A such that the linear span of the set AB is dense in B. Then B has the property $\mathbb{B}$ ($\mathrm{O}\mathbb{B}$).*

We now restrict our attention to a variety of significant Banach algebras that come from a locally compact group G. Let G be a locally compact group, and let $p \in]1, \infty[$. Then $A_p(G)$ is the *Figà–Talamanca–Herz algebra* of G. Also $A_p(G)$ is a regular, Tauberian, semisimple, commutative Banach algebra whose character space is identified with G by point evaluation. It should be pointed out that $A_2(G)$ agrees with the *Fourier algebra* $A(G)$ of G. If $q \in [1, \infty[$, then $A_p^q(G) = A_p(G) \cap L^q(G)$ is the *Figà–Talamanca–Herz–Lebesgue algebra* of G. Note that $A_p^q(G)$ is an abstract Segal algebra in $A_p(G)$, and it is a regular, semisimple, commutative Banach algebra whose character space is G (see [7, Theorem 1]). Let $E \subset G$ be closed. Then $A_p(E)$ and $A_p^q(E)$ denote the usual quotient algebras $A_p(G)/I(E)$ and $A_p^q(G)/I(E)$, respectively. These algebras can be thought of as the algebras obtained from $A_p(G)$ and $A_p^q(G)$, respectively, by restriction to E.

Since the dual of $A(G)$ can be identified with the group von Neumann algebra $VN(G)$ of G, it follows that $A(G)$ is an operator space in a natural manner. Further, with this structure, $A(G)$ becomes a quantized (actually, completely contractive) Banach algebra (see [2, Sections 16.1 and 16.2]). There have been several attempts to equip $A_p(G)$ with an operator space structure. Here we consider the structure defined in [11] which turns $A_p(G)$ into a quantized Banach algebra (though the multiplication is not known to be completely contractive). In [6], it is shown that $A_p^q(G)$ admits an operator space structure under which it is an operator abstract Segal algebra in $A_p(G)$.

Theorem 2.12. *Let G be a locally compact group, and let E be a closed subset of G. Then the algebras $A_p(E)$ for $p \in]1, \infty[$ and $A_2^1(E)$ have the property $\mathrm{O}\mathbb{B}$. Furthermore, if the principal component of G is abelian, then they have the property $\mathbb{B}$.*

Proof. By [15, Theorem 28], $A_p(G)$ is operator hyper-Tauberian, and Proposition 2.5 shows that $A_p(G)$ has the property $\mathrm{O}\mathbb{B}$. In the case where the principal component of G is abelian, [15, Theorem 22] and Proposition 2.4 show that $A_p(G)$ has the property $\mathbb{B}$. Corollary 2.8 now gives the required property for $A_p(E)$.

By [6, Corollary 2.4], the linear span of the set $A(G)A_2^1(G)$ is dense in $A_2^1(G)$. Thus [6, Theorem 4.4] shows that $A_2^1(G)$ is always operator hyper-Tauberian, and it is hyper-Tauberian in the case when the principal component of G is abelian. Then the claimed property for $A_2^1(E)$ follows from Proposition 2.4, Proposition 2.5, and Corollary 2.8. □

Theorem 2.13. *Let G be a locally compact group, and let E be a closed subset of G. Suppose that $A_p(G)$ has an approximate identity $(u_\lambda)_{\lambda\in\Lambda}$ such that*

$$\sup\{\|u_\lambda f\| : f \in A_p(G), \|f\| \le 1, \lambda \in \Lambda\} < \infty.$$

Then the algebra $A_p^q(E)$ for $p \in]1, \infty[$ and $q \in [1, \infty[$ has the property $\mathrm{O}\mathbb{B}$. Furthermore, if the principal component of G is abelian, then it has the property $\mathbb{B}$.

Proof. On account of Theorem 2.12, $A_p(G)$ has the property $\mathrm{O}\mathbb{B}$. By [7, Corollary 2], the set $A_p(G)A_p^q(G)$ is dense in $A_p^q(G)$. Consequently, Corollary 2.11 shows that $A_p^q(G)$ has the property $\mathrm{O}\mathbb{B}$, and Corollary 2.8 yields that property for $A_p^q(E)$. An obvious adjustment in the preceding argument proves that $A_p^q(E)$ has the property $\mathbb{B}$ in the case where the principal component of G is abelian. □

Remark 2.14. Let G an amenable locally compact group, and let $p \in]1, \infty[$. Then $A_p(G)$ has an approximate identity of bound 1 (see [9, Theorem 6]), and hence it satisfies the requirement in Theorem 2.13. On account of Corollary 2.8, every closed ideal I of $A_p(G)$ has the property $\mathrm{O}\mathbb{B}$ (actually, the property $\mathbb{B}$ in the case where the principal component of G is abelian). By [8, Theorem 2.1], the Fourier algebra $A(\mathbb{F}_2)$ of the free group on two generators has an approximate identity that satisfies the requirement in Theorem 2.13 even though $\mathbb{F}_2$ is not amenable.

