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SHARP WEAK ESTIMATES FOR HARDY-TYPE OPERATORS

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Abstract. In this article, we prove the weak bound for an n-dimensional Hardy operator on a central Morrey space. Meanwhile, we obtain the precise operator norm, and we give the weak bounds for the conjugate Hardy operator on Lebesgue space with power weights. The corresponding operator norms are also computed. As an application, we obtain an estimate for the gamma function.

1. Introduction

Let h and h^* be the Hardy averaging operator and its conjugate Hardy operator,

$$hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \qquad h^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt, \quad x > 0,$$

respectively. The classical Hardy inequalities are

$$\|hf\|_{L^p(\mathbb{R}^+)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}, \quad 1 < p \le \infty,$$

and

$$\|h^*f\|_{L^p(\mathbb{R}^+)} \le p\|f\|_{L^p(\mathbb{R}^+)}, \quad 1 \le p < \infty.$$

In addition, Hardy proved that the constants p and $\frac{p}{p-1}$ above are the best possible; hence,

$$\|h\|_{L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)} = \frac{p}{p-1}, \quad 1 < p \le \infty,$$

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and

$$\|h^*\|_{L^p(\mathbb{R}^+)\to L^p(\mathbb{R}^+)} = p, \quad 1 \le p < \infty.$$

In [3], Bliss computed the best possible constant C_{pq} in the inequality

$$\|hf\|_{L^q(x^\alpha)} \le C_{pq}\|f\|_{L^p(\mathbb{R}^+)}$$

to be

$$C_{pq} = \Big(\frac{p'}{q}\Big)^{\frac{1}{q}} \Big[\frac{\frac{q-p}{p}\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{p}{q-p})\Gamma(\frac{p(q-1)}{q-p})}\Big]^{\frac{1}{p}-\frac{1}{q}},$$

where $1 < p < q < \infty$ and $\alpha = \frac{q}{p} - 1$. There is a continuity in the sharp constant C_{pq} as $q \to p$; that is,

$$C_{pq} \to \frac{p}{p-1} \quad \text{as } q \to p$$

(see [16] for details). If we define the 1-dimensional fractional Hardy operator as

$$h_\beta f(x) = \frac{1}{x^{1-\beta}} \int_0^x f(t)\,dt, \quad 0 < \beta < 1,$$

then the above Bliss result can be written by

$$\|h_\beta f\|_{L^q(\mathbb{R}^+)} \le C_{pq}\|f\|_{L^p(\mathbb{R}^+)}$$

for $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \beta$. Here C_{pq} is the best possible constant.

The Hardy inequality and its generalizations play important roles in various branches of analysis. Some developments and their applications are given in [8], [9], and [14]. In 1995, Christ and Grafakos [4] studied the n-dimensional Hardy operator

$$Hf(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f(y)\,dy$$

and obtained the following Hardy inequality:

$$\|H\|_{L^p(\mathbb{R}^n)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } 1 < p < \infty,$$

and here the constant $\frac{p}{p-1}$ is the best option possible. The conjugate operator H^* of H is

$$H^* f(x) = \int_{B(0,|x|)^c} \frac{f(y)}{|B(0,|y|)|}\,dy.$$

By duality, we obtain

$$\|H^*\|_{L^p(\mathbb{R}^n)} \le p\|f\|_{L^p(\mathbb{R}^n)}$$

and

$$\|H^*\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \|H\|_{L^{p'}(\mathbb{R}^n)\to L^{p'}(\mathbb{R}^n)} = p$$

for $1 < p < \infty$. Recently, Zhao, Fu, and Lu in [17] proved the sharp weak estimates for the n-dimensional Hardy operator H with operator norm

$$\|H\|_{L^p(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)} = 1$$

for $1 \le p \le \infty$. In [5], Fu et al. considered the sharp bound for n-dimensional Hardy operator H on central Morrey space $\dot{B}^{p,\lambda}(\mathbb{R}^n)$. They proved that

$$\|H\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)\to\dot{B}^{p,\lambda}(\mathbb{R}^n)} = \frac{1}{1+\lambda}$$

for $1 < p < \infty$ and $-\frac{1}{p} \le \lambda \le 0$.

In [6], Fu et al. defined a fractional Hardy operator H_β

$$H_\beta f(x) = \frac{1}{|B(0,|x|)|^{1-\frac{\beta}{n}}} \int_{B(0,|x|)} f(y)\,dy, \quad x \in \mathbb{R}^n\backslash\{0\}$$

for $0 \le \beta < n$. Then its conjugate operator reads as

$$H_\beta^* f(x) = \int_{B(0,|x|)^c} \frac{f(y)}{|B(0,|y|)|^{1-\frac{\beta}{n}}}\,dy.$$

If $\beta = 0$, we denote H_0 and H_0^* by H and H^*, respectively. In [10], Lu, Yan, and Zhao obtained the boundedness of H_β^* from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\beta}}(\mathbb{R}^n)$ with

$$\|H_\beta\|_{L^1(\mathbb{R}^n)\to L^{\frac{n}{n-\beta}}(\mathbb{R}^n)} = 1.$$

In [18], Zhao and Lu extended Bliss's result to H_β and proved that

$$\|H_\beta\|_{L^q(\mathbb{R}^n)} \le A\|f\|_{L^p(\mathbb{R}^n)}$$

with

$$A = \Big(\frac{p'}{q}\Big)^{\frac{1}{q}} \Big[\frac{n}{q\beta} B\Big(\frac{n}{q\beta}, \frac{n}{q'\beta}\Big)\Big]^{-\frac{\beta}{n}}$$

being sharp. Here $0 < \beta < n$, $1 < p < q < \infty$, and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. By duality, we will calculate the sharp norm of H_β^* on the Lebesgue space in Section 3.