Remark 2.15. Let $\mathbb{S}^2$ be the 2-dimensional sphere. According to the preceding remark, the closed ideal $I(\mathbb{S}^2)$ of $A(\mathbb{R}^3)$ has the property $\mathbb{B}$. By [15, Proposition 18], the algebra $A(\mathbb{R}^3)$ is hyper-Tauberian, and a famous theorem of Schwartz states that $\mathbb{S}^2$ is not a set of synthesis for the Fourier algebra $A(\mathbb{R}^3)$. Therefore, [15, Theorem 26(v)] shows that $I(\mathbb{S}^2)$ is not operator hyper-Tauberian.

3. Disjointness-preserving maps

Properties $\mathbb{B}$ and $\mathrm{O}\mathbb{B}$ are useful in studying disjointness-preserving maps. Since property $\mathbb{B}$ has already been extensively discussed in [1], we now focus attention on property $\mathrm{O}\mathbb{B}$. Recall that a linear map $\Phi\colon A \to B$ between commutative Banach algebras A and B is said to be *disjointness-preserving* if $\Phi(a)\Phi(b) = 0$ whenever $a, b \in A$ are such that $ab = 0$.

Lemma 3.1. *Let A be a commutative Banach algebra, and let ϕ be a nonzero continuous linear functional on A. Suppose that*

$$\phi(ab)\phi(c) = \phi(a)\phi(bc) \quad (a, b, c \in A).$$

Then ϕ can be uniquely expressed in the form $\phi = \alpha\gamma$, where $\alpha \in \mathbb{C} \setminus \{0\}$ and $\gamma \in \Omega(A)$.

Proof. Take $c \in A$ with $\phi(c) = 1$. If $a \in \ker(\phi)$ and $b \in A$, then

$$\phi(ab) = \phi(ab)\phi(c) = \phi(a)\phi(bc) = 0.$$

Consequently, $\ker(\phi)$ is a closed 1-codimensional ideal of A, and therefore there exists $\gamma \in \Omega(A)$ such that $\ker(\gamma) = \ker(\phi)$. This implies that there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\phi = \alpha\gamma$.

We proceed to show the uniqueness of the representation. Suppose that $\phi = \beta\tau$, where $\beta \in \mathbb{C}$ and $\tau \in \Omega(A)$. Then $\gamma = \alpha^{-1}\beta\tau$ with $\gamma, \tau \in \Omega(A)$, and this implies that $\alpha^{-1}\beta = 1$ and $\gamma = \tau$. □

Lemma 3.2. *Let A be a commutative Banach algebra, and let B be a semisimple, commutative Banach algebra. Let $\Phi\colon A \to B$ be a nonzero bounded linear map, and let $\mathcal{O}(\Phi) = \{\gamma \in \Omega(B) : \gamma \circ \Phi \neq 0\}$. Suppose that*

$$\Phi(ab)\Phi(c) = \Phi(a)\Phi(bc) \quad (a, b, c \in A).$$

Then there exist a continuous function $\mu\colon \mathcal{O}(\Phi) \to \mathbb{C}\setminus\{0\}$ *and a continuous map* $\sigma\colon \mathcal{O}(\Phi) \to \Omega(A)$ *such that* $\gamma \circ \Phi = \mu(\gamma)\sigma(\gamma)$ *for each* $\gamma \in \mathcal{O}(\Phi)$*. Furthermore, the following statements hold:*

(i) *if* Φ *is bijective, then* σ *is a homeomorphism from* $\Omega(B)$ *onto* $\Omega(A)$*,*
(ii) *if* Φ *is surjective, then* σ *is a homeomorphism from* $\Omega(B)$ *onto* $h(\ker(\Phi))$*.*

Proof. Let $\gamma \in \mathcal{O}(\Phi)$. The composition $\gamma \circ \Phi$ yields a nonzero continuous linear functional on A satisfying the requirement in Lemma 3.1, and therefore it can be written in a unique way in the form $\gamma \circ \Phi = \mu(\gamma)\sigma(\gamma)$, where $\mu(\gamma) \in \mathbb{C} \setminus \{0\}$ and $\sigma(\gamma) \in \Omega(A)$. Hence there exist a function $\mu\colon \mathcal{O}(\Phi) \to \mathbb{C} \setminus \{0\}$ and a map $\sigma\colon \mathcal{O}(\Phi) \to \Omega(A)$ such that

$$\gamma \circ \Phi = \mu(\gamma)\sigma(\gamma) \tag{3.1}$$

for each $\gamma \in \mathcal{O}(\Phi)$.