Recently, Persson and Samko in [16] proved the estimate

$$\big\||x|^n H f\big\|_{L^q(|x|^\alpha)} \le C_{pqn}\|f\|_{L^p(|x|^\beta)},$$

where $1 < p < q < \infty$, $\beta < n(p-1)$, $\frac{\alpha+n}{q} = \frac{\beta+n}{p} - n$, and the sharp constant

$$C_{pqn} = \nu_n^{1/q-\frac{1}{p}} \Big(\frac{p'}{q}\Big)^{\frac{1}{q}} \Big[\frac{n(p-1)}{np-1-\beta}\Big]^{\frac{1}{p'}} \Big[\frac{\frac{q-p}{p}\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{p}{q-p})\Gamma(\frac{p(q-1)}{q-p})}\Big]^{\frac{1}{p}-\frac{1}{q}}.$$

They also obtained the corresponding boundedness for H^*. Similar weighted estimates also hold for h and h^* (see [16]). Gao and Zhao in [7] obtained the sharp weak bound for H on $L^1(|x|^\alpha)$ to be 1, which is independent of the dimension n and the index α. But for H^*, they obtained

$$\|H^*\|_{L^1(|x|^\gamma)\to L^{1,\infty}(|x|^\gamma)} = \frac{n}{n+\gamma}$$

for $\gamma > -n$. They also obtained

$$\|H_\beta^*\|_{L^1(\mathbb{R}^n)\to L^{\frac{n}{n-\beta},\infty}(\mathbb{R}^n)} = 1$$

and

$$\|H_\beta^*\|_{L^p(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)}=\left(\frac{q}{p'}\right)^{\frac{1}{p'}}$$

for $1<p<q<\infty$, $0<\beta<n$, and $\frac{1}{q}=\frac{1}{n}-\frac{\beta}{n}$.

Weighted weak-type inequalities for the Hardy-type operators play important roles in the research of Hilbert transforms, maximal functions, and singular integral operators. (We refer the reader to [2], [11]–[13], and [15].) In those articles, the authors only gave weighted weak-type estimates and did not give the sharp constants in these estimates. In the present paper, we will obtain the sharp weak bound for the n-dimensional Hardy operator on a central Morrey space. Then we will work on the sharp weak bounds for the conjugate Hardy operator on Lebesgue space with power weights. As an application, we will obtain an estimate for the gamma function.

In what follows, we introduce some definitions and notation. Throughout this article, the set $B(0,|x|)$ is the ball in $\mathbb{R}^n$ centered at the origin with radius $|x|$. Let $|B(0,|x|)|$ and $|S^{n-1}|$ denote the volume of the ball $B(0,|x|)$ and the measure of the unit sphere S^{n-1}, respectively, and let $(B(0,r))^c=\mathbb{R}^n\backslash B(0,|x|)$. For $1<p<\infty$, we denote by p' the conjugate index of p; that is, $\frac{1}{p}+\frac{1}{p'}=1$. Let $w(x)$ be a nonnegative measurable function on $\mathbb{R}^n$. A measurable function f belongs to $L^p(w)$ if

$$\|f\|_{L^p(w)}=\left(\int_{\mathbb{R}^n}\left|f(x)\right|^p w(x)\,dx\right)^{\frac{1}{p}}<\infty.$$

A measurable function f belongs to $L^{p,\infty}(w)$ if

$$\|f\|_{L^{p,\infty}(w)}=\sup_{\lambda>0}\lambda w\left(\left\{x\in\mathbb{R}^n:\left|f(x)\right|>\lambda\right\}\right)^{\frac{1}{p}}<\infty,$$

where

$$w\left(\left\{x\in\mathbb{R}^n:\left|f(x)\right|>\lambda\right\}\right)=\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}}w(x)\,dx.$$

Let z_1 and z_2 be complex numbers with positive real parts, with the beta function $B(z_1,z_2)$ defined by $B(z_1,z_2)=\int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,dt$ and with the Gamma function $\Gamma(z_1)=\int_0^\infty t^{z_1-1}e^{-t}\,dt$. When z_1 and z_2 have positive real parts, $B(z_1,z_2)\Gamma(z_1+z_2)=\Gamma(z_1)\Gamma(z_2)$. We denote

$$\nu_n=\left|B(0,1)\right|=\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})}.$$

In [1, p. 5], Alvarez, Guzmán-Partida, and Lakey defined the central Morrey space.

Definition 1.1. Let $1\le p<\infty$ and let $-\frac{1}{p}\le\lambda<0$. A function $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the central Morrey space $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)}=\sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda p}}\int_{B(0,R)}\left|f(x)\right|^p dx\right)^{\frac{1}{p}}<\infty.$$

When $\lambda = -\frac{1}{p}$, then $\dot{B}^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. It is easy to check that $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ spaces reduce to $\{0\}$ when $\lambda < -\frac{1}{p}$.

Definition 1.2. Let $1 \le p < \infty$ and $-\frac{1}{p} \le \lambda < 0$. The weak central Morrey space $W\dot{B}^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$W\dot{B}^{p,\lambda}(\mathbb{R}^n) = \big\{f : \|f\|_{W\dot{B}^{p,\lambda}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|f\|_{W\dot{B}^{p,\lambda}(\mathbb{R}^n)} = \sup_{R>0}\big|B(0,R)\big|^{-\lambda-\frac{1}{p}}\|f\|_{WL^p(B(0,R))},$$

and where $\|f\|_{WL^p(B(0,R))}$ is the local weak L^p-norm of $f(x)$ restricted to the ball $B(0,R)$; that is,

$$\|f\|_{WL^p(B(0,R))} = \sup_{\lambda>0}\lambda\big|\big\{x \in B(0,R) : \big|f(x)\big| > \lambda\big\}\big|^{\frac{1}{p}}.$$

Obviously, if $\lambda = -\frac{1}{p}$, then $W\dot{B}^{p,-\frac{1}{p}}(\mathbb{R}^n) = L^{p,\infty}(\mathbb{R}^n)$ is the weak L^p space. It is also clear that $\dot{B}^{p,\lambda}(\mathbb{R}^n) \subseteq W\dot{B}^{p,\lambda}(\mathbb{R}^n)$ for $1 \le p < \infty$ and $-\frac{1}{p} \le \lambda < 0$.

2. Main results

Theorem 2.1. *Let $1 \le p < \infty$ and let $-\frac{1}{p} \le \lambda < 0$. If $f \in \dot{B}^{p,\lambda}(\mathbb{R}^n)$, then*

$$\|Hf\|_{W\dot{B}^{p,\lambda}(\mathbb{R}^n)} \le 1 \cdot \|f\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)},$$

and the constant 1 *is optimal.*

Proof. By Hölder's inequality, we obtain

$$\begin{aligned}\big|Hf(x)\big| &\le \frac{1}{|B(0,|x|)|}\Big(\int_{B(0,|x|)}\big|f(y)\big|^p\,dy\Big)^{\frac{1}{p}}\Big(\int_{B(0,|x|)}dy\Big)^{\frac{1}{p'}}\\ &\le \nu_n^{\lambda}|x|^{n\lambda}\|f\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

Let $A = \nu_n^{\lambda}\|f\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)}$. Since $\lambda < 0$, we have

$$\begin{aligned}\|Hf\|_{W\dot{B}^{p,\lambda}(\mathbb{R}^n)} &\le \sup_{R>0}\sup_{t>0} t\big|B(0,R)\big|^{-\lambda-\frac{1}{p}}\big|\big\{x \in B(0,R) : A|x|^{n\lambda} > t\big\}\big|^{\frac{1}{p}}\\ &= \sup_{R>0}\sup_{t>0} t\big|B(0,R)\big|^{-\lambda-\frac{1}{p}}\Big|\Big\{|x| < R : |x| < \Big(\frac{t}{A}\Big)^{\frac{1}{n\lambda}}\Big\}\Big|^{\frac{1}{p}}.\end{aligned}$$

If $0 < R \le (\frac{t}{A})^{\frac{1}{n\lambda}}$, then, for $\lambda < 0$, we get

$$\begin{aligned}&\sup_{t>0}\sup_{0<R\le(\frac{t}{A})^{1/n\lambda}} t\big|B(0,R)\big|^{-\lambda-\frac{1}{p}}\Big|\Big\{|x| < R : |x| < \Big(\frac{t}{A}\Big)^{\frac{1}{n\lambda}}\Big\}\Big|^{\frac{1}{p}}\\ &\quad\le \nu_n^{-\lambda}\sup_{t>0}\sup_{0<R\le(\frac{t}{A})^{1/n\lambda}} tR^{-n\lambda}\\ &\quad= \|f\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

If $R > (\frac{t}{A})^{\frac{1}{n\lambda}}$, then, for $\lambda \geq -\frac{1}{p}$, we have

$$\begin{aligned}
&\sup_{t>0} \sup_{R>(\frac{t}{A})^{1/n\lambda}} t\big|B(0,R)\big|^{-\lambda-\frac{1}{p}}\Big|\Big\{|x|<R:|x|<\Big(\frac{t}{A}\Big)^{\frac{1}{n\lambda}}\Big\}\Big|^{\frac{1}{p}}\\
&\quad\leq \nu_n^{-\lambda}\sup_{t>0}\sup_{R>(\frac{t}{A})^{1/n\lambda}} tR^{-n(\lambda+\frac{1}{p})}\Big(\frac{t}{A}\Big)^{\frac{1}{p\lambda}}\\
&\quad= \|f\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, we obtain

$$\|Hf\|_{W\dot{B}^{p,\lambda}(\mathbb{R}^n)} \leq \|f\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)}.$$