Let us observe that

$$\gamma\big(\Phi(ab)\big) = \sigma(\gamma)(a)\gamma\big(\Phi(b)\big) \tag{3.2}$$

and

$$\mu(\gamma)\gamma\big(\Phi(ab)\big) = \gamma\big(\Phi(a)\big)\gamma\big(\Phi(b)\big) \tag{3.3}$$

for all $a, b \in A$ and $\gamma \in \mathcal{O}(\Phi)$. Indeed, by (3.1), we have

$$\gamma\big(\Phi(ab)\big) = \mu(\gamma)\sigma(\gamma)(ab) = \mu(\gamma)\sigma(\gamma)(a)\sigma(\gamma)(b) = \sigma(\gamma)(a)\gamma\big(\Phi(b)\big),$$

which gives (3.2), and, multiplying by $\mu(\gamma)$, we obtain (3.3).

Our next goal is to prove the continuity of both μ and σ. Let $\gamma_0 \in \mathcal{O}(\Phi)$. Pick $b \in A$ with $\gamma_0(\Phi(b)) \neq 0$, and let W be the open neighborhood of γ_0 defined by $W = \{\gamma \in \mathcal{O}(\Phi) : \gamma(\Phi(b)) \neq 0\}$. By (3.2), we have $\sigma(\gamma)(a) = \gamma(\Phi(ab))/\gamma(\Phi(b))$ for all $\gamma \in W$ and $a \in A$. Since the functions $\gamma \mapsto \gamma(\Phi(ab))$ and $\gamma \mapsto \gamma(\Phi(b))$ are continuous at γ_0, we see that the function $\gamma \mapsto \sigma(\gamma)(a)$ is continuous at γ_0 for each $a \in A$. This proves that the map σ is continuous at γ_0 because $\Omega(A)$ is equipped with the w^*-topology. On account of (3.1), $\sigma(\gamma)(b) \neq 0$ and $\mu(\gamma) = \gamma(\Phi(b))/\sigma(\gamma)(b)$ for each $\gamma \in W$. Since the functions $\gamma \mapsto \gamma(\Phi(b))$ and $\gamma \mapsto \sigma(\gamma)(b)$ are continuous at γ_0, it follows that μ is continuous at γ_0.

Suppose that Φ is bijective. We claim that $\mathcal{O}(\Phi) = \Omega(B)$. Indeed, since B is semisimple, it follows that $\Omega(B) \setminus \mathcal{O}(\Phi) = h(\Phi(A)) = h(B) = \varnothing$. Our objective is to show that the conditions in the lemma hold for Φ^{-1}. We begin by proving that A is semisimple. Assume that a lies in the radical of A. Then $\gamma(\Phi(a)) = \mu(\gamma)\sigma(\gamma)(a) = 0$ for each $\gamma \in \Omega(B)$, and therefore $\Phi(a) = 0$. Since Φ is injective, we conclude that $a = 0$ as desired. By the open mapping theorem, Φ^{-1} is bounded. We now proceed to show that $\Phi^{-1}(uv)\Phi^{-1}(w) = \Phi^{-1}(u)\Phi^{-1}(vw)$ for all u, v, $w \in B$. Let u, v, $w \in B$, and let $\gamma \in \Omega(B)$. From (3.3), we have

$$\mu(\gamma)\gamma\big(\Phi\big(\Phi^{-1}(uv)\Phi^{-1}(w)\big)\big) = \gamma\big(\Phi\big(\Phi^{-1}(uv)\big)\Phi\big(\Phi^{-1}(w)\big)\big) = \gamma(uvw),$$

and similarly we obtain $\mu(\gamma)\gamma(\Phi(\Phi^{-1}(u)\Phi^{-1}(vw))) = \gamma(uvw)$. We thus get $\gamma(\Phi(\Phi^{-1}(uv)\Phi^{-1}(w))) = \gamma(\Phi(\Phi^{-1}(u)\Phi^{-1}(vw)))$. Since γ is arbitrary and B is semisimple, it may be concluded that

$$\Phi\big(\Phi^{-1}(uv)\Phi^{-1}(w)\big) = \Phi\big(\Phi^{-1}(u)\Phi^{-1}(vw)\big),$$

and hence that $\Phi^{-1}(uv)\Phi^{-1}(w) = \Phi^{-1}(u)\Phi^{-1}(vw)$. From what has already been proved, it follows that there exist a continuous function $\nu\colon \Omega(A) \to \mathbb{C}\setminus\{0\}$ and a continuous map $\tau\colon \Omega(A) \to \Omega(B)$ such that $\xi\circ\Phi^{-1} = \nu(\xi)\tau(\xi)$ for each $\xi \in \Omega(A)$. For every $\gamma \in \Omega(B)$, we have