It remains to prove that the constant 1 is optimal. Choosing $f_0(x) = \chi_{[0,1]}(|x|)$, then, for $-\frac{1}{p} \leq \lambda < 0$, we obtain $f_0 \in \dot{B}^{p,\lambda}(\mathbb{R}^n)$ and $\|f_0\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)} = \nu_n^{-\lambda}$. In fact, when $0 < R \leq 1$,

$$\int_{B(0,R)} \big|f_0(y)\big|^p\,dy = \nu_n R^n.$$

When $R > 1$,

$$\int_{B(0,R)} \big|f_0(y)\big|^p\,dy = \nu_n.$$

Let $E = \int_{B(0,R)} |f_0(y)|^p\,dy$. Since $-\frac{1}{p} \leq \lambda < 0$, we have

$$\begin{aligned}
\|f_0\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)} &= \max\Big\{\sup_{0<R\leq 1}\Big(\frac{E}{|B(0,R)|^{1+\lambda p}}\Big)^{\frac{1}{p}}, \sup_{R>1}\Big(\frac{E}{|B(0,R)|^{1+\lambda p}}\Big)^{\frac{1}{p}}\Big\}\\
&= \max\big\{\sup_{0<R\leq 1}(\nu_n R^n)^{-\lambda}, \sup_{R>1}\nu_n^{-\lambda}R^{-n(\lambda+\frac{1}{p})}\big\}\\
&= \nu_n^{-\lambda}.
\end{aligned}$$

Moreover,

$$Hf_0(x) = \begin{cases} |x|^{-n}, & |x| > 1,\\ 1, & |x| \leq 1.\end{cases}$$

Obviously, $|Hf_0(x)| \leq 1$. On the other hand, when $0 < R \leq 1$, we obtain

$$\|Hf_0\|_{WL^p(B(0,R))} = \sup_{0<t\leq 1} t\big|\big\{x \in B(0,R) : 1 \geq t\big\}\big|^{\frac{1}{p}} = (\nu_n R^n)^{\frac{1}{p}}.$$

When $R > 1$, we have

$$\|Hf_0\|_{WL^p(B(0,R))} = \sup_{0<t\leq 1} t\big|\big\{x \in B(0,1) : 1 \geq t\big\} \cup \big\{1 \leq |x| < R : |x|^{-n} > t\big\}\big|^{\frac{1}{p}}.$$

If $1 < R \leq t^{-\frac{1}{n}}$, then

$$\|Hf_0\|_{WL^p(B(0,R))} = \sup_{0<t\leq 1} t(\nu_n + \nu_n R^n - \nu_n)^{\frac{1}{p}} = \sup_{0<t\leq 1}(\nu_n R^n)^{\frac{1}{p}}.$$

If $1 < t^{-\frac{1}{n}} < R$, then

$$\|Hf_0\|_{WL^p(B(0,R))} = \sup_{0<t\leq 1} t(\nu_n + \nu_n t^{-1} - \nu_n)^{\frac{1}{p}} = \sup_{0<t\leq 1} \nu_n^{1/p} t^{1-\frac{1}{p}}.$$

It follows from $1 \le p < \infty$ and $-\frac{1}{p} \le \lambda < 0$ that

$$\begin{aligned}\|Hf_0\|_{W\dot{B}^{p,\lambda}(\mathbb{R}^n)} &= \nu_n^{-\lambda} \max\Big\{\sup_{0<t\le 1} t \sup_{1<R\le(\frac{1}{t})^{\frac{1}{n}}} R^{-n\lambda}, \sup_{0<t\le 1} t^{1-\frac{1}{p}} \sup_{1<(\frac{1}{t})^{\frac{1}{n}}<R} R^{-n(\lambda+\frac{1}{p})}\Big\} \\ &= \nu_n^{-\lambda} \max\Big\{\sup_{0<t\le 1} t^{1-\lambda}, \sup_{0<t\le 1} t^{1-\lambda}\Big\} \\ &= \nu_n^{-\lambda} = \|f_0\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

Therefore,

$$\|H\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)\to W\dot{B}^{p,\lambda}(\mathbb{R}^n)} = 1. \qquad \square$$

Remark 2.2. If $\lambda = -\frac{1}{p}$, then $\dot{B}^{p,\lambda}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $W\dot{B}^{p,\lambda}(\mathbb{R}^n) = L^{p,\infty}(\mathbb{R}^n)$. Therefore, we extend the result in [17].

Theorem 2.3. *Let $0 \le \beta < n$, let $1 < q < \infty$, let $\min\{\alpha,\gamma\} > -n$, and let $\frac{\gamma+n}{q} = \alpha + n - \beta$. If $f \in L^1(|x|^\alpha)$, then*

$$\|H_\beta^* f\|_{L^{q,\infty}(|x|^\gamma)} \le \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{q}} \nu_n^{\frac{1}{n}(\alpha-\frac{\gamma}{q})} \|f\|_{L^1(|x|^\alpha)},$$

and the constant $(\frac{n}{n+\gamma})^{\frac{1}{q}} \nu_n^{\frac{1}{n}(\alpha-\frac{\gamma}{q})}$ is optimal.

Proof. Because of $\frac{\gamma+n}{q} = \alpha + n - \beta$, we get

$$H_\beta^* f(x) = \int_{B(0,|x|)^c} \frac{f(y)|y|^\alpha |y|^{-\alpha}}{|B(0,|y|)|^{1-\frac{\beta}{n}}}\, dy = \nu_n^{\frac{\beta}{n}-1} \int_{B(0,|x|)^c} |y|^{-\frac{n+\gamma}{q}} f(y)|y|^\alpha\, dy.$$

When $\gamma > -n$ and $y \in B(0,|x|)^c$, we have $|y|^{-\frac{n+\gamma}{q}} \le |x|^{-\frac{n+\gamma}{q}}$. Therefore,

$$\big|H_\beta^* f(x)\big| \le \nu_n^{\frac{\beta}{n}-1} \|f\|_{L^1(|x|^\alpha)} |x|^{-\frac{n+\gamma}{q}}.$$

Let $C_f = \nu_n^{\frac{\beta}{n}-1} \|f\|_{L^1(|x|^\alpha)}$. Then, for any $\lambda > 0$,

$$\big\{x \in \mathbb{R}^n : \big|H_\beta^* f(x)\big| > \lambda\big\} \subset \Big\{x \in \mathbb{R}^n : |x| < \Big(\frac{C_f}{\lambda}\Big)^{\frac{q}{n+\gamma}}\Big\}.$$

Thus,

$$\begin{aligned}\|H_\beta^* f\|_{L^{q,\infty}(|x|^\gamma)} &\le \sup_{\lambda>0} \lambda \Big(\int_{\mathbb{R}^n} |x|^\gamma \chi_{\{x\in\mathbb{R}^n: |x|<(\frac{C_f}{\lambda})^{\frac{q}{n+\gamma}}\}}(x)\, dx\Big)^{\frac{1}{q}} \\ &= \sup_{\lambda>0} \lambda \Big(\int_{S^{n-1}} d\sigma \int_0^{(\frac{C_f}{\lambda})^{\frac{q}{n+\gamma}}} r^{n+\gamma-1}\, dr\Big)^{\frac{1}{q}}.\end{aligned}$$

Noting that $\gamma > -n$ and $\frac{\gamma+n}{q} = \alpha + n - \beta$, we obtain

$$\|H_\beta^* f\|_{L^{q,\infty}(|x|^\gamma)} \le \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{q}} \nu_n^{1/n(\alpha-\frac{\gamma}{q})} \|f\|_{L^1(|x|^\alpha)}.$$

Now we prove the constant $(\frac{n}{n+\gamma})^{\frac{1}{q}}\nu_n^{1/n(\alpha-\frac{\gamma}{q})}$ is optimal. For $0\le\beta<n$, we choose

$$f_\epsilon(x)=\begin{cases}|x|^{-(\beta+n+\alpha)/\epsilon}, & |x|\ge 1,\\ 0, & |x|<1,\end{cases}$$

where $0<\epsilon<1$. Then $f_\epsilon\in L^1(|x|^\alpha)$ and

$$\|f_\epsilon\|_{L^1(|x|^\alpha)}=\int_{S^{n-1}}d\sigma\int_1^\infty r^{n+\alpha-(\beta+n+\alpha)/\epsilon-1}\,dr.$$

Since $\alpha>-n$ and $0<\epsilon<1$, we reach

$$\|f_\epsilon\|_{L^1(|x|^\alpha)}=\frac{n\nu_n}{(\beta+n+\alpha)/\epsilon-n-\alpha}.$$

On the other hand,

$$H^*_\beta f_\epsilon(x)=\nu_n^{(\beta/n)-1}\int_{|y|\ge|x|}|y|^{-(\beta+n+\alpha)/\epsilon-n+\beta}\chi_{\{|y|\ge 1\}}(y)\,dy.$$

When $|x|\ge 1$, noting that $\alpha>-n$ and $0<\epsilon<1$, we have

$$\begin{aligned}H^*_\beta f_\epsilon(x)&=\nu_n^{(\beta/n)-1}\int_{|y|\ge|x|}|y|^{-(\beta+n+\alpha)/\epsilon-n+\beta}\,dy\\&=\frac{n\nu_n^{\beta/n}|x|^{\beta-(\beta+n+\alpha)/\epsilon}}{(\beta+n+\alpha)/\epsilon-\beta}\end{aligned}$$

by the spherical coordinates. Note also that $0<\epsilon<1$, $0\le\beta<n$, and $|x|\ge 1$. Therefore,

$$H^*_\beta f_\epsilon(x)\le\frac{n\nu_n^{\beta/n}}{(\beta+n+\alpha)/\epsilon-\beta}.$$

If $|x|<1$, again by the spherical coordinates, we similarly have

$$H^*_\beta f_\epsilon(x)=\frac{n\nu_n^{\beta/n}}{(\beta+n+\alpha)/\epsilon-\beta}.$$

Therefore, we have obtained

$$\begin{aligned}&\left\{x\in\mathbb{R}^n:\left|H^*_\beta f_\epsilon(x)\right|>\lambda\right\}\\&\quad=\left\{|x|<1:\frac{n\nu_n^{\beta/n}}{(\beta+n+\alpha)/\epsilon-\beta}>\lambda\right\}\cup\left\{|x|>1:\frac{n\nu_n^{\beta/n}|x|^{\beta-(\beta+n+\alpha)/\epsilon}}{(\beta+n+\alpha)/\epsilon-\beta}>\lambda\right\}.\end{aligned}$$