$$\gamma = (\gamma\circ\Phi)\circ\Phi^{-1} = \mu(\gamma)\sigma(\gamma)\circ\Phi^{-1} = \mu(\gamma)\nu\big(\sigma(\gamma)\big)\tau\big(\sigma(\gamma)\big),$$

which shows that γ and $\tau(\sigma(\gamma))$ are proportional characters, and therefore that $\tau(\sigma(\gamma)) = \gamma$. Likewise, we check that $\sigma(\tau(\xi)) = \xi$ for each $\xi \in \Omega(A)$. Consequently, σ is bijective with $\sigma^{-1} = \tau$, and therefore σ is a homeomorphism.

Finally, suppose that Φ is surjective. Write $I = \ker(\Phi)$. We claim that I is an ideal of A. Let $a \in I$, and let $b \in A$. By (3.3), we have $\mu(\gamma)\gamma(\Phi(ab)) = \gamma(\Phi(a)\Phi(b)) = 0$, and so $\gamma(\Phi(ab)) = 0$ for each $\gamma \in \Omega(B)$. Since B is semisimple, it follows that $\Phi(ab) = 0$, which establishes the claim. Since Φ is continuous, I is a closed ideal of A. Then Φ drops to a continuous bijective linear map $\widetilde{\Phi}\colon A/I \to B$ so that $\widetilde{\Phi}\circ Q = \Phi$, where Q denotes the quotient homomorphism from A onto A/I. Let us recall that the map $\zeta \mapsto \zeta\circ Q$ yields a homeomorphism from $\Omega(A/I)$ onto $h(I) = \{\xi \in \Omega(A) : \xi(I) = \{0\}\}$. We can now apply what has previously been proved to get a continuous function $\widetilde{\mu}\colon \Omega(B) \to \mathbb{C}\setminus\{0\}$ and a homeomorphism $\widetilde{\sigma}\colon \Omega(B) \to \Omega(A/I)$ such that $\gamma\circ\widetilde{\Phi} = \widetilde{\mu}(\gamma)\widetilde{\sigma}(\gamma)$ for each $\gamma \in \Omega(B)$. Therefore, $(\gamma\circ\widetilde{\Phi})\circ Q = \widetilde{\mu}(\gamma)\widetilde{\sigma}(\gamma)\circ Q$, and, on the other hand, $(\gamma\circ\widetilde{\Phi})\circ Q = \gamma\circ\Phi = \mu(\gamma)\sigma(\gamma)$ for each $\gamma \in \Omega(B)$. This shows that $\widetilde{\mu}(\gamma) = \mu(\gamma)$ and $\widetilde{\sigma}(\gamma)\circ Q = \sigma(\gamma)$ for each $\gamma \in \Omega(B)$. Consequently, σ is a homeomorphism from $\Omega(B)$ onto $h(I)$. □

Theorem 3.3. *Let A be a quantized commutative Banach algebra with the property* $\mathbb{OB}$, *and let B be a semisimple, commutative Banach algebra. Let $\Phi\colon A \to B$ be a nonzero bounded disjointness-preserving linear map, and let $\mathcal{O}(\Phi) = \{\gamma \in \Omega(B) : \gamma\circ\Phi \neq 0\}$. Then there exist a continuous function $\mu\colon \mathcal{O}(\Phi) \to \mathbb{C}\setminus\{0\}$ and a continuous map $\sigma\colon \mathcal{O}(\Phi) \to \Omega(A)$ such that $\gamma\circ\Phi = \mu(\gamma)\sigma(\gamma)$ for each $\gamma \in \mathcal{O}(\Phi)$. Furthermore, the following statements hold:*

(i) *if Φ is bijective, then σ is a homeomorphism from $\Omega(B)$ onto $\Omega(A)$;*
(ii) *if Φ is surjective, then σ is a homeomorphism from $\Omega(B)$ onto $h(\ker(\Phi))$.*

Proof. Let $\gamma \in \Omega(B)$. Then the bilinear functional $\varphi\colon A\times A \to \mathbb{C}$ defined by $\varphi(a,b) = \gamma(\Phi(a))\gamma(\Phi(b))$ for all $a,b \in A$ is easily seen to be orthosymmetric. Further, since the continuous linear functional $\gamma\circ\Phi$ is automatically completely bounded, it follows that φ is completely bounded. From Proposition 2.5 it follows that $\gamma(\Phi(ab)\Phi(c)) = \gamma(\Phi(a)\Phi(bc))$ for all $a,b,c \in A$. Since γ is arbitrary and B is semisimple, it may be concluded that Φ satisfies the condition in Lemma 3.2, which establishes all the statements in the theorem. □

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