Obviously, if $\lambda>\frac{n\nu_n^{\beta/n}}{(\beta+n+\alpha)/\epsilon-\beta}$, then $\{x\in\mathbb{R}^n:|H^*_\beta f_\epsilon(x)|>\lambda\}=\emptyset$. Let $C_\epsilon=\frac{n\nu_n^{\beta/n}}{(\beta+n+\alpha)/\epsilon-\beta}$. For $0<\epsilon<1$ and $0\le\beta<n$,

$$\begin{aligned}\|H^*_\beta f_\epsilon\|_{L^{q,\infty}(|x|^\gamma)}&=\sup_{0<\lambda<C_\epsilon}\lambda\Big(\int_{\{x\in\mathbb{R}^n:|H^*_\beta f_\epsilon(x)|>\lambda\}}|x|^\gamma\,dx\Big)^{\frac{1}{q}}\\&=\sup_{0<\lambda<C_\epsilon}\lambda\Big(\int_{\mathbb{R}^n}|x|^\gamma\chi_{\{|x|^{(\beta+n+\alpha)/\epsilon-\beta}<\frac{C_\epsilon}{\lambda}\}}(x)\,dx\Big)^{\frac{1}{q}}.\end{aligned}$$

Noting that $\gamma > -n$, we get

$$\|H^*_\beta f_\epsilon\|_{L^{q,\infty}(|x|^\gamma)} = C_\epsilon^{\frac{n+\alpha-\beta}{(\beta+n+\alpha)/\epsilon-\beta}} \Big(\frac{n\nu_n}{n+\gamma}\Big)^{\frac{1}{q}} \sup_{0<\lambda<C_\epsilon} \lambda^{1-\frac{n+\alpha-\beta}{(\beta+n+\alpha)/\epsilon-\beta}}.$$

Since $\alpha > -n$ and $0 < \epsilon < 1$, we have

$$\begin{aligned}\|H^*_\beta f_\epsilon\|_{L^{q,\infty}(|x|^\gamma)} &= \Big(\frac{n\nu_n}{n+\gamma}\Big)^{\frac{1}{q}} C_\epsilon \\ &= \nu_n^{1/n(\alpha-\frac{\gamma}{q})} \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{q}} \frac{(\beta+n+\alpha)/\epsilon-n-\alpha}{(\beta+n+\alpha)/\epsilon-\beta} \|f_\epsilon\|_{L^1(|x|^\alpha)}.\end{aligned}$$

Letting $\epsilon \to 0^+$, we have

$$\lim_{\epsilon\to 0^+} \frac{(\beta+n+\alpha)/\epsilon-n-\alpha}{(\beta+n+\alpha)/\epsilon-\beta} = 1.$$

In summary, we conclude that

$$\|H^*_\beta\|_{L^1(|x|^\alpha)\to L^{q,\infty}(|x|^\gamma)} = \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{q}} \nu_n^{1/n(\alpha-\frac{\gamma}{q})}.$$

□

Theorem 2.4. *Let $0 \le \beta < n$, let $1 < p < \frac{n+\alpha}{\beta}$, and let $\frac{n+\gamma}{q} = \frac{n+\alpha}{p} - \beta$. If $f \in L^p(|x|^\alpha)$, then*

$$\|H^*_\beta\|_{L^{q,\infty}(|x|^\gamma)} \le \Big(\frac{q}{p'}\Big)^{\frac{1}{p'}} \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{p'}+\frac{1}{q}} \nu_n^{1/n(\frac{\alpha}{p}-\frac{\gamma}{q})} \|f\|_{L^p(|x|^\alpha)},$$

and the constant $(\frac{q}{p'})^{\frac{1}{p'}} (\frac{n}{n+\gamma})^{\frac{1}{p'}+\frac{1}{q}} \nu_n^{1/n(\frac{\alpha}{p}-\frac{\gamma}{q})}$ is optimal.

Proof. First, using the Hölder inequality and $1 < p < \frac{n+\alpha}{\beta}$, we have

$$\begin{aligned}\big|H^*_\beta f(x)\big| &\le \Big(\int_{B(0,|x|)^c} \big|f(y)\big|^p |y|^\alpha\,dy\Big)^{\frac{1}{p}} \Big(\int_{B(0,|x|)^c} \big|B\big(0,|y|\big)\big|^{p'(\frac{\beta}{n}-1)} |y|^{-\frac{\alpha p'}{p}}\,dy\Big)^{\frac{1}{p'}} \\ &\le \Big(\frac{np-n}{n+\alpha-p\beta}\Big)^{\frac{1}{p'}} \nu_n^{\beta/n-\frac{1}{p}} |x|^{\frac{p\beta-n-\alpha}{p}} \|f\|_{L^p(|x|^\alpha)}.\end{aligned}$$

Noting that $\frac{\gamma+n}{q} = \frac{\alpha+n}{p} - \beta$, we reach

$$\big|H^*_\beta f(x)\big| \le \Big[\frac{qn}{p'(n+\gamma)}\Big]^{\frac{1}{p'}} \nu_n^{\beta/n-\frac{1}{p}} |x|^{-\frac{n+\gamma}{q}} \|f\|_{L^p(|x|^\alpha)}.$$

Let $C_f = [\frac{qn}{p'(n+\gamma)}]^{\frac{1}{p'}} \nu_n^{\beta/n-\frac{1}{p}} \|f\|_{L^p(|x|^\alpha)}$. Then, for any $\lambda > 0$,

$$\big\{x\in\mathbb{R}^n : \big|H^*_\beta f(x)\big| > \lambda\big\} \subset \Big\{x\in\mathbb{R}^n : |x| < \Big(\frac{C_f}{\lambda}\Big)^{\frac{q}{n+\gamma}}\Big\}.$$

Consequently,

$$\begin{aligned}\|H^*_\beta f\|_{L^{q,\infty}(|x|^\gamma)} &\le \sup_{\lambda>0} \lambda \Big(\int_{\mathbb{R}^n} |x|^\gamma \chi_{\{x\in\mathbb{R}^n:|x|<(\frac{C_f}{\lambda})^{\frac{q}{n+\gamma}}\}}(x)\,dx\Big)^{\frac{1}{q}} \\ &= \sup_{\lambda>0} \lambda \Big(\int_{S^{n-1}} d\sigma \int_0^{(\frac{C_f}{\lambda})^{q/n+\gamma}} r^{n+\gamma-1}\,dr\Big)^{\frac{1}{q}}.\end{aligned}$$

Noting that $p < \frac{n+\alpha}{\beta}$ and $\frac{\gamma+n}{q} = \frac{\alpha+n}{p} - \beta$, we obtain

$$\|H_\beta^* f\|_{L^{q,\infty}(|x|^\gamma)} \leq \Big(\frac{q}{p'}\Big)^{\frac{1}{p'}} \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{p'}+\frac{1}{q}} \nu_n^{1/n(\frac{\alpha}{p}-\frac{\gamma}{q})} \|f\|_{L^p(|x|^\alpha)}.$$

Below we will prove that the constant $(\frac{q}{p'})^{\frac{1}{p'}}(\frac{n}{n+\gamma})^{\frac{1}{p'}+\frac{1}{q}}\nu_n^{1/n(\frac{\alpha}{p}-\frac{\gamma}{q})}$ is optimal. For $0 \leq \beta < n$, we choose a function $f_0(x) = |x|^{\frac{\beta-n-\alpha}{p-1}} \chi_{\{|x|>1\}}(x)$. Because $1 < p < \frac{n+\alpha}{\beta}$, we have $f_0 \in L^p(|x|^\alpha)$ and

$$\begin{aligned}\|f_0\|_{L^p(|x|^\alpha)}^p &= \int_1^\infty \int_{S^{n-1}} r^{\frac{p(\beta-n-\alpha)}{p-1}+\alpha+n-1}\, d\sigma\, dr \\ &= \frac{p-1}{n+\alpha-p\beta}|S^{n-1}|.\end{aligned}$$

Note that $\frac{n+\gamma}{q} = \frac{n+\alpha}{p} - \beta$; hence,

$$\|f_0\|_{L^p(|x|^\alpha)}^p = \frac{q}{p'} \cdot \frac{n}{n+\gamma}\nu_n.$$

On the other hand, $H_\beta^* f_0(x) = \nu_n^{\beta/n-1} \int_{|y|\geq|x|} |y|^{\frac{p(\beta-n)-\alpha}{p-1}} \chi_{\{|y|>1\}}(y)\, dy$. When $|x| > 1$, we have

$$\begin{aligned}H_\beta^* f_0(x) &= \nu_n^{\beta/n-1} \int_{|y|\geq|x|} |y|^{\frac{p(\beta-n)-\alpha}{p-1}}\, dy \\ &= \nu_n^{\beta/n-1}|S^{n-1}|\frac{p-1}{n+\alpha-p\beta}|x|^{\frac{p\beta-n-\alpha}{p-1}} \\ &= \frac{q}{p'} \cdot \frac{n}{n+\gamma}\nu_n^{\beta/n}|x|^{-\frac{p'(n+\gamma)}{q}}.\end{aligned}$$

When $|x| \leq 1$, we have

$$H_\beta^* f_0(x) = \nu_n^{\beta/n-1} \int_{|y|\geq 1} |y|^{\frac{p(\beta-n)-\alpha}{p-1}}\, dy = \frac{q}{p'} \cdot \frac{n}{n+\gamma}\nu_n^{\beta/n}.$$

Therefore, we reach

$$\begin{aligned}&\big\{x \in R^n : \big|H_\beta^* f_0(x)\big| > \lambda\big\} \\ &\quad = \Big\{|x| \leq 1 : \frac{q}{p'} \cdot \frac{n}{n+\gamma}\nu_n^{\beta/n} > \lambda\Big\} \cup \Big\{|x| > 1 : \frac{q}{p'} \cdot \frac{n}{n+\gamma}\nu_n^{\beta/n}|x|^{-\frac{p'(n+\gamma)}{q}} > \lambda\Big\}.\end{aligned}$$

In fact, we see that

$$E := \big\{x \in R^n : \big|H_\beta^* f_0(x)\big| > \lambda\big\} = \emptyset$$

if $\lambda > \frac{q}{p'} \cdot \frac{n}{n+\gamma}\nu_n^{\beta/n}$. Let $A = \frac{q}{p'} \cdot \frac{n}{n+\gamma}\nu_n^{\beta/n}$. Then, when $0 < \lambda < A$, we obtain

$$\int_E |x|^\gamma\, dx = \frac{|S^{n-1}|}{n+\gamma} \cdot \Big(\frac{A}{\lambda}\Big)^{\frac{q}{p'}}.$$

Therefore, we have

$$\begin{aligned}\|H_\beta^* f_0\|_{L^{q,\infty}(|x|^\gamma)} &= \sup_{0<\lambda<A} \lambda\Big(\int_E |x|^\gamma\,dx\Big)^{\frac{1}{q}} \\ &= \sup_{0<\lambda<A} \lambda\Big(\frac{|S^{n-1}|}{n+\gamma}\Big)^{\frac{1}{q}} \cdot \Big(\frac{A}{\lambda}\Big)^{\frac{1}{p'}} \\ &= A\nu_n^{1/q}\Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{q}} = \frac{q}{p'} \cdot \Big(\frac{n}{n+\gamma}\Big)^{1+\frac{1}{q}} \nu_n^{\beta/n+\frac{1}{q}}.\end{aligned}$$

Note that $\|f_0\|_{L^p(|x|^\alpha)}^p = \frac{q}{p'} \cdot \frac{n}{n+\gamma}\nu_n$. Thus,

$$\begin{aligned}\|H_\beta^* f_0\|_{L^{q,\infty}(|x|^\gamma)} &= \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{p'}+\frac{1}{q}} \Big(\frac{q}{p'}\Big)^{\frac{1}{p'}} \nu_n^{\beta/n+\frac{1}{q}-\frac{1}{p}} \|f_0\|_{L^p(|x|^\alpha)} \\ &= \Big(\frac{q}{p'}\Big)^{\frac{1}{p'}} \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{p'}+\frac{1}{q}} \nu_n^{1/n(\frac{\alpha}{p}-\frac{\gamma}{q})} \|f_0\|_{L^p(|x|^\alpha)}.\end{aligned}$$

Therefore, the constant $(\frac{q}{p'})^{\frac{1}{p'}}(\frac{n}{n+\gamma})^{\frac{1}{p'}+\frac{1}{q}}\nu_n^{1/n(\frac{\alpha}{p}-\frac{\gamma}{q})}$ is optimal. □

Remark 2.5. When $p \to 1^+$, we have

$$\Big(\frac{q}{p'}\Big)^{\frac{1}{p'}} \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{p'}+\frac{1}{q}} \nu_n^{1/n(\frac{\alpha}{p}-\frac{\gamma}{q})} \to \Big(\frac{n}{n+\gamma}\Big)^{\frac{1}{q}} \nu_n^{1/n(\alpha-\frac{\gamma}{q})}.$$

Therefore,

$$\lim_{p\to 1^+} \|H_\beta^*\|_{L^p(|x|^\alpha)\to L^{q,\infty}(|x|^\gamma)} = \|H_\beta^*\|_{L^1(|x|^\alpha)\to L^{q,\infty}(|x|^\gamma)}.$$

Remark 2.6. Choosing $\alpha = \gamma = 0$ in Theorem 2.3 and Theorem 2.4, we have the same results in [7].

Remark 2.7. Our theorems apply for the 1-dimensional Hardy operator h and h^*.

3. Application

By duality and the boundedness of H_β from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, we have

$$\|H_\beta^*\|_{L^{q'}(\mathbb{R}^n)\to L^{p'}(\mathbb{R}^n)} = \|H_\beta\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}$$

(see [16] or [18]). Therefore,

$$\begin{aligned}\|H_\beta^*\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} &= \|H_\beta\|_{L^{q'}(\mathbb{R}^n)\to L^{p'}(\mathbb{R}^n)} \\ &= \Big(\frac{q}{p'}\Big)^{\frac{1}{p'}} \Big[\frac{n}{p'\beta} B\Big(\frac{n}{p'\beta}, \frac{n}{p\beta}\Big)\Big]^{-\frac{\beta}{n}} \\ &= \Big(\frac{q}{p'}\Big)^{\frac{1}{p'}} \left[\frac{\frac{q-p}{q(p-1)}\Gamma\big(\frac{pq}{q-p}\big)}{\Gamma\big(\frac{q}{q-p}\big)\Gamma\big(\frac{q(p-1)}{q-p}\big)}\right]^{\frac{1}{p}-\frac{1}{q}}\end{aligned}$$

for $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ (please see [16] for more details).

On the other hand, we have

$$\|H_\beta^*\|_{L^p(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} = \Big(\frac{q}{p'}\Big)^{\frac{1}{p'}}.$$

Since $L^q(\mathbb{R}^n) \subseteq L^{q,\infty}(\mathbb{R}^n)$ for $1 \le q < \infty$, we have

$$\|H_\beta^*\|_{L^p(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} \le \|H_\beta^*\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)};$$

that is,

$$\Gamma\Big(\frac{q}{q-p}\Big)\Gamma\Big(\frac{q(p-1)}{q-p}\Big) \le \frac{q-p}{q(p-1)}\Gamma\Big(\frac{pq}{q-p}\Big),$$

and so we have the following.

Proposition 3.1. *Let $1 < p < q < \infty$. Then*

$$\Gamma\Big(\frac{q}{q-p}\Big)\Gamma\Big(\frac{q(p-1)}{q-p}\Big) \le \frac{q-p}{q(p-1)}\Gamma\Big(\frac{pq}{q-p}\Big).$$

Remark 3.2. Combined with Persson and Samko's results in [16, Theorem 4.2] and Theorem 2.4, we can obtain another inequality for the gamma function by a similar argument.

